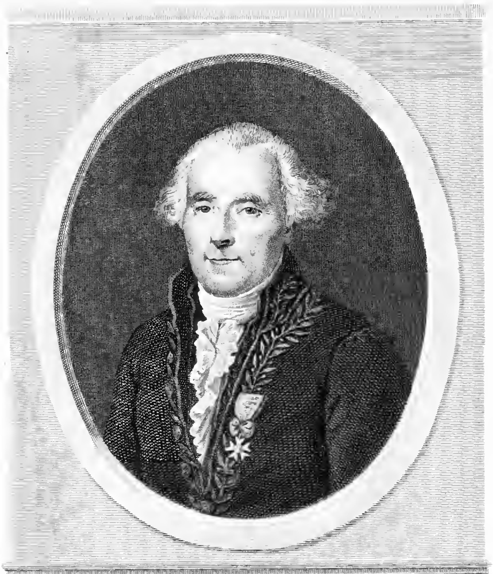
MÉCANIQUE CÉLESTE.



P. S. LAPLACE.

from the tra

MÉCANIQUE CÉLESTE.

BY THE

MARQUIS DE LA PLACE,

PEER OF FRANCE; GRAND CROSS OF THE LEGION OF HONOR; MEMBER OF THE FRENCH ACADEMY, OF THE ACADEMY OF SCIENCES OF PARIS, OF THE BOARD OF LONGITUDE OF FRANCE, OF THE ROYAL SOCIETIES OF LONDON AND GÖTTINGEN, OF THE ACADEMIES OF SCIENCES OF RUSSIA, DENMARK, SWEDEN, PRUSSIA, HOLLAND, AND ITALY; MEMBER OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES; ETC.

TRANSLATED, WITH A COMMENTARY,

BY

NATHANIEL BOWDITCH, LL. D.

FELLOW OF THE ROYAL SOCIETIES OF LONDON, EDINBURGH, AND DUBLIN; OF THE ASTRONOMICAL SOCIETY OF LONDON; OF THE PHILOSOPHICAL SOCIETY HELD AT PHILADELPHIA; OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES; ETC.

VOLUME III.

BOSTON:

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M DCCC XXXIV.

TO

BONAPARTE,

MEMBER OF THE NATIONAL INSTITUTE.

Citizen First Consul,

You have permitted me to dedicate this work to you. It is gratifying and honorable to me to present it to the Hero, the Pacificator of Europe,* to whom France owes her prosperity, her greatness, and the most brilliant epoch of her glory; to the enlightened Protector of the Sciences, who, himself distinguished in them, perceives, in their cultivation, the source of the most noble enjoyment, and, in their progress, the perfection of all useful arts and social institutions. May this work, consecrated to the most sublime of the natural sciences, be a durable monument of the gratitude inspired in those who cultivate them, by your kindness, and by the rewards of the government. Of all the truths which this work contains, the expression of this sentiment will ever be the most precious to me.

Salutation and Respect,

LA PLACE.

[* This volume was published, by La Place, in 1802, soon after the peace of Amiens.]

ADVERTISEMENT.

This volume contains the numerical values of the secular and periodical inequalities of the motions of the planets and moon; the numbers, given in the original work, having been reduced from centesimal to sexagesimal seconds, to render them more convenient for reference. The Appendix contains many important formulas and tables, which are useful to astronomers in computing the motions of the planets and comets. Some of these tables are new, and the others have been varied in their forms, to render them more simple in their uses and applications: none of them have heretofore been published in this country. Several of the formulas have been introduced into the calculations of modern astronomy, since the commencement of the first part of the original work. The portrait of the author, accompanying this volume, was obtained in France, and is an impression from the original plate, which was engraved under his direction, for the *Système du Monde*. The fourth volume of the work will be put to press in the course of a few weeks.

PREFACE.

We have given, in the first part of this work, the general principles of the equilibrium and motion of bodies. The application of these principles to the motions of the heavenly bodies, has conducted us, by geometrical reasoning, without any hypothesis, to the law of universal attraction; the action of gravity, and the motions of projectiles on the surface of the earth, being particular cases of this law. We have then taken into consideration, a system of bodies subjected to this great law of nature; and have obtained, by a singular analysis, the general expressions of their motions, of their figures, and of the oscillations of the fluids which cover them. From these expressions, we have deduced all the known phenomena of the flow and ebb of the tide; the variations of the degrees, and of the force of gravity at the surface of the earth; the precession of the equinoxes; the libration of the moon; and the figure and rotation of Saturn's Rings. We have also pointed out the cause, why these rings remain, permanently, in the plane of the equator of Saturn. Moreover, we have deduced, from the same theory of gravity, the principal equations of the motions of the planets; particularly those of Jupiter and Saturn, whose great inequalities have a period of above nine hundred years. The inequalities in the motions of Jupiter and Saturn, presented, at first, to astronomers, nothing but anomalies, whose laws and causes were unknown; and, for a long time, these irregularities appeared to be inconsistent with the theory of gravity; but a more thorough examination has shown, that they can be deduced from it; and now, these motions are

one of the most striking proofs of the truth of this theory. We have developed the secular variations of the elements of the planetary system, which do not return to the same state till after the lapse of many centuries. In the midst of all these changes we have discovered the constancy of the mean motions, and of the mean distances of the bodies of this system; which nature seems to have arranged, at its origin, for an eternal duration, upon the same principles as those which prevail, so admirably, upon the earth, for the preservation of individuals, and for the perpetuity of the species. From the single circumstance, that the motions are all in the same direction, and in planes but little inclined to each other, it follows, that the orbits of the planets and satellites must always be nearly circular, and but little inclined to each other. Thus, the variations of the obliquity of the ecliptic, which are always included within narrow limits, will never produce an eternal spring upon the earth. We have proved that the attraction of the terrestrial spheroid, by incessantly drawing towards its centre the hemisphere of the moon, which is directed towards the earth, transfers to the rotatory motion of this satellite, the great secular variations of its motion of revolution; and, by this means, keeps always from our view, the other hemisphere. Lastly, we have demonstrated, in the motions of the three first satellites of Jupiter, the following remarkable law, namely, that, in consequence of their mutual attractions, *the mean longitude of the first satellite, seen from the centre of Jupiter, minus three times that of the second satellite, plus twice that of the third satellite, is always exactly equal to two right angles;* so that they cannot all be eclipsed at the same time. It remains now to consider particularly the perturbations of the motions of the planets and comets about the sun; of the moon about the earth; and of the satellites about their primary planets. This is the object of the second part of this work, which is particularly devoted to the improvement of astronomical tables.

The tables have followed the progress of the science, which serves as their basis ; and this progress was, at first, extremely slow. During a very long time, the apparent motions only of the planets were observed. This interval, which commenced in the most remote antiquity, may be considered as the infancy of Astronomy. It comprises the labors of Hipparchus and Ptolemy ; also, those of the Indians, the Arabs, and the Persians. The system of Ptolemy, which they successively adopted, is, in fact, nothing more than a method of representing the apparent motions ; and, on this account, it was useful to science. Such is the weakness of the human mind, that it often requires the aid of a theory, to connect together a series of observations. If we restrict the theory to this use, and take care not to attribute to it a reality which it does not possess, and afterwards frequently rectify it, by new observations, we may finally discover the true cause, or, at least, the laws of the phenomena. The history of Philosophy affords us more than one example, of the advantages which may be derived from an assumed theory ; and, of the errors to which we are exposed, in considering it to be the true representation of nature. About the middle of the sixteenth century, Copernicus discovered, that the apparent motions of the heavenly bodies indicated a real motion of the earth about the sun, with a rotatory motion about its own axis : by this means, he showed to us the universe in a new point of view, and completely changed the face of Astronomy. A remarkable concurrence of discoveries will forever render memorable, in the history of science, the century immediately following this discovery ; a period which is also illustrious, by many master-pieces of literature and the fine arts. Kepler discovered the laws of the elliptical motion of the planets ; the telescope, which was invented by the most fortunate accident, and was immediately improved by Galileo, enabled him to see, in the heavens, new inequalities and new worlds. The application of the pendulum to clocks, by Huygens, and that

of telescopes to the astronomical quadrant, gave more accurate measures of angles and times, and thus rendered sensible the least inequalities in the celestial motions. At the same time that observations presented to the human mind new phenomena, it created, to explain them, and to submit them to calculation, new instruments of thought. Napier invented logarithms: the analysis of curves, and the science of dynamics, were formed by the hands of Descartes and Galileo: Newton discovered the differential calculus, decomposed a ray of light, and penetrated into the general principle of gravity. In the century which has just passed, the successors of this great man have finished the superstructure, of which he laid the foundation. They have improved the analysis of infinitely small quantities, and have invented the calculus of partial differences, both infinitely small and finite: and have reduced the whole science of mechanics to formulas. In applying these discoveries to the law of gravity, they have deduced from it all the celestial phenomena; and have given to the theories and to astronomical tables an unexpected degree of accuracy; which is to be attributed, in a great measure, to the labors of French mathematicians, and to the prizes proposed by the Academy of Sciences. To these discoveries in the last century, we must add those of Bradley, on the aberration of the stars, and on the nutation of the earth's axis: the numerous measures of the degrees of the meridian, and of the lengths of the pendulum; of which operations, the first example was given by France, in sending academicians to the north, to the equator, and to the southern hemisphere, to observe the lengths of these degrees, and the intensity of gravity: the measure of the arc of the meridian, comprised between Dunkirk and Barcelona; which has been determined by very accurate observation, and is used as the basis of the most simple and natural system of measures: the numerous voyages of discovery, undertaken to explore the different parts of the globe, and to observe the transits of

Venus over the sun's disc; by which means, the exact determination of the dimensions of the solar system has been obtained, as the fruit of these voyages: the discoveries, by Herschel, of the planet Uranus, its satellites, and two new satellites of Saturn: finally, if we add to all these discoveries, the admirable invention of the instrument of reflexion, so useful at sea; that of the achromatic telescope; also the repeating circle, and chronometer; we must be satisfied, that the last century, considered with respect to the progress of the human mind, is worthy of that which preceded it. The century we have now entered upon, commenced under the most favorable auspices for Astronomy. Its first day was remarkable, by the discovery of the planet *Ceres*; followed, almost immediately afterwards, by that of the planet *Pallas*, having nearly the same mean distance from the sun. The proximity of Jupiter to these two extremely small bodies; the greatness of the excentricities and of the inclinations of their mutually intersecting orbits, must produce, in their motions, considerable inequalities, which will throw new light on the theory of the celestial attractions, and must give rise to farther improvements in Astronomy.

It is chiefly in the application of *analysis* to the system of the world, that we perceive the power of this wonderful instrument; without which, it would have been impossible to have discovered a mechanism which is so complicated in its effects, while it is so simple in its cause. The mathematician now includes in his formulas, the whole of the planetary system, and its successive variations; he looks back, in imagination, to the several states, which the system has passed through, in the most remote ages; and foretells what time will hereafter make known to observers. He sees this sublime spectacle, whose period includes several millions of years, repeated in a few centuries, in the system of the satellites of

Jupiter, by means of the rapidity of their revolutions; which produce remarkable phenomena, similar to those which had been suspected, by astronomers, in the planetary motions; but had not been determined, because they were either too complex, or too slow, for an accurate determination of their laws. The theory of gravity, which, by so many applications, has become a means of discovery, as certain as by observation itself, has made known to him several new inequalities, in the motions of the heavenly bodies, and enabled him to predict the return of the comet of 1759, whose revolutions are rendered very unequal, by the attractions of Jupiter and Saturn. He has been enabled, by this means, to deduce, from observation, as from a rich mine, a great number of important and delicate elements, which, without the aid of analysis, would have been forever hidden from his view: such as the relative values of the masses of the sun, the planets and satellites, determined by the revolutions of these bodies, and by the development of their periodical and secular inequalities: the velocity of light, and the ellipticity of Jupiter; which are given, by the eclipses of its satellites, with greater accuracy, than by direct observation: the rotation and oblateness of Uranus and Saturn; deduced from the consideration, that the different bodies which revolve about those two planets, are in the same plane, respectively: the parallaxes of the sun and moon: and, also, the figure of the earth, deduced from some lunar inequalities: for, we shall see hereafter, that the moon, by its motion, discloses to modern astronomy, the small ellipticity of the terrestrial spheroid, whose roundness was made known to the first observers by the eclipses of that luminary. Lastly, by a fortunate combination of analysis with observation, that body, which seems to have been given to the earth, to enlighten it, during the night, becomes also the most sure guide of the navigator; who is protected by it from the dangers, to which he was for a long time exposed, by the errors of his *reckoning*.

The perfection of the theory, and of the lunar tables, to which he is indebted for this important object, and for that of determining, with accuracy, the position of the places he falls in with, is the fruit of the labors of mathematicians and astronomers, during the last fifty years: it unites all that can give value to a discovery; the importance and usefulness of the object, its various applications, and the merit of the difficulty which is overcome. It is thus, that the most abstract theories, diffused by numerous applications to nature and to the arts, have become inexhaustible sources of comfort and enjoyment, even to those who are wholly ignorant of the nature of these theories.

CONTENTS OF THE THIRD VOLUME.

PARTICULAR THEORIES OF THE MOTIONS OF THE HEAVENLY BODIES.

SIXTH BOOK.

THEORY OF THE PLANETARY MOTIONS.

SEVENTH BOOK.

THEORY OF THE MOON.

APPENDIX BY THE AUTHOR.

The chief object of this appendix is to demonstrate a theorem, discovered by Mr. Poisson, that the mean motions of the planets are invariable, when we notice only the terms depending on the first and second powers of the disturbing forces [5744, &c.] This is done by giving new forms to some of the differential expressions of the elements of the orbits, as is observed in [5743, &c]. Forms of these differentials, including all the terms depending upon the first power of the disturbing masses [5786—5791]. Expressions of the mean motion [5794]; of the periodical inequalities in the elements [5873—5879]; and of the secular inequalities of the elements [5882--5888].

Investigation of the mutual action of two planets upon each other, referring their inequalities to an intermediate invariable plane [5905, &c].

New method of computing the lunar inequalities, depending upon the oblateness of the earth [5937—5973].

On the two great inequalities of Jupiter and Saturn; correcting for the mistake in the signs of the functions $N^{(0)}$, $N^{(1)}$ &c. [5974—5981].

IN THE COMMENTARY.

Among the subjects treated of in the Notes, we may mention the following:

Correction to be made in the formula $m\int \mathrm{d}R + m'\int \mathrm{d}R' = 0$, [1202], in some of the terms of the order of the square of the disturbing masses [4004e, &c]. The necessity of this correction was first made known by Mr. Plana [4006u, &c.]. Results of the discussion upon this subject, by Messrs. Plana, Pontecoulant, Poisson and La Place [4005b'—4008z]. New formula by La Place, relative to some of these terms [4008x]. This formula has been called "the last gift of La Place to Astronomy," being the last work he ever published.

On the values of the constant quantities $f_{,}$, $f'_{,}$, g, &c.; introduced into the integral expressions of δr, δv, δs, by La Place [4058e, &c.]; which were objected to by Mr. Plana. The results of La Place's calculation proved to be correct by him, and by Mr. Poisson, in [4058e—4060k].

Corrected values of the masses of the planets, finally adopted by the author [4061d].

Elements of the newly discovered planets Vesta, Juno, Pallas and Ceres; corresponding to the 23d July, 1831, as given by Encke [4079i].

Elements of the orbits of the comets of Halley, Olbers, Encke and Biela [4079m].

Inequalities in the motions of Venus and the Earth, having a period of 239 years, and depending on terms of the fifth order of the excentricities and inclinations; discovered and computed by Professor Airy [4296 a — q, 4310 c — f].

Mr. Pontecoulant's table of the part of the great inequality of the motion of Jupiter, depending on the square of the disturbing force [4431f]. Similar table for the inequalities of the motion of Saturn [4489e].

Results of the calculations of Professor Hansen [4489 n — p].

The action of the fixed stars affects the accuracy of the equation $e^2.m.\sqrt{a}+e'^2.m'.\sqrt{a'}+\&c.=0$ [4685g].

Results of the calculations of several authors relative to the sun's parallax, by means of the parallactic inequality in the moon's longitude, and by the transits of Venus over the sun's disc [5589 a — m].

Inequality in the moon's longitude, whose period is about 179 years. It is found to be insensible [5611 a — q]; instead of being $15^s,39$ at its maximum, as the author supposes in [5665].

The planets and comets move in a resisting medium, according to the observations of Encke's comet [5667 a — c].

Notice of the papers published by La Grange and Poisson, relative to the invariableness of the mean motions of the planets, which is treated of in the appendix to this volume [5741 a — l].

It appears from the calculations of Nicolai, Encke and Airy, that the estimated value of the mass of Jupiter, adopted by La Place from Bouvard's calculations of its action on Saturn and Uranus, must be increased, to satisfy the observed perturbations of the planets Juno and Vesta; as well as those of Encke's comet, [5980 i — p].

APPENDIX BY THE TRANSLATOR.

The method usually employed in this calculation requires some modification, when M appears under the form of a fraction, in which the numerators and denominators are both very small [5994(257)]. These methods are explained in [5994(387--392)].

Mr. Lubbock's method of computing the orbit of a comet [5994(405—458)].

Method of computing the elements of the orbit of a heavenly body; there being given the two radii r, r', the intermediate angle $v'-v=2f$, and the time $t'-t$ of describing the angle $2f$ [5995].

Collection of formulas for solving this problem, in an elliptical orbit [5995(4—67)]; with their demonstrations [5995(68—174)]. Examples of the uses of these formulas [5995(175—193)].

Collection of formulas for solving this problem in a parabolic orbit [5996(2—25)]; with their demonstrations [5996(26—50)]; illustrated by an example in [5996(51—53)].

Collection of formulas for solving this problem in a hyperbolic orbit [5997(1—59)]; with their demonstrations [5997(60—172)]. Example of the uses of these formulas [5997(173—183)].

Gauss's method of correcting for the effect of the parallax and aberration of any newly discovered planet or comet, in computing its orbit by means of three geocentric observations, with the intervals of time between them [5998].

Corrections in the places of the earth, on account of the planet's parallax [5998(47—50)].

Method of calculating the longitude and latitude of the zenith [5998(67—71) &c.]; also the longitude and latitude of the planet from its right ascension and declination [5998(97—107)], with examples.

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Example for illustrating the calculations relative to the parallax and aberration [5998(118—126)].

Gauss's method of computing the orbit of a planet or comet, by means of three geocentric longitudes and latitudes, together with the times of observation [5999.]

Table of the symbols and formulas which are used in this method [5999(9—54)].

Demonstrations of these formulas [5999(58, &c.)].

Example, containing the whole calculation of the elements of the orbit of Juno, from three observations of Maskelyne [5999(274—650)].

CATALOGUE OF THE TABLES IN THE APPENDIX.

TABLE I. Contains the square roots of the numbers from 0,001 to 10,1; to be used in Olbers's method of computing the orbit of a comet; in finding r, r'', c; from r^2, r''^2, c^2; which are given by three fundamental equations of this method [5994(31, 32, 33)].

TABLE II. To find the time T of describing a parabolic arc, by a comet; there being given the sum of the radii $r+r''$, and the chord c, connecting the two extreme parts of the arc. This table is computed by Lambert's formula [750], namely,

$$T=9^{\text{days}},688724.\ \{(r+r''+c)^{\frac{3}{2}}-(r+r''-c)^{\frac{3}{2}}\};$$

and the numbers are given to the nearest unit in the third decimal place, expressed in days and parts of a day. This degree of accuracy being abundantly sufficient for the purpose of computing the orbit of a comet, by Dr Olbers's method; and the table serves to facilitate this part of the calculation.

The Tables V—X, include all those which Gauss published in his *Theoria Motus*, etc. We have altered, in some respects, the arrangement and forms of these tables, to render them more convenient for use; and upon comparison it will be found, that this appendix contains the most important of the methods which are given in that great work, as well as in that of Dr Olbers. The methods of Gauss being somewhat simplified, by reducing many of the processes to the common operations of spherical trigonometry, instead of using a great number of unusual auxiliary formulas, expressed in an analytical manner; and Olbers's calculations are abridged by the use of Tables I, II.

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ERRATA.

CORRECTIONS AND ADDITIONS

IN VOLUME I.

Page.	Line.	
119	6 bot.	*For* dw *read* $d\varpi$.
120	13, 19, 21	*For* $(zdx-xdy)$ *read* $(zdx-xdz)$.
125	12	*For* $dZ+dy_{\prime}$, *read* $dZ+dz_{\prime}$.
134	7 bot.	*For* $-\Sigma.m\,\delta\,v.ds$, *read* $-\Sigma\int.m.\delta\,v.ds$.
147	7 bot.	*For* $-y'ddx'$, *read* $-yddx'$.
147	4 bot.	*For* Y *read* y.
147	3 bot.	Insert dm in the last term.
159	7	Insert (after x'.
182	9 bot.	*For* $\frac{4}{5}$ *read* $\frac{4}{3}$.
183	4 bot.	*For* $\frac{2}{5}$ *read* $\frac{3}{5}$.
209	9 bot.	*For* axis of z, *read* axis of x.
215	16	*For* dy *read* $\delta\,y$.
220	10 bot.	*For* (dp) *read* $(\delta\,p)$.
230	3 bot.	*For* dr' *read* dr.
234	4 bot.	*For* ag *read* αg.
235	8 bot.	*For* αu *read* $\alpha u'$.
280	7	Change the accents in the denominator of V.
281	1 bot.	*For* β^2, *read* β^{-2}.
301	7 bot.	*For* ^{2}, *read* r^2.
371	12	*For* sin.mt, *read* sin.mnt.
371	13	*For* cos.mt, *read* cos.mnt.
378	3	*For* sin.$2nt$, *read* 2.sin.$2nt$.
378	11	*For* [688a], *read* [668a].
381	1	*For* $\sin.^4.(v_{\prime}-\theta)$, *read* $\sin.4.(v_{\prime}-\theta)$.
398	10 bot.	*For* $\frac{2}{r}$, *read* $\frac{2}{r'}$.
413	3	*For* $0'',5$, *read* $0^s,5$.
455	1,2 bot.	*For* logarithm, *read* logarithmic.
464	8 bot.	*For* tang.$(\beta''-j)$, tang. $(\beta'''-j)$; *read* sin.$(\beta''-j)$, sin $(\beta'''-j)$.
475	7 bot.	*For* $d^{\prime}y$, *read* $d^{\prime}y'$.
478	6	*For* c, *read* c'.
480	4	*For* y', y', &c., *read* y, y', &c.
487	18	*For* y', y', &c., *read* y, y', &c.
495	5 bot.	*For* θ' *read* φ'.
499	6 bot.	*For* $c=V'$, *read* $c'=V'$.
542	4 bot.	*For* A', *read* $A^{(1)}$.
581	8 bot.	*For* [1034a], *read* [1069a].
585	1 bot.	*For* the exponent $-\frac{1}{2}$, *read* $\frac{1}{2}$.
593	5 bot.	*For* [1098a], *read* [1097b'].
608	16	*For* B, *read* B_0.
618	15	*For* spherical angle, *read* spherical triangle.
679	5 bot.	*For* $m'p$, *read* $m'p'$; and *for* $m'q$, *read* $m'q'$.
698	4 bot.	*For* m, *read* m'.
715	15 bot.	*For* $andt$, *read* an, in both formulas.

IN VOLUME II.

Page.	Line.	
370	16	*For* [1581a], *read* [1851a].
510	11 bot.	*For* λ *read* e.
780	4 bot.	*For* $\frac{L'}{r^3}$, *read* $\frac{L'}{r_{\prime}'^3}$.
781	5 bot.	*For* $\frac{L'}{r'^3}$, *read* $\frac{L}{r_{\prime}'^3}$.

IN VOLUME III.

The same measures have been used for correcting the mistakes of the press in Volume III, as in printing the preceding volumes. The reader will also omit the third line from the bottom in page 501, which is unnecessarily repeated; and at the end of the paragraph, page 556, line 16, will make the following addition of a paragraph which was accidentally omitted. "The function [5082s] contains also the terms depending on $120m^2.A_1^{(8)}$, $120m^2.A_1^{(9)}$ [5261c, e, line 1], which are derived from the part $-\frac{1}{2}a$. funct. [4931p] contained in [5082q]. For by combining the term $A_1^{(8)}ee'.\cos.(cv+c'mv)$ in [4931p, col. 1] with $-\frac{5}{2}e.\sin.(2v-2mv-cv)$, in col. 2, we get the first of these terms; and by combining the term $A_1^{(9)}.ee'.\cos.(cv-c'mv)$, with $-\frac{5}{2}e.\sin.(2v-2mv-cv)$, in col. 2, we get the second of these terms." Lastly, in page 458, line 3, we may add, that the function [4957] must be multiplied by the chief term of [4890], or $\frac{1}{a}$, to obtain the corresponding terms of [4961 or 4960c].

SECOND PART.

PARTICULAR THEORIES OF THE MOTIONS OF THE HEAVENLY BODIES.

SIXTH BOOK.

THEORY OF THE PLANETARY MOTIONS.

The motions of the planets are sensibly disturbed by their mutual attractions, and it is important to determine accurately the inequalities which result from this cause; for the purposes of verifying the law of universal gravitation, improving the accuracy of astronomical tables, and discovering whether any cause, foreign from the planetary system, produces a change in its constitution or its motions. The object of this book is to apply to the bodies of this system, the methods and general formula given in the first part of this work. We have developed in the second book, only those inequalities which are independent of the excentricities or inclinations of the orbits, and those which depend upon the first power of these quantities. But it is often indispensable to extend the approximation to the square and to the higher powers of these elements; and sometimes it is also necessary to consider the terms depending on the square of the disturbing force. We shall first give the formulas relative to these inequalities; and shall then substitute in these formulas, and in those of the second book, the numbers or values of the elements corresponding to each planet. By this means we shall obtain the numerical expressions of the radius vector, and the motions of the planet in longitude and in latitude. Bouvard has willingly undertaken the calculation of these substitutions, and the zeal with which he has prosecuted this laborious work, deserves the acknowledgment of all astronomers. Several mathematicians had previously calculated the greater part of the planetary inequalities; and their results have been useful in verifying those of Bouvard; for when any difference has been found, he has examined into the source of

the error, in order to satisfy himself of the accuracy of his own calculation. Lastly, he has reviewed with particular care, the calculation of those inequalities which had not been before computed; and by means of several equations of condition, which obtain between these inequalities, I have been enabled to verify many of them. Notwithstanding all these precautions, there may possibly be found in the following results, some errors, which almost inevitably occur in such long calculations; but there is reason to believe that they amount only to insensible quantities, and that they cannot be detrimental to the general accuracy of the tables founded upon them. These results, on account of their importance in the planetary astronomy, of which they are the basis, deserve to be verified with the same care that has been taken in the calculation of the tables of logarithms and of sines.

The theories of Mercury, Venus, the Earth, and Mars, produce only periodical equations of small moment; they are, however, very sensible, by modern observations, with which they agree in a remarkable manner. The development of the secular equations of the planets and of the moon will make known accurately the masses of these bodies, which is the only part of their theory that remains yet somewhat imperfect. It is chiefly in the motions of Jupiter and Saturn, the two greatest bodies of the planetary system, that the mutual attraction of the planets is sensible. Their mean motions are nearly commensurable; so that five times that of Saturn is nearly equal to twice that of Jupiter, and the great inequalities in the motions of these two bodies arise from this circumstance. When the laws and causes of these motions were unknown, they seemed, for a long time, to form an exception to the law of universal gravitation, and now they are one of the most striking proofs of its correctness. It is extremely curious to see with what precision the two principal equations of the motions of these planets, whose period includes more than nine hundred years, satisfy ancient and modern observations. The development of these equations in future ages, will more and more prove this agreement of the theory and observation. To facilitate the comparison with distant observations, we have carried on the approximation to terms depending on the square of the disturbing force, and it is hoped that the values here assigned to these equations will vary but very little from those found by a long series of observations continued during an entire period. These equations have a great influence upon the secular variations of the orbits of Jupiter and Saturn, and we have developed the analytical and numerical expressions arising from this source. Lastly, the

planet Uranus is subjected to sensible inequalities, which we have determined, and which have been confirmed by observation.

The first day of this century is remarkable for the discovery of a new planet, situated between the orbits of Jupiter and Mars,* and to which the name of *Ceres* has been given. It appears as a star of the eighth or ninth magnitude; its excessive smallness renders its action insensible on the planetary system; but it must suffer considerable perturbation from the attractions of the other planets, particularly Jupiter and Saturn, which ought to be ascertained. It is what we propose to do in the course of this work, after the elements of the orbit have been determined by observation to a sufficient degree of accuracy.

It is hardly three centuries since Copernicus first introduced into astronomical tables the motion of the planets about the sun. A century afterwards, Kepler made known the laws of the elliptical motion, which he had discovered by observation; and from these laws, Newton was led to the discovery of universal gravitation. Since these three memorable epochs in the history of the sciences, the progress of the infinitesimal analysis has enabled us to submit to calculation the numerous inequalities of the planets depending upon their reciprocal action; and by this means the tables have acquired an unexpected degree of accuracy. It is believed that the following results will give to them a much greater degree of precision.

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* (2341) This volume was published by the author shortly after the discovery of Ceres, January 1, 1801; and before the discovery of the planets Pallas, Juno, and Vesta. He did [3698a]
not compute the numerical values of the perturbations of their motions as he had intended.

CHAPTER I.

FORMULAS FOR THE INEQUALITIES OF THE MOTIONS OF THE PLANETS WHICH DEPEND UPON THE SQUARES AND HIGHER POWERS OF THE EXCENTRICITIES AND INCLINATIONS OF THE ORBITS.

ON THE INEQUALITIES WHICH DEPEND UPON THE SQUARES AND PRODUCTS OF THE EXCENTRICITIES AND INCLINATIONS.

Differential equation in $r\,\delta r$. First form

1. To determine these inequalities, we shall resume the formula [926],*

[3699] $$0=\frac{d^2.r\,\delta r}{d\,t^2}+\frac{\mu.r\,\delta r}{r^3}+2\int \mathrm{d}\,R+r.\left(\frac{d\,R}{d\,r}\right).$$

We have, as in [605′, 669],†

[3700] $$\frac{\mu}{a^3}=n^2;$$

Radius

[3701] $$r=a.\{1+\tfrac{1}{2}e^2-e.\cos.(n\,t+\varepsilon-\varpi)-\tfrac{1}{2}e^2.\cos.2.(n\,t+\varepsilon-\varpi)\};$$

hence the preceding differential equation becomes,‡

Differential equation in $r\,\delta r$. Second form

[3702] $$0=\frac{d^2.r\,\delta r}{d\,t^2}+n^2.r\,\delta r+3\,n^2a.\delta r.\{e.\cos.(n\,t+\varepsilon-\varpi)+e^2.\cos.2.(n\,t+\varepsilon-\varpi)\}$$
$$+2\int \mathrm{d}\,R+r.\left(\frac{d\,R}{d\,r}\right).$$

[3699a] * (2342) Substituting, in [926], the value of $r\,R'$ [928′], it becomes as in [3699].

[3700a] † (2343) The equation [3700] is easily deduced from [605′]; and the value of r [3701] is the same as that in [669], neglecting terms of the order e^3.

‡ (2344) If we use, for brevity, the same symbols as in [1018a], namely,

[3702a] $$T=n'\,t-n\,t+\varepsilon'-\varepsilon,\qquad W=n\,t+\varepsilon-\varpi,\qquad b=\tfrac{1}{2}e^2-e.\cos.W-\tfrac{1}{2}e^2.\cos.2\,W,$$

[3702b] we shall have $r=a.(1+b)$ [3701]; hence $r^{-3}=a^{-3}.(1+b)^{-3}=a^{-3}.(1-3b+6b^2)$; neglecting b^3 and the higher powers of b; or, in other words, neglecting e^3, e^4, &c. Now, by

Now all the terms of the expression of R, depending on the squares and products of the excentricities and inclinations of the orbits, may be reduced to the one or the other of these two forms,* [3702′]

Terms of R depending on angles of two different forms.

$$R = M \,.\, \cos. \{ i \,.\, (n' t - n t + \varepsilon' - \varepsilon) + 2 n t + K \} \,; \qquad \text{[First form.]} \tag{3703}$$

$$R = N \,.\, \cos. \{ i \,.\, (n' t - n t + \varepsilon' - \varepsilon) + L \} \,; \qquad \text{[Second form.]} \tag{3704}$$

in which i includes all integral numbers, positive or negative, comprehending also $i = 0$ [954″]. We shall, in the first place, consider the term [3703]. [3704′]

It produces, in $2 \int \mathrm{d} R + r \,.\, \left(\frac{d R}{d r}\right)$, the function† [3704″]

$$\left\{ \frac{2 \,.\, (2 - i) \,.\, n}{i n' + (2 - i) \,.\, n} \,.\, M + a \,.\, \left(\frac{d M}{d a}\right) \right\} . \cos. \{ i . (n' t - n t + \varepsilon' - \varepsilon) + 2 n t + K \}. \tag{3705}$$

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retaining terms of the order e^2, we get, successively, $6 b^2 = 6 e^2 . \cos.^2 W = 3 e^2 + 3 e^2 . \cos. 2 W$; hence $1 - 3 b + 6 b^2 = 1 + \frac{3}{2} e^2 + 3 e \,.\, \cos.\, W + \frac{9}{2} e^2 \cos.\, 2\, W$. Substituting this in r^{-3}. [3702b], and then multiplying by $\mu \,.\, r\, \delta r$, we get [3702d]; which is easily reduced to the form [3702e], by the substitution of n^2 [3700] and $r = a . (1 - e . \cos. W)$ [3701] in the last term of the second member. Now we have $-3 e^2 . \cos.^2 W = -\frac{3}{2} e^2 - \frac{3}{2} e^2 . \cos. 2 W$; hence [3702$e$] becomes as in [3702$f$], [3702$c$]

$$\frac{\mu . r \delta r}{r^3} = \frac{\mu}{a^3} . r \delta r + \frac{\mu}{a^3} . r \delta r . \{ \tfrac{3}{2} e^2 + 3 e \,.\, \cos.\, W + \tfrac{9}{2} e^2 . \cos.\, 2\, W \} \tag{3702d}$$

$$= n^2 . r \delta r + n^2 . a \delta r . \{ \tfrac{3}{2} e^2 + 3 e \,.\, \cos.\, W + \tfrac{9}{2} e^2 . \cos.\, 2\, W \} . \{ 1 - e \,.\, \cos.\, W \} \tag{3702e}$$

$$= n^2 . r \delta r + n^2 . a \delta r . \{ 3 e . \cos.\, W + 3 e^2 . \cos.\, 2\, W \}. \tag{3702f}$$

Substituting this in [3699], we get [3702].

* (2345) This will be evident by the substitution of $u_{,}$, $v_{,}$, &c. [1009, 669] in [957]. It also appears from [957$^{\text{viii}}$, &c.]; for in [3703], the coefficients of $n' t$, $-n t$, are i, $i - 2$, respectively; their difference 2 expresses the order of the coefficient k [957$^{\text{viii}}$, &c.], or that of M [3703]; which must therefore be of the order 2 or e^2. In like manner, the coefficients of $n' t$, $-n t$ [3704] being both equal to i; the coefficient N may contain terms of the orders 0, 2, 4, &c. [957$^{\text{viii}}$, &c.], which include those of the order e^2; and a very little attention to the remarks in [957$^{\text{v}}$, &c.] will show, that these are the only forms of this kind containing e^2. [3704a] [3704b]

† (2346) Substituting the expression $r . \left(\frac{d R}{d r}\right) = a . \left(\frac{d R}{d a}\right)$ [962], in the function [3704″], we get $2 \int \mathrm{d} R + r . \left(\frac{d R}{d r}\right) = 2 \int \mathrm{d} R + a . \left(\frac{d R}{d a}\right)$. In finding $\mathrm{d} R$, we must suppose, as in [916′], *the ordinates of the body m to be the only variable quantities; or, in other words, we must consider $n t$ as variable, and $n' t$ constant,* as is done in finding $\mathrm{d} R$ [1012a—c]. Now in taking for R the form [3703], $R = M . \cos. \{ i . (n' t - n t + \varepsilon' - \varepsilon) + 2 n t + K \}$, [3705$a$] [3705$b$] [3705$c$]

We have seen, in the second book [1016], that the parts of $\frac{\delta r}{a}$ depending on the angles $i.(n't-nt+\varepsilon'-\varepsilon)$ and $i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon$, are of the following forms,

Terms of $\frac{\delta r}{a}$ depending on angles of the first form.

[3706] $$\frac{\delta r}{a}=F.\cos.i.(n't-nt+\varepsilon'-\varepsilon)+eG.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\} \\ +e'H.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\};$$

hence the function

[3707] $$3n^2.a\delta r.\{e.\cos.(nt+\varepsilon-\varpi)+e^2.\cos.(2nt+2\varepsilon-2\varpi)\}$$

will produce, in [3702], the following terms,*

[3708] $$\tfrac{3}{2}n^2a^2.\left\{\begin{array}{l}(F+G).e^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi\} \\ \quad+H.ee'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'\}\end{array}\right\}.$$

Therefore, if we notice only the terms depending on the angle

$$i.(n't-nt+\varepsilon'-\varepsilon)+2nt,$$

[3709] and put $\mu=1$; *which is equivalent to the supposition that the sun's mass is*
[3709′] *equal to unity, neglecting the mass of the planet;*† *we shall have* $n^2a^3=1$;

[3705d] we obtain $\mathrm{d}R=-(2-i).n.M.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}.dt.$ Integrating this, and multiplying by 2, we get

[3705e] $$2\int\mathrm{d}R=\frac{2.(2-i).n}{in'+(2-i).n}.M.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}.$$

The partial differential of R [3705c], relative to a, being multiplied by a, gives

[3705f] $$a.\left(\frac{dR}{da}\right)=a.\left(\frac{dM}{da}\right).\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}.$$

Adding this to the expression [3705e], we get $2\int\mathrm{d}R+a.\left(\frac{dR}{da}\right)$, as in [3705].

* (2347) The forms of the terms of $\frac{\delta r}{a}$, assumed in [3706], are the same as those computed in [1016]; the constant part corresponding to $i=0$; and the secular
[3708a] terms being made to disappear, as in [1036^v, &c.]. Substituting these in [3707], and reducing by formula [20] Int., retaining only the terms depending on the angle $i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K$ [3703], we get [3708].

† (2348) M being the mass of the sun, and m that of the planet, we have $M+m=\mu$
[3709a] [914′]. If we put $M=1$, and neglect m on account of its smallness, we shall have $\mu=1$; and then from [3700], we shall get [3709′].

and then the differential equation [3702] will become*

$$0=\frac{d^2.(r\delta r)}{dt^2}+n^2.r\delta r+\tfrac{3}{2}n^2a^2.\left\{\begin{array}{l}(F+G).e^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi\}\\ \quad+H.ee'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'\}\end{array}\right\}$$
$$+n^2a^2.\left\{\frac{2.(2-i).n}{in'+(2-i).n}.aM+a^2.\left(\frac{dM}{da}\right)\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}. \quad [3710]$$

Hence we get, by integration,†

Values of $r\delta r$ depending on angles of the first form.

$$\frac{r\delta r}{a^2}=\frac{\left\{\begin{array}{l}\tfrac{3}{2}n^2.\left\{\begin{array}{l}(F+G).e^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi\}\\ \quad+H.ee'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'\}\end{array}\right\}\\ +\left\{\frac{2.(2-i).n}{in'+(2-i).n}.aM+a^2.\left(\frac{dM}{da}\right)\right\}.n^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}\end{array}\right\}(A)}{\{i.n'+(3-i).n\}.\{in'+(1-i).n\}}. \quad [3711]$$

If this expression of $\frac{r\delta r}{a^2}$ be considerable, and one of its divisors $in'+(3-i).n$, $in'+(1-i).n$, be very small, as is the case in the theory of Jupiter, disturbed by Saturn, when we suppose $i=5$; $2n$ being [3712]
nearly equal to $5n'$;‡ the variableness of the elements of the orbit will

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* (2349) Substituting, in [3702], the value of its third and fourth terms [3708], also the values of the fifth and sixth terms [3705], multiplied by $n^2a^3=1$, for the sake of [3710a]
homogeneity; it becomes as in [3710].

† (2350) If we put, in [865, 870′], $y=r\delta r$, $a=n$, $\alpha Q=\Sigma.\alpha K.\frac{\sin.}{\cos.}(m_{,}t+\varepsilon_{,})$, [3711a]
the letters m, ε being accented to prevent confusion in the notation, and Σ denoting the sign of finite integrals; we shall have the differential equation [3711b], whose integral [871] is as in [3711c],

$$0=\frac{d^2.(r\delta r)}{dt^2}+n^2.r\delta r+\Sigma.\alpha K.\tfrac{\sin.}{\cos.}(m_{,}t+\varepsilon_{,}); \quad [3711b]$$

$$r\delta r=\Sigma.\frac{\alpha K}{m_{,}^2-n^2}.\tfrac{\sin.}{\cos.}(m_{,}t+\varepsilon_{,})=\frac{\alpha Q}{m_{,}^2-n^2}. \quad [3711c]$$

Comparing the coefficient of t in the expressions [3710, 3711b], we get $m_{,}=i.(n'-n)+2n$; [3711d]
hence $m_{,}^2-n^2=(m_{,}+n).(m_{,}-n)=\{in'+(3-i).n\}.\{in'+(1-i).n\}$; substituting [3711e]
this in [3711c], and then dividing by a^2, we get [3711].

‡ (2351) We have, in [4077], for Saturn $n'=43997^s$; and for Jupiter $n=109256^s$ [3711f]
nearly; hence $5n'-2n=1473^s$; which is quite small in comparison with n or n', being only $\frac{1}{74}$ part of n.

have a sensible influence on this expression; it is important, therefore, to notice this circumstance. For this purpose we shall put the differential equation [3710] under the following form,*

[3713] $$0=\frac{d^2.(r\delta r)}{dt^2}+n^2.r\delta r+n^2a^2.P.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}$$
$$+n^2a^2.P'.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}.$$

Integrating this, and neglecting the terms depending on the second and higher differentials of P, P', we shall obtain†

Another value of $r\delta r$, noticing the secular variation of the elements.

[3714] $$\frac{r\delta r}{a^2}=\frac{n^2}{\{in'+(3-i).n\}.\{in'+(1-i).n\}}$$
$$\times\left\{\begin{array}{l}\left\{P+\dfrac{2.\{i.(n'-n)+2n\}.\dfrac{dP'}{dt}}{\{in'+(3-i).n\}.\{in'+(1-i).n\}}\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}\\[2ex] +\left\{P'-\dfrac{2.\{i.(n'-n)+2n\}.\dfrac{dP}{dt}}{\{in'+(3-i).n\}.\{in'+(1-i).n\}}\right\}.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}\end{array}\right\}.\ (B)$$

[3711g] * (2352) If we put, for brevity, $T_{,}=i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon$, the term depending on F, in [3710], will become

[3711h] $$\tfrac{3}{2}n^2a^2Fe^2.\cos.(T_{,}-2\varpi)=\tfrac{3}{2}n^2a^2Fe^2.\{\cos.T_{,}.\cos.2\varpi+\sin.T_{,}.\sin.2\varpi\};$$

[3711i] if we put $\tfrac{3}{2}Fe^2.\cos.2\varpi=P$; $\tfrac{3}{2}Fe^2.\sin.2\varpi=P'$, it becomes $n^2a^2.\{P.\cos.T_{,}+P'.\sin.T_{,}\}$, as in [3713]. In like manner, the terms of [3710], depending on G, H, M, may be
[3711k] reduced to the forms [3711i]; P, P' being functions of the variable elements e, ϖ, &c., and T, T' independent of these variable elements; observing, that n, a, ε [1045', 1044''] are considered as constant, as well as the similar elements of the planet m'.

† (2353) Using the abridged symbols $m_{,}$. $T_{,}$ [3711d, g], and substituting, in [3711b],
[3714a] the function [3711i], instead of the terms under the sign Σ, this differential equation becomes of the form [3714b], and the integral [3711c], taken in the hypothesis that P, P' are constant, becomes as in [3714c],

[3714b] $$0=\frac{d^2.(r\delta r)}{dt^2}+n^2.r\delta r+n^2a^2.\{P.\cos.T_{,}+P'.\sin.T_{,}\};$$

[3714c] $$r\delta r=\frac{n^2a^2.\{P.\cos.T_{,}+P'.\sin.T_{,}\}}{m_{,}^2-n^2}.$$

We shall suppose $r\delta r$, to be increased by the quantity $[r\delta r]$, in consequence of the secular variation of P, P', so that instead of [3714c], we shall have, generally,

[3714d] $$r\delta r=\frac{n^2a^2.\{P.\cos.T_{,}+P'.\sin.T_{,}\}}{m_{,}^2-n^2}+[r\delta r].$$

The formula [931] becomes, by putting $\mu = 1$,*

$$\delta v = \frac{\left\{\begin{array}{l} \frac{2d.(r\delta r)}{a^2 n\, dt} - \tfrac{1}{2}.\left\{\begin{array}{l}(F+G).e^2.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi\} \\ +H.ee'.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'\}\end{array}\right\} \\ +\left\{\frac{(6-3i).n^2}{\{in'+(2-i).n\}^2}.aM + \frac{2na^2\left(\frac{dM}{da}\right)}{in'+(2-i).n}\right\}.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}\end{array}\right\}\ (C)}{\sqrt{1-e^2}};$$ [3715]

Values of δv depending on angles of the first form.

and by giving to i all positive and negative values, including zero [3704'], [3715']
we shall obtain all the inequalities, in which the coefficient of nt differs from that of $n't$ by two.

Now as the value of $r\delta r$ [3714c] satisfies the equation [3714b], supposing P, P' to be constant, and by hypothesis the value [3714d] satisfies the same equation [3714b], when P, P' are [3714d']
variable by reason of the secular inequalities, we may substitute [3714d] in [3714b], and then, from the resulting expression subtract the equation [3714b], and we shall obtain an equation of the form [3714f], observing, that we must retain only the terms depending on the first and second differentials of P, P', namely, dP, dP', d^2P, d^2P', to the exclusion of P, P', [3714e]

$$0=\frac{d^2.[r\delta r]}{dt^2}+n^2.[r\delta r]+n^2a^2.\frac{d^2.\{P.\cos.T_{,}+P'.\sin.T_{,}\}}{(m_{,}^2-n^2).dt^2}.$$ [3714f]

Now we have, generally, $d^2.(P.\cos.T_{,})=d^2P.\cos.T_{,}+2dP.d.(\cos.T_{,})+Pd^2.(\cos.T_{,})$; [3714g]
in which the term containing P is to be rejected [3714e]; and if we neglect the term depending on d^2P, on account of its smallness, we shall obtain

$$d^2.(P.\cos.T_{,})=2dP.d.\cos.T_{,}=-2dP.m_{,}dt.\sin.T_{,}\quad [3711d, g].$$ [3714g']

In like manner we have

$$d^2.(P'.\sin.T_{,})=2dP'.d.\sin.T_{,}=\quad 2dP'.m_{,}dt.\cos.T_{,};$$ [3714h]

hence [3714f] becomes

$$0=\frac{d^2.[r\delta r]}{dt^2}+n^2.[r\delta r]+\frac{n^2a^2}{m_{,}^2-n^2}.\left\{2m_{,}.\frac{dP'}{dt}.\cos.T_{,}-2m_{,}.\frac{dP}{dt}.\sin.T_{,}\right\}.$$ [3714i]

This is similar to the equation [3711b], changing $r\delta r$ into $[r\delta r]$, representing by αQ the terms depending on dP', dP. These terms being divided by $m_{,}^2-n^2$, give, as in [3711e], the following value of $[r\delta r]$;

$$[r\delta r]=\frac{n^2a^2}{m_{,}^2-n^2}.\left\{\frac{2m_{,}}{m_{,}^2-n^2}.\frac{dP'}{dt}.\cos.T_{,}-\frac{2m_{,}}{m_{,}^2-n^2}.\frac{dP}{dt}.\sin.T_{,}\right\}.$$ [3714k]

Substituting this in [3714d], connecting together the terms depending on $\cos.T_{,}$, also those depending on $\sin.T_{,}$, then substituting the value of $m_{,}^2-n^2$ [3711e], and dividing by a^2, we get [3714].

* (2354) We have $2r.d\delta r+dr.\delta r=2d.(r\delta r)-dr.\delta r$, as is easily [3715a]
proved by developing the first term of the second member, and reducing. Substituting

If the coefficient $i n' + (2 - i) . n$ be very small, and this inequality be very sensible, as is the case in the theory of Uranus, disturbed

[3715″] by Saturn [4527]; we must put the part of R depending on the

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this and [3705a] in [931], we obtain

[3715b]
$$\delta v = \frac{\frac{2 d.(r \delta r)}{a^2 n d t} - \frac{d r . \delta r}{a^2 n d t} + \int \left\{ 3 a \int n d t . \mathrm{d} R + 2 a n d t . a . \left(\frac{d R}{d a}\right) \right\}}{\sqrt{(1 - e^2)}} .$$

The differential of [3701], being multiplied by $-\frac{\delta r}{a^2 n d t}$, becomes

[3715c]
$$-\frac{d r . \delta r}{a^2 n d t} = -\frac{\delta r}{a} . \{ e . \sin. (n t + \varepsilon - \varpi) + e^2 . \sin. 2 . (n t + \varepsilon - \varpi) \} .$$

This is to be reduced, as in [3708a], by substituting the value of $\frac{\delta r}{a}$ [3706], using the formula [18] Int., and retaining only the terms depending on the angle $T_{,}$ [3711g]; hence we get

[3715d]
$$-\frac{d r . \delta r}{a^2 n d t} = -\tfrac{1}{2} . (F + G) . e^2 . \sin. (T_{,} - 2 \varpi) - \tfrac{1}{2} H e e' . \sin. (T_{,} - \varpi - \varpi') .$$

[3715e] Again, if we put, for brevity, $T_2 = i . (n' t - n t + \varepsilon' - \varepsilon) + 2 n t + K$, the term of R [3703] will become $R = M . \cos. T_2$; hence the differential $\mathrm{d} R$, found as in [916′], upon the

[3715f] supposition that $n t$ is the variable quantity, is $\mathrm{d} R = -(2 - i) . n d t . M . \sin. T_2$. Multiplying this by $3 a . n d t$, integrating and using $m_{,}$ [3711d], we get

$$3 a \int n d t . \mathrm{d} R = \frac{(6 - 3 i) . n^2 d t}{m_{,}} . a M . \cos. T_2 .$$

[3715g]

To this we must add $2 a n d t . a . \left(\frac{d R}{d a}\right) = 2 a n d t . a . \left(\frac{d M}{d a}\right) . \cos. T_2$; and then, by integrating the sum, we obtain

[3715h]
$$\int \left\{ 3 a . \int n d t . \mathrm{d} R + 2 a n d t . a . \left(\frac{d R}{d a}\right) \right\} = \left\{ \frac{(6 - 3 i) . n^2}{m_{,}^2} . a M + \frac{2 n a^2 . \left(\frac{d M}{d a}\right)}{m_{,}} \right\} . \sin. T_2 .$$

Substituting this and [3715d], in [3715b], we get [3715].

[3715i] In the great inequalities of Jupiter and Saturn, the most important parts of δr, $\delta v'$ [3715b, &c.] are those depending on the double integration of $\mathrm{d} R$, $\mathrm{d}' R'$, which introduces the divisor $(5 n' - 2 n)^2$. These parts are to be applied to the mean motions

[3715k] of the planets, as is shown in [1066″, 1070″]. As we must frequently refer to these parts δv, $\delta v'$, of the mean motions ζ, ζ' of the planets m, m', we shall here give their values, deduced from [1183, 1204, 3709a], or from the appendix [5794], representing the chief parts of δv, $\delta v'$ [3715b, &c.];

[3715l] δv deduced from $\zeta = 3 a n . \int d t . \int \mathrm{d} R$;

[3715m] $\delta v'$ deduced from $\zeta' = 3 a' n' . \int d t . \int \mathrm{d}' R'$.

angle $i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon$ [3703], under the following form,*

$$R = Q.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\} + Q'.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}; \qquad [3716]$$

and we shall have,†

$$3a.\int\int ndt.\mathrm{d}R = \frac{(6-3i).n^2a}{\{in'+(2-i).n\}^2}.\left\{Q+\frac{2.\frac{dQ'}{dt}}{in'+(2-i).n}\right\}.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}$$
$$-\frac{(6-3i).n^2a}{\{in'+(2-i).n\}^2}.\left\{Q'-\frac{2.\frac{dQ}{dt}}{in'+(2-i).n}\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}. \qquad [3717]$$

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* (2355) Using, for brevity, $K_{\prime}=K-2\varepsilon$, and $T_{\prime}$ [3711g], the expression of R [3703] becomes $R=M.\cos.(T_{\prime}+K_{\prime})=M.\cos.K_{\prime}.\cos.T_{\prime}-M.\sin.K_{\prime}.\sin.T_{\prime}$; and by putting $M.\cos.K_{\prime}=Q$, $-M.\sin.K_{\prime}=Q'$; it changes into $R=Q.\cos.T_{\prime}+Q'.\sin.T_{\prime}$, as in [3716]; Q, Q' being like P, P' [3711k], functions of the variable elements of the orbits, and $T_{\prime}$ independent of them. Now we have, in [4077], for Uranus $n=15425^s$; for Saturn $n'=43997^s$ nearly; hence $3n-n'=2278^s$; which is much smaller than n or n'; and by putting $i=-1$, in the divisor $in'+(2-i).n$, it becomes $3n-n'$; therefore this small divisor must occur in computing the perturbations of Uranus by Saturn, as is observed in [3715″]. [3716a] [3716b] [3716c] [3716d]

† (2356) The differential $\mathrm{d}R$, deduced from [3716b], considering nt as the variable quantity, as in [3715f], is

$$\mathrm{d}R = -(2-i).ndt.Q.\sin.T_{\prime}+(2-i).ndt.Q'.\cos.T_{\prime}; \qquad [3717a]$$

hence we have

$$3a.\int\int ndt.\mathrm{d}R = \int\int an^2.dt^2.\{(-6+3i).Q.\sin.T_{\prime}+(6-3i).Q'.\cos.T_{\prime}\}. \qquad [3717b]$$

If the integral of the second member of this expression be taken, supposing Q, Q' to be constant, it will produce the terms independent of dQ, dQ' in [3717]. The terms depending on dQ, dQ' may be estimated by means of the general formula [1209b], which, by changing A, B into Q, A, respectively, and neglecting d^2Q, d^3Q, &c., becomes

$$\int\int AQdt^2 = Q\int\int Adt^2-2.\frac{dQ}{dt}.\int\int\int Adt^3. \qquad [3717c]$$

From this formula, it appears, that the term depending on $\frac{dQ}{dt}$, is easily deduced from that depending on Q, by changing Q into $-2.\frac{dQ}{dt}.dt$, and then integrating relatively to t, supposing $\frac{dQ}{dt}$ to be constant. In this way we easily deduce the term depending [3717d] on dQ [3717] from that of Q; and in like manner we get the term depending on dQ' from that of Q'.

Hence the formula [3715b] will give*

$$\delta v = \frac{2d.(r\delta r)}{a^2 n d t} - \tfrac{1}{2}\cdot\left\{\begin{array}{l}(F+G).e^2.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi\} \\ +H.ee'.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'\}\end{array}\right\} \quad (D)$$

Another form of this value of δv, [3718] the elements being variable.

$$+\left\{\frac{(6-3i).n^2}{\{in'+(2-i).n\}^2}\cdot\left[aQ+\frac{2a.\frac{dQ'}{dt}}{in'+(2-i).n}\right]+\frac{2na^2.\left(\frac{dQ}{da}\right)}{in'+(2-i).n}\right\}.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}$$

$$-\left\{\frac{(6-3i).n^2}{\{in'+(2-i).n\}^2}\cdot\left[aQ'-\frac{2a.\frac{dQ}{dt}}{in'+(2-i).n}\right]+\frac{2na^2.\left(\frac{dQ'}{da}\right)}{in'+(2-i).n}\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon\}.$$

[3718'] For greater accuracy, we have neglected the divisor $\sqrt{1-e^2}$ in this expression of δv; because it does not affect the part of this expression which has the square of $in'+(2-i).n$ for a divisor, as we have seen
[3719] in [1197]; and in the present case, this part is much greater than the others. Moreover, we must, as in [1197viii, 1066'', 1070''], apply this part
[3719'] of δv to the mean motion of m†; and as it is very nearly equal to the

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* (2357) Using the value of R [3716], or rather [3716b, 3711g]; taking its partial differential, relatively to a, which will affect only Q, Q'; multiplying by $2a^2.ndt$, and then integrating, we get

[3718a] $$\int 2andt.a.\left(\frac{dR}{da}\right)=\frac{2na^2}{m_{,}}.\left(\frac{dQ}{da}\right).\sin.T_{,}-\frac{2na^2}{m_{,}}.\left(\frac{dQ'}{da}\right).\cos.T_{,};$$

$m_{,}$ being, as in [3711d]. Substituting this in [3715b], also the values of the terms [3717, 3715d], it becomes as in [3718]; except that the divisor $\sqrt{(1-e^2)}$ is neglected,
[3718b] for the reason mentioned in [3718'], namely, that the *chief part* of δv or ζ [1195 or 1197] does not contain this divisor; and as the other terms are very small, it may also be neglected in them.

† (2358) The terms of δv [3718], having for divisor the square of $in'+(2-i).n$,
[3719a] are those depending on $3a\iint ndt.\mathrm{d}R$, computed in [3717]; and it is evident, that this part of δv much exceeds the other parts depending on F, G, H, &c. Now, by [1066'', 1070''], or by [1197viii], the parts depending on $3a\iint ndt.\mathrm{d}R$, must be applied to the *mean* motion, and as the other parts, depending on the same angle, are much
[3719b] smaller, we may suppose that the whole of this equation is to be applied to the mean motion, as in [3720]. We may remark incidentally, that the expression of r [1066], as well as that of v [1070], contains the double integral $\iint ndt.\mathrm{d}R$; hence, at the first view, it would seem that if v contain terms depending on this double integral with the small divisor $\{in'+(2-i).n\}^2$, as in [3718], r would contain similar terms of the same order. But we must observe, that these terms of r, v [1066, 1070] are multiplied,
[3719c] respectively, by $\left(\frac{dr}{ndt}\right)$, $\left(\frac{dv}{ndt}\right)$, or by their equivalent values $ae.\sin.(nt+\varepsilon-\varpi)$, $1+2e.\cos.(nt+\varepsilon-\varpi)$ [669]. Hence these terms of v will be multiplied by 1,

whole term depending on the angle $i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon$, *we may apply this whole inequality to the mean motion of* m. [3720]

We shall obtain the values of $\frac{dP}{dt}$, $\frac{dP'}{dt}$, $\frac{dQ}{dt}$, $\frac{dQ'}{dt}$, by taking the differentials of the expressions P, P', Q, Q', relative to the excentricities and inclinations of the orbits, the positions of their perihelia and nodes, and then substituting the values of the differentials of these quantities. But we may obtain these values of $\frac{dP}{dt}$, &c. more simply in the following manner. [3721]
Find the value of P, for an epoch which is distant by two hundred years from the epoch taken for the origin of the time t; then putting $P_{,}$ for this value, and T for the interval of two hundred years, we shall have* [3722]

Formula for the determination of dP, dP, &c.

$$T.\frac{dP}{dt}=P_{,}-P. \qquad [3723]$$

In the same manner, we may find the values of $\frac{dP'}{dt}$, $\frac{dQ}{dt}$, $\frac{dQ'}{dt}$.

To deduce the expression of $\frac{\delta r}{a}$ from that of $\frac{r\delta r}{a^2}$, we shall denote by $\frac{\delta_{,} r}{a}$, the part of $\frac{\delta r}{a}$ depending on the angle $i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon$, [3724]
and we shall have†

$$\frac{r\delta r}{a^2}=\frac{r}{a}.\left\{\begin{array}{l}\frac{\delta_{,}r}{a}+F.\cos.i.(n't-nt+\varepsilon'-\varepsilon)+Ge.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}\\ +He'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}\end{array}\right\}. \qquad [3725]$$

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and those of r by the small quantity e, which will make it of a less order; it will also be of [3719d]
a different form from those contained in this article, by reason of the factor $\sin.(nt+\varepsilon-\varpi)$.

* (2359) From Taylor's theorem [617], we have $P_{,}=P+T.\frac{dP}{dt}+\frac{1}{2}T^2.\frac{d^2P}{dt^2}+\&c.$; and if we neglect the square and higher powers of T, on account of the smallness of the [3723a]
terms, it becomes as in [3723].

† (2360) Adding $\frac{\delta_{,}r}{a}$ to the part of $\frac{\delta r}{a}$ [3706], we shall obtain all the terms of $\frac{\delta r}{a}$ depending upon the angles $i.(n't-nt+\varepsilon'-\varepsilon)$, $i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon$, [3725a]
$i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon$. Multiplying this by $\frac{r}{a}$, we get [3725].

Value of $\delta_{,} r$, for the angles of the first form.

Hence we deduce*

[3726] $$\frac{\delta_{,} r}{a} = \frac{r\,\delta r}{a^2} + \tfrac{1}{4}\,.\,(F+2G).e^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi\} \atop + \tfrac{1}{2}\,H.ee'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'\}.$$

[3726'] 2. *We shall compute, in the same manner, the terms depending on the angle* $i.(n't-nt+\varepsilon'-\varepsilon)$; and shall suppose, that, by carrying on the approximation to the first power only of the excentricities, we shall have†

Computation for angles of the second form.

[3727] $$\begin{aligned}\frac{\delta r}{a} = F.\cos.i.(n't-nt+\varepsilon'-\varepsilon) &+ Ge.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}\\ &+ G'e.\cos.\{-i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}\\ &+ He'.\cos.\{\ \ i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}\\ &+ H'e'.\cos.\{-i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\};\end{aligned}$$

[3726a] * (2361) Using the symbols [3702a], namely, $T=n't-nt+\varepsilon'-\varepsilon$, $W=nt+\varepsilon-\varpi$, $W'=n't+\varepsilon'-\varpi'$, the expressions [3725] will give, by transposing the terms depending on F, G, H; putting also $W_{,}' = nt + \varepsilon - \varpi';$

[3726b] $$\frac{r}{a}.\frac{\delta_{,} r}{a} = \frac{r\,\delta r}{a^2} - \frac{r}{a}.F.\cos.iT - \frac{r}{a}.Ge.\cos.(iT+W) - \frac{r}{a}.He'.\cos.(iT+W_{,}');$$

[3726c] and from [3701] we get $\frac{r}{a} = 1+\frac{1}{2}e^2-e.\cos.W-\frac{1}{2}e^2.\cos.2W$; which is to be substituted in [3726b]. In making this substitution, we have, by hypothesis, only to notice terms of the order e^2, ee', e'^2, &c. [3702', &c.], and of the same form as [3703]. Now

[3726d] the term $\frac{\delta_{,} r}{a}$ [3724] being already of the second order, we may substitute for the factor $\frac{r}{a}$, by which it is multiplied, the first term of its value [3726c], namely 1; in the coefficient

[3726e] of F, we may use the term $-\frac{1}{2}e^2.\cos.2W$; and in the coefficients of G, H, the term $-e.\cos.W$; by this means it will become as in [3726g]. Reducing this

[3726f] expression by means of [20] Int., and retaining only terms of the form [3703], it becomes as in [3726h], which is of the same form as in [3726].

[3726g] $$\frac{\delta_{,} r}{a} = \frac{r\,\delta r}{a^2} + (\tfrac{1}{2}e^2.\cos.2W).F.\cos.iT + (e.\cos.W).Ge.\cos.(iT+W) \atop + (e.\cos.W).He'.\cos.(iT+W_{,}')$$

[3726h] $$= \frac{r\,\delta r}{a^2} + \tfrac{1}{4}Fe^2.\cos.(iT+2W) + \tfrac{1}{2}Ge^2.\cos.(iT+2W) + \tfrac{1}{2}Hee'.\cos.(iT+W+W_{,}').$$

† (2362) The expression of $\frac{\delta r}{a}$ [3727] is the same as [3706], making the alteration

[3727a] required by the supposition, that i is positive [3727']. If we use, for brevity, the symbols [3726a], this formula will become

[3727b] $$\frac{\delta r}{a} = F.\cos.iT + Ge.\cos.(iT+W) + G'e.\cos.(-iT+W) + He'.\cos.(iT+W_{,}') + H'e'.\cos.(-iT+W_{,}').$$

The case of $i=0$, is separately considered in [3755^iv, &c.].

i being positive [3727*a*, *b*]. We shall then get* [3727′]

$$\frac{r\,\delta r}{a^2}=\frac{\left\{\begin{array}{l}\tfrac{3}{2}n^2.\left\{\begin{array}{l}(G+G').e^2.\cos.i.(n't-nt+\varepsilon'-\varepsilon)\\ \quad+He\,e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+\varpi-\varpi'\}\\ \quad+H'e\,e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)-\varpi+\varpi'\}\end{array}\right\}\\ +n^2.\left\{a^2.\left(\frac{dN}{da}\right)-\frac{2n}{n'-n}.aN\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+L\}\end{array}\right\}}{\{in'-(i+1).n\}.\{in'-(i-1).n\}};\quad (E)$$ [3728] Value of $r\delta r$.

* (2363) In finding the part of $r\delta r$ depending on the angle $i.(n't-nt+\varepsilon'-\varepsilon)$, or iT, by means of the formula [3702], it is necessary to compute the part of $2f\mathrm{d}R+r.\left(\frac{dR}{dr}\right)$, [3728*a*] depending upon the same angle, or upon $R=N.\cos.(iT+L)$ [3704]. This [3728*b*] gives for $\mathrm{d}R$, similarly to [3705*d*], the expression $\mathrm{d}R=nN.i.\sin.(iT+L).dt$; hence $2f\mathrm{d}R=-\frac{2n}{n'-n}.N.\cos.(iT+L)$; also from [3705*a*], we obtain [3728*c*]

$$r.\left(\frac{dR}{dr}\right)=a.\left(\frac{dR}{da}\right)=a.\left(\frac{dN}{da}\right).\cos.(iT+L).$$ [3728*d*]

Multiplying the sum of these two expressions by $1=n^2a^3$ [3709′], we get

$$2f\mathrm{d}R+r.\left(\frac{dR}{dr}\right)=n^2a^2.\left\{a^2.\left(\frac{dN}{da}\right)-\frac{2n}{n'-n}.aN\right\}.\cos.(iT+L).$$ [3728*e*]

Again, if we multiply [3727*b*] by $3n^2a^2.\{e.\cos.W+e^2.\cos.2W\}$, we shall obtain [3728*f*] the terms of [3702], which are multiplied by $3n^2a.\delta r$; and as we have to notice only the terms depending on angles of the form iT [3726′], we may neglect the second [3728*g*] term of this factor $e^2.\cos.2W$ [3728*f*], and then it will become $3n^2a^2.e.\cos.W$. In multiplying [3727*b*], by this last factor, and reducing by [20] Int., the term F produces no term of the required form, and each of the other terms G, G', H, H', produces one; hence we finally obtain

$$\begin{aligned}3n^2a.\delta r.\{e.\cos.W+e^2.\cos.2W\}&=\tfrac{3}{2}n^2a^2.\{(G+G').e^2.\cos.iT+He\,e'.\cos.(iT+W_{\prime}'-W)\\&\qquad+H'ee'.\cos.(iT-W_{\prime}'+W)\end{aligned}$$ [3728*h*]

$$\begin{aligned}&=\tfrac{3}{2}n^2a^2.\{(G+G').e^2.\cos.iT+He\,e'.\cos.(iT+\varpi-\varpi')\\&\qquad+H'e\,e'.\cos.(iT-\varpi+\varpi').\end{aligned}$$ [3728*i*]

The sum of the second members of the expressions [3728*e*, *i*], being represented by $\alpha\,Q$ for brevity, the differential equation [3702] becomes $0=\frac{d^2.(r\delta r)}{dt^2}+n^2.r\delta r+\alpha\,Q$; and [3728*k*] we find by inspection, that $\alpha\,Q$ is equal to the numerator of the second member of [3728], [3728*l*] multiplied by a^2. This equation, being solved as in [3711*b*, *c*], gives $r\delta r=\frac{\alpha\,Q}{m_{\prime}^2-n^2}$, using $m_{\prime}$ [3711*a*]; hence we get $\frac{r\delta r}{a^2}=\frac{\alpha\,Q}{a^2.(m_{\prime}^2-n^2)}=\frac{\alpha\,Q}{a^2.(m_{\prime}-n).(m_{\prime}+n)}$, as in [3728]. [3728*m*]

Value of δv.

[3729]
$$\delta v = \frac{\left\{ \begin{array}{l} \dfrac{2d.(r\delta r)}{a^2 n\, dt} + \tfrac{1}{2}.\left\{ \begin{array}{l} (G-G').e^2.\sin.\ i.(n't-nt+\varepsilon'-\varepsilon) \\ +He e'.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+\varpi-\varpi'\} \\ -He e'.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)-\varpi+\varpi'\} \end{array} \right\} \\ + \left\{ \dfrac{2n}{in'-in}.a^2.\left(\dfrac{dN}{da}\right) - \dfrac{3n^2 i}{(in'-in)^2}.aN \right\}.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+L\} \end{array} \right\}}{\sqrt{1-e^2}} .^*(F)$$

[3730] If we put $\frac{\delta_{\prime} r}{a}$ for the part of $\frac{\delta r}{a}$, which depends on angles of the form $i.(n't-nt+\varepsilon'-\varepsilon)$,† and is also of the order of the square of

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* (2364) The value of δv [3729] is easily deduced from [3715b]; since the denominator $\sqrt{(1-e^2)}$ is the same in both, also the first term of the numerator; and the other terms may be obtained by a calculation similar to that in [3728a—i]. For if we multiply the expression [3728c] by $\frac{3}{2}a n\,dt$, and [3728d] by $2an\,dt$, and take the sum of the products, we shall get

[3729a]
$$3a.fn\,dt.\mathrm{d}R+2a.n\,dt.a.\left(\frac{dR}{da}\right)=\left\{2n.a^2.\left(\frac{dN}{da}\right)-\frac{3n^2}{n'-n}.aN\right\}.\cos.(iT+L).dt.$$

Integrating this we get the two last terms of [3715b], which are the same as the two last terms of the numerator of [3729], or those depending on N, dN. The only remaining term of [3715b] is the second, which is found by multiplying the differential of r [3701]

[3729b] by $-\frac{\delta r}{a^2 n\,dt}$; whence we get $-\frac{dr.\delta r}{a^2 n\,dt}=-\frac{\delta r}{a}.\{e.\sin.W+e^2.\sin.2W\}$.

Substituting $\frac{\delta r}{a}$ [3727], we may neglect the term $e^2.\sin.2W$, and the term F, as in [3728g, &c.]; the other terms being reduced as in [18, 19] Int., retaining only angles of the form iT; we get, in like manner, as in [3728h, &c.];

[3729c]
$$-\frac{dr.\delta r}{a^2 n\,dt}=-\frac{\delta r}{a}.e.\sin.W=\tfrac{1}{2}.\{(G-G').e^2.\sin.iT+Hee'.\sin.(iT+W'_{\prime}-W)-Hee'.\sin.(iT-W'_{\prime}+W)\}$$
$$=\tfrac{1}{2}.\{(G-G').e^2.\sin.iT+Hee'.\sin.(iT+\varpi-\varpi')-Hee'.\sin.(iT-\varpi+\varpi')\};$$

being the same as the terms depending on G, G', H, H', [3729]. We may remark,

[3729d] that from the formulas [3728, 3729], we may deduce others similar to [3714, 3718], in which the secular variations of the elements e, ϖ, &c. are noticed.

† (2365) The second member of [3727] being denoted by F', it will include all the

[3731a] terms of $\frac{\delta r}{a}$, depending on the angle iT, as far as the first power of the excentricities [3726']. Adding to this the expression $\frac{\delta_{\prime} r}{a}$, depending on the same angle, and on terms

[3731b] of the order e^2, ee', &c., we get $\frac{\delta r}{a}=F'+\frac{\delta_{\prime} r}{a}$, for the expression of $\frac{\delta r}{a}$, containing

the excentricities or inclinations, we shall have

Value of $\delta_{\prime} r$ [3731] for angles of the second form.

$$\frac{\delta_{\prime} r}{a} = \frac{r\,\delta r}{a^2} + \tfrac{1}{2}\cdot\{G+G'-F\}.e^2.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$$
$$+\tfrac{1}{2}.He\,e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+\varpi-\varpi'\}$$
$$+\tfrac{1}{2}.H'e\,e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)-\varpi+\varpi'\}.$$

In these three expressions i *must be supposed positive* [3727']. [3731']

3. The great number of inequalities depending on the squares of the excentricities, and of the inclinations, makes it troublesome to compute all of them; and we must be guided in the selection of those which are of a sensible magnitude, by the following considerations. *First.* If the quantity $i\,n'+(2-i).n$ differ but little from $\pm n$; then the one or the other of the divisors $i\,n'+(3-i).n$, $i\,n'+(1-i).n$, in the formula [3711], [3732] will be quite small, and by this means the expression may acquire a sensible value. *Second.* If the quantity $i\,n'+(2-i).n$ be small, those terms of [3733] the formula [3715], having this quantity for a divisor, may become sensible. *Third.* If the quantity $i.(n'-n)$ differ but little from $\pm n$, the one or the other of the divisors $i\,n'-(i+1).n$, $i\,n'-(i-1).n$, of the [3734] formula [3728], will be small, consequently this expression may acquire a sensible value. *Fourth.* If the quantity $i.(n-n')$ be small, the terms

Method of selecting the most important terms.

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terms as far as the order e^2, $e\,e'$, &c. inclusively. Multiplying this by $\frac{r}{a}$, we get $\frac{r}{a}.\frac{\delta_{\prime} r}{a}=\frac{r\,\delta r}{a^2}-\frac{r}{a}.F'$. In the first member of this expression, we may put $\frac{r}{a}=1$, [3731b'] as in [3726d], and in the factor of F', we may use the value [3726c]; hence we shall get

$$\frac{\delta_{\prime} r}{a}=\frac{r\,\delta r}{a^2}+F'.\{-1-\tfrac{1}{2}e^2+e.\cos.W+\tfrac{1}{2}e^2.\cos.2W\} \qquad [3731c]$$

$$=\frac{r\,\delta r}{a^2}-\tfrac{1}{2}e^2.F.\cos.i\,T+F'.e.\cos.W; \qquad [3731d]$$

the second of these expressions being easily deduced from the first, by observing, that of the four terms comprising the factor of F' [3731c], the first term, -1, produces nothing [3731e] of the order e^2, when the value of F' [3727] is substituted; the second term, $-\frac{1}{2}e^2$, produces the term depending on F in [3731d]; the third produces the term depending on F' [3731d]; and the fourth term, $\frac{1}{2}e^2.\cos.2W$, produces nothing of the proposed *form* and *order*. Now substituting, in the term $F'.e.\cos.W$ [3731d], the value of F', [3731f] or the second member of [3727], reducing the products by [20] Int., and retaining only angles of the form $i\,T$, it becomes as in [3731].

[3735] of the formula [3729], which have this divisor, may become sensible. We must therefore estimate carefully all the inequalities subjected to either of these four conditions.

4. The quantities F, G, G', H, H', are determined by the approximative methods in the second book [1016, &c., 3727]. We shall now determine M, N; and for this purpose we shall resume the value of R [913, &c.];*

General value of R.

[3736] First form.

$$R=\frac{m'.(xx'+yy'+zz')}{r'^3}-\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}};$$

[3737] r' being the radius vector of m'. *We shall take, for the fixed plane, the primitive orbit of* m, *and for the axis of* x, *the line of nodes of the orbit*

[3738] *of* m' *upon this plane.* If we put v for the angle formed by the radius r and the axis x; v' for the angle formed by the same axis and by r';

[3739] also γ for the tangent of the inclination of the two orbits to each other, we shall have†

Values of x, y, z,

[3740]

$$x=r.\cos.v;\qquad y=r.\sin.v;\qquad z=0;$$

x', y', z'.

[3740′]

$$x'=r'.\cos.v';\qquad y'=\frac{r'.\sin.v'}{\sqrt{1+\gamma^2}};\qquad z'=\frac{r'.\gamma.\sin.v'}{\sqrt{1+\gamma^2}}.$$

---

* (2366) As there are only two bodies m, m', the value of R, λ [913, 914] become

[3736a]

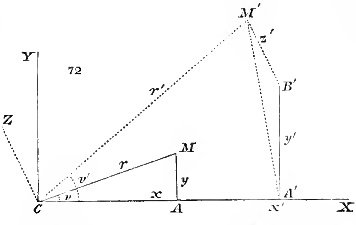
$$R=\frac{m'.(xx'+yy'+zz')}{\{x'^2+y'^2+z'^2\}^{\frac{3}{2}}}-\frac{\lambda}{m},\qquad \frac{\lambda}{m}=\frac{m'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}};$$

[3736b] and by using $r'^2=x'^2+y'^2+z'^2$ [914′], we get [3736].

† (2367) In the annexed figure 72, C is the origin of the co-ordinates, or centre of the sun; CX, CY, CZ, the axes of X, Y, Z, respectively; M the place of the body m, supposing it to be situated nearly upon the

[3740a] plane of xy [3737]; M' the place of the body m'. The co-ordinates of m are $CA=x$, $AM=y$, $z=0$ nearly; those of m' are $CA'=x'$, $A'B'=y'$, $B'M'=z'$. Moreover

[3740b] angle $MCA=v$ [924b], $M'CA'=v'$, tang. $M'A'B'=\gamma$, $CM=r$, $CM'=r'$.



Then in the rectangular triangle CAM, we have $CA=CM.\cos.ACM$, $AM=CM.\sin.ACM$, or in symbols, $x=r.\cos.v$, $y=r.\sin.v$ [3740]. In the

Hence we get, by neglecting the fourth powers of γ,* [3741]

Second form of R.

$$R=\frac{m'r}{r'^2}\cdot\cos.(v'-v)-\frac{m'.\gamma^2}{4}\cdot\frac{r}{r'^2}\cdot\{\cos.(v'-v)-\cos.(v'+v)\}$$
$$-\frac{m'}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{1}{2}}}+\frac{m'.\gamma^2}{4}\cdot\frac{rr'.\{\cos.(v'-v)-\cos.(v'+v)\}}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{3}{2}}}. \quad [3742]$$

We shall suppose, as in [954, 956],

$$\frac{a}{a'^2}.\cos.(n't-nt+\varepsilon'-\varepsilon)-\{a^2-2aa'.\cos.(n't-nt+\varepsilon'-\varepsilon)+a'^2\}^{-\frac{1}{2}} \quad [3743]$$
$$=\tfrac{1}{2}\Sigma.A^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$$

$A^{(i)}, B^{(i)}$.

$$\{a^2-2aa'.\cos.(n't-nt+\varepsilon'-\varepsilon)+a'^2\}^{-\frac{3}{2}}=\tfrac{1}{2}\Sigma.B^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon); \quad [3744]$$

---

rectangular triangle $CA'M'$, we have $CA'=CM'.\cos.A'CM'$, $A'M'=CM'.\sin.A'CM$; or in symbols, $x'=r'.\cos.v'$ [3740'], $A'M'=r'.\sin.v'$. In the rectangular triangle $A'B'M'$, we have, $A'B'=A'M'.\cos.B'A'M'$, $B'M'=A'M'.\sin.B'A'M'$; substituting in these [3740c] the preceding value of $A'M'$, also $\cos.B'A'M'=\frac{1}{\sqrt{(1+\gamma^2)}}$, $\sin.B'A'M'=\frac{\gamma}{\sqrt{(1+\gamma^2)}}$, we get y', z' [3740'].

* (2368) If we neglect γ^4, as in [3741], we shall have $(1+\gamma^2)^{-\frac{1}{2}}=1-\frac{1}{2}\gamma^2$; hence we obtain from [3740'], $y'=r'.\sin.v'-\frac{1}{2}\gamma^2.r'.\sin.v'$; $z'^2=\gamma^2.r'^2.\sin.^2v'$; [3742a] substituting these and the other values [3740, 3740'], in the first member of [3742b], and then reducing by [24, 17] Int., we get [3742c];

$$xx'+yy'+zz'=rr'.(\cos.v'.\cos.v+\sin.v.\sin.v')-\tfrac{1}{2}\gamma^2.rr'.\sin.v.\sin.v' \quad [3742b]$$
$$=rr'.\cos.(v'-v)-\tfrac{1}{4}\gamma^2.rr'.\{\cos.(v'-v)-\cos.(v'+v)\}. \quad [3742c]$$

Substituting this last expression in the first term of R [3736], we get the two first terms of [3742]. Again, if we develop the first member of [3742e], and substitute $r^2=x^2+y^2+z^2$, $r'^2=x'^2+y'^2+z'^2$ [3740, 3740'], also the expression [3742c], we get [3742d]

$$(x'-x)^2+(y'-y)^2+(z'-z)^2=(x^2+y^2+z^2)-2.(xx'+yy'+zz')+(x'^2+y'^2+z'^2) \quad [3742e]$$
$$=\{r^2-2rr'.\cos.(v'-v)+r'^2\}+\tfrac{1}{2}\gamma^2.rr'.\{\cos.(v'-v)-\cos.(v'+v)\}. \quad [3742f]$$

Involving this to the power $-\frac{1}{2}$, we get

$$\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{-\frac{1}{2}}=\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{-\frac{1}{2}}$$
$$-\tfrac{1}{4}\gamma^2.rr'.\{\cos.(v'-v)-\cos.(v'+v)\}.\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{-\frac{3}{2}}; \quad [3742g]$$

substituting this in the last term of [3736], we get the two last terms of [3742].

[3744′] and shall represent $R = M.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}$ [3703], by the following function;

[3745] Terms of R [3745′] of the first form. [3745″] [3745‴]

$$R = M^{(0)}.e^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi\}$$
$$+M^{(1)}.ee'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'\}$$
$$+M^{(2)}.e'^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi'\}$$
$$+M^{(3)}.\gamma^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\Pi\};$$

Π. [3746] Π *being the longitude of the ascending node of the orbit of* m' *upon that of* m, counted from the line which is taken for the origin of the angle $nt+\varepsilon$. We have, as in [669],

[3747] $$\frac{r}{a} = 1+\tfrac{1}{2}e^2-e.\cos.(nt+\varepsilon-\varpi)-\tfrac{1}{2}e^2.\cos.2.(nt+\varepsilon-\varpi);$$

[3748] $$v = nt+\varepsilon-\Pi+2e.\sin.(nt+\varepsilon-\varpi)+\tfrac{5}{4}e^2.\sin.2.(nt+\varepsilon-\varpi).$$

From these we get the values of $\frac{r'}{a'}$, v', by marking with one accent, the quantities n, e, ε, &c. Then we have, as in [955], the product of

$$\Sigma.A^{(i)}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)\},$$

by the sine or cosine of any angle $ft+I$; which is equal to

[3749] $$\Sigma.A^{(i)}.\begin{smallmatrix}\sin.\\ \cos.\end{smallmatrix}\{i.(n't-nt+\varepsilon'-\varepsilon)+ft+I\}.$$

Hence we easily obtain*

[3750] $$M^{(0)} = \frac{m'}{8}.\left\{i.(4i-5).A^{(i)}+2.(2i-1).a.\left(\frac{d.A^{(i)}}{da}\right)+a^2.\left(\frac{dd.A^{(i)}}{da^2}\right)\right\};$$

Values corresponding to the first form.

[3750′] $$M^{(1)} = -\frac{m'}{4}.\left\{4.(i-1)^2.A^{(i-1)}+2.(i-1).a.\left(\frac{d.A^{(i-1)}}{da}\right)-2.(i-1).a'.\left(\frac{d.A^{(i-1)}}{da'}\right)-aa'.\left(\frac{dd.A^{(i-1)}}{dada'}\right)\right\};$$

[3750″] $$M^{(2)} = \frac{m'}{8}.\left\{(i-2).(4i-3).A^{(i-2)}-2.(2i-3).a'.\left(\frac{d.A^{(i-2)}}{da'}\right)+a'^2.\left(\frac{dd.A^{(i-2)}}{da'^2}\right)\right\};$$

[3750‴] $$M^{(3)} = -\frac{m'}{8}.aa'.B^{(i-1)}.$$

[3750a] * (2369) In [952, 953] we have $r=a.(1+u_{,})$; $v=nt+\varepsilon-\Pi+v_{,}$; the term Π being added to conform to the present notation. Comparing these with [3747, 3748],

[3750b] we get the following values of $u_{,}$, $v_{,}$, also the similar ones of $u_{,}'$, $v_{,}'$, using the abridged symbols [3726a];

[3750c] $$u_{,} = -e.\cos.W+\tfrac{1}{2}e^2-\tfrac{1}{2}e^2.\cos.2W; \qquad v_{,}=2e.\sin.W+\tfrac{5}{4}.e^2.\sin.2W;$$

[3750d] $$u_{,}' = -e'.\cos.W'+\tfrac{1}{2}e'^2-\tfrac{1}{2}e'^2.\cos.2W'; \qquad v_{,}'=2e'.\sin.W'+\tfrac{5}{4}.e'^2.\sin.2W';$$

and in the case of $i=1$ [3750y, y'], we have

$$M^{(3)}=\frac{m'}{4}\cdot\frac{a}{a'^2}-\frac{m'}{8}\cdot aa'.B^{(0)}.$$ [3751]

Finding the squares and products of these quantities, then reducing them by [17—20] Int., retaining merely the terms of the second degree in e, e', γ, which are the only terms now under consideration [3702'], we obtain the following system of equations. In these expressions we have substituted for W' its value $W'=T+W+\varpi-\varpi'$ [3726a], [3750e] in order that the quantity $n't+\varepsilon'$ may not appear in the terms of R, except in connexion with i, as in the assumed form of these terms of R, given in [3745, &c., 957]. The numbers prefixed to the formulas [3750f] express the order of the terms in the value of R [957].

$$
\begin{array}{lll}
2 & u_{,} = & \tfrac{1}{2}e^2-\tfrac{1}{2}e^2.\cos.2W; \\
3 & u'_{,} = & \tfrac{1}{2}e'^2-\tfrac{1}{2}e'^2.\cos.2.(T+W+\varpi-\varpi'); \\
4 & v'_{,} = & \tfrac{5}{4}e'^2.\sin.2.(T+W+\varpi-\varpi'); \\
5 & v_{,} = & \tfrac{5}{4}e^2.\sin.2W; \\
6 & u_{,}^2 = & \tfrac{1}{2}e^2+\tfrac{1}{2}e^2.\cos.2W; \\
7 & u_{,}u'_{,} = & \tfrac{1}{2}ee'.\cos.(T+\varpi-\varpi')+\tfrac{1}{2}ee'.\cos.(T+2W+\varpi-\varpi'); \\
8 & u'^2_{,} = & \tfrac{1}{2}e'^2+\tfrac{1}{2}e'^2.\cos.2.(T+W+\varpi-\varpi'); \\
9 & u_{,}v'_{,} = & -ee'.\sin.(T+\varpi-\varpi')-ee'.\sin.(T+2W+\varpi-\varpi'); \\
10 & u_{,}v_{,} = & -e^2.\sin.2W; \\
11 & u'_{,}v'_{,} = & -e'^2.\sin.2.(T+W+\varpi-\varpi'); \\
12 & u'_{,}v_{,} = & ee'.\sin.(T+\varpi-\varpi')-ee'.\sin.(T+2W+\varpi-\varpi'); \\
13 & v'^2_{,} = & 2e'^2-2e'^2.\cos.2.(T+W+\varpi-\varpi'); \\
14 & v_{,}v'_{,} = & 2ee'.\cos.(T+\varpi-\varpi')-2e'^2.\cos.(T+2W+\varpi-\varpi'); \\
15 & v_{,}^2 = & 2e^2-2e^2.\cos.2W.
\end{array}
$$ [3750f]

Substituting these in [957], we shall obtain the terms of R depending upon $M^{(0)}$, $M^{(1)}$, $M^{(2)}$, [3745, &c.]. The terms of the form $M^{(3)}$, arising from the terms of z, z', in the two [3750g] lower lines of the value of R [957], will be considered hereafter in [3750u, &c.]. In making these substitutions, we must use the following formulas, which are the same as those in [954c, 955a, 955f], changing W into $W_{,}$, to prevent confusion in the notation.

$$\cos.W_{,}.\tfrac{1}{2}\Sigma.A^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.A^{(i)}.\cos.(iT+W_{,});$$ [3750h]

$$\sin.W_{,}.\tfrac{1}{2}\Sigma.i.A^{(i)}.\sin.iT=-\tfrac{1}{2}\Sigma.i.A^{(i)}.\cos.(iT+W_{,});$$ [3750i]

$$\cos.W_{,}.\tfrac{1}{2}\Sigma.i^2.A^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.i^2.A^{(i)}.\cos.(iT+W_{,}).$$ [3750k]

We shall represent $R = N . \cos. \{ i . (n't - nt + \varepsilon' - \varepsilon) + L \}$ [3704]

We shall consider the terms depending on each of the factors $M^{(0)}$, $M^{(1)}$, $M^{(2)}$ [3745, &c.] separately; and in the first place, shall take the terms of the form $M^{(0)}. e^2. \cos. (iT + 2W)$. These are evidently produced by the factors $\sin. 2W$, $\cos. 2W$, which occur in the terms

[3750l] of [3750f], marked 2, 5, 6, 10, 15; reducing the products by the formulas [3750h—k]. These five terms, marked in the order in which they occur, without reduction, supposing them all to have the common factor $\frac{m'}{8} . e^2 . \cos. (iT + 2W)$, and omitting Σ for brevity, are

[3750m] $$-2a . \left(\frac{dA^{(i)}}{da}\right) - 5iA^{(i)} + a^2 . \left(\frac{ddA^{(i)}}{da^2}\right) + 4ia . \left(\frac{dA^{(i)}}{da}\right) + 4i^2 . A^{(i)}.$$

This expression is easily reduced to the form of the coefficient of $\frac{m'}{8}$, in the value of $M^{(0)}$ [3750]. Proceeding in the same manner with the parts of the terms 7, 9, 12, 14 [3750f], depending on the angle $T + 2W + \varpi - \varpi'$, we find that they produce in R [957] terms of the form $M^{(1)} e e' . \cos. \{(i+1) . T + 2W + \varpi - \varpi'\}$, which may

[3750n] be represented by $-\frac{m'}{4} . ee' . \cos. \{(i+1) . T + 2W + \varpi - \varpi'\}$, multiplied by the following expression, which includes the terms as they occur, without any reduction;

[3750o] $$-aa' . \left(\frac{ddA^{(i)}}{da\,da'}\right) + 2ia . \left(\frac{dA^{(i)}}{da}\right) - 2ia' . \left(\frac{dA^{(i)}}{da'}\right) + 4i^2 . A^{(i)}.$$

We may change in this i into $i-1$ [3715'], and then we get for the coefficient

[3750p] of $-\frac{m'}{4} . ee' . \cos. (iT + 2W + \varpi - \varpi')$, or $-\frac{m'}{4} . ee' . \cos. (iT + 2nt + 2\varepsilon - \varpi - \varpi')$, an expression which is the same as the coefficient of $-\frac{m'}{4}$, in the value of $M^{(1)}$ [3750']. Again, the terms 3, 4, 8, 11, 13 [3750f], depending on the angle $2 . (T + W + \varpi - \varpi')$, produce in R [957], terms of the form $M^{(2)} . e'^2 . \cos. \{(i+2) . T + 2W + 2\varpi - 2\varpi'\}$;

[3750q] which may be expressed by $\frac{m'}{8} . e'^2 . \cos. \{(i+2) . T + 2W + 2\varpi - 2\varpi'\}$, multiplied by the following function, which includes all these terms as they occur, without reduction;

[3750r] $$-2a' . \left(\frac{dA^{(i)}}{da'}\right) + 5iA^{(i)} + a'^2 . \left(\frac{ddA^{(i)}}{da'^2}\right) - 4ia' . \left(\frac{dA^{(i)}}{da'}\right) + 4i^2 . A^{(i)};$$

or, as it may be written,

[3750r'] $$i . (4i+5) . A^{(i)} - 2 . (2i+1) . a' . \left(\frac{dA^{(i)}}{da'}\right) + a'^2 . \left(\frac{ddA^{(i)}}{da'^2}\right).$$

We may change in this i into $i-2$ [3715'], and then we have for the coefficient

[3750s] of $\frac{m'}{8} . e'^2 . \cos. (iT + 2W + 2\varpi - 2\varpi')$, or $\frac{m'}{8} . \cos. (iT + 2nt + 2\varepsilon - 2\varpi')$, the

[3750t] same quantity as the coefficient of $\frac{m'}{8}$, in the value of $M^{(2)}$ [3750''].

by the following terms;*

Terms of R depending on angles of the second form.

$$R = N^{(0)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon) \qquad [3752]$$

$$+N^{(1)}.ee'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+\varpi-\varpi'\} \qquad [3752']$$

$$+N^{(2)}.ee'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)-\varpi+\varpi'\}; \qquad [3752'']$$

We shall now notice the terms depending on z, z', which were neglected in [3750g]; these are the same as those depending on γ^2, in the value of R [3742]. As we neglect terms of a higher order than γ^2, we may substitute, in these terms, the values $r=a$; $r'=a'$; $v=nt+\varepsilon-\Pi$; $v'=n't+\varepsilon'-\Pi$; $v'-v=n't-nt+\varepsilon'-\varepsilon=T$; [3750u]
$v'+v=n't+nt+\varepsilon'+\varepsilon-2\Pi=T+2nt+2\varepsilon-2\Pi$; hence this part of R [3742] becomes

$$R=-\frac{m'\gamma^2}{4}\cdot\frac{a}{a'^2}.\{\cos.T-\cos.(T+2nt+2\varepsilon-2\Pi)\}$$
$$+\frac{m'\gamma^2}{4}\cdot\frac{aa'}{\{a^2-2aa'.\cos.T+a'^2\}^{\frac{3}{2}}}.\{\cos.T-\cos.(T+2nt+2\varepsilon-2\Pi)\}. \qquad [3750v]$$

Substituting, in the last term, the value of the denominator [3744], namely $\frac{1}{2}\Sigma.B^{(i)}.\cos.iT$, and reducing by means of the formula [3750h], it becomes

$$R=\frac{m'\gamma^2}{4}.\left\{\begin{array}{l}-\frac{a}{a'^2}.\cos.T+\frac{a}{a'^2}.\cos.(T+2nt+2\varepsilon-2\Pi) \\ +\frac{1}{2}aa'.\Sigma.B^{(i)}.\cos.(i+1).T-\frac{1}{2}aa'.\Sigma.B^{(i)}.\cos.\{(i+1).T+2nt+2\varepsilon-2\Pi\}\end{array}\right\}. \qquad [3750w]$$

The last term of this expression, changing i into $i-1$ [3715'], becomes

$$-\frac{m'\gamma^2}{8}.aa'.\Sigma.B^{(i-1)}.\cos.(iT+2nt+2\varepsilon-2\Pi); \qquad [3750x]$$

which is of the same form as [3745'''], and is equal to it by putting $M^{(3)}=-\frac{m'}{8}.aa'.\Sigma.B^{(i-1)}$, as in [3750''']. In the case of $i=1$, the term [3750x] becomes

$$-\frac{m'\gamma^2}{8}.aa'.B^{(0)}.\cos.(T+2nt+2\varepsilon-2\Pi); \qquad [3750y]$$

connecting this with the second term of [3750w], namely,

$$\frac{m'\gamma^2}{4}.\frac{a}{a'^2}.\cos.(T+2nt+2\varepsilon-2\Pi); \qquad [3750y']$$

and putting the whole equal to this value [3745'''], we get, for this case, the same value of $M^{(3)}$, as in [3751].

* (2370) By proceeding as in the last note, we shall find, that the substitution of the values [3750f] in R [957], produces terms depending on the angle iT, $iT+\varpi-\varpi'$, [3752a]
$iT-\varpi+\varpi'$, as in [3752—3752''], without W, which occurs in the forms [3745—3745'''].

and we shall have

Coefficients depending on angles of the second form.

[3753] $$\mathcal{N}^{(0)}=-\frac{m'}{4}\cdot\Big\{(e^2+e'^2)\cdot\Big[4i^2.\mathcal{A}^{(i)}-2a.\Big(\frac{d\mathcal{A}^{(i)}}{da}\Big)-a^2.\Big(\frac{dd\mathcal{A}^{(i)}}{da^2}\Big)\Big]-\frac{\gamma^2}{2}.aa'.[B^{(i-1)}+B^{(i+1)}]\Big\};$$

[3753'] $$\mathcal{N}^{(1)}=\frac{m'}{4}\cdot\Big\{4.(i-1)^2.\mathcal{A}^{(i-1)}-2.(i-1).a.\Big(\frac{d\mathcal{A}^{(i-1)}}{da}\Big)-2.(i-1).a'.\Big(\frac{d\mathcal{A}^{(i-1)}}{da'}\Big)+aa'.\Big(\frac{dd\mathcal{A}^{(i-1)}}{da\,da'}\Big)\Big\};$$

[3753''] $$\mathcal{N}^{(2)}=\frac{m'}{4}\cdot\Big\{4.(i+1)^2.\mathcal{A}^{(i+1)}+2.(i+1).a.\Big(\frac{d\mathcal{A}^{(i+1)}}{da}\Big)+2.(i+1).a'.\Big(\frac{d\mathcal{A}^{(i+1)}}{da'}\Big)+aa'.\Big(\frac{dd\mathcal{A}^{(i+1)}}{da\,da'}\Big)\Big\}.$$

We shall calculate these terms separately, commencing with the angle iT, which is
[3752b] produced in R [957], by the substitution of the terms $\frac{1}{2}e^2$, $\frac{1}{2}e'^2$, occurring in the terms of [3750f], marked 2, 3, 6, 8, 13, 15. These quantities produce in R, the
[3752b'] expression $-\frac{m'}{4}.\cos.iT$, multiplied by the following terms, written down in the order in which they appear, without reduction, and omitting Σ for brevity;

[3752c] $$-e^2.a.\Big(\frac{d\mathcal{A}^{(i)}}{da}\Big)-e'^2.a'.\Big(\frac{d\mathcal{A}^{(i)}}{da'}\Big)-\tfrac{1}{2}e^2.a^2.\Big(\frac{dd\mathcal{A}^{(i)}}{da^2}\Big)-\tfrac{1}{2}e'^2.a'^2.\Big(\frac{dd\mathcal{A}^{(i)}}{da'^2}\Big)+2e'^2.i^2.\mathcal{A}^{(i)}+2e^2.i^2.\mathcal{A}^{(i)}.$$

Now if we multiply the first of the equations [1003] by -1, and the third of these equations by $-\frac{1}{2}$; the sum of their products will give

[3752d] $$-a'.\Big(\frac{d\mathcal{A}^{(i)}}{da'}\Big)-\tfrac{1}{2}a'^2.\Big(\frac{dd\mathcal{A}^{(i)}}{da'^2}\Big)=-a.\Big(\frac{d\mathcal{A}^{(i)}}{da}\Big)-\tfrac{1}{2}a^2.\Big(\frac{dd\mathcal{A}^{(i)}}{da^2}\Big);$$

substituting this in [3752c], we find, that the coefficient of e'^2 is the same as that of e^2, and the whole expression becomes

[3752e] $$-\frac{m'}{4}.(e^2+e'^2).\Big\{2i^2.\mathcal{A}^{(i)}-a.\Big(\frac{d\mathcal{A}^{(i)}}{da}\Big)-\tfrac{1}{2}a^2.\Big(\frac{dd\mathcal{A}^{(i)}}{da^2}\Big)\Big\}.\cos.iT.$$

To this we must add the third term of [3750w], depending on $\cos.(i+1).T$, which,
[3752f] by changing i into $i-1$, as in [3750x], becomes $\frac{m'\gamma^2}{4}.\frac{1}{2}aa'.\Sigma.B^{(i-1)}.\cos.iT$. The
[3752f'] expression [3752e] is the same for $-i$, as for $+i$; because $\mathcal{A}^{(-i)}=\mathcal{A}^{(i)}$ [954'']. Moreover, the term [3752f], by the same change of i, using $B^{(-i-1)}=B^{(i+1)}$ [956'],
[3752g] becomes $\frac{m'\gamma^2}{4}.\frac{1}{2}aa'.\Sigma.B^{(i+1)}.\cos.iT$. Hence, if we use only positive values of i, we must double the function [3752e], and add to it the two expressions [3752f, g]; the sum of these three functions, being put equal to $\mathcal{N}^{(0)}.\cos.iT$ [3752], gives the same value of $\mathcal{N}^{(0)}$, as in [3753]. In the case of $i=1$, this sum must be increased by the
[3752h] first term of [3750w]; by which means $\mathcal{N}^{(0)}$ is increased by the quantity given in [3754]. The case of $i=0$, which is separately considered in [3755$^{\text{iv}}$], produces, in R, the following expression, which is deduced from [3752c, f], by putting $i=0$;

[3752i] $$\frac{m'}{8}.e^2.\Big\{2a.\Big(\frac{d\mathcal{A}^{(0)}}{da}\Big)+a^2.\Big(\frac{dd\mathcal{A}^{(0)}}{da^2}\Big)\Big\}+\frac{m'}{8}.e'^2.\Big\{2a'.\Big(\frac{d\mathcal{A}^{(0)}}{da'}\Big)+a'^2.\Big(\frac{dd\mathcal{A}^{(0)}}{da'^2}\Big)\Big\}+\frac{m'}{8}.aa'.B^{(1)}.\gamma^2.$$

In these three last expressions i is supposed to be positive and greater than [3753‴]
zero. In case $i=1$, we must add to $N^{(0)}$ the term $-\frac{m'\gamma^2}{4}\cdot\frac{a}{a'^2}$ [3752h]. [3754]

It is more convenient, for numerical calculations, to have the differentials relative to only one of the two quantities a, a', in these formulas.*

Proceeding in the same manner with the angle $iT+\varpi-\varpi'$ [3752′], we find, that terms of this form are produced in R [957], by the substitution of the parts of the terms of [3750f] depending on the angle $T+\varpi-\varpi'$, and marked 7, 9, 12, 14; reducing them by means of the formulas [954c, 955a, f]. Hence this part of R becomes equal to $\frac{m'}{4}.ee'.\cos.\{(i+1).T+\varpi-\varpi'\}$, multiplied by the following expression, retaining [3752k]
the terms according to the order of the numbers, without any reduction;

$$aa'.\left(\frac{dd.A^{(i)}}{da\,da'}\right)-2ia.\left(\frac{d.A^{(i)}}{da}\right)-2ia'.\left(\frac{d.A^{(i)}}{da'}\right)+4i^2.A^{(i)}. \qquad [3752l]$$

Changing i into $i-1$, in [3752k, l], we find, that this part of R may be represented by $\;.ee'.N^{(1)}.\cos.(iT+\varpi-\varpi')$ [3753′]; observing, that this change in the value [3752m]
of i, reduces the expression [3752l] to the same form as the factor of $\frac{m'}{4}$, in the value of $N^{(1)}$ [3753′]. We must retain only the positive values of i in [3752′, 3753′]; for if we change the sign of i, the expression $\cos.(iT+\varpi-\varpi')$, becomes $\cos.(-iT+\varpi-\varpi')$ [3752n]
or $\cos.(iT-\varpi+\varpi')$, which is of the same form as [3752″]. Hence it appears, that we may deduce $N^{(2)}$ [3752″] from $N^{(1)}$ [3752′], by changing the sign of i. Performing [3752o]
this operation on [3753′], we get [3753″], using $A^{(-i-1)}=A^{(i+1)}$ [3752f']. Finally, the case of $i=0$, is found by putting $i=0$ in [3752m], or in the similar terms depending [3752o']
on $N^{(2)}$ [3752o]; observing, that when $i=0$, the expressions $N^{(1)}$, $N^{(2)}$ [3753′, 3753″] become equal to each other; and this part of R becomes

$$\frac{m'}{4}.ee'.\left\{4A^{(1)}+2a.\left(\frac{d.A^{(1)}}{da}\right)+2a'.\left(\frac{d.A^{(1)}}{da'}\right)+aa'.\left(\frac{dd.A^{(1)}}{da\,da'}\right)\right\}.\cos.(\varpi-\varpi'). \qquad [3752p]$$

* (2371) In making the reduction of $M^{(1)}$ from [3750′] to [3755], it will be convenient to use the abridged symbols $a^m.\left(\frac{d^m.A^{(n)}}{da^m}\right)=A^{(n)}_m$; $a'^m.\left(\frac{d^m.A'^{(n)}}{da'^m}\right)=A'^{(n)}_m$; and as the [3755$a$]
index n is the same for all the terms depending on $M^{(1)}$, we may neglect it, and put simply

$$A^{(n)}=A_0,\qquad a.\left(\frac{d.A^{(n)}}{da}\right)=A_1,\qquad a^2.\left(\frac{dd.A^{(n)}}{da^2}\right)=A_2,\ \text{\&c.};$$

$$A'^{(n)}=A'_0,\qquad a'.\left(\frac{d.A'^{(n)}}{da'}\right)=A'_1,\qquad a'^2.\left(\frac{dd.A'^{(n)}}{da'^2}\right)=A'_2,\ \text{\&c.}; \qquad [3755b]$$

This is obtained by means of [1003], from which we get

[3755] $$M^{(1)}=-\frac{m'}{4}\cdot\left\{(2i-2)\cdot(2i-1)\cdot A^{(i-1)}+2\cdot(2i-1)\cdot a\cdot\left(\frac{dA^{(i-1)}}{da}\right)+a^2\cdot\left(\frac{ddA^{(i-1)}}{da^2}\right)\right\};$$

[3755'] $$M^{(2)}=\frac{m'}{8}\cdot\left\{(4i^2-7i+2)\cdot A^{(i-2)}+2\cdot(2i-1)\cdot a\cdot\left(\frac{dA^{(i-2)}}{da}\right)+a^2\cdot\left(\frac{ddA^{(i-2)}}{da^2}\right)\right\};$$

Reduced values.

[3755''] $$N^{(1)}=\frac{m'}{4}\cdot\left\{(2i-2)\cdot(2i-1)\cdot A^{(i-1)}-2a\cdot\left(\frac{dA^{(i-1)}}{da}\right)-a^2\cdot\left(\frac{ddA^{(i-1)}}{da^2}\right)\right\};$$

[3755'''] $$N^{(2)}=\frac{m'}{4}\cdot\left\{(2i+2)\cdot(2i+1)\cdot A^{(i+1)}-2a\cdot\left(\frac{dA^{(i+1)}}{da}\right)-a^2\cdot\left(\frac{ddA^{(i+1)}}{da^2}\right)\right\}.$$

[3755iv] 5. The case of $i=0$ deserves particular attention. We shall resume the expression [923], and shall consider, in the first place, the

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and the same symbols may be used in the reduction of $M^{(2)}$, $N^{(1)}$, $N^{(2)}$. Then the coefficient of $-\frac{1}{4}m'$, in the value of $M^{(1)}$ [3750'], will become, by the substitution of the first and second formulas [1003],

[3755c]
$$\begin{aligned}&4\cdot(i-1)^2\cdot A_0+2\cdot(i-1)\cdot A_1+2\cdot(i-1)\cdot\{A_0+A_1\}+\{2\cdot A_1+A_2\}\\&=2\cdot(i-1)\cdot\{2\cdot(i-1)+1\}\cdot A_0+\{4i-2\}\cdot A_1+A_2\\&=(2i-2)\cdot(2i-1)\cdot A_0+2\cdot(2i-1)\cdot A_1+A_2;\end{aligned}$$

which is the same as the coefficient of $-\frac{1}{4}m'$ in [3755]. In like manner, the coefficient of $\frac{m'}{8}$, in [3750''], becomes, by using the first and third of the formulas [1003],

[3755d]
$$\begin{aligned}&(i-2)\cdot(4i-3)\cdot A_0+2\cdot(2i-3)\cdot\{A_0+A_1\}+\{2A_0+4A_1+A_2\}\\&=\{(i-2)\cdot(4i-3)+4i-4\}\cdot A_0+2\cdot(2i-1)\cdot A_1+A_2;\end{aligned}$$

which is easily reduced to the form of the factor of $\frac{m'}{8}$, in $M^{(2)}$ [3755']. Again, the factor of $\frac{1}{4}m'$, in the value of $N^{(1)}$ [3753'] becomes, by the substitution of the values in the first and second formulas [1003];

[3755e]
$$\begin{aligned}&4\cdot(i-1)^2\cdot A_0-2\cdot(i-1)\cdot A_1+2\cdot(i-1)\cdot\{A_0+A_1\}+\{-2A_1-A_2\}\\&=2\cdot(i-1)\cdot\{2\cdot(i-1)+1\}\cdot A_0-2A_1-A_2;\end{aligned}$$

which is the same as the coefficient of $\frac{1}{4}m'$, in the value of $N^{(1)}$ [3755'']. From this we may easily obtain $N^{(2)}$, by merely changing the sign of i, as in [3752o].

* (2372) The terms of R depending on $i=0$, are given in [3752i, 3752p]; they are independent of nt, $n't$, and produce in δv a secular equation [3773]; and on this account, they are carefully computed, though it is finally found, in [4446, 4505], that

term $\frac{d.(2r.d\delta r+dr.\delta r)}{r^2 dv}$, of the expression of $d\delta v$, given by this formula. We have, as in [1037], by *noticing only the terms affected with the arc of a circle* nt,* [3755v]

On the secular part of δv.

$$\frac{r}{a}=1-h.\sin.(nt+\varepsilon)-l.\cos.(nt+\varepsilon); \qquad [3756]$$

$$\frac{\delta r}{a}=\tfrac{1}{2}m'.(lC+l'D).nt.\sin.(nt+\varepsilon) \qquad [3756']$$
$$-\tfrac{1}{2}m'.(hC+h'D).nt.\cos.(nt+\varepsilon).$$

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they are insensible. To reduce these terms of R to the form [3761], we may use the following symbols, given in [1022, 1033];

$$h=e.\sin.\varpi; \quad l=e.\cos.\varpi; \quad h'=e'.\sin.\varpi'; \quad l'=e'.\cos.\varpi'; \qquad [3756a]$$

$$e^2=h^2+l^2; \quad e'^2=h'^2+l'^2; \qquad [3756b]$$

$$\gamma.\sin.\Pi=p'-p; \quad \gamma.\cos.\Pi=q'-q; \quad \gamma^2=(p'-p)^2+(q'-q)^2. \qquad [3756c]$$

Now substituting, in [3752i], the values of e^2, e'^2, γ^2 [3756b, c], they will produce, respectively, the first, second, and fourth lines of the expression of R or δR [3761]; observing, that, by using the sign δ, as in [917'], these terms of R may be represented by δR. The term [3752p] produces the third line of the same value of δR; for we have, by using [3756a], [3756d]

$$ee'.\cos.(\varpi-\varpi')=ee'.(\sin.\varpi.\sin.\varpi'+\cos.\varpi.\cos.\varpi')=hh'+ll'; \qquad [3756e]$$

substituting this in [3752p], it produces this term of δR [3761], having the factor $hh'+ll'$. This value of δR is to be used in the formula [923], to compute the part of δv, which is independent of the angles nt, $n't$; and of the second degree in h, h', l, l', &c. [3756f]

* (2373) The object of the present computation is merely to ascertain the part of δv, mentioned in [3756f], by means of the expression of $d\delta v$ [923]. This may be reduced to the form [3757d], by observing, that $rR'=r.\left(\frac{dR}{dr}\right)=a.\left(\frac{dR}{da}\right)$ [928', 962], [3757a] and that we have, identically, $2r.\delta R'+R'.\delta r=2\delta.(rR')-R'.\delta r$. From the first [3757b] of these equations, we see that R' is of the same order as R, or of the order m'; and by rejecting terms of the order m'^2, as in [3768'], we may neglect the term $-R'.\delta r$, and then this expression [3757b], by the substitution of rR' [3757a], becomes

$$2r.\delta R'+R'.\delta r=2\delta.(rR')=2a.\left(\frac{d\delta R}{da}\right). \qquad [3757c]$$

Substituting this in [923], also the value of r^2dv [3759], we get

$$d\delta v=\frac{d.(2r.d\delta r+dr.\delta r)+dt^2.\left\{3f\delta\,dR+2a.\left(\frac{d\delta R}{da}\right)\right\}}{a^2.ndt.\sqrt{(1-e^2)}}. \qquad [3757d]$$

[3757] These give, by *noticing only the terms depending on the squares and products of* h, l, h', l', *independent of the sines and cosines of* $nt+\varepsilon$, *and its multiples*,*

[3758] $$d.(2r.d\delta r+dr.\delta r)=-\frac{m'.n^2a^2.dt^2}{4}.\{(h^2+l^2).C+(hh'+ll').D\}.$$

In this we must substitute δR [3764], and those terms of dr, δr, which produce quantities of the form and order mentioned in [3756f]. Now these quantities will be obtained by selecting, from the general value [1037], the three terms contained in the
[3757e] second member of [3756], for $\frac{r}{a}$; and the terms in the second member of [3756'], for $\frac{\delta r}{a}$. It is unnecessary to use any other terms of a higher order in h, l, &c.; for if we retain, in $\frac{r}{a}$, any term of the order h^2, hl, l^2, connected with $\sin.2.(nt+\varepsilon)$ or $\cos.2.(nt+\varepsilon)$, it must also be connected, in [3757d], with terms of $\frac{\delta r}{a}$, or of its differential, of the same
[3757f] forms and order, producing terms of the *fourth* order in h, l, and independent of the angles nt, $n't$, which are neglected in this article. The same remarks will apply to other terms of $\frac{r}{a}$, depending on higher multiples of the angle $nt+\varepsilon$. Having adopted
[3757g] this form of $\frac{r}{a}$, it will be unnecessary to retain any terms of $\frac{\delta r}{a}$ [1023, 1037], except those in the second member of [3756]; for, though other terms in $\frac{\delta r}{a}$ [1023], of the
[3757h] forms P, $P'.\sin.(nt+\varepsilon)$, $P''.\cos.(nt+\varepsilon)$, might produce, in $2r.d\delta r+dr.\delta r$, quantities independent of the sine or cosine of the angle $nt+\varepsilon$, or its multiples; yet if we notice only terms of the order m', they will vanish in its differential, which occurs in [3757d, 3760]; and this does not happen with the arcs of a circle retained in [3756'], as is shown in [3760].

* (2374) In finding the terms of $2r.d\delta r+dr.\delta r$, of the order m', it is only
[3758a] necessary to notice quantities of the form $Q.nt.dt$, containing the arc of a circle nt, Q being constant; for if the function contain any constant term, or elements of the planet's orbit, it will either vanish from its differential [3760] or become of the order m'^2, &c.; and terms depending on the sine and cosine of $nt+\varepsilon$, are neglected [3757]. Substituting r [3756], and its differential, in the first member of the following expression, we get

[3758b] $$\begin{aligned}2r.d\delta r+dr.\delta r=&\{2a-2ah.\sin.(nt+\varepsilon)-2al.\cos.(nt+\varepsilon)\}.d\delta r\\&+\{-ah.\cos.(nt+\varepsilon)+al.\sin.(nt+\varepsilon)\}.ndt.\delta r;\end{aligned}$$

in which we must substitute the values of δr, $d\delta r$. Now if, for a moment, we
[3758c] put $\frac{1}{2}m'.a.(lC+l'D)=L$, $\frac{1}{2}m'.a.(hC+h'D)=H$, we shall get, from [3756']

We then have $r^2\,d\,v = a^2\,n\,d\,t\,.\sqrt{1-e^2}$ [1057]; hence we shall obtain [3759]

$$\frac{d\,.(2\,r\,.\,d\,\delta\,r + d\,r\,.\,\delta\,r)}{r^2\,d\,v} = -\frac{m'.\,n\,d\,t}{4}\,.\,\{(h^2+l^2)\,.\,C + (h\,h' + l\,l')\,.\,D\}. \qquad [3760]$$

We have, in [1071],

$$(0,1) = -\tfrac{1}{2}\,m'\,n\,C\,; \qquad \boxed{0,1} = \tfrac{1}{2}\,m'\,n\,D\,; \qquad [3761]$$

therefore *

$$\frac{d\,.(2\,r\,.\,d\,\delta\,r + d\,r\,.\,\delta\,r)}{r^2\,d\,v} = \tfrac{1}{2}\,d\,t\,.\,\{(0,1)\,.\,(h^2+l^2) - \boxed{0,1}\,.\,(h\,h' + l\,l')\}. \qquad [3762]$$

We shall now consider the term $\dfrac{3\,d\,t^2.\int\delta\,\mathrm{d}\,R}{r^2\,d\,v}$, of the same formula [923]

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and from its differential, the following expressions, retaining only the terms which contain the arc of a circle, as in [3755v];

$$\begin{aligned}\delta\,r &= L\,.\,n\,t\,.\sin.(n\,t+\varepsilon) - H\,.\,n\,t\,.\cos.(n\,t+\varepsilon)\,;\\ d\,\delta\,r &= L\,.\,n^2\,.\,t\,d\,t\,.\cos.(n\,t+\varepsilon) + H\,.\,n^2\,.\,t\,d\,t\,.\sin.(n\,t+\varepsilon).\end{aligned} \qquad [3758d]$$

Substituting these values of $\delta\,r$, $d\,\delta\,r$, in the first members of the equations [3758e], reducing by [17—20] Int., retaining only the terms containing the arc of a circle, independent of the sine or cosine of $n\,t+\varepsilon$, we get

$$\begin{aligned} 2\,a\,.\,d\,\delta\,r &= 0\,;\\ -2\,a\,h\,.\sin.(n\,t+\varepsilon)\,.\,d\,\delta\,r &= -a\,h\,H\,.\,n^2\,t\,d\,t\,;\\ -2\,a\,l\,.\cos.(n\,t+\varepsilon)\,.\,d\,\delta\,r &= -a\,l\,L\,.\,n^2\,t\,d\,t\,;\\ -\;a\,h\,.\cos.(n\,t+\varepsilon)\,.\,n\,d\,t\,.\,\delta\,r &= \tfrac{1}{2}\,a\,h\,H\,.\,n^2\,t\,d\,t\,;\\ +\;a\,l\,.\sin.(n\,t+\varepsilon)\,.\,n\,d\,t\,.\,\delta\,r &= \tfrac{1}{2}\,a\,l\,L\,.\,n^2\,t\,d\,t.\end{aligned} \qquad [3758e]$$

The sum of the terms in the first members of [3758e] is equal to the second member of [3758b]; consequently the first member of [3758b] is equal to the sum of the second members of [3758e]; hence we get

$$2\,r\,.\,d\,\delta\,r + d\,r\,.\,\delta\,r = -\tfrac{1}{2}\,a\,h\,H\,.\,n^2\,t\,d\,t - \tfrac{1}{2}\,a\,l\,L\,.\,n^2\,t\,d\,t. \qquad [3758f]$$

The differential of this expression becomes, by resubstituting [3758c],

$$\begin{aligned} d\,.\{2\,r\,.\,d\,\delta\,r + d\,r\,.\,\delta\,r\} &= -\tfrac{1}{2}\,n^2\,a\,.\,d\,t^2.\,(h\,H + l\,L)\\ &= -\tfrac{1}{4}\,m'.\,n^2\,a^2\,.\,d\,t^2.\,\{(h^2+l^2)\,.\,C + (h\,h' + l\,l')\,.\,D\}.\end{aligned} \qquad [3758g]$$

Dividing this by the expression of $r^2\,d\,v$ [3759], neglecting the divisor $\sqrt{(1-e^2)}$, which only produces terms of the fourth degree in h, h', e, &c., it becomes as in [3760].

* (2375) Substituting the values [3761] in [3760], we get [3762]. [3762a]

or [3757*d*]. If we notice only the secular quantities depending on the
[3763] squares and products of the excentricities and inclinations of the orbits, we shall have, by the analysis of the preceding article [3756*d*—*f*],

[3764]

Part of δR, corresponding to $i=0$.

$$\begin{aligned}\delta R = {} & \frac{m'}{8}\cdot(h^2+l^2)\cdot\left\{2a\cdot\left(\frac{dA^{(0)}}{da}\right)+a^2\cdot\left(\frac{ddA^{(0)}}{da^2}\right)\right\}\\ & +\frac{m'}{8}\cdot(h'^2+l'^2)\cdot\left\{2a'\cdot\left(\frac{dA^{(0)}}{da'}\right)+a'^2\cdot\left(\frac{ddA^{(0)}}{da'^2}\right)\right\}\\ & +\frac{m'}{4}\cdot(hh'+ll')\cdot\left\{4A^{(1)}+2a\cdot\left(\frac{dA^{(1)}}{da}\right)+2a'\cdot\left(\frac{dA^{(1)}}{da'}\right)+aa'\cdot\left(\frac{ddA^{(1)}}{da\,da'}\right)\right\}\\ & +\frac{m'}{8}\cdot aa'\cdot B^{(1)}\cdot\{(p'-p)^2+(q'-q)^2\};\end{aligned}$$

p, p', q, q', denoting the same quantities as in [1032]. Hence we easily obtain, from Book II, § 55, 59,*

[3765]
$$\begin{aligned}an\cdot\delta R = {} & -\tfrac{1}{2}\cdot(0,1)\cdot\{h^2+l^2+h'^2+l'^2\}+\boxed{0,1}\cdot\{hh'+ll'\}\\ & +\tfrac{1}{2}\cdot(0,1)\cdot\{(p'-p)^2+(q'-q)^2\};\end{aligned}$$

which gives†

[3766]
$$\begin{aligned}an\cdot\mathrm{d}\delta R = {} & dh\cdot\{-(0,1)\cdot h+\boxed{0,1}\cdot h'\}-dl\cdot\{(0,1)\cdot l-\boxed{0,1}\cdot l'\}\\ & -(0,1)\cdot dp\cdot(p'-p)-(0,1)\cdot dq\cdot(q'-q).\end{aligned}$$

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* (2376) If we multiply [3764] by an, we shall get the value of $an\cdot\delta R$, which may be easily reduced to the form [3765] by the following considerations. The coefficient
[3765*a*] of h^2+l^2 is equal to $-\frac{(0,1)}{2}$ [1073], and the coefficient of $h'^2+l'^2$ is of the same value; as evidently appears by the substitution of the expression [3752*d*]. The coefficient of $(p'-p)^2+(q'-q)^2$, in this product, is $\frac{1}{8}m'n\cdot a^2a'\cdot B^{(1)}=\frac{1}{2}\cdot(0,1)$ [1130]. Lastly, the coefficient of $hh'+ll'$ in this product, is evidently equal to $\frac{1}{2}m'n$, multiplied
[3765*b*] by the expression of D [1013], and this is shown in [1071] to be equal to $\boxed{0,1}$, as in [3765].

† (2377) In taking the differential of [3765], relatively to the characteristic d [3705*b*], we must consider h, l, p, q as the variable quantities, and h', l', p', q' as constant; and then we shall get

[3766*a*]
$$\begin{aligned}an\cdot\mathrm{d}\delta R = {} & -(0,1)\cdot(h\,dh+l\,dl)+\boxed{0,1}\cdot(h'dh+l'dl)\\ & +(0,1)\cdot\{-(p'-p)\cdot dp-(q'-q)\cdot dq\};\end{aligned}$$

being the same as in [3766], with a slight alteration in the arrangement of the terms.

The second member of this equation becomes nothing, in virtue of the equations [1089, 1132]; therefore we have*

$$a n \,.\, \mathrm{d}\,\delta R = 0\,; \qquad [3767]$$

hence we deduce, by observing that $n^2 a^3 = 1$ [3709'],† [3767']

$$\frac{3\,dt\,.\int dt\,.\,\mathrm{d}\,\delta R}{r^2\,dv} = \frac{3\,m'.\,g\,dt}{n\,a^2.\sqrt{1-e^2}}\,; \qquad [3768]$$

$m'g$ being the arbitrary constant quantity added to the integral $\int \mathrm{d}\,\delta R$ [1012'].

It now remains to consider the function $\frac{dt^2.\{2\,r\,\delta R' + R'\,\delta r\}}{r^2\,dv}$, which occurs in the expression of $d\,\delta v$ [923]. If we neglect the square of [3768'] the disturbing force, this function will be reduced to $\frac{2\,\delta\,.\,(r\,R')\,.\,dt^2}{r^2\,dv}$,

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* (2378) Taking into consideration only two bodies, m, m', we get, as in [1072],

$$\frac{dh}{dt} = (0,1)\,.\,l - \boxed{0,1}\,.\,l'\,; \qquad \frac{dl}{dt} = -(0,1)\,.\,h + \boxed{0,1}\,.\,h'. \qquad [3767a]$$

Multiplying the first of these equations by $-dl$, the second by dh, and adding the products, we find, that the sum of the terms of the *first* member vanishes; consequently [3767b] the sum of the terms in the *second* member, being the same as the terms depending on dh, dl, in [3766], must also vanish. Again, we have, in [1131],

$$\frac{dp}{dt} = (0,1)\,.\,(q'-q)\,; \qquad \frac{dq}{dt} = -(0,1)\,.\,(p'-p)\,; \qquad [3767c]$$

multiplying these, respectively, by $-dq$, dp, and taking the sum of the products; the *first* member becomes identically nothing, and the *second* member is the same as the terms [3767d] depending on dp, dq [3766], which are therefore equal to nothing, as in [3767]. We may incidentally remark, that the quantities $(0,1)$, $\boxed{0,1}$, &c. [3761]; also dh, dl, &c. [1102, 1102a], are of the order m'; consequently the second member of [3766] is of the order m'^2; but its integration, in [3768], introduces divisors of the order g, g_1, g_2, &c. [1102, 1102a], which are of the order m' [1097c]; by this means, the [3767e] integral $\int dt\,.\,\mathrm{d}\,\delta R$ [3768], is reduced to terms of the order m', like the other terms computed in this article.

† (2379) The integral of [3767], using the constant g [1012'], is $a n\,.\int \mathrm{d}\delta R = a n\,.\,m'g$; multiplying this by $\frac{3\,dt^2}{a n}$, and then dividing by $r^2\,dv = a^2 n\,dt\,.\sqrt{(1-e^2)}$ [3759], [3768a] we get [3768].

[3769] or by [928, 962], to $\frac{2a^2.\left(\frac{d\,\delta R}{da}\right).n\,dt}{\sqrt{1-e^2}}$.* This quantity produces,

[3769'] in the first place, the term $\frac{m'.n\,dt.a^2.\left(\frac{d.A^{(0)}}{da}\right)}{\sqrt{1-e^2}}$,† which is to be added

[3769''] to $\frac{3m'.g\,dt}{na^2.\sqrt{1-e^2}}$ [3768], or to the equivalent expression $\frac{3m'.ag\,n\,dt}{\sqrt{1-e^2}}$, deduced from $n^2a^3=1$ [3767']; and the sum vanishes by the

[3770] substitution of $g=-\frac{1}{3}a.\left(\frac{d\,A^{(0)}}{da}\right)$ [1017].

Resuming the expression of δR [3764], we shall observe, that the function

[3771] $$\frac{m'}{8}.aa'.B^{(1)}.\{(p'-p)^2+(q'-q)^2\}+\&c.‡$$

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* (2380) We have, in [3757b, c], by neglecting the square of the disturbing

[3769a] force, $2r\delta R'+R'\delta r=2\delta.(rR')=2a.\left(\frac{d\delta R}{da}\right)$. Multiplying this by dt^2 and by $1=n^2a^3$ [3767'], and then dividing by $r^2dv=a^2n\,dt.\sqrt{(1-e^2)}$ [3759], we

[3769b] get $\frac{2a^2.\left(\frac{d\delta R}{da}\right).n\,dt}{\sqrt{(1-e^2)}}$ [3769], for the corresponding term of $d\delta v$.

† (2381) The value of R [957], or rather [1011], gives, for the case of $i=0$,

[3770a] and for terms independent of nt, $n't$, $\delta R=\frac{1}{2}m'.A^{(0)}$. Substituting this in the term of $d\delta v$ [3769b], it becomes as in [3769']. Now if we substitute $g=-\frac{1}{3}a.\left(\frac{d.A^{(0)}}{da}\right)$

[3770b] [1017], in the term of $d\delta v$, [3769''], it becomes $-\frac{m'.n\,dt.a^2}{\sqrt{(1-e^2)}}.\left(\frac{d.A^{(0)}}{da}\right)$, and this is destroyed by the equal and opposite term obtained in [3769']; so that this sum becomes

[3770c] nothing, as in [3770]. The calculation [3767—3770] is in some respects a repetition of that in [1016'', &c.]; and we see that the value of g, assumed in [1017], suffices even when we notice the parts of R contained in [3764].

‡ (2382) Taking into consideration only two bodies m, m', the differential of [3771]

[3771a] will be $\frac{1}{4}m'.aa'.B^{(1)}.\{(p'-p).(dp'-dp)+(q'-q).(dq'-dq)\}$; observing that $B^{(1)}$ [956] is a function of the constant quantities a, a' [1044'']. Now the first and second of the equations [1132] become as in [3767c], and the third and fourth of those

[3771b] equations give $\frac{dp'}{dt}=-(1,0).(q'-q)$; $\frac{dq'}{dt}=(1,0).(p'-p)$. Hence the differential expression [3771a] becomes

[3771c] $$\tfrac{1}{4}m'.aa'.B^{(1)}.(p'-p).(q'-q).\{-(1,0)-(0,1)+(1,0)+(0,1)\}.dt;$$

is equal to a constant quantity independent of the time t, because its differential becomes nothing, in virtue of the equations [1132]; and if we consider only two planets, m, m', as we shall hereafter do, $(p'-p)^2+(q'-q)^2$ [3771′] will be a quantity independent of the time, in consequence of the same equations. Therefore the preceding function [3771] can produce in

$\dfrac{2\,n\,dt\,.\,a^2.\left(\frac{d\,\delta R}{da}\right)}{\sqrt{1-e^2}}$ [3769], only a quantity independent of $t\,dt$, &c., which [3771″] may therefore be neglected, since it may be supposed to be included in the value of $n\,dt$. Hence we shall have, by eliminating the partial differentials [3771‴] of $A^{(0)}$ and $A^{(1)}$, relatively to a', by means of their values [1003],*

$$2\,n\,dt\,.\,a^2.\left(\frac{d\,\delta R}{da}\right)=\frac{m'\,n\,dt}{2}.\{h^2+l^2+h'^2+l'^2\}.\left\{a^2.\left(\frac{dA^{(0)}}{da}\right)+2a^3.\left(\frac{ddA^{(0)}}{da^2}\right)+\tfrac{1}{2}a^4.\left(\frac{d^3A^{(0)}}{da^3}\right)\right\}$$
$$-m'\,n\,dt\,.\,(h\,h'+l\,l').\left\{2a^3.\left(\frac{ddA^{(1)}}{da^2}\right)+\tfrac{1}{2}a^4.\left(\frac{d^3A^{(1)}}{da^3}\right)\right\}. \qquad [3772]$$

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in which the terms between the braces mutually destroy each other, and render this quantity equal to nothing; therefore the expression [3771] must be constant, and may be represented by G, and it will introduce into δR [3764] the constant quantity G. Now [3771d] as this quantity, considered as a function of a, produces in [3771″], only a term which may be included in the expression of $n\,dt$, we may neglect it, and reject the term [3771e] depending on $B^{(1)}$ in [3764].

* (2383) It appears from [3752d], that the coefficients of $\frac{1}{8}m'.(h^2+l^2)$, $\frac{1}{8}m'.(h'^2+l'^2)$, are equal in the value of δR [3764]; these terms may therefore be connected together, as in [3772b]. Now if we put the two expressions of $N^{(2)}$ [3753″, 3755‴] equal to each other, then divide by $\frac{1}{4}m'$, we shall have, for the case of $i=0$,

$$4.A^{(1)}+2a.\left(\frac{dA^{(1)}}{da}\right)+2a'.\left(\frac{dA^{(1)}}{da'}\right)+a\,a'.\left(\frac{ddA^{(1)}}{da\,da'}\right)=2.A^{(1)}-2a.\left(\frac{dA^{(1)}}{da}\right)-a^2.\left(\frac{ddA^{(1)}}{da^2}\right); \qquad [3772a]$$

substituting this in the coefficient of $\frac{1}{4}m'.(h\,h'+l\,l')$ [3764], it becomes as in [3772b]; hence we get

$$\delta R=\tfrac{1}{8}m'.(h^2+l^2+h'^2+l'^2).\left\{2a.\left(\frac{dA^{(0)}}{da}\right)+a^2.\left(\frac{ddA^{(0)}}{da^2}\right)\right\}$$
$$+\tfrac{1}{4}m'.(h\,h'+l\,l').\left\{2A^{(1)}-2a.\left(\frac{dA^{(1)}}{da}\right)-a^2.\left(\frac{ddA^{(1)}}{da^2}\right)\right\}. \qquad [3772b]$$

Taking the partial differential of this expression, relatively to a, and multiplying it by $2\,n\,dt\,.\,a^2$, we get [3772].

Now if we collect together all these terms, we shall obtain,*

Expression of $d\,\delta\,v$ depending on $i=0$.

[3773]
$$d\,\delta\,v = \frac{m'.n\,d\,t}{8}.(h^2+l^2).\left\{2\,a^2.\left(\frac{d.A^{(0)}}{d\,a}\right)+7\,a^3.\left(\frac{d\,d.A^{(0)}}{d\,a^2}\right)+2\,a^4.\left(\frac{d^3.A^{(0)}}{d\,a^3}\right)\right\}$$
$$+\frac{m'.n\,d\,t}{4}.(h'^2+l'^2).\left\{2\,a^2.\left(\frac{d.A^{(0)}}{d\,a}\right)+4\,a^3.\left(\frac{d\,d.A^{(0)}}{d\,a^2}\right)+a^4.\left(\frac{d^3.A^{(0)}}{d\,a^3}\right)\right\}$$
$$-\frac{m'.n\,d\,t}{8}.(h\,h'+l\,l').\left\{2\,a.A^{(1)}-2\,a^2.\left(\frac{d.A^{(1)}}{d\,a}\right)+15\,a^3.\left(\frac{d\,d.A^{(1)}}{d\,a^2}\right)+4\,a^4.\left(\frac{d^3.A^{(1)}}{d\,a^3}\right)\right\}.$$

[3774] In this expression we may neglect the terms independent of the time t [3773e]. Hence it is easy to deduce the expression of $d\,\delta\,v'$, by changing what relates to m into the corresponding terms of m' and the contrary; and observing,
[3775] that, though the value of $A^{(i)}$ [997], relative to the action of m' upon m, is different from its value relative to the action of m upon m', yet we may
[3775'] use, in the preceding expression, either of these values at pleasure.† But

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* (2384) The value of $d\,\delta\,v$ [3773] is found, by adding together the several parts of the expression [3757d], computed in this article; and as the terms [3768'—3771''] destroy
[3773a] each other, there will remain only the terms [3762, 3772], to be connected together. The expression [3762], by the substitution of the values of (0.1), $[\overline{0.1}]$ [1073] becomes

[3773b]
$$\frac{m'.n\,d\,t}{2}.(h^2+l^2).\left\{-\tfrac{1}{2}\,a^2.\left(\frac{d.A^{(0)}}{d\,a}\right)-\tfrac{1}{4}\,a^3.\left(\frac{d\,d.A^{(0)}}{d\,a^2}\right)\right\}$$
$$-m'.n\,d\,t.(h\,h'+l\,l').\left\{\tfrac{1}{4}\,a\,A^{(1)}-\tfrac{1}{4}\,a^2.\left(\frac{d.A^{(1)}}{d\,a}\right)-\tfrac{1}{8}\,a^3.\left(\frac{d\,d.A^{(1)}}{d\,a^2}\right)\right\};$$

and as the factors without the braces are the same as in [3772], the sum of the two expressions [3772, 3773b] is easily found to be as in [3773]; which is a function of the
[3773c] elements of the orbits similar to that mentioned in [1345^{viii}]. If all the terms of this function were constant, they might be included in the expression of the mean motion $n\,d\,t$.
[3773d] But $e^2=h^2+l^2$, $e'^2=h'^2+l'^2$, &c. [1108, 1109], are composed of *constant* quantities, and of others depending on the secular periodical variations of e, e', &c.; and it is evident,
[3773e] that the constant quantities produce in $d\,\delta\,v$ terms of the same form as the mean motion; they may therefore be neglected, as in [3771''', 3774].

† (2385) Substituting [964] in [963^v], and then putting $s=\frac{1}{2}$, we get

[3775a]
$$(a^2-2\,a\,a'.\cos.\theta+a'^2)^{-\frac{1}{2}}=a'^{-1}.\{\tfrac{1}{2}\,b_{\frac{1}{2}}^{(0)}+b_{\frac{1}{2}}^{(1)}.\cos.\theta+b_{\frac{1}{2}}^{(2)}.\cos.2\,\theta+\&c.\}.$$

Now the first member of this equation is symmetrical in a, a'; therefore its second member must also be symmetrical; so that we shall have, generally, $a'^{-1}.b_{\frac{1}{2}}^{(i)}$ equal to a symmetrical function of a, a'; and if we refer to the formulas [996, 997], we shall see, that for all
[3775b] values of i, except $i=1$, the function $A^{(i)}$ is likewise symmetrical. In the case of $i=1$,

we may obtain $d\,\delta\,v'$ more easily by the following considerations. If we [3775''] add the value of $d\,\delta\,v$, multiplied by $m\sqrt{a}$, to the value of $d\,\delta\,v'$, multiplied by $m'\sqrt{a'}$, we shall have, by substituting the partial differentials of $A^{(0)}$, $A^{(1)}$, relative to a, instead of those relative to a',*

$$m\sqrt{a}.d\,\delta\,v+m'\sqrt{a'}.d\,\delta\,v'=-\frac{3\,m\,m'.dt}{4}.\{h^2+l^2+h'^2+l'^2\}.\left\{a.\left(\frac{d.A^{(0)}}{da}\right)+\tfrac{1}{2}a^2.\left(\frac{dd.A^{(0)}}{da^2}\right)\right\}$$
$$-\frac{3\,m\,m'.dt}{2}.\{h\,h'+l\,l'\}.\left\{A^{(1)}-a.\left(\frac{d.A^{(1)}}{da}\right)-\tfrac{1}{2}a^2.\left(\frac{dd.A^{(1)}}{da^2}\right)\right\}. \quad [3776]$$

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corresponding to the action of m' upon m, we have $A^{(1)}=\frac{a}{a'^2}-\frac{1}{a'}.b_{\frac{1}{2}}^{(1)}$ [997]; and in the action of m upon m', it becomes $A^{(1)}=\frac{a'}{a^2}-\frac{1}{a'}.b_{\frac{1}{2}}^{(1)}$; but we may neglect the [3775c] parts $\frac{a}{a'^2}$, $\frac{a'}{a^2}$, because they produce nothing in $d\,\delta\,v$, $d\,\delta\,v'$. To prove this, we shall [3775d] observe, that by noticing only the part $A^{(1)}=\frac{a}{a'^2}$, we shall get

$$\left(\frac{d.A^{(1)}}{da}\right)=\frac{1}{a'^2}; \qquad \left(\frac{dd.A^{(1)}}{da^2}\right)=0; \qquad \left(\frac{dd.A^{(1)}}{da^3}\right)=0; \quad [3775e]$$

substituting these in [3773], the terms mutually destroy each other; so that we may neglect this part of $A^{(1)}$, and for similar reasons we may neglect the part $A^{(1)}=\frac{a'}{a^2}$, in [3775f] computing the action of m upon m', and then the two expressions [3775c] become symmetrical in a, a', as in [3775'].

* (2386) Multiplying [3773] by $m\sqrt{a}$, and dividing the second member by $n\,a\sqrt{a}=1$ [3709'], we get, by reducing the factors without the braces to a symmetrical form,

$$m\sqrt{a}.d\,\delta\,v=\ \tfrac{1}{4}m\,m'.dt.(h^2+l^2).\left\{a\ .\left(\frac{d.A^{(0)}}{da}\right)+\tfrac{7}{2}a^2.\left(\frac{dd.A^{(0)}}{da^2}\right)+a^3.\left(\frac{d^3.A^{(0)}}{da^3}\right)\right\}$$
$$+\tfrac{1}{4}m\,m'.dt.(h'^2+l'^2).\left\{2\,a.\left(\frac{d.A^{(0)}}{da}\right)+4\,a^2.\left(\frac{dd.A^{(0)}}{da^2}\right)+a^3.\left(\frac{dd.A^{(0)}}{da^3}\right)\right\} \quad [3776a]$$
$$+\tfrac{1}{4}m\,m'.dt.(hh'+ll').\left\{-A^{(1)}+a.\left(\frac{d.A^{(1)}}{da}\right)-\tfrac{15}{2}a^2.\left(\frac{d^2.A^{(1)}}{da^2}\right)-2\,a^3.\left(\frac{d^3.A^{(1)}}{da^3}\right)\right\}.$$

Changing the elements m, a, v, h, l, &c. into m', a', v', h', l', &c. and the contrary; which does not, in the present case, alter the values of $A^{(0)}$ or $A^{(1)}$ [3775], we obtain [3776b] the expression of $m'\sqrt{a'}.d\,\delta\,v'$. The factors between the braces corresponding to the first, second, and third lines of [3776a], become, respectively, as in the first members of [3776d, f, h], and by means of the expressions [1003], they may be reduced to the

[3776′] If we consider only two planets, m and m',* the differential of the second

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[3776c] forms [3776e, g, i]. In making these reductions, we may use the abridged symbols A_0, A_1, A_2, A_3 [3755b], observing, that the index of $A^{(0)}$ or $A^{(1)}$ remains unchanged;

[3776d]
$$a'.\left(\frac{d.A^{(0)}}{da'}\right)+\tfrac{7}{2}a'^2.\left(\frac{dd.A^{(0)}}{da'^2}\right)+a'^3.\left(\frac{d^3.A^{(0)}}{da'^3}\right)=\{-A_0-A_1\}+\tfrac{7}{2}.\{2A_0+4A_1+A_2\}$$
$$+\{-6A_0-18A_1-9A_2-A_3\}$$

[3776e]
$$=-5A_1-\tfrac{11}{2}A_2-A_3\,;$$

[3776f]
$$2a'.\left(\frac{d.A^{(0)}}{da'}\right)+4a'^2.\left(\frac{dd.A^{(0)}}{da'^2}\right)+a'^3.\left(\frac{d^3.A^{(0)}}{da'^3}\right)=2.\{-A_0-A_1\}+4.\{2A_0+4A_1+A_2\}$$
$$+\{-6A_0-18A_1-9A_2-A_3\}$$

[3776g]
$$=-4A_1-5A_2-A_3\,;$$

[3776h]
$$-A^{(1)}+a'.\left(\frac{d.A^{(1)}}{da'}\right)-\tfrac{15}{2}a'^2.\left(\frac{d^2.A^{(1)}}{da'^2}\right)-2a'^3.\left(\frac{d^3.A^{(1)}}{da'^3}\right)=-A_0+\{-A_0-A_1\}-\tfrac{15}{2}.\{2A_0+4A_1+A_2\}$$
$$-2.\{-6A_0-18A_1-9A_2-A_3\}$$

[3776i]
$$=-5A_0+5A_1+\tfrac{21}{2}.A_2+2A_3\,.$$

Now substituting the values corresponding to [3776e, g, i] in the value of $m'\sqrt{a'}.d\delta v'$, deduced from [3776a], by the change of the elements [3776b], we get

$$m'\sqrt{a'}.d\delta v'=\tfrac{1}{4}mm'.dt.(h'^2+l'^2).\left\{-5a.\left(\frac{d.A^{(0)}}{da}\right)-\tfrac{11}{2}a^2.\left(\frac{dd.A^{(0)}}{da^2}\right)-a^3.\left(\frac{d^3.A^{(0)}}{da^3}\right)\right\}$$

[3776k]
$$+\tfrac{1}{4}mm'.dt.(h^2+l^2).\left\{-4a.\left(\frac{d.A^{(0)}}{da}\right)-5a^2.\left(\frac{dd.A^{(0)}}{da^2}\right)-a^3.\left(\frac{d^3.A^{(0)}}{da^3}\right)\right\}$$
$$+\tfrac{1}{4}mm'.dt.(hh'+ll').\left\{-5A^{(1)}+5a.\left(\frac{d.A^{(1)}}{da}\right)+\tfrac{21}{2}a^2.\left(\frac{d^2.A^{(1)}}{da^2}\right)+2a^3.\left(\frac{d^3.A^{(1)}}{da^3}\right)\right\}.$$

Adding together the two expressions [3776a, k], we obtain [3776], observing, that in this sum, the coefficient of h^2+l^2 is found to be the same as that of $h'^2+l'^2$. We
[3776l] may remark, that the factor $-\frac{3mm'.dt}{2}$, in the second line of [3776], is erroneously printed $-\frac{3mm'.dt}{4}$ in the original work. If we multiply the second member of [3776] by $na^{\frac{3}{2}}=1$ [3709′], and substitute the expressions $(0,1)$, $\boxed{0,1}$ [1073], we shall get, instead of [3776], the following equation;

[3776m]
$$m\sqrt{a}.d\delta v+m'\sqrt{a'}.d\delta v'=\tfrac{3}{2}m\sqrt{a}.dt.(0,1).(h^2+l^2+h'^2+l'^2)$$
$$-3m\sqrt{a}.dt.\boxed{0,1}.(hh'+ll').$$

* (2387) The differential of the equation [3776m], may be put under the following form;

[3777a]
$$d.\{m\sqrt{a}.d\delta v+m'\sqrt{a'}.d\delta v'\}=3m\sqrt{a}.dt.\{(0,1).(hdh+ldl)-\boxed{0,1}.(h'dh+l'dl)\}$$
$$+3m\sqrt{a}.dt.\{(0,1).(h'dh'+l'dl')-\boxed{0,1}.(hdh'+ldl')\}.$$

member of this equation will be nothing, in virtue of the equations [1089]; therefore *we have, by noticing only secular periodical quantities,*

Formula for $\delta v'$.

$$0 = m\sqrt{a} \cdot d\,\delta v + m'\sqrt{a'} \cdot d\,\delta v'; \qquad [3777]$$

which immediately gives $d\,\delta v'$, *when* $d\,\delta v$ *is known.*

The *value of* $d\,\delta v$ is *relative to the angle formed by the two radii vectores* r *and* $r+dr$. *To obtain its value relative to a fixed plane,* we shall [3778] observe, that if we put $d\,v_{,}$ for the projection of $d\,v$ upon this plane, and neglect the fourth power of the inclination of the orbit, we shall find, as in [925],*

$$d\,v_{,} = d\,v \cdot \left\{1 + \tfrac{1}{2}\,s^2 - \tfrac{1}{2} \cdot \frac{d\,s^2}{d\,v^2}\right\}. \qquad [3779]$$

We have, as in [1051],

$$s = q \cdot \sin.(nt+\varepsilon) - p \cdot \cos.(nt+\varepsilon) + \&c.; \qquad [3780]$$

which gives †

$$ds = \left(nq - \frac{dp}{dt}\right) . dt . \cos.(nt+\varepsilon) + \left(np + \frac{dq}{dt}\right) . dt . \sin.(nt+\varepsilon) + \&c.; \qquad [3781]$$

Substituting, in the first line of the second member of this expression, the values dh, dl [3767a], it vanishes, because the terms mutually destroy each other. The second line of the second member becomes, by the substitution of the formulas [1093, 1094], equal to $3\,m'\sqrt{a'}.dt.\{(1,0).(h'dh'+l'dl') - \boxed{1,0}.(h\,dh'+l\,dl')\}$, which vanishes also by [3777b] the substitution of $dh' = \{(1,0).l' - \boxed{1,0}.l\}.dt$, $dl' = \{-(1,0).h' + \boxed{1,0}.h\}.dt$, [3777c] deduced from the third and fourth of the equations [1089]. This is also evident from the consideration, that the expression [3777b] may be derived from the first line of [3777a], by changing the elements relative to m into those corresponding to m', and the contrary; [3777d] and as that line is found to vanish by the substitution of the values of dh, dl [3767a], the other will in like manner vanish by the substitution of the values of dh', dl' [3777c]. Now the second member of [3777a] being equal to nothing, we have, by integration, $m\sqrt{a}.d\,\delta v + m'\sqrt{a'}.d\,\delta v' = G\,dt$; G being a constant quantity independent of the secular periodical equations. This quantity $G\,dt$ may be supposed, as in [3771'''], [3777e] to be connected with $n\,dt$, $n'dt$; so that by noticing only the secular periodical equations, we may put the first member of the preceding equation equal to nothing, as in [3777].

* (2388) The equation [925] may be put under the form $dv_{,} = dv\sqrt{\left\{1+s^2 - \frac{ds^2}{(1+ss).dv^2}\right\}}$. [3778a] Developing this, and neglecting terms of the fourth degree in s or ds, we get [3779].

† (2389) The differential of s [3780], considering p, q, t as variable, becomes as in [3781]. The squares of these expressions, which enter into the function [3779], are

[3781′] hence we shall find, by neglecting the periodical quantities depending on $n t$, and observing that $d v = n\, d t$, very nearly,

[3782] $$d v_{,} = d v + \tfrac{1}{2} \cdot (q\, d p - p\, d q) \,;$$

therefore to obtain the value of $d \delta v_{,}$, we must add the quantity

[3783] $\tfrac{1}{2} \cdot (q\, d p - p\, d q)$ to the preceding value of $d \delta v$ [3773].

If we only consider two planets m, m', we shall have, by means of [1132, 1130],*

[3784] $$(q d p - p d q) \,.\, m \sqrt{a} + (q' d p' - p' d q') \,.\, m' \sqrt{a'} = -\tfrac{1}{4} m m' \,.\, d t \,.\, a a' \,.\, B^{(1)} \,.\, \{(p'-p)^2 + (q'-q)^2\} \,;$$

[3779a] of the order of the terms computed in this article [3702′], and by neglecting terms of a higher order, we may omit, in $\frac{d s^2}{d v^2}$ [3779], the terms of $d v$ [3748] depending on e,

[3779b] and put $d v = n\, d t$, by which means we shall get $d v_{,} = d v \,.\, \left\{1 + \tfrac{1}{2} s^2 - \tfrac{1}{2} \,.\, \frac{d s^2}{n^2 d t^2}\right\}$, in which we must substitute s, $d s$ [3780, 3781]. In making these substitutions, and noticing the terms independent of the sine and cosine of $n t$ or its multiples, as is done in this article, where the secular periodical terms only are retained, we may, as in [3651a], put

[3779c] $$\sin.^2 (n t + \varepsilon) = \tfrac{1}{2}, \qquad \cos.^2 (n t + \varepsilon) = \tfrac{1}{2}, \qquad \sin. (n t + \varepsilon) \,.\, \cos. (n t + \varepsilon) = 0 \,;$$

then the squares of [3780, 3781] will give, by neglecting $d q^2$, $d p^2$, which are of the order of the square of the disturbing forces,

[3779d] $$\tfrac{1}{2} s^2 = \tfrac{1}{4} \cdot (q^2 + p^2) \,;$$

[3779e] $$-\tfrac{1}{2} \cdot \frac{d s^2}{n^2 d t^2} = -\frac{1}{4 n^2} \cdot \left(n q - \frac{d p}{d t}\right)^2 - \frac{1}{4 n^2} \cdot \left(n p + \frac{d q}{d t}\right)^2 = -\tfrac{1}{4} \cdot \left(q - \frac{d p}{n d t}\right)^2 - \tfrac{1}{4} \cdot \left(p + \frac{d q}{n d t}\right)^2$$
$$= -\tfrac{1}{4} \cdot (q^2 + p^2) + \frac{q d p - p d q}{2 n d t} = -\tfrac{1}{4} \cdot (q^2 + p^2) + \frac{q d p - p d q}{2 d v}.$$

Substituting these in [3779], we get [3782].

* (2390) Substituting the values of $d p$, $d q$, $d p'$, $d q'$ [3767c, 3771b] in the first members of [3784a, b], and reducing the second expression by means of [1093, 1094], we get the second members [3784a, c];

[3784a] $$m \sqrt{a} \,.\, (q\, d p - p\, d q) = m \sqrt{a} \,.\, (0, 1) \,.\, d t \,.\, \{q \,.\, (q' - q) + p \,.\, (p' - p)\} \,;$$

[3784b] $$m' \sqrt{a'} \,.\, (q' d p' - p' d q') = m' \sqrt{a'} \,.\, (1, 0) \,.\, d t \,.\, \{-q' \,.\, (q' - q) - p' \,.\, (p' - p)\}$$

[3784c] $$= m \sqrt{a} \,.\, (0, 1) \,.\, d t \,.\, \{-q' \,.\, (q' - q) - p' \,.\, (p' - p)\}.$$

The sum of the two equations [3784a, c] gives the value of [3784] under the form

[3784d] $-m \sqrt{a} \,.\, (0, 1) \,.\, d t \,.\, \{(q' - q)^2 + (p' - p)^2\}$; substituting $(0, 1)$ [1130], and dividing by $n a^{\frac{3}{2}} = 1$ [3709′], it becomes as in the second member of [3784].

and the second member of this equation is equal to dt, multiplied by a constant quantity;* therefore *by noticing only the secular periodical quantities, we shall have* [3784']

The same formula for $\delta v'$, reduced to the fixed plane.

$$0 = m\sqrt{a} \,.\, d\,\delta v_{,} + m'\sqrt{a'} \,.\, d\,\delta v_{,}' \,; \qquad [3785]$$

$\delta v_{,}$ and $\delta v_{,}'$ *being relative to the fixed plane.*

6. *We shall now consider the inequalities in the motion in latitude, depending on the products of the excentricities and inclinations of the orbits.* [3785'] For this purpose we shall resume the third of the equations [915];

Differential equation for the latitude.

$$0 = \frac{ddz}{dt^2} + \frac{\mu z}{r^3} + \left(\frac{dR}{dz}\right). \qquad [3786]$$

We shall take for the fixed plane the primitive orbit of m, *in consequence of which we may put* $z = 0$ in the expression of $\left(\frac{dR}{dz}\right)$. We shall have, [3786'] by [3736—3741], observing that $z' = r's'$,† [3787]

$$\left(\frac{dR}{dz}\right) = \frac{m's'}{r'^2} - \frac{m'r's'}{\{r^2 - 2rr'.\cos.(v'-v) + r'^2\}^{\frac{3}{2}}} \,; \qquad [3788]$$

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* (2391) The differential of the second member of [3784], being divided by $-2m\,dt$, becomes as in [3771a], and is therefore equal to nothing, as is shown in [3771c]; hence [3785a] we find, as in [3771d], that the first member of [3784] is equal to dt, multiplied by a constant quantity G, which may be neglected as in [3771e]; so that by noticing only the secular periodical equations, we shall have $(q\,dp - p\,dq).m\sqrt{a} + (q'dp' - p'dq').m'\sqrt{a'} = 0$. [3785$b$] Now we have found, in [3782], that by reducing v to a fixed plane, the value of dv or $d\delta v$ must be augmented by $\frac{1}{2}.(q\,dp - p\,dq)$; and in like manner, the quantity $d\delta v'$ must be increased by $\frac{1}{2}.(q'dp' - p'dq')$. Multiplying these by $m\sqrt{a}$, $m'\sqrt{a'}$, respectively, [3785c] and adding the products, we get the increment of the function [3777], or the quantity to be added to it, to obtain the value of $m\sqrt{a}\,.\,d\,\delta v_{,} + m'\sqrt{a'}\,.\,d\,\delta v_{,}'$. Now this increment vanishes by means of the equation [3785b]; consequently the function [3777], varied in this manner, becomes as in [3785].

† (2392) The latitude of the body m', neglecting terms of the third order, being represented by s', and the radius vector by r', we shall have, by the principles of orthographic projection, $z' = r's'$, as in [3787]. Now r' [3736b] being independent of z, the partial differential of R [3736], relative to z, becomes [3787a]

$$\left(\frac{dR}{dz}\right) = \frac{m'z'}{r'^3} - \frac{m'.(z'-z)}{\{(x'-x)^2 + (y'-y)^2 + (z'-z)\}^{\frac{3}{2}}} \,; \qquad [3787a']$$

the differential equation in z, will by this means become*

[3789]
$$0=\frac{d\,d\,z}{d\,t^2}+n^2 z\,.\{1+3\,e\,.\cos.\,(n\,t+\varepsilon-\varpi)\}+\frac{m'.\,n^2\,a^3.\,s'}{r'^2}-\frac{m'.\,n^2\,a^3.\,r'\,s'}{\{r^2-2\,r\,r'.\cos.\,(v'-v)+r'^2\}^{\frac{3}{2}}}.$$

We shall now put†

[3790]
$$\left(\frac{d\,R}{d\,z}\right)=M.\sin.\{i.(n't-n\,t+\varepsilon'-\varepsilon)+2n\,t+K\}+N.\sin.\{i.(n't-n\,t+\varepsilon'-\varepsilon)+L\},$$

for the part of [3788]

[3791]
$$\left(\frac{d\,R}{d\,z}\right)=\frac{m'\,s'}{r'^2}-\frac{m'.\,r'\,s'}{\{r^2-2\,r\,r'.\cos.\,(v'-v)+r'^2)\}^{\frac{3}{2}}};$$

depending on the angles $i.(n't-n\,t+\varepsilon'-\varepsilon)+2n\,t$ and $i.(n't-n\,t+\varepsilon'-\varepsilon)$;*
[3792] and shall suppose, that by noticing only the inequalities of z, depending on

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and if we neglect quantities of the order z'^3, we may reject terms of the order z'^2 or γ^2 in the denominator; then, as in [3742g], we shall have

[3787b]
$$(x'-x)^2+(y'-y)^2+(z'-z)^2=r^2-2\,r\,r'.\cos.\,(v'-v)+r'^2;$$

substituting this and $z=0$, $z'=r'\,s'$ [3786', 3787] in [3787a'], we get [3788].

We may here remark, that the method used in this article, in finding the motion in
[3787c] latitude, depending on terms of the order of the product of the excentricity by the inclination of the orbit, is different from that proposed in [948], and used in [1025, &c.] in finding the terms independent of the excentricity. This last method may, however, be applied without any difficulty to terms depending on the excentricity, and we shall obtain the same
[3787d] result as in [3795—3797]; as has been shown by Mr. Plana, in Vol. XII, page 449, &c. of Zach's *Correspondance Astronomique*, &c.

* (2393) We have, by means of [3702b, c, 3700],

[3789a]
$$\mu\,r^{-3}=\mu\,a^{-3}.\{1+3\,e\,.\cos.\,(n\,t+\varepsilon-\varpi)+\&c.\}=n^2.\{1+3\,e\,.\cos.\,(n\,t+\varepsilon-\varpi)+\&c.\}.$$

Substituting this in [3786], also the expression [3788], multiplied by $n^2\,a^3=1$ [3709'], we get [3789].

† (2394) The reasons for assuming these forms are evident from [3704a—b], observing
[3790a] that the object proposed at the commencement of this book, is to notice merely the terms depending on the squares and products of the excentricities and inclinations.

the first power of the inclination of the orbits, the part of z, relative to the angle $i.(n't-nt+\varepsilon'-\varepsilon)+nt$, will be*

$$z=\gamma a F.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\Pi\}. \qquad [3793]$$

We then have, by retaining only the terms depending on the products of the excentricities and inclinations,†

$$\begin{aligned}0=\frac{ddz}{dt^2}+n^2z+\tfrac{3}{2}n^2.e\gamma.aF.&\left\{\begin{array}{l}\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\Pi\}\\+\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+\varpi-\Pi\}\end{array}\right\}\\&+n^2a^3.M.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}\\&+n^2a^3.N.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+L\};\end{aligned} \qquad [3794]$$

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* (2395) Putting, for brevity,

$$T_3=i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon, \qquad F=\frac{m'.n^2.a^2a'}{2}\cdot\frac{B^{(i-1)}}{n^2-\{n-i.(n-n')\}^2}, \qquad [3792a]$$

we shall have, for the terms of s [1034] depending on $B^{(i-1)}$, the expression

$$F.\{(q'-q).\sin.T_3-(p'-p).\cos.T_3\}; \qquad [3792b]$$

substituting in this the values $p'-p=\gamma.\sin.\Pi$, $q'-q=\gamma.\cos.\Pi$ [1033], it becomes [3792b′]

$$\begin{aligned}F\gamma.\{\sin.T_3.\cos.\Pi-\cos.T_3.\sin.\Pi\}&=F\gamma.\sin.(T_3-\Pi)\\&=F\gamma.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\Pi\}.\end{aligned} \qquad [3792c]$$

Multiplying this by r, we get the corresponding part of $z=rs$ [3787, 3796], to be [3792d] substituted in the term $3n^2ez.\cos.(nt+\varepsilon-\varpi)$ [3789]. Now this term is of the [3792e] second order, or of the same order as the terms now under consideration [3702′]; and by neglecting those of a higher order, we may substitute a for r, in the expression of z [3792d], and we shall have $z=as$; hence the term of s, computed in [3792c], produces in z [3792f] the quantity [3793]. Substituting this in [3792e], and reducing by means of [18] Int., we get the terms depending on F in [3794]. In computing the value of the term [3792e], and neglecting quantities of the order m'^2 or e^2, it is not necessary to notice any other terms of s [1034], except those depending on $B^{(i-1)}$ or F, which we have used above. [3792g] For the terms depending on the arc of a circle nt, in the second and third lines of [1034], vanish, as in [1051], in consequence of the secular variations of p, q. Again, having taken the primitive orbit of m for the fixed plane, we have $z=0$ or $s=0$ [3786′], at the commencement of the motion, corresponding to $p=0$, $q=0$ [1034, 1032]; so that [3792h] these terms may be neglected in computing [3792e]. Lastly, the terms of s depending on $\sin.(n't+\varepsilon')$, $\cos.(n't+\varepsilon')$, in the fourth line of s [1034], may be considered as included in the term of T_3 or of F [3792a], depending on $i=1$; consequently the [3792i] function [3792e] is rightly expressed by the terms depending on F in [3794]; the quantity F being of the order m' [3792a], as well as M, N [3790, 3791]. [3792k]

† (2396) The equation [3794] is easily deduced from [3789]; for the two first terms are identically the same in each; the third term depending on e, reduced as in [3792f, &c.],

hence we get, by integration,*

[3795]
$$z=\frac{\left\{\begin{array}{l}\tfrac{3}{2}n^2.e\gamma.aF.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\Pi\}\\ +n^2a^3.M.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}\end{array}\right\}}{\{in'-(i-1).n\}.\{in'-(i-3).n\}}$$
$$+\frac{\left\{\begin{array}{l}\tfrac{3}{2}n^2.e\gamma.aF.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+\varpi-\Pi\}\\ +n^2a^3.N.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+L\}\end{array}\right\}}{\{in'-(i+1).n\}.\{in'-(i-1).n\}}.$$

We have the latitude s, by observing that†

[3796]
$$s=\frac{z}{r}=\frac{z}{a}+\frac{z}{a}.e.\cos.(nt+\varepsilon-\varpi);$$

therefore s may be obtained by dividing the preceding expression of z by a, and adding to it the quantity‡

[3797]
$$\tfrac{1}{2}e\gamma.F.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\Pi\}$$
$$+\tfrac{1}{2}e\gamma.F.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+\varpi-\Pi\}.$$

[3794a] produces the terms depending on F [3794]; the two remaining terms, comprised in the second line of the second member of [3789], are represented by the function [3791], or by the equivalent expression [3790] multiplied by $n^2a^3=1$, as in the two last lines of [3794].

* (2397) The equation [3794] is of the same form as [865a], putting $y=z$, $a=n$;
[3795a] then any term of [3794] depending on F, M, or N, being represented by $\alpha K.\sin.(m_{,}t+\varepsilon_{,})$,
[3795b] the corresponding term of z will be represented by $\frac{\alpha K\,\sin.(m_{,}t+\varepsilon_{,})}{(m_{,}+n).(m_{,}-n)}$, as in [871]; the letters $m_{,}$, $\varepsilon_{,}$, being accented to distinguish them from the similar letters of the present
[3795c] article. Now putting $m_{,}=i.(n'-n)+2n$ in the first and third of *these terms* of [3794], and $m_{,}=i.(n'-n)$ in the second and fourth, we get, successively, the terms of z [3795];
[3795d] all of which are of the order m' [3792k].

† (2398) We get, in like manner as in [3787], $rs=z$; dividing this by r, or its
[3796a] equivalent expression $a.\{1-e.\cos.(nt+\varepsilon-\varpi)\}$ [3701], we get the two values of s [3796], neglecting, in the last of them, the terms of the third order in e and z.

‡ (2399) Substituting, in $\frac{z}{a}.e.\cos.(nt+\varepsilon-\varpi)$ [3796], the term of z of the first order γ,
[3797a] assumed in [3793], and reducing the product by means of [18] Int., we obtain the corresponding values [3797]. Adding these to the term of $\frac{z}{a}$ [3796], deduced from [3795], we get the terms of s, of the proposed forms and order. These terms are neglected

Nothing more is required but to ascertain the values of M and N; which may be easily found by the analysis in §4. *We shall, however, dispense with this calculation, because the inequalities of this order in latitude are* [3797′]
insensible except in the orbits of Jupiter and Saturn, whose mean motions are nearly commensurable, and we shall give, in [3884—3888], a very [3798]
simple method for the determination of these inequalities.

If we refer the motion of m to a fixed plane, which is but very slightly inclined to that of its primitive orbit, putting φ for the inclination of the orbit to this plane, and θ for the longitude of its ascending node; we shall [3799]
have the reduction of the motion in the orbit to this plane, by the method explained in Book II, §22 [675, &c.],*

$$-\tfrac{1}{4}\,.\,\text{tang.}^2\varphi\,.\,\sin.\,(2\,v_{\prime}-2\,\theta)-\text{tang.}\,\varphi\,.\,\delta s\,.\,\cos.\,(v_{\prime}-\theta)\,; \qquad [3800]$$

$v_{\prime}$ being the motion v referred to the fixed plane. Hence the motion in [3800′]
latitude produces in the motion in longitude, upon the ecliptic, inequalities depending on the squares and higher powers of the excentricities and

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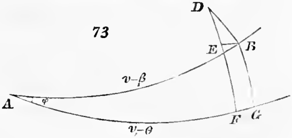
by the author in [3797′] on account of their smallness. The most important terms of the perturbation in latitude, of the second order, computed in [3885, 3886], are reduced to numbers [3797b]
in [4458, 4513], for Jupiter and Saturn, in whose orbits these terms have a sensible value.

* (2400) In the annexed figure 73, AB is the primitive orbit of the planet m, AG the fixed plane, D the place of the planet, $BD=\delta s$ the perturbation in latitude now [3800a]
under consideration, which is perpendicular to AB; lastly, the arcs BG, DEF are perpendicular to AG, and BE perpendicular to DF. Then by using the notation [669″], we have $AB=v-\beta$, $AG=v_{\prime}-\theta$, [3800b]
$BAG=\varphi$; and in [676′], by neglecting φ^4, $AB=AG+\text{tang.}^2\tfrac{1}{2}\varphi\,.\,\sin.\,(2\,v_{\prime}-2\,\theta)$; but [3800c]
on account of the smallness of φ, we may put $\text{tang.}^2\tfrac{1}{2}\varphi=(\tfrac{1}{2}\,\text{tang.}\,\varphi)^2=\tfrac{1}{4}\,\text{tang.}^2\varphi$; [3800d]



so that to reduce AB to AG, we must apply the correction $-\tfrac{1}{4}\,\text{tang.}^2\varphi\,.\sin.\,(2\,v_{\prime}-2\,\theta)$, as in the first term of [3800]. Again, since BD is perpendicular to AB, and BE perpendicular to DF or BG, we have nearly, the angle $ABG=$ angle DBE; [3800e]
moreover, in the spherical triangle ABG, we have $\cos.\,ABG=\sin.\,BAG\,.\cos.\,AG$ [1345[32]], or in symbols, $\cos.\,DBE=\sin.\,\varphi\,.\cos.\,(v_{\prime}-\theta)$. Now in the right-angled [3800f]
triangle BED, we have, very nearly, $BE=BD.\cos.\,DBE=\delta s\,.\,\sin.\,\varphi\,.\,\cos.\,(v_{\prime}-\theta)$; and on account of the smallness of φ, we may change $\sin.\,\varphi$ into $\text{tang.}\,\varphi$, also BE into FG; hence $FG=\delta s\,.\,\text{tang.}\,\varphi\,.\cos.\,(v_{\prime}-\theta)$. Subtracting this from AG, we get AF; and [3800g]
in this way we obtain the second term of [3800].

[3800''] inclinations of the orbits; but these inequalities are insensible except for Jupiter and Saturn.

If we notice only the secular quantities, and put, as in [1032],

[3801] $$\text{tang.}\,\varphi\,.\,\sin.\,\theta = p\,; \qquad \text{tang.}\,\varphi\,.\,\cos.\,\theta = q\,;$$

we shall have*

[3802] $$\delta s = t\,.\,\frac{dq}{dt}\,.\,\sin.\,(nt+\varepsilon) - t\,.\,\frac{dp}{dt}\,.\,\cos.\,(nt+\varepsilon).$$

[3803] The term $-\text{tang.}\,\varphi\,.\,\delta s\,.\,\cos.\,(v_{,}-\theta)$ produces the following expression,

[3804] $\dfrac{t\,.\,(q\,dp - p\,dq)}{2\,dt}$; so that we shall have†

[3805] $$v_{,} = v + t\,.\,\frac{(q\,dp - p\,dq)}{2\,dt}\,;$$

which agrees with what we have found in the preceding article [3782].‡

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* (2401) If we suppose s to be a function of t, which becomes S, when $t=0$, we

[3801a] shall have, by the theorem [607, &c.], $s = S + t\,.\left(\frac{dS}{dt}\right) + \frac{t^2}{1.2}\,.\left(\frac{d^2S}{dt^2}\right) + \&c.$ If we neglect t^2 and the higher powers of t, and notice only the secular inequalities, we shall get $s - S = t\,.\left(\frac{dS}{dt}\right)$. Now $s-S$, being the variation of s in the time t, is what is

[3801b] represented above by δs; hence $\delta s = t\,.\left(\frac{dS}{dt}\right)$; and by noticing only the secular inequalities depending on dp, dq, in [3781], we obtain

[3801c] $$\left(\frac{dS}{dt}\right) = \frac{dq}{dt}\,.\,\sin.\,(nt+\varepsilon) - \frac{dp}{dt}\,.\,\cos.\,(nt+\varepsilon)\,;$$

consequently δs becomes as in [3802].

† (2402) Developing $\cos.\,(v_{,}-\theta)$ by [24] Int., and then substituting the values [3801], we get

[3804a] $$-\text{tang.}\,\varphi\,.\,\cos.\,(v_{,}-\theta) = -\text{tang.}\,\varphi\,.\{\cos.\,\theta\,.\cos.\,v_{,} + \sin.\,\theta\,.\,\sin.\,v_{,}\} = -q\,.\cos.\,v_{,} - p\,.\,\sin.\,v_{,}$$

[3804b] $$= -q\,.\cos.\,(nt+\varepsilon) - p\,.\,\sin.\,(nt+\varepsilon)\,;$$

observing, that as this quantity is of the order p, q, and is to be multiplied by δs, in [3800], which is also of the same order [3802, 3767c], we may put $v_{,} = nt+\varepsilon$, neglecting, as usual, the terms of a higher order in p, q. Multiplying together the expressions [3802, 3804b],

[3804c] and retaining only the quantities independent of the periodical angle $2nt+2\varepsilon$, we may use the values [3779c], and we shall get, for $-\text{tang.}\,\varphi\,.\,\delta s\,.\,\cos.\,(v_{,}-\theta)$, the same

[3804d] expression as in [3804]. This represents the secular change of v, arising from the last term of [3800]; and by adding it to v, it gives $v_{,}$, as in [3805]. We may observe, that

[3804e] the first term of [3800] produces no secular terms, or such as are independent of $2v_{,}-2\theta$, and it is therefore neglected in this estimate of $v_{,}$ [3805].

[3805a] ‡ (2403) Neglecting terms of the order t^2 or m'^2, we may suppose $\frac{1}{2}\,.\,(q\,dp - p\,dq)$

ON THE INEQUALITIES DEPENDING ON THE CUBES AND PRODUCTS OF THREE DIMENSIONS OF THE EXCENTRICITIES AND INCLINATIONS OF THE ORBITS AND THEIR HIGHER POWERS. [3806]

7. The inequalities depending on the cubes and products of three dimensions of the excentricities and inclinations of the orbits, are susceptible of two forms,* [3806']

Two forms of R of the third order.

$$R = M.\sin.\{i.(n't - nt + \varepsilon' - \varepsilon) + 3nt + K\}; \qquad \text{[First form.]} \qquad [3807]$$

$$R = N.\sin.\{i.(n't - nt + \varepsilon' - \varepsilon) + nt + L\}. \qquad \text{[Second form.]} \qquad [3807']$$

We may determine them by the analysis employed in the preceding articles; but as they become sensible only when they increase very slowly, we can make use of this circumstance to simplify the calculation. We shall resume the expression [3715b], and shall neglect the term $\frac{2d.(r\delta r)}{a^2 n\,dt}$, which is [3808] then insensible,† because of the smallness of the coefficient of t, in the inequalities now under consideration. Then this formula becomes

$$\delta v = -\frac{dr.\delta r}{a^2.n dt} + 3a.\textstyle\int\int n\,dt.\,\mathrm{d}R + 2\int n\,dt..a^2.\left(\frac{dR}{da}\right).\ddagger \qquad [3809]$$

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to be equal to $C dt$, C being a constant quantity; then [3782] becomes $dv_{\prime} = dv + C dt$, [3805b] whose integral is $v_{\prime} = v + Ct$, as in [3805].

* (2404) The reason for assuming these forms is evident from the principles used in [3704a—b]. For the coefficients of $n't$, $-nt$, in [3807], are i, $i-3$, respectively; [3807a] their difference 3 expresses the order of the coefficient k [957$^{\text{viii}}$, &c.], or that of M [3807], which must therefore be of the order e^3. Again, the coefficients of $n't$, $-nt$ [3807'] are i, $i-1$; their difference is 1, consequently N may contain terms of the order [3807b] 1, 3, 5, &c. [957$^{\text{vii}}$, &c.]; which include those of the order e^3; and it is evident from [957$^{\text{v}}$, &c.], that these forms embrace all these terms of the third order.

† (2405) This remark applies exclusively to terms of the form [3807], like those [3808a] in the three first lines of the second member of [3819], depending on the angles $i.(n't - nt + \varepsilon' - \varepsilon) + 3nt$, whose differential introduces the very small factor $i.(n'-n) + 3n$ [3818d]. But this small factor is not produced in the differential of the terms of the form [3807'], contained in the last line of the second member [3819]; and then the [3808b] term [3808] is not neglected, but is computed in [3822c].

‡ (2406) In the terms treated of in §7, and depending on the cubes of the excentricities, no quantities are finally retained except those which have the *small divisor* $i.(n'-n) + 3n$, [3809a] or its powers; and as the expression of δv [3715b] contains the function $2d.(r\delta r)$, divided by $a^2.n\,dt$; we must examine whether this function contains the small divisor we have just mentioned. Now by the inspection of the value of $r\delta r$, or rather of δr [1016], [3809b]

[3809′] The divisor $\sqrt{1-e^2}$ [3715b] must be neglected for greater accuracy, as in Book II, § 54 or [3718′]. *We must also, by the same article, apply these inequalities to the mean motion of the planet* m, *in computing its elliptical motion* [3720]. This being premised, if we suppose

[3810]
$$\begin{aligned} R = {} & m'P.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\} \\ & + m'P'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\}; \end{aligned}$$

which comprises all the terms of R, where the coefficient of nt is greater

[3811] or less than that of $n't$ by the number 3; we shall get, as in [1209],*

[3812]
$$3a.\textstyle\iint n\,dt.\mathrm{d}R = \dfrac{3.(3-i).m'n^2.a}{\{i.(n'-n)+3n\}^2}$$
$$\times\left\{\begin{array}{l} \left\{P'+\dfrac{2dP}{\{i.(n'-n)+3n\}.dt}-\dfrac{3ddP'}{\{i.(n'-n)+3n\}^2.dt^2}\right\}.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\} \\ -\left\{P-\dfrac{2dP'}{\{i.(n'-n)+3n\}.dt}-\dfrac{3ddP}{\{i.(n'-n)+3n\}^2.dt^2}\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\} \end{array}\right\}.$$

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we shall not find, in the preceding function, any term depending on the *first* power of e,

[3809c] and having the divisor $i.(n'-n)+3n$. In quantities of the *second* order in e, e', given in [3711, 3714], we find such terms having the first power of that divisor; but these terms depend upon angles of the form $i.(n't-nt+\varepsilon'-\varepsilon)+2nt$, which are different from those under consideration in this article [3806′—3807′]; so that they may be neglected. To investigate the similar terms of the order e^3, which depend on the angle $i.(n't-nt+\varepsilon'-\varepsilon)+3nt$, we may go through a calculation similar to that in

[3809d] [3703—3714], changing, however, the angle $i.(n't-nt+\varepsilon'-\varepsilon)+2nt$ into $i.(n't-nt+\varepsilon'-\varepsilon)+3nt$; which is the same as to increase the *integral number* $2-i$, connected with nt by unity; by which means the divisors $in'+(1-i).n$, $in'+(2-i).n$, $in'+(3-i).n$, which occur in [3705, 3710, 3711, 3714], are changed, respectively, into $in'+(2-i).n$, $in'+(3-i).n$, $in'+(4-i).n$. Hence the quantity $r\delta r$,

[3809e] similar to [3711], will contain a term of the order e^3, depending on the form [3807], and having for divisor the *first power* of the small quantity $in'+(3-i).n$, as is hereafter found in [3819]; but this divisor will vanish from the differential $d.(r\delta r)$; therefore it may be neglected, as in [3809a]; and then the formula [3715b] becomes as in [3809]; omitting the divisor, $\sqrt{(1-e^2)}$, for the reasons given in [3718′].

* (2407) Substituting, in the first member of [1209], the assumed value of

[3812a] $k.\sin.(i'n't-int+A)$ [1208vi], it becomes

[3812b]
$$\iint an^2.dt^2.\{Q.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)+Q'.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)\} =$$
$$\frac{n^2a.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)}{(i'n'-in)^2}.\left\{-Q+\frac{2dQ'}{(i'n'-in).dt}+\frac{3d^2Q}{(i'n'-in)^2.dt^2}-\frac{4d^3Q'}{(i'n'-in)^3.dt^3}-\&c.\right\}$$
$$+\frac{n^2a.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)}{(i'n'-in)^2}.\left\{-Q'-\frac{2dQ}{(i'n'-in).dt}+\frac{3d^2Q'}{(i'n'-in)^2.dt^2}+\frac{4d^3Q}{(i'n'-in)^3.dt^3}-\&c.\right\}.$$

Then we shall have*

$$2.\int n\,d\,t\,.\,a^2.\left(\frac{d\,R}{d\,a}\right)=-\frac{2\,m'\,n}{i.(n'-n)+3\,n}.\left\{\begin{array}{l} a^2.\left(\frac{d\,P}{d\,a}\right).\cos.\{i.(n't-n\,t+\varepsilon'-\varepsilon)+3n\,t+3\varepsilon\} \\ -a^2.\left(\frac{d\,P'}{d\,a}\right).\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\}\end{array}\right\}. \qquad [3813]$$

Lastly, we shall suppose, that by noticing only the angle

$$i\,.\,(n't-n\,t+\varepsilon'-\varepsilon)+2\,n\,t+2\,\varepsilon,$$

we have†

$$\frac{r\,\delta\,r}{a^2}=H\,.\cos.\{i\,.\,(n't-n\,t+\varepsilon'-\varepsilon)+2\,n\,t+2\,\varepsilon+A\}\,; \qquad [3814]$$

Now if we take the differential of [3810], relatively to d, then multiply it by $3\,a\,.\,n\,d\,t$, and prefix the double sign of integration, we shall get, by using for brevity, $T=n'\,t-n\,t+\varepsilon'-\varepsilon$ [3702a], [3812c]

$$3\,a\,.\!\int\!\!\int n\,d\,t\,.\,\mathrm{d}\,R=\int\!\!\int a\,n^2.\,d\,t^2.\left\{\begin{array}{l}-3\,.\,(3-i)\,.\,m'.\,P'.\sin.\,(i\,T+3\,n\,t+3\,\varepsilon) \\ +3\,.\,(3-i)\,.\,m'.\,P\,.\cos.(i\,T+3\,n\,t+3\,\varepsilon)\end{array}\right\}. \qquad [3812d]$$

The second member of this expression is of the same form as the first member of [3812b], as is easily perceived by changing, in [3812b], i' into i, and i into $i-3$; also putting $Q=-3\,.\,(3-i)\,.\,m'.\,P'$, $Q'=3\,.\,(3-i)\,m'.\,P$; then making the same [3812e] changes in the second member of [3812b], we obtain for $3\,a\,.\!\int\!\!\int n\,d\,t\,.\,\mathrm{d}\,R$, the same expression as in [3812]. We may observe, as in [3714d′], that the secular variations of [3812f] the elements are noticed by the introduction of the differentials $d\,P$, $d\,P'$, $d\,d\,P$, $d\,d\,P'$, which are computed in [4415, &c., 4484, &c.].

* (2408) The partial differential of R [3810], taken relatively to a, being multiplied, [3813a] by $2\,n\,d\,t\,.\,a^2$, and then integrated, gives [3813].

† (2409) The expression [3814] is equivalent to that in [3711]; H being taken for the coefficient of any one of the terms of this formula, and A representing that one of the [3814a] quantities $-2\,\varpi$, $-\varpi-\varpi'$, $K-2\,\varepsilon$, which is connected with this coefficient H; [3814b] observing that H is of the second dimension in e, e'. The differential of [3701], is $d\,r=a\,e\,.\,n\,d\,t\,.\sin.\,(n\,t+\varepsilon-\varpi)+$ &c.; multiplying this by [3814], and neglecting [3814c] terms of the fourth order, we get, by using T [3812c],

$$\begin{aligned}\frac{r\,\delta\,r}{a^2}\,.\,d\,r= &\quad H\,a\,e\,.\,n\,d\,t\,.\cos.(i\,T+2\,n\,t+2\,\varepsilon+A)\,.\sin.\,(n\,t+\varepsilon-\varpi) \\ = &\quad \tfrac{1}{2}\,H\,a\,e\,.\,n\,d\,t\,.\sin.\,(i\,T+3\,n\,t+3\,\varepsilon-\varpi+A) \\ &-\tfrac{1}{2}\,H\,a\,e\,.\,n\,d\,t\,.\sin.\,(i\,T+\;\;n\,t+\;\;\varepsilon+\varpi+A).\end{aligned} \qquad [3814d]$$

As this is of the third order [3814b], we may, in the first member, put $r=a$, and then dividing by $-a\,n\,d\,t$, we get

$$\begin{aligned}-\frac{d\,r\,.\,\delta\,r}{a^2\,n\,d\,t}= &-\tfrac{1}{2}\,H\,e\,.\sin.\,(i\,T+3\,n\,t+3\,\varepsilon-\varpi+A) \\ &+\tfrac{1}{2}\,H\,e\,.\sin.\,(i\,T+\;\;n\,t+\;\;\varepsilon+\varpi+A).\end{aligned} \qquad [3814e]$$

[3814′] H being determined as in [3814a], and having the very small divisor $i.(n'-n)+3n$; then the first term of δv [3809] gives the following expression;

[3815] $$-\frac{dr.\delta r}{a^2.ndt}=-\tfrac{1}{2}He.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon-\varpi+A\}.$$

Hence we shall find, by noticing only terms which have the divisor

[3816] $i.(n'-n)+3n$,*

[3817] $$\delta v=\frac{3.(3-i).m'.n^2}{\{i.(n'-n)+3n\}^2}.\left\{\begin{array}{l}\left\{aP'+\frac{2a.dP}{\{i.(n'-n)+3n\}.dt}-\frac{3a.ddP'}{\{i.(n'-n)+3n\}^2.dt^2}\right\}.\sin.\left\{\begin{array}{l}i.(n't-nt+\varepsilon'-\varepsilon)\\+3nt+3\varepsilon\end{array}\right\}\\-\left\{aP-\frac{2a.dP'}{\{i.(n'-n)+3n\}.dt}-\frac{3a.ddP}{\{i.(n'-n)+3n\}^2.dt^2}\right\}.\cos.\left\{\begin{array}{l}i.(n't-nt+\varepsilon'-\varepsilon)\\+3nt+3\varepsilon\end{array}\right\}\end{array}\right\}$$

Terms of δv of the third order.

$$-\frac{2m'n}{i.(n'-n)+3n}.\left\{\begin{array}{l}a^2.\left(\frac{dP}{da}\right).\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\}\\-a^2.\left(\frac{dP'}{da}\right).\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\}\end{array}\right\}$$

$$-\tfrac{1}{2}He.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon-\varpi+A\}.$$

The differential equation [3699]

[3818] $$0=\frac{d^2.(r\delta r)}{dt^2}+\frac{\mu.r\delta r}{r^3}+2\int\mathrm{d}R+r.\left(\frac{dR}{dr}\right),\dagger$$

[3814f] The first term of the second member is the same as in [3815]; the second term is noticed in [3822d]. We may observe, that it is not necessary to notice terms of the order e^2 in dr [3814c], because they depend on the elliptical motion, and have no divisor of the form $i.(n'-n)+3n$; moreover they must be multiplied by terms of the order e,

[3814g] which occur in $\frac{\delta r}{a}$ [1023], to produce terms of the third order now under consideration; and these terms of [1023] do not contain the small divisor just mentioned.

[3816a] * (2410) Substituting, in the expression of δv [3809], the values of the terms in its second member, given in [3815, 3812, 3813], we get [3817].

† (2411) The expression [3818] is the same as [3699], from which we have deduced [3702], and by using [3705a], it becomes

[3818a] $$0=\frac{d^2.(r\delta r)}{dt^2}+n^2.r\delta r+\left\{3n^2a.\delta r.[e.\cos.(nt+\varepsilon-\varpi)+e^2.\cos.2.(nt+\varepsilon-\varpi)]+2\int\mathrm{d}R+a.\left(\frac{dR}{da}\right)\right\}.$$

This is solved as in [3711b, c], and if any term of the expression between the braces be

[3818a′] represented, as in [3711b], by $\alpha K.\sin.(m_{,}t+\varepsilon_{,})$, or $\alpha K.\cos.(m_{,}t+\varepsilon_{,})$, the

[3818a″] corresponding terms of $r\delta r$ [3711c] will contain the divisor $m_{,}^2-n^2$, or rather the two divisors $(m_{,}+n)$, $(m_{,}-n)$. To find the values of $m_{,}$ producing the divisor $i.(n'-n)+3n$ [3818′].

gives, by noticing only the terms which have the divisor $i.(n'-n)+3n$, [3818']

$$\frac{r\,\delta r}{a^2}=\frac{2.(i-3).m'n}{i.(n'-n)+3n}.\left\{\begin{array}{l} aP.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\} \\ +aP'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\}\end{array}\right\}$$ [3819]

Terms of $r\,\delta r$.

$$-\tfrac{3}{2}He.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon-\varpi+A\}$$
$$+\tfrac{1}{2}He.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon+\varpi+A\}.$$

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we shall put it successively equal to $m_{,}+n$ and $m_{,}-n$; and we shall get $m_{,}=i.(n'-n)+2n$, $m_{,}=i.(n'-n)+4n$; but we may neglect the last, because the coefficients of n', n differ by 4, and the terms depending on it must be of the fourth dimension in e, e' [3704a, &c.], which are here neglected. Therefore, in finding $r\,\delta r$, we need notice only the following terms. *First.* Where $m_{,}=i.(n'-n)+2n$. *Second.* Where the quantity R, or rather $\int\mathrm{d}R$, contains the divisor $i.(n'-n)+3n$ [3818']. Hence it is evident, that we may neglect $a.\left(\frac{dR}{da}\right)$, which produces no such terms. The part of R, given in [3810], produces in $2\int\mathrm{d}R$, the following terms, [3818b]

$$\frac{-2.(i-3).m'.n}{i.(n'-n)+3n}.\left\{\begin{array}{l} P.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\} \\ +P'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\}\end{array}\right\}.$$ [3818c]

These come under the second form [3818b], in which αK has the divisor $i.(n'-n)+3n$. The part of $r\,\delta r$ [3818a''], depending on these terms, is found by dividing them by $m_{,}^2-n^2$; $m_{,}$ being in this case equal to $i.(n'-n)+3n$; and by hypothesis it is very small in comparison with n. Thus, for Jupiter and Saturn, where $i=5$, it becomes $m_{,}=i.(n'-n)+3n=5n'-2n=\frac{1}{74}n$ [3711f]; so that $m_{,}^2$ is less than $\frac{n^2}{5000}$, and for the divisor $m_{,}^2-n^2$, we may write simply $-n^2=-a^{-3}$ [3709']. Therefore, by multiplying [3818c], by $-a^3$, we get the part of $r\,\delta r$ corresponding to these terms of $2\int\mathrm{d}R$; and then dividing this result by a^2, we obtain the corresponding terms of $\frac{r\,\delta r}{a^2}$. The terms thus computed agree with those in [3819], depending on P, P'. It is not necessary to notice the terms of $2\int\mathrm{d}R$, like those depending on [3703, 3704], because they will produce in $\frac{r\,\delta r}{a^2}$, terms depending on different angles from those proposed in [3807, 3807'], or else such as have not the small divisor mentioned in [3818']. The next term of αK [3818a'], which we shall notice, is that depending on the quantity $3n^2a.\delta r.e^2.\cos.2.(nt+\varepsilon-\varpi)$ [3818a]; and as we retain merely the terms of the third dimension in e, e', &c., it will only be necessary to notice terms of the first dimension in δr. Now if we examine [1023], we shall find, that none of its terms, of that order, have the small divisor [3818']; therefore we may neglect this part, and then the only remaining quantity in [3818a], producing terms of αK, is $3n^2a.\delta r.e.\cos.(nt+\varepsilon-\varpi)$. As this contains the factor e, we may notice in δr only terms of the second dimension, in [3818d] [3818e] [3818f] [3818g]

Adding this expression to that in [3814],

[3820] $$\frac{r\delta r}{a^2}=H.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon+A\},$$

we obtain*

Terms of δr.

[3821] $$\begin{aligned}\frac{\delta r}{a}= &\quad H.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon+A\}\\ &-He.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon-\varpi+A\}\\ &+He.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon+\varpi+A\}\\ &+\frac{2.(i-3).m'n}{i.(n'-n)+3n}.\left\{\begin{array}{l} aP.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\}\\ +aP'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon\}\end{array}\right\}.\end{aligned}$$

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order to procure those of the third dimension, which are the only ones investigated in this article. The terms of the second dimension, which can produce the angles proposed in [3807, 3807'], are evidently included in the form [3814] or [3820]; multiplying this by $3n^2a^2.e.\cos.(nt+\varepsilon-\varpi)$, and reducing by [20] Int., it becomes

[3818h] $$3n^2a.\delta r.e.\cos.(nt+\varepsilon-\varpi)\times\frac{r}{a}=\tfrac{3}{2}He.n^2a^2.\left\{\begin{array}{l}\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon-\varpi+A\}\\ +\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon+\varpi+A\}\end{array}\right\}.$$

Now He [3814b] is of the third dimension in e, e', &c., and by neglecting higher dimensions, we may put $\frac{r}{a}=1$ [3701], and then we shall have for the remaining terms of $\alpha K.\cos.(m_,t+\varepsilon_,)$ [3818a],

[3818i] $$\begin{aligned}&\tfrac{3}{2}He.n^2a^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon-\varpi+A\}\\ &+\tfrac{3}{2}He.n^2a^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon+\varpi+A\}.\end{aligned}$$

Dividing this by $m_,^2-n^2$ [3818a''], we get the corresponding terms of $r\delta r$. Now for the first of these angles $i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon-\varpi+A$, we have $m_,=i.(n'-n)+3n$, and as this is very small [3818d], it may be neglected; and then the divisor becomes $-n^2$.
[3818k] In the second angle [3818i], the value of $m_,$ is $i.(n'-n)+n$ or $\{i.(n'-n)+3n\}-2n$, which is nearly equal to $-2n$; hence $m_,^2-n^2$ is nearly $3n^2$; consequently this divisor is nearly equal to $3n^2$. Therefore if we divide these terms of [3818i] by $-n^2$ and $3n^2$, respectively, we shall obtain the corresponding terms of $r\delta r$; lastly, dividing these results
[3818l] by a^2, we get the terms of $\frac{r\delta r}{a^2}$ depending on He, as in [3819].

* (2412) None of the terms of $\frac{r\delta r}{a^2}$ or $\frac{\delta r}{a}$, of the order $m'e$, contain the small divisor [3818'], as is evident from the inspection of the formula [1016]; so that the terms
[3821a] of $\frac{r\delta r}{a^2}$, containing this divisor, and which must be noticed, are included in the functions of the second members of [3819, 3820]. Adding these quantities together, and multiplying

This value of $\frac{\delta r}{a}$ produces in δv, an inequality depending on the angle [3822]
$i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon$, which has $i.(n'-n)+3n$ for a divisor. [3822']
To determine it, we shall resume the expression of δv, given by the formula [931].* The part $\frac{2r.d\delta r+dr.\delta r}{a^2.ndt}$ of this expression produces [3822'']
in δv the term

$$\delta v=\tfrac{5}{2}He.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon+\varpi+A\};\qquad [3823]$$

which is the only one of this kind having the divisor $i.(n'-n)+3n$.
The inequality of δv depending on the angle $i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon$, [3824]
noticing only the terms having the divisor $i.(n'-n)+3n$, is, by [3715, 3814], very nearly equal to

$$2H.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon+A\}.\qquad [3825]$$

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their sum by $\frac{a}{r}$, which, by [3701], is equal to $1+e.\cos.(nt+\varepsilon-\varpi)+\&c.$, we [3821b]
get the corresponding terms of $\frac{\delta r}{a}$. The quantities produced by this multiplication are equal to the sum of the terms [3819, 3820], with the additional term produced by multiplying the function [3820] by $e.\cos.(nt+\varepsilon-\varpi)$, and this term is

$$He.\cos.(nt+\varepsilon-\varpi).\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon+A\},\qquad [3821c]$$

which, by [20] Int., becomes

$$\begin{aligned}&\tfrac{1}{2}He.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+3nt+3\varepsilon-\varpi+A\}\\ &+\tfrac{1}{2}He.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon+\varpi+A\}.\end{aligned}\qquad [3821d]$$

Connecting this with the other terms [3819, 3820], we obtain, by reduction, the function $\frac{\delta r}{a}$ [3821].

* (2413) This formula, by the substitution of [3715a, 3705a], becomes as in [3715b], the part mentioned in [3822''] being represented by $\frac{2d.(r\delta r)}{a^2.ndt}-\frac{dr.\delta r}{a^2.ndt}$. Now the last [3822a]
term of the second member of [3819] depends on the angle $iT+nt+\varepsilon+\varpi+A$ [3702a], mentioned in [3822'], and if we substitute it in the first term of the preceding expression $\frac{2d.(r\delta r)}{a^2.ndt}$, it produces the term

$$-\{i.(n'-n)+n\}.\frac{He}{n}.\sin.\{iT+nt+\varepsilon+\varpi+A\};\qquad [3822b]$$

and as we have, very nearly, $-\{i.(n'-n)+n\}=2n$ [3818k]; it becomes $2He.\sin.\{iT+nt+\varpi+A)$. Again, the second term of [3822a] has already been [3822c]
computed in [3814e], and contains the quantity $\frac{1}{2}He.\sin.(iT+nt+\varepsilon+\varpi+A)$; [3822d]
connecting this with the preceding [3822c], the sum becomes as in [3823].

Therefore, if we denote this inequality by

[3826] $$K.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon+B\},^*$$

Terms of δv. we shall have, in δv, the following expression,

[3827] $$\delta v=\tfrac{5}{4}Ke.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon+\varpi+B\}.$$

8. *It is chiefly in the theory of Jupiter and Saturn that these different inequalities are sensible.* If we suppose $i=5$, the function

[3828] $$i.(n'-n)+3n=5n'-2n,$$

becomes very small [3818d], in consequence of the nearly commensurable ratio which obtains between the mean motions of these planets; and from this cause the corresponding terms of δr, δv acquire great values. To determine them, we shall resume the expression of R [3742]. The part†

[3829] $$\frac{m'r}{r'^2}.\cos.(v'-v)-\frac{m'.\gamma^2}{4}.\frac{r}{r'^2}.\{\cos.(v'-v)-\cos.(v'+v)\}+\frac{m'.\gamma^2}{4}.\frac{rr'.\cos.(v'-v)}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{3}{2}}},$$

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* (2414) The parts of R [957, 1011], represented by M, N [3703, 3704], do not contain the small divisor $i.(n'-n)+3n$, as is evident from inspection. Moreover,

[3826a] F, G, H [3706], being the parts of $\frac{\delta r}{a}$ [1016], depending on terms of the first degree in e, e', do not contain this divisor, as appears by the inspection of [1016]. Therefore no part of δv [3715], except the first term $\frac{2d.(r\delta r)}{a^2.ndt}$, contains this divisor; and if we substitute in this term the value of $r\delta r$ [3814], we shall obtain, in δv, the term

[3826b] $$-\frac{2}{n}.\{i.(n'-n)+2n\}.H.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+\varepsilon+A\};$$

substituting $-\{i.(n'-n)+2n\}=n$ [3822c], it becomes as in [3825]. If we now compare the expressions [3825, 3823], we find, that [3823] may be derived from [3825]

[3826c] by multiplying its coefficient by $\frac{5}{4}e$, and decreasing the argument by $nt+\varepsilon-\varpi$. The same process of derivation being used upon the assumed form [3826], produces the expression [3827]; which is computed in [4439] for Jupiter, by this very simple process.

† (2415) We shall suppose, as in [1009, 956c, 963$^{\text{iv}}$, 1018a], for the sake of brevity,

[3829a] $$r=a(1+u_{,});\qquad r'=a'.(1+u_{,}');\qquad v=nt+\varepsilon+v_{,};\qquad v'=n't+\varepsilon'+v_{,}';$$

[3829b] $$\alpha_0=au_{,};\qquad \alpha'=a'u_{,}';\qquad \alpha''=v_{,}'-v_{,};\qquad \alpha=\frac{a}{a'};$$

[3829b'] $$T=n't-nt+\varepsilon'-\varepsilon;\qquad dT=(n'-n).dt;$$

[3829c] $$W=nt+\varepsilon-\varpi;\qquad W'=n't+\varepsilon'-\varpi';$$

[3829c'] $u_{,}$, $u_{,}'$ $v_{,}'-v$ are of the order of the excentricities, and α is changed into α_0, to

produces no term of the third order of the excentricities and inclinations, [3830]

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distinguish it from α [963^iv]. If we represent the function [3829] by u, and suppose U to be the part of this value independent of $u_{,}$, $u'_{,}$, $v_{,}$, $v'_{,}$, we shall have U as in [3829f]; [3829d]
observing that the last term of [3829] becomes in this case, by using [3744, 3749],

$$\begin{aligned}\tfrac{1}{4} m'.\gamma^2.a a'.\cos.T.\{a^2-2aa'.\cos.T+a'^2\}^{-\frac{3}{2}} &= \tfrac{1}{4} m'.\gamma^2.a a'.\cos.T.\tfrac{1}{2}\Sigma.B^{(i)}.\cos.iT \\ &= \tfrac{1}{4} m'.\gamma^2.a a'.\tfrac{1}{2}\Sigma.B^{(i)}.\cos.(i+1).T \\ &= \tfrac{1}{8} m'.\gamma^2.a a'.\Sigma.B^{(i-1)}.\cos.iT;\end{aligned} \qquad [3829e]$$

$$\begin{aligned}U = \frac{m'a}{a'^2}.\cos.T - \tfrac{1}{4} m'.\gamma^2.\frac{a}{a'^2}.\cos.T + \tfrac{1}{4} m'.\gamma^2.\frac{a}{a'^2}.\cos.(n't+nt+\varepsilon'+\varepsilon) \\ + \tfrac{1}{8} m'.\gamma^2.a a'.\Sigma.B^{(i-1)}.\cos.iT;\end{aligned} \qquad [3829f]$$

i being as in [3715']. The development of u, as far as the second powers of α_0, α', α'', being found as in [957e], is

$$\begin{aligned}u = U + \alpha_0.\left(\frac{dU}{da}\right) + \alpha'.\left(\frac{dU}{da'}\right) + \alpha''.\left(\frac{dU}{dT}\right) + \tfrac{1}{2}\alpha_0^2.\left(\frac{ddU}{da^2}\right) + \alpha_0\alpha'.\left(\frac{ddU}{da\,da'}\right) \\ + \tfrac{1}{2}\alpha'^2.\left(\frac{ddU}{da'^2}\right) + \alpha_0\alpha''.\left(\frac{ddU}{da\,dT}\right) + \alpha'\alpha''.\left(\frac{ddU}{da'dT}\right) + \tfrac{1}{2}\alpha''^2.\left(\frac{ddU}{dT^2}\right);\end{aligned} \qquad [3829g]$$

the terms of the third order, obtained in the same manner, are

$$\begin{aligned}\tfrac{1}{6}\alpha_0^3.\left(\frac{d^3U}{da^3}\right) + \tfrac{1}{2}\alpha_0^2\alpha'.\left(\frac{d^3U}{da^2da'}\right) + \tfrac{1}{2}\alpha_0\alpha'^2.\left(\frac{d^3U}{da\,da'^2}\right) + \tfrac{1}{6}\alpha'^3.\left(\frac{d^3U}{da'^3}\right) \\ + \tfrac{1}{2}\alpha_0^2\alpha''.\left(\frac{d^3U}{da^2dT}\right) + \tfrac{1}{2}\alpha_0\alpha''^2.\left(\frac{d^3U}{da\,dT^2}\right) + \tfrac{1}{2}\alpha'^2\alpha''.\left(\frac{d^3U}{da'^2dT}\right) \\ + \tfrac{1}{2}\alpha'\alpha''^2.\left(\frac{d^3U}{da'dT^2}\right) + \alpha_0\alpha'\alpha''.\left(\frac{d^3U}{da\,da'dT}\right) + \tfrac{1}{6}\alpha''^3.\left(\frac{d^3U}{dT^3}\right).\end{aligned} \qquad [3829h]$$

We have given this full development of u, because it will hereafter be of use in the notes on this article; and for the same purpose, we shall also insert the following expressions, deduced [3829i]
from the comparison of the values of α_0, α', α'' [3829b, a] with [659, 668, 669];

$$\alpha_0 = a.\{\tfrac{1}{2}e^2 - (e-\tfrac{3}{8}e^3).\cos.W - \tfrac{1}{2}e^2.\cos.2W - \tfrac{3}{8}e^3.\cos.3W\} = a\,u_{,}; \qquad [3829k]$$

$$\alpha' = a'.\{\tfrac{1}{2}e'^2 - (e'-\tfrac{3}{8}e'^3).\cos.W' - \tfrac{1}{2}e'^2.\cos.2W' - \tfrac{3}{8}e'^3.\cos.3W'\} = a'u'_{,}; \qquad [3829l]$$

$$\alpha'' = \left\{\begin{array}{l}(2e'-\tfrac{1}{4}e'^3).\sin.W' + \tfrac{5}{4}e'^2.\sin.2W' + \tfrac{13}{12}e'^3.\sin.3W' \\ -(2e-\tfrac{1}{4}e^3).\sin.W - \tfrac{5}{4}e^2.\sin.2W - \tfrac{13}{12}e^3.\sin.3W\end{array}\right\} = v'_{,} - v_{,}. \qquad [3829m]$$

From these values it appears, by a slight examination, that none of the terms of U [3829f] produce quantities of the *third* order, depending on the angle $5n't-2nt$, now under consideration. For the terms of [3829f], multiplied by γ^2, of the *second* order, depend [3829n]
on the angles T, $n't+nt+\varepsilon'+\varepsilon$, iT; and when we combine these with terms of the

depending on the angle $5n't-2nt$; such terms can therefore only arise from the remaining part*

Value of R for this case.

[3831] $$R=-\frac{m'}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{1}{2}}}-\frac{m'.\gamma^2}{4}.\frac{rr'.\cos.(v'+v)}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{3}{2}}};$$

[3832] and then the expressions of P and P' [3810] will be the same, whether we consider the action of m' on m, or that of m on m'. We shall now investigate these values of P, P'.

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first order in α_0, α', α'' [3829*k*—*m*], they will not produce the angle $5n't-2nt$. The

[3829*o*] only remaining term of U [3829*f*] is the first, depending on $\cos.T$ or $\cos.(n't-nt+\varepsilon'-\varepsilon)$; and if this were multiplied by a term depending on the angle $4n't-nt$, it would produce a quantity of the required form; but none of the powers and products of α_0, α', α'' [3829*k*—*m*], retained in [3829*g*, *h*] contain terms of the third order depending on this angle; therefore we may also reject this term, as in [3830].

* (2416) If we reject the terms of R [3742], mentioned in [3829], which we have proved, in the last note, not to contain terms of the required form and order, we shall obtain

[3831*a*] for R the function [3831]. This expression is not altered by changing r, v into r', v', respectively, and the contrary; so that it will be of the same form, whether we compute the

[3831*a'*] action of m' upon m, or that of m upon m'; but in the first case it will be multiplied by m', in the second by m. Supposing, as in [3829*d*], that the general value of the function R [3831] is represented by u, and that it becomes equal to U, by putting

[3831*b*] $$r=a,\quad r'=a'\quad v=nt+\varepsilon,\quad v'=n't+\varepsilon',\quad v'-v=n't-nt+\varepsilon'-\varepsilon=T,$$

we shall get the first of the following expressions of U [3831*c*]. The second expression

[3831*b'*] [3831*d*] is deduced from the first by the substitution of the values [3743, 3744], neglecting, however, the first term of [3743], which makes an exception in the value of $A^{(i)}$, in the case of $i=1$; because this term produces no effect in the present calculation, as we have seen in [3829*o*];

[3831*c*] $$U=-m'.\{a^2-2aa'.\cos.T+a'^2\}^{-\frac{1}{2}}-\tfrac{1}{4}m'.\gamma^2.aa'.\cos.(n't+nt+\varepsilon'+\varepsilon).\{a^2-2aa'.\cos.T+a'^2\}^{-\frac{3}{2}}$$

[3831*d*] $$=\tfrac{1}{2}m'.\Sigma.A^{(i)}.\cos.iT-\tfrac{1}{8}m'.\gamma^2.aa'.\cos.(n't+nt+\varepsilon'+\varepsilon).\Sigma.B^{(i)}.\cos.iT$$

[3831*e*] $$=\tfrac{1}{2}m'.\Sigma.A^{(i)}.\cos.iT-\tfrac{1}{8}m'.\gamma^2.aa'.\Sigma.B^{(i-1)}.\cos.(iT+2nt+2\varepsilon-2\Pi).$$

We may remark, that, in reducing [3831*d*] to the form [3831*e*], we obtain, in the first place, from [3749],

[3831*e'*] $$\cos.(n't+nt+\varepsilon'+\varepsilon).\Sigma.B^{(i)}.\cos.iT=\Sigma.B^{(i)}.\cos.(iT+n't+nt+\varepsilon'+\varepsilon)$$
$$=\Sigma.B^{(i)}.\cos.\{(i+1).T+2nt+2\varepsilon\};$$

and by changing i into $i-1$, it becomes $\Sigma.B^{(i-1)}.\cos.\{iT+2nt+2\varepsilon\}$; but as this

[3831*f*] quantity is to be multiplied by γ^2, we must change $2nt+2\varepsilon$ into $2nt+2\varepsilon-2\Pi$, as in [3745'''—3748], and then the value of U becomes as in [3831*e*].

We have, in Book II, §22, by carrying on the approximation to terms of the third order of the excentricities [659, 668, 669],*

Values of r, v.

$$\frac{r}{a}=1+\tfrac{1}{2}e^2-(e-\tfrac{3}{8}e^3).\cos.(nt+\varepsilon-\varpi)-\tfrac{1}{2}e^2.\cos.(2nt+2\varepsilon-2\varpi)$$
$$-\tfrac{3}{8}e^3.\cos.(3nt+3\varepsilon-3\varpi)\,; \qquad [3833]$$

$$v=nt+\varepsilon+(2e-\tfrac{1}{4}e^3).\sin.(nt+\varepsilon-\varpi)+\tfrac{5}{4}e^2.\sin.(2nt+2\varepsilon-2\varpi)$$
$$+\tfrac{13}{12}e^3.\sin.(3nt+3\varepsilon-3\varpi). \qquad [3834]$$

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* (2417) We shall now commence the investigation of the part of R depending upon the first term of [3831e], namely, $U=\frac{1}{2}m'.\Sigma.A^{(i)}.\cos.i\,T$; the other terms depending on $B^{(i-1)}$, being computed in [3840a, &c.]. Substituting this value of U, in the terms [3829g, h], we get the following value of R, [3834a]

Terms of R depending on $A^{(i)}$.

1 $R=\tfrac{1}{2}m'.\Sigma.A^{(i)}.\cos.i\,T$

2, 3 / 4
$$+\left\{\begin{array}{l}+\tfrac{1}{2}m'.\alpha_0.\Sigma.\left(\frac{dA^{(i)}}{da}\right).\cos.i\,T+\tfrac{1}{2}m'.\alpha'.\Sigma.\left(\frac{dA^{(i)}}{da'}\right).\cos.i\,T\\ -\tfrac{1}{2}m'.\alpha''.\Sigma\,i.A^{(i)}.\sin.i\,T\end{array}\right\}$$

5, 6 / 7, 8 / 9, 10
$$+\left\{\begin{array}{l}+\tfrac{1}{4}m'.\alpha_0^2.\Sigma.\left(\frac{ddA^{(i)}}{da^2}\right).\cos.i\,T+\tfrac{1}{2}m'.\alpha_0\alpha'.\Sigma.\left(\frac{ddA^{(i)}}{da\,da'}\right).\cos.i\,T\\ +\tfrac{1}{4}m'.\alpha'^2.\Sigma.\left(\frac{ddA^{(i)}}{da'^2}\right).\cos.i\,T-\tfrac{1}{2}m'.\alpha_0\alpha''.\Sigma i.\left(\frac{dA^{(i)}}{da}\right).\sin.i\,T\\ -\tfrac{1}{2}m'.\alpha'\alpha''.\Sigma i.\left(\frac{dA^{(i)}}{da'}\right).\sin.i\,T-\tfrac{1}{4}m'.\alpha''^2.\Sigma\,i^2.A^{(i)}.\cos.i\,T\end{array}\right\}$$

[3834b]

11, 12 / 13, 14 / 15, 16 / 17, 18 / 19, 20
$$+\left\{\begin{array}{l}+\tfrac{1}{12}m'.\alpha_0^3.\Sigma.\left(\frac{d^3A^{(i)}}{da^3}\right).\cos.i\,T+\tfrac{1}{4}m'.\alpha_0^2\alpha'.\Sigma.\left(\frac{d^3A^{(i)}}{da^2da'}\right).\cos.i\,T\\ +\tfrac{1}{4}m'.\alpha_0\alpha'^2.\Sigma.\left(\frac{d^3A^{(i)}}{da\,da'^2}\right).\cos.i\,T+\tfrac{1}{12}m'.\alpha'^3.\Sigma.\left(\frac{d^3A^{(i)}}{da'^3}\right).\cos.i\,T\\ -\tfrac{1}{4}m'.\alpha_0^2\alpha''.\Sigma i.\left(\frac{ddA^{(i)}}{da^2}\right).\sin.i\,T-\tfrac{1}{4}m'.\alpha_0\alpha''^2.\Sigma i^2.\left(\frac{dA^{(i)}}{da}\right).\cos.i\,T\\ -\tfrac{1}{4}m'.\alpha'^2\alpha''.\Sigma i.\left(\frac{ddA^{(i)}}{da'^2}\right).\sin.i\,T-\tfrac{1}{4}m'.\alpha'\alpha''^2.\Sigma i^2.\left(\frac{dA^{(i)}}{da'}\right).\cos.i\,T\\ -\tfrac{1}{2}m'.\alpha_0\alpha'\alpha''.\Sigma i.\left(\frac{ddA^{(i)}}{da\,da'}\right).\sin.i\,T+\frac{m'}{12}.\alpha''^3.\Sigma\,i^3.A^{(i)}.\sin.i\,T\end{array}\right\}.$$

We must substitute, in this expression, the values of α_0, α', α'' [3829k—m], and retain only the terms of the third dimension, and of the form $5n't-2nt$ [3834″], in which the coefficients of $n't$, nt differ by 3. Now as these coefficients are equal in the angle $i\,T$, which occurs in [3834b], this difference in the coefficients of $n't$, nt must arise from the [3834c]

[3834'] This being premised, if we develop R [3831] according to the order of the

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powers and products of α_0, α', α''; and it is evident, from [957$^{\text{viii}}$, &c.], that such terms
[3834d] must have for a factor, some one of the four quantities e'^3, $e'^2 e$, $e' e^2$, e^3. If we take the powers and products of the quantities α_0, α', α'' [3829k—m], of the third dimension, and reduce them by means of [17—20] Int., we shall find, that the *greatest* angles connected
[3834e] with these factors e'^3, $e'^2 e$, $e' e^2$, e^3, are, respectively, $3\,W'$, $2\,W'+W$, $W'+2\,W$, $3\,W$; it is not necessary to notice the *smaller* angles W, W', $2\,W'-W$, &c., because they do not produce terms of the form $5\,n't-2\,n\,t$ [3834c]; substituting $W'=T+n\,t+\varepsilon-\varpi'$,
[3834e'] $W=n\,t+\varepsilon-\varpi$ [3829c]; they become, respectively,

[3834f]
$$3\,T+3\,n\,t+3\,\varepsilon-3\,\varpi'; \qquad 2\,T+3\,n\,t+3\,\varepsilon-2\,\varpi'-\varpi;$$
$$T+3\,n\,t+3\,\varepsilon-\varpi'-2\,\varpi; \qquad 3\,n\,t+3\,\varepsilon-3\,\varpi.$$

Now we perceive, by inspection, that the cosine of any one of these angles is multiplied, in [3834b], by a term of the form $A_1^{(i)}.\cos.\,i\,T$; and its sine by a term of the form $A_1^{(i)}.\sin.\,i\,T$; the products reduced by the formula [3749], are found to depend, respectively, upon the angles

[3834g]
$$(i+3)\,.\,T+3\,n\,t+3\,\varepsilon-3\,\varpi'; \qquad (i+2)\,.\,T+3\,n\,t+3\,\varepsilon-2\,\varpi'-\varpi;$$
$$(i+1)\,.\,T+3\,n\,t+3\,\varepsilon-\varpi'-2\,\varpi; \qquad i\,T+3\,n\,t+3\,\varepsilon-3\,\varpi.$$

In order to reduce all the angles to the form $i\,T$, we must change, in the first, i into $i-3$; in the second, i into $i-2$; in the third, i into $i-1$; and make the same changes in the index of $A_1^{(i)}$; by this means the terms in question become of the forms

[3834h]
$$e'^3.\,\Sigma\,.\,A_1^{(i-3)}.\cos.\,(i\,T+3\,n\,t+3\,\varepsilon-3\,\varpi');$$
$$e'^2 e\,.\,\Sigma\,.\,A_1^{(i-2)}.\cos.\,(i\,T+3\,n\,t+3\,\varepsilon-2\,\varpi'-\varpi);$$
$$e' e^2.\,\Sigma\,.\,A_1^{(i-1)}.\cos.\,(i\,T+3\,n\,t+3\,\varepsilon-\varpi'-2\,\varpi);$$
$$e^3\,.\,\Sigma\,.\,A_1^{(i)}.\cos.\,(i\,T+3\,n\,t+3\,\varepsilon-3\,\varpi).$$

Putting $i=5$, as in [3828], these expressions become of the same forms as the four first
[3834i] terms of R [3835], depending on $M^{(0)}$, $M^{(1)}$, $M^{(2)}$, $M^{(3)}$, respectively. The two remaining terms $M^{(4)}$, $M^{(5)}$, depend on $B^{(-1)}$, which was neglected in [3834a], and will be computed in [3840a, &c.]. We may remark, that the exponent of e, in any one of the terms [3834h], being increased by $i-3$, gives the corresponding index of A_1,
[3834k] and when $i=5$, we have for this increment $i-3=2$.

We shall now proceed to the computation of the values of the powers and products of α_0, α', α'', which occur in the expression of R [3834b], retaining only the terms
[3834l] depending on e'^3, $e'^2 e$, $e' e^2$, e^3, which are wanted in finding the values of $M^{(0)}$,

terms depending on the angle $5n't-2nt$, we shall obtain an expression [3834l]
of the following form,

$M^{(1)}$, $M^{(2)}$, $M^{(3)}$. These quantities are arranged in the following table, in the order in [3834m]
which they occur in [3834b], noticing only the greatest angles mentioned in [3834c];

2 $\alpha_0 = -\frac{3}{8}a.e^3.\cos.3W$;

3 $\alpha' = -\frac{3}{8}a'.e'^3.\cos.3W'$;

4 $\alpha'' = \frac{13}{12}e'^3.\sin.3W'-\frac{13}{12}e^3.\sin.3W$;

5 $\alpha_0^2 = \frac{1}{2}a^2.e^3.\cos.3W$;

6 $\alpha_0\alpha' = \frac{1}{4}a'a.e'^2e.\cos.(2W'+W)+\frac{1}{4}a'a.e'e^2.\cos.(W'+2W)$;

7 $\alpha'^2 = \frac{1}{2}a'^2.e'^3.\cos.3W'$;

8 $\alpha_0\alpha'' = \frac{9}{8}a.e^3.\sin.3W-\frac{5}{8}ae'^2e.\sin.(2W'+W)-\frac{1}{2}a.e'e^2.\sin.(W'+2W)$;

9 $\alpha'\alpha'' = -\frac{9}{8}a'.e'^3.\sin.3W'+\frac{5}{8}a'.e'e^2.\sin.(W'+2W)+\frac{1}{2}a'.e'^2e.\sin.(2W'+W)$;

10 $\alpha''^2 = -\frac{5}{2}e^3.\cos.3W+\frac{5}{2}e'e^2.\cos.(W'+2W)+\frac{5}{2}e'^2e.\cos.(2W'+W)-\frac{5}{2}e'^3.\cos.3W'$;

[3835a]

11 $\alpha_0^3 = -\frac{1}{4}a^3.e^3.\cos.3W$;

12 $\alpha_0^2\alpha' = -\frac{1}{4}a'a^2.e'e^2.\cos.(W'+2W)$;

13 $\alpha_0\alpha'^2 = -\frac{1}{4}a'^2a.e'^2e.\cos.(2W'+W)$;

14 $\alpha'^3 = -\frac{1}{4}a'^3.e'^3.\cos.3W'$;

15 $\alpha_0^2\alpha'' = -\frac{1}{2}a^2.e^3.\sin.3W+\frac{1}{2}a^2.e'e^2.\sin.(W'+2W)$;

16 $\alpha_0\alpha''^2 = a.e^3.\cos.3W-2a.e'e^2.\cos.(W'+2W)+a.e'^2e.\cos.(2W'+W)$;

17 $\alpha'^2\alpha'' = \frac{1}{2}a'^2.e'^3.\sin.3W'-\frac{1}{2}a'^2.e'^2e.\sin.(2W'+W)$;

18 $\alpha'\alpha''^2 = a'.e'^3.\cos.3W'-2a'.e'^2e.\cos.(2W'+W)+a'.e'e^2.\cos.(W'+2W)$;

19 $\alpha_0\alpha'\alpha'' = -\frac{1}{2}aa'.e'e^2.\sin.(W'+2W)+\frac{1}{2}aa'.e'^2e.\sin.(2W'+W)$;

20 $\alpha''^3 = 2e^3.\sin.3W-6e'e^2.\sin.(W'+2W)+6e'^2e.\sin.(2W'+W)-2e'^3.\sin.3W'$.

We shall use these expressions in the following notes, in computing $M^{(0)}$, $M^{(1)}$, &c.; and we shall also make use of the following formulas, which are deduced from [955e—h], by taking the differentials relative to T, and dividing by $\pm dT$, changing also W into $W_{,}$, as in [3750h, &c.];

$$\sin.W_{,}.\tfrac{1}{2}\Sigma.i^3.A^{(i)}.\sin.iT=-\tfrac{1}{2}\Sigma.i^3.A^{(i)}.\cos.(iT+W_{,});$$ [3835b]

$$\cos.W_{,}.\tfrac{1}{2}\Sigma.i^3.A^{(i)}.\sin.iT=\tfrac{1}{2}\Sigma.i^3.A^{(i)}.\sin.(iT+W_{,});$$ [3835c]

$$\sin.W_{,}.\tfrac{1}{2}\Sigma.i^3.A^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.i^3.A^{(i)}.\sin.(iT+W_{,});$$ [3835d]

$$\cos.W_{,}.\tfrac{1}{2}\Sigma.i^3.A^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.i^3.A^{(i)}.\cos.(iT+W_{,}).$$ [3835e]

General form of R for terms of the third order.

[3835]
$$\begin{aligned} R = \; & M^{(0)}.\,e'^3.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-3\varpi') \\ & + M^{(1)}.e'^2e.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi'-\varpi) \\ & + M^{(2)}.e'e^2.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'-2\varpi) \\ & + M^{(3)}.\,e^3.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-3\varpi) \\ & + M^{(4)}.e'\gamma^2.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'-2\Pi) \\ & + M^{(5)}.\,e\gamma^2.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-2\Pi); \end{aligned}$$

and we shall find, after all the reductions,*

[3836]
$$a'M^{(0)} = -\frac{m'}{48}.\left\{389\,b_{\frac{1}{2}}^{(2)} + 201\,\alpha.\frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} + 27\,\alpha^2.\frac{ddb_{\frac{1}{2}}^{(2)}}{d\alpha^2} + \alpha^3.\frac{d^3b_{\frac{1}{2}}^{(2)}}{d\alpha^3}\right\};$$

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* (2418) The part of R [3835], depending on e'^3, may be put under the form
[3836a] $M^{(0)}.c'^3.\cos.(iT+3W')$ or $M^{(0)}.e'^3.\cos.(2T+3W')$, using T, W, &c. [3829b', c]; the coefficient of T being $i=2$. Terms of this kind are produced in R, by multiplying the quantities which are connected with e'^3 in [3835a], by the corresponding terms with which they are combined in [3831b], and then reducing the products by means of the formulas [955, 955a—h, 3835b]. The terms depending on $A^{(i)}$ and its differentials, are
[3836b] given in the value of $M^{(0)}$ [3836d], in the order in which they occur, without any reduction, and omitting Σ for brevity; so that the terms of [3835a], marked 4, 10, 20, are connected with $A^{(i)}$; 3, 9, 18 with $\left(\frac{d.A^{(i)}}{da'}\right)$; 7, 17 with $\left(\frac{d^2.A^{(i)}}{da'^2}\right)$; 14 with $\left(\frac{d^3.A^{(i)}}{da'^3}\right)$. Substituting $i=2$ [3836a] in this first value of $M^{(0)}$, we get the second value of [3836e];
[3836c] and this, by using the values [1003], becomes as in [3836f], or by reduction, as in [3836g]. Lastly, substituting in this the values [996—1001], we get [3836h], which is easily reduced to the form [3836];

[3836d]
$$M^{(0)} = m'.A^{(i)}.\{\tfrac{13}{24}i+\tfrac{5}{8}i^2+\tfrac{1}{6}i^3\} + m'.a'.\left(\frac{d.A^{(i)}}{da'}\right).\{-\tfrac{3}{16}-\tfrac{9}{16}i-\tfrac{1}{4}i^2\}$$
$$+ m'.a'^2.\left(\frac{d^2.A^{(i)}}{da'^2}\right).\{\tfrac{1}{8}+\tfrac{1}{8}i\} - m'.a'^3.\left(\frac{d^3.A^{(i)}}{da'^3}\right).\tfrac{1}{48}$$

[3836e]
$$= \tfrac{236}{48}m'.A^{(2)} - \tfrac{111}{48}m'.a'.\left(\frac{d.A^{(2)}}{da'}\right) + \tfrac{18}{48}m'.a'^2.\left(\frac{d^2.A^{(2)}}{da'^2}\right) - \tfrac{1}{48}m'.a'^3.\left(\frac{d^3.A^{(2)}}{da'^3}\right)$$

[3836f]
$$= \tfrac{236}{48}m'.A^{(2)} + \tfrac{111}{48}m'.\left\{A^{(2)}+a.\left(\frac{d.A^{(2)}}{da}\right)\right\} + \tfrac{18}{48}m'.\left\{2A^{(2)}+4a.\left(\frac{d.A^{(2)}}{da}\right)+a^2.\left(\frac{d^2.A^{(2)}}{da^2}\right)\right\}$$
$$+\frac{m'}{48}.\left\{6A^{(2)}+18a.\left(\frac{d.A^{(2)}}{da}\right)+9a^2.\left(\frac{dd.A^{(2)}}{da^2}\right)+a^3.\left(\frac{d^3.A^{(2)}}{da^3}\right)\right\}$$

[3836g]
$$= \tfrac{389}{48}m'.A^{(2)} + \tfrac{201}{48}m'.a.\left(\frac{d.A^{(2)}}{da}\right) + \tfrac{27}{48}m'.a^2.\left(\frac{d^2.A^{(2)}}{da^2}\right) + \frac{m'}{48}.a^3.\left(\frac{d^3.A^{(2)}}{da^3}\right)$$

[3836h]
$$= \frac{m'}{48a'}.\left\{-389\,b_{\frac{1}{2}}^{(2)} - 201\,\alpha.\frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} - 27\,\alpha^2.\frac{d^2b_{\frac{1}{2}}^{(2)}}{d\alpha^2} - \alpha^3.\frac{d^3b_{\frac{1}{2}}^{(2)}}{d\alpha^3}\right\}.$$

$$a' M^{(1)} = \frac{m'}{16} \cdot \left\{ 402\, b_{\frac{1}{2}}^{(3)} + 193\, \alpha \,.\, \frac{d b_{\frac{1}{2}}^{(3)}}{d\alpha} + 26\, \alpha^2 . \frac{d d b_{\frac{1}{2}}^{(3)}}{d\alpha^2} + \alpha^3 . \frac{d^3 b_{\frac{1}{2}}^{(3)}}{d\alpha^3} \right\} ;^* \qquad [3837]$$

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* (2419) Proceeding as in the last note, we find, that the part of R [3835] depending on $e'^2 e$, may be put under the form $M^{(1)}. e'^2 e \,.\, \cos. (i\,T + 2\,W' + W)$ [3829b', c], [3837a] in which the coefficient of T is $i = 3$. Substituting the values [3835a] in [3834b], we obtain the first of the following values of $M^{(1)}$; observing, that the terms of [3835a] depending on $e'^2 e$, marked 10, 20, are connected with $A^{(i)}$; the terms 8, 16, with $\left(\frac{d A^{(i)}}{d a}\right)$; the terms 9, 18, with $\left(\frac{d A^{(i)}}{d a'}\right)$; the terms 6, 19, with $\left(\frac{d d A^{(i)}}{d a\, d a'}\right)$; the term 17 with $\left(\frac{d d A^{(i)}}{d a'^2}\right)$; and the term 13 with $\left(\frac{d^3 A^{(i)}}{d a'^2 d a}\right)$. Substituting $i = 3$ in [3837c], [3837b] we get [3837d]; and this, by using the values [1003], becomes as in [3837e], or by reduction, as in [3837f]. Lastly, substituting in this the values [996—1001], we get [3837g], which is equivalent to [3837];

$$M^{(1)} = m'.A^{(i)}.\{-\tfrac{5}{8} i^2 - \tfrac{1}{2} i^3\} + m'.a.\left(\frac{d A^{(i)}}{d a}\right).\{-\tfrac{5}{16} i - \tfrac{1}{4} i^2\} + m'.a'.\left(\frac{d A^{(i)}}{d a'}\right).\{\tfrac{1}{4} i + \tfrac{1}{2} i^2\}$$
$$+ m'.a'a.\left(\frac{d d A^{(i)}}{d a'\, d a}\right).\{\tfrac{1}{8} + \tfrac{1}{4} i\} - \tfrac{i}{8} m'.a'^2.\left(\frac{d^2 A^{(i)}}{d a'^2}\right) - \tfrac{1}{16} m'.a'^2 a.\left(\frac{d^3 A^{(i)}}{d a'^2 d a}\right) \qquad [3837c]$$

$$= -\tfrac{306}{16} m'.A^{(3)} - \tfrac{51}{16} m'.a.\left(\frac{d A^{(3)}}{d a}\right) + \tfrac{84}{16} m'.a'.\left(\frac{d A^{(3)}}{d a'}\right)$$
$$+ \tfrac{14}{16} m'.a'a.\left(\frac{d d A^{(3)}}{d a' d a}\right) - \tfrac{6}{16} m'.a'^2.\left(\frac{d^2 A^{(3)}}{d a'^2}\right) - \tfrac{1}{16} m'.a'^2 a.\left(\frac{d^3 A^{(3)}}{d a'^2 d a}\right) \qquad [3837d]$$

$$= -\tfrac{306}{16} m'.A^{(3)} - \tfrac{51}{16} m'.a.\left(\frac{d A^{(3)}}{d a}\right) + \tfrac{84}{16} m'.\left\{-A^{(3)} - a.\left(\frac{d A^{(3)}}{d a}\right)\right\}$$
$$+ \tfrac{14}{16} m'.\left\{-2a.\left(\frac{d A^{(3)}}{d a}\right) - a^2.\left(\frac{d^2 A^{(3)}}{d a^2}\right)\right\} - \tfrac{6}{16} m'.\left\{2A^{(3)} + 4a.\left(\frac{d A^{(3)}}{d a}\right) + a^2.\left(\frac{d^2 A^{(3)}}{d a^2}\right)\right\} \qquad [3837e]$$
$$- \tfrac{1}{16} m'.\left\{6a.\left(\frac{d A^{(3)}}{d a}\right) + 6a^2.\left(\frac{d^2 A^{(3)}}{d a^2}\right) + a^3.\left(\frac{d^3 A^{(3)}}{d a^3}\right)\right\}$$

$$= \frac{m'}{16}.\left\{-402\, A^{(3)} - 193\, a.\left(\frac{d A^{(3)}}{d a}\right) - 26\, a^2.\left(\frac{d^2 A^{(3)}}{d a^2}\right) - a^3.\left(\frac{d^3 A^{(3)}}{d a^3}\right)\right\} \qquad [3837f]$$

$$= \frac{m'}{16 a'} \cdot \left\{ 402\, b_{\frac{1}{2}}^{(3)} + 193\, \alpha \,.\, \frac{d b_{\frac{1}{2}}^{(3)}}{d\alpha} + 26\, \alpha^2 . \frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\alpha^2} + \alpha^3 . \frac{d^3 b_{\frac{1}{2}}^{(3)}}{d\alpha^3} \right\} . \qquad [3837g]$$

[3838] $$a'M^{(2)} = -\frac{m'}{16}\cdot\left\{396\,b_{\frac{1}{2}}^{(4)} + 184\,\alpha\,.\,\frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} + 25\,\alpha^2.\,\frac{d\,d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha^2} + \alpha^3.\,\frac{d^3 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^3}\right\};*$$

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* (2420) We may compute [3838, 3839] as in the two last notes, but it is rather less laborious to derive them from $M^{(0)}$, $M^{(1)}$, by changing the symbols as below, namely,

[3838a] For $i,\ n't,\ nt,\ \varepsilon',\ \varepsilon,\ \varpi',\ \varpi,\ e',\ e,\ a',\ a;\ \ \alpha',\ \alpha_0,\ \ T;$

[3838b] Write $-i,\ nt,\ n't,\ \varepsilon,\ \varepsilon',\ \varpi,\ \varpi',\ e,\ e',\ a,\ a';\ \ \alpha_0,\ \alpha',\ -T.$

The changes in these three last values of α', α_0, T, evidently follow from those proposed in the other symbols, using [3829k, l]. The value α'' [3829m] is not altered, except in its sign, because $e.\sin.W$ changes into $e'.\sin.W'$, and $e'.\sin.W'$ into $e.\sin.W$, &c.; moreover, $A^{(i)}$ is not altered, because we have $A^{(-i)} = A^{(i)}$ [954″]; we also have, as
[3838c] in [3831c, d], $-\{a^2 - 2\,a\,a'.\cos.T + a'^2\}^{-\frac{1}{2}} = \frac{1}{2}\,\Sigma\,.\,A^{(i)}.\cos.i\,T$; and as the first member is symmetrical in a, a', the second, or $A^{(i)}$, must also be symmetrical, and will
[3838d] not be varied by putting a, a' for a', a, respectively; lastly, the expression of R [3834b] is not altered by making these changes; observing, that the quantities $i\,\alpha''$, $i\,T$ remain unchanged. Now the part of R [3835] depending on $e'e^2$, may be put under the form
[3838e] $M^{(2)}.\,e'e^2.\cos.(i\,T + 2\,W + W')$, in which the coefficient of T is $i = 4$. Comparing this with [3837a], we find, that by making the changes [3838a, b], the expression [3837a], corresponding to $i = -4$, will become like [3838e], and $M^{(1)}$ will change into $M^{(2)}$; we may therefore obtain the values of $M^{(2)}$ [3838h], by changing a, a', i into a', a, $-i$,
[3838f] respectively; then putting $i = 4$, we get [3838h′]. This value may be reduced to the form [3838i], by the substitution of the values [1003], and also the partial differential of the second of this system of equations, taken relatively to a, which gives

[3838g] $$a'.\left(\frac{d^3.A^{(4)}}{d\,a'\,d\,a^2}\right) = -3.\left(\frac{d^2.A^{(4)}}{d\,a^2}\right) - a\,.\left(\frac{d^3.A^{(4)}}{d\,a^3}\right).$$

Reducing the expression [3838i], we get [3838k]; and by the substitution of the values [996—1001], it becomes as in [3838l], being the same as [3838];

[3838h] $$\begin{aligned} M^{(2)} = {} & m'.A^{(i)}.\{-\tfrac{5}{8}\,i^2 + \tfrac{1}{2}\,i^3\} + m'.a'.\left(\frac{d\,A^{(i)}}{d\,a'}\right).\{\tfrac{5}{16}\,i - \tfrac{1}{4}\,i^2\} + m'.a\,.\left(\frac{d\,A^{(i)}}{d\,a}\right).\{-\tfrac{1}{4}\,i + \tfrac{1}{2}\,i^2\} \\ & + m'.\,a\,a'.\left(\frac{d^2.A^{(i)}}{d\,a\,d\,a'}\right).\{\tfrac{1}{8} - \tfrac{1}{4}\,i\} + \tfrac{1}{8}\,m'.\,a^2.\left(\frac{d^2.A^{(i)}}{d\,a^2}\right) - \tfrac{1}{16}\,m'.\,a^2\,a'.\left(\frac{d^3.A^{(i)}}{d\,a^2\,d\,a'}\right) \end{aligned}$$

[3838h′] $$\begin{aligned} = {} & \tfrac{352}{16}\,m'.A^{(4)} - \tfrac{44}{16}\,m'.a'.\left(\frac{d\,A^{(4)}}{d\,a'}\right) + \tfrac{112}{16}\,m'.\,a\,.\left(\frac{d\,A^{(4)}}{d\,a}\right) - \tfrac{14}{16}\,m'.\,a\,a'.\left(\frac{d\,d.A^{(4)}}{d\,a\,d\,a'}\right) \\ & + \tfrac{8}{16}\,m'.a^2.\left(\frac{d^2.A^{(4)}}{d\,a^2}\right) - \tfrac{1}{16}\,m'.\,a^2\,a'.\left(\frac{d^3.A^{(4)}}{d\,a^2\,d\,a'}\right) \end{aligned}$$

[3838i] $$\begin{aligned} = {} & \tfrac{352}{16}\,m'.A^{(4)} + \tfrac{44}{16}\,m'.\left\{A^{(4)} + a.\left(\frac{d\,A^{(4)}}{d\,a}\right)\right\} + \tfrac{112}{16}\,m'.a.\left(\frac{d\,A^{(4)}}{d\,a}\right) + \tfrac{14}{16}\,m'.\left\{2\,a.\left(\frac{d\,A^{(4)}}{d\,a}\right) + a^2.\left(\frac{d^2.A^{(4)}}{d\,a^2}\right)\right\} \\ & + \tfrac{8}{16}\,m'.\,a^2.\left(\frac{d^2.A^{(4)}}{d\,a^2}\right) + \tfrac{1}{16}\,m'.\left\{3\,a^2.\left(\frac{d^2.A^{(4)}}{d\,a^2}\right) + a^3.\left(\frac{d^3.A^{(4)}}{d\,a^3}\right)\right\} \end{aligned}$$

$$a'\,M^{(3)} = \frac{m'}{48}\cdot\left\{380\,b_{\frac{1}{2}}^{(5)} + 174\,\alpha\,.\,\frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} + 24\,\alpha^2.\,\frac{d\,d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha^2} + \alpha^3.\,\frac{d^3\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha^3}\right\};* \qquad [3839]$$

$$a'\,M^{(4)} = -\frac{m'\alpha}{16}\cdot\left\{10\,b_{\frac{3}{2}}^{(3)} + \alpha\,.\,\frac{d\,b_{\frac{3}{2}}^{(3)}}{d\,\alpha}\right\};\dagger \qquad [3840]$$

$$M^{(2)} = \frac{m'}{16}\cdot\left\{396\,A^{(4)} + 184\,a\,.\left(\frac{d.A^{(4)}}{d\,a}\right) + 25\,a^2.\left(\frac{d^2.A^{(4)}}{d\,a^2}\right) + a^3.\left(\frac{d^3.A^{(4)}}{d\,a^3}\right)\right\} \qquad [3838k]$$

$$= \frac{m'}{16a'}\cdot\left\{-396\,b_{\frac{1}{2}}^{(4)} - 184\,\alpha\,.\,\frac{d\,b_{\frac{1}{2}}^{(4)}}{d\alpha} - 25\,\alpha^2.\,\frac{d^2 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^2} - \alpha^3.\,\frac{d^3 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^3}\right\}. \qquad [3838l]$$

* (2421) The part of R [3835] depending on e^3, may be put under the form $M^{(3)}.e^3.\cos.(i\,T+3\,W)$, in which the coefficient of T is $i=5$. Comparing this with [3836a], we find, that by making the changes a, a', i, &c. into a', a, $-i$, &c., respectively, as in [3838a, b], the expression [3836d] will become as in [3839b]. This represents the value of $M^{(3)}$, or the coefficient of e^3 in [3835]; and by putting $i=5$, it becomes as in [3839b']; which, by means of [996—1001], is easily reduced to the form [3839]; [3839a]

$$M^{(3)} = m'.A^{(i)}.\{-\tfrac{13}{24}\,i + \tfrac{5}{8}\,i^2 - \tfrac{1}{6}\,i^3\} + m'.a\,.\left(\frac{d\,.A^{(i)}}{d\,a}\right).\{-\tfrac{3}{16} + \tfrac{9}{16}\,i - \tfrac{1}{4}\,i^2\}$$

$$+\,m'.a^2.\left(\frac{d^2.A^{(i)}}{d\,a^2}\right).\{\tfrac{1}{8} - \tfrac{1}{8}\,i\} - m'.a^3.\left(\frac{d^3.A^{(i)}}{d\,a^3}\right).\tfrac{1}{48} \qquad [3839b]$$

$$= \frac{m'}{48}\cdot\left\{-380\,A^{(5)} - 174\,a\,.\left(\frac{d.A^{(5)}}{d\,a}\right) - 24\,a^2.\left(\frac{d^2.A^{(5)}}{d a^2}\right) - a^3.\left(\frac{d^3.A^{(5)}}{d\,a^3}\right)\right\} \qquad [3839b']$$

$$= \frac{m'}{48\,a'}\cdot\left\{380\,b_{\frac{1}{2}}^{(5)} + 174\,\alpha\,.\,\frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} + 24\,\alpha^2.\,\frac{d^2 b_{\frac{1}{2}}^{(5)}}{d\,\alpha^2} + \alpha^3.\,\frac{d^3\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha}\right\}. \qquad [3839c]$$

† (2422) The values of $M^{(4)}$, $M^{(5)}$ [3840, 3841] depend on the second term of [3831e]; and by retaining only this term, we shall have $U = -\tfrac{1}{8}m'.\gamma^2.\,a\,a'.\,\Sigma\,.\,B^{(i-1)}.\cos.\,T_1$, [3840$a$] supposing, for a moment, that $T_1 = i\,.\,(n'\,t - n\,t + \varepsilon' - \varepsilon) + 2\,n\,t + 2\,\varepsilon - 2\,\Pi$. [3840$a'$] As this expression is multiplied by γ^2, of the second order, we need only notice terms of the first order in α_0, α', α'', in the development of u or R, and we shall get for this part of R, the following expression [3829g],

$$R = \alpha_0\,.\left(\frac{d\,U}{d\,a}\right) + \alpha'.\left(\frac{d\,U}{d\,a'}\right) + \alpha''.\left(\frac{d\,U}{d\,T_1}\right); \qquad [3840b]$$

observing, that we notice in this article only terms of the third dimension. The values of α_0, α', to be substituted in this expression, are the same as in [3829k, l]; and by retaining terms of the first order, we have $\alpha_0 = -a\,e\,.\cos.\,W$, $\alpha' = -a'\,e'.\cos.\,W'$. [3840$c$] The angle T_1 represents the mean value of $i\,.\,(v'-v) + 2\,v$; its increment, depending [3840d]

[3841] $$a'M^{(5)}=\frac{m'\alpha}{16}\cdot\left\{7\,b_{\frac{3}{2}}^{(4)}+\alpha\cdot\frac{d\,b_{\frac{3}{2}}^{(4)}}{d\alpha}\right\}.$$

[3840d'] on $v_{,}$, $v_{,}'$ [3829a], is $\alpha''=i\,.(v_{,}'-v_{,})+2\,v_{,}=i\,v_{,}'-(i-2)\,.\,v_{,}$, and by substituting $v_{,}'=2\,e'.\sin.W'$, $v_{,}=2\,e\,.\sin.W$ [669], we get α'', and then [3840b] becomes

[3840e] $$R=-e'.\left\{a'.\cos.W'.\left(\frac{d\,U}{d\,a'}\right)-2\,i\,.\sin.W'.\left(\frac{d\,U}{d\,T_4}\right)\right\}$$
$$-e\,.\left\{a\,.\cos.W.\left(\frac{d\,U}{d\,a}\right)+(2\,i-4)\,.\sin.W.\left(\frac{d\,U}{d\,T_4}\right)\right\};$$

and by substituting the partial differentials of U [3840a], we obtain, without any reduction,

[3840f] $$R=\tfrac{1}{8}\,m'.\,e'\gamma^2.\cos.W'.\left\{a'\,a\,.\Sigma\,.\,B^{(i-1)}.\cos.T_4+a'^2\,a\,.\,\Sigma\,.\left(\frac{d\,B^{(i-1)}}{d\,a'}\right).\cos.T_4\right\}$$
$$+\tfrac{1}{4}\,m'.\,e'\gamma^2.\sin.W'.\,a'\,a\,.\,\Sigma\,i\,.\,B^{(i-1)}.\sin.T_4$$
$$+\tfrac{1}{8}\,m'.\,e\,\gamma^2.\cos.W.\left\{a'\,a\,.\Sigma\,.\,B^{(i-1)}.\cos.T_4+a^2a'.\,\Sigma\,.\left(\frac{d\,B^{(i-1)}}{d\,a}\right).\cos.T_4\right\}$$
$$-\tfrac{1}{8}\,m'.\,e\,\gamma^2.\sin.W.\,a'\,a\,.\,\Sigma\,.(2\,i-4)\,.\,B^{(i-1)}.\sin.T_4.$$

The terms of this expression, depending on $e'\gamma^2$, contain the factors $\cos.W'.\cos.T_4$,
[3840g] and $\sin.W'.\sin.T_4$, both of which, as in [17, 20] Int., produce the terms $\frac{1}{2}\cos.(T_4+W')$, which, by putting $i=4$, becomes $\frac{1}{2}\cos.(5\,n't-2\,n\,t+5\,\varepsilon'-2\,\varepsilon-\varpi'-2\,\Pi)$ [3840a']. Comparing this with the term depending on $M^{(4)}$ in [3835], we get the first of the following expressions, omitting Σ for brevity, and then by successive reductions, using [963^iv, 1006—1008], we finally obtain [3840l], which is easily reduced to the form [3840];

[3840h] $$M^{(4)}=\tfrac{1}{16}\,m'.\left\{a'\,a\,.B^{(i-1)}+a'^2\,a\,.\left(\frac{d\,B^{(i-1)}}{d\,a'}\right)\right\}-\tfrac{1}{8}\,m'.\,a'\,a\,.\,i\,.\,B^{(i-1)}$$

[3840i] $$=\tfrac{1}{16}\,m'.a'a.\left\{-7\,B^{(3)}+a'.\left(\frac{d\,B^{(3)}}{d\,a'}\right)\right\}=\tfrac{1}{16}\,m'.a'a.\left\{-7\,B^{(3)}+\left[-3\,B^{(3)}-a\,.\left(\frac{d\,B^{(3)}}{d\,a}\right)\right]\right\}$$

[3840k] $$=\tfrac{1}{16}\,m'.a'a.\left\{-10B^{(3)}-a.\left(\frac{d\,B^{(3)}}{d\,a}\right)\right\}=\tfrac{1}{16}\,m'.a'a.\left\{-\frac{10}{a'^3}.b_{\frac{3}{2}}^{(3)}-\frac{a}{a'^4}.\frac{d\,b_{\frac{3}{2}}^{(3)}}{d\alpha}\right\}$$

[3840l] $$=-\frac{m'.\alpha}{16\,a'}.\left\{10\,b_{\frac{3}{2}}^{(3)}+\alpha\,.\frac{d\,b_{\frac{3}{2}}^{(3)}}{d\alpha}\right\}.$$

In like manner, the terms of [3840f], depending on $e\gamma^2$, contain the factors $\cos.W.\cos.T_4$, $\sin.W.\sin.T_4$, producing the term $\frac{1}{2}\cos.(T_4+W)$, which,
[3840m] by putting $i=5$, becomes $\frac{1}{2}\cos.(5\,n'\,t-2\,n\,t+5\,\varepsilon'-2\,\varepsilon-\varpi-2\,\Pi)$ [3840a']. Comparing this with the term depending on $M^{(5)}$ [3835], we get the first of the following

Hence we deduce*

General form of P;

$$m'.a'P = a'M^{(0)}.e'^3.\sin.3\varpi' + a'M^{(1)}.e'^2e.\sin.(2\varpi'+\varpi)$$
$$+a'M^{(2)}.e'e^2.\sin.(\varpi'+2\varpi)+a'M^{(3)}.e^3.\sin.3\varpi$$
$$+a'M^{(4)}.e'\gamma^2.\sin.(2\Pi+\varpi')+a'M^{(5)}.e\gamma^2.\sin.(2\Pi+\varpi). \qquad [3842]$$

We shall get $m'.a'P'$, *by changing the sines into cosines, in this expression of* $m'.a'P$; and it will be easy to deduce the values of aP, aP', by [3843]

and of P'.

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expressions, in which we must put $i=5$, and then, by reducing as above, it becomes as in [3840p]; whence we easily deduce [3841],

$$M^{(5)}=\tfrac{1}{16}m'.\left\{a'a.B^{(i-1)}+a^2a'.\left(\frac{dB^{(i-1)}}{da}\right)\right\}+\tfrac{1}{16}m'.a'a.(2i-4).B^{(i-1)} \qquad [3840n]$$

$$=\tfrac{1}{16}m'.a'a.\left\{7B^{(4)}+a.\left(\frac{dB^{(4)}}{da}\right)\right\}=\tfrac{1}{16}m'.a'a.\left\{\frac{7}{a'^3}.b_{\frac{3}{2}}^{(4)}+\frac{a}{a'^4}.\frac{db_{\frac{3}{2}}^{(4)}}{d\alpha}\right\} \qquad [3840o]$$

$$=\frac{m'.\alpha}{16a'}.\left\{7b_{\frac{3}{2}}^{(4)}+\frac{\alpha\, db_{\frac{3}{2}}^{(4)}}{d\alpha}\right\}. \qquad [3840p]$$

* (2423) In the case of $i=5$, if we use, for a moment, the abridged symbol $T_5=5n't-2nt+5\varepsilon'-2\varepsilon$, the value of R [3810] becomes [3842a]

$$R=m'.P.\sin.T_5+m'.P'.\cos.T_5. \qquad [3842a']$$

Now each term of R [3835] may be easily reduced to the form [3842a']; since, if we take, for example, the first $M^{(0)}.e'^3.\cos.(T_5-3\varpi')$, and develop it by [24] Int., it becomes $M^{(0)}.e'^3.\sin.3\varpi'.\sin.T_5+M^{(0)}.e'^3.\cos.\varpi'.\cos.T_5$. Comparing this with [3842a'], we get for the parts of $m'.P$, $m'.P'$, the following expressions, [3842b]

$$m'.P=M^{(0)}.e'^3.\sin.3\varpi', \qquad m'.P'=M^{(0)}.e'^3.\cos.3\varpi', \qquad [3842b']$$

as in [3842, 3843]. In like manner, we obtain the other terms of [3842] from [3835]. The values of P, P', deduced from [3842, 3843], may be put under the following forms, which will be of use hereafter,

Expressions of P, P'.

$$P=\Sigma.M'.e'^{b'}.e^b.\gamma^{2c}.\sin.(b'\varpi'+b\varpi+2c\Pi),$$
$$P'=\Sigma.M'.e'^{b'}.e^b.\gamma^{2c}.\cos.(b'\varpi'+b\varpi+2c\Pi); \qquad [3842c]$$

Σ being the characteristic of finite integrals, and b', b, c, integral numbers, including zero, satisfying the equation $b'+b+2c=3$. [3842d]

[3843'] multiplying $a' P$, $a' P'$, by $\frac{a}{a'}$ or α. We shall then find, by putting $i=5$, in the expressions of δv and $\frac{\delta r}{a}$ [3817, 3827, 3821],*

Expression of the terms of δv of the third order.

[3844]
$$\delta v = \frac{-6m'.n^2}{(5n'-2n)^2}\cdot\left\{\begin{array}{l}\left\{aP' + \frac{2a.dP}{(5n'-2n).dt} - \frac{3a.ddP'}{(5n'-2n)^2.dt^2}\right\}.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon) \\ -\left\{aP - \frac{2a.dP'}{(5n'-2n).dt} - \frac{3a.ddP}{(5n'-2n)^2.dt^2}\right\}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\}$$
$$-\frac{2m'n}{5n'-2n}\cdot\left\{\begin{array}{l}a^2.\left(\frac{dP}{da}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon) \\ -a^2.\left(\frac{dP'}{da}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\}$$
$$-\tfrac{1}{2}He.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+A)$$
$$+\tfrac{5}{4}Ke.\sin.(5n't-4nt+5\varepsilon'-4\varepsilon+\varpi+B);$$

Expression of the terms of δr of the third order.

[3845]
$$\frac{\delta r}{a} = H.\cos.(5n't-3nt+5\varepsilon'-3\varepsilon+A) - He.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+A)$$
$$+ He.\cos.(5n't-4nt+5\varepsilon'-4\varepsilon+\varpi+A)$$
$$+\frac{4m'.n}{5n'-2n}.\{aP.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)+aP'.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\}.$$

[3845'] If we suppose $i=-2$,† and change the elements of m into

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[3844a] * (2424) Adding the terms of δv [3817, 3827], and putting $i=5$, we get [3844]. Putting $i=5$, in [3821], we obtain [3845].

† (2425) By restricting ourselves to terms of the first order of the masses, and of the

[3846a] third dimension in e, e', γ, the expression of $\frac{R}{m'}$ [3831] becomes symmetrical in the elements of m, m', so that these elements may be interchanged without altering this value of $\frac{R}{m'}$ [3831a, a']. The same symmetry obtains in the expression of $\frac{R}{m'}$ [3810]; for

[3846b] if we put, for a moment, $T_5 = 5n't-2nt+5\varepsilon'-2\varepsilon$, $T_6 = 5nt-2n't+5\varepsilon-2\varepsilon'$, and retain, in [3810], only the two terms arising from the successive substitution of the values $i=5$, $i=-2$, it becomes

[3846c]
$$\frac{R}{m'} = P.\sin.T_5 + P'.\cos.T_5 + P_0.\sin.T_6 + P'_0.\cos.T_6;$$

[3846d] P_0, P'_0, T_6, being, respectively, the values of P, P', T_5, when the elements a, n, e, &c. are changed into a', n', e', &c., and the contrary, this being necessary to preserve the

[3846d'] symmetry [3846a]. In computing the action of m' upon m, it is not necessary to notice

the corresponding ones, relative to m', and the contrary, we shall obtain

Expression of the terms of $\delta v'$ of the third order.

$$\delta v' = \frac{15\,m\,.\,n'^2}{(5n'-2n)^2}\cdot\left\{\begin{array}{l}\left\{a'P' + \frac{2a'.dP}{(5n'-2n).dt} - \frac{3a'.ddP'}{(5n'-2n)^2.dt^2}\right\}.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\\ -\left\{a'P - \frac{2a'.dP'}{(5n'-2n).dt} - \frac{3a'.ddP}{(5n'-2n)^2.dt^2}\right\}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\}$$

$$-\frac{2m.n'}{5n'-2n}\cdot\left\{\begin{array}{l} a'^2.\left(\frac{dP}{da'}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\\ -a'^2.\left(\frac{dP'}{da'}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\} \qquad [3846]$$

$$-\tfrac{1}{2}H'e'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'+A')$$

$$+\tfrac{5}{4}K'e'.\sin.(3n't-2nt+3\varepsilon'-2\varepsilon+\varpi'+B')\,;$$

Expression of the terms of $\delta r'$ of the third order.

$$\frac{\delta r'}{a'} = H'.\cos.(4n't-2nt+4\varepsilon'-2\varepsilon+A') - H'e'.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'+A')$$

$$+\ H'e'.\cos.(3n't-2nt+3\varepsilon'-2\varepsilon+\varpi'+A')$$

$$-\frac{10m.n'}{5n'-2n}.\{a'P.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)+a'P'.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\}\,; \qquad [3847]$$

$H'.\cos.(4n't-2nt+4\varepsilon'-2\varepsilon+A')$ being the part of $\frac{r'\delta r'}{a'^2}$ depending [3848]
on the angle $4n't-2nt$,* and $K'.\sin.(4n't-2nt+4\varepsilon'-2\varepsilon+B')$

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the angle T_6, because it does not produce terms having the small divisor $5n'-2n$. [3846d'']
In making the change of the elements of m into those of m', according to the directions [3845'], the value of $\frac{R}{m}$, corresponding to the action of m upon m', becomes

$$\frac{R}{m} = P_0.\sin.T_6 + P'_0.\cos.T_6 + P.\sin.T_5 + P'.\cos.T_5. \qquad [3846e]$$

The second members of [3846c, e], are evidently identical; but in this last expression the terms depending on the angle T_5, are derived from those of [3846c], which depend on $i=-2$; by changing the elements m, a, e, &c. into those of m', a', e', &c., as in [3845']. [3846f]
Lastly, we may observe, that the quantities P, P', connected, respectively, with $\sin.T_5$, $\cos.T_5$, are the same in [3846c, e]. Hence we may derive $\delta v'$ from δv, by taking the sum of the two parts of δv [3817, 3827], putting $i=-2$, then changing m, a, n, e, H, K, &c. into m', a', n', e', H', K', &c., respectively; by which means we get [3846]. In like manner, we may derive [3847] from [3821]. [3846g]

* (2426) These terms correspond to [3814, 3826], putting $i=-2$, and changing the elements as in [3845']. [3848a]

being the part of $\delta v'$ relative to the same angle. In these various inequalities, *we shall, for greater simplicity, refer the origin of the angles to*
[3849] *the common intersection of the orbits of Jupiter and Saturn;* as we have already done in the development of the expression of R [3736—3738], and shall continue to do in the following article. For the sake of symmetry, we shall retain the angle Π, which must be supposed equal to nothing.

Computation of dP, ddP, &c.

We shall determine the differentials $\frac{dP}{dt}$, $\frac{ddP}{dt^2}$, $\frac{dP'}{dt}$, $\frac{ddP'}{dt^2}$, in the following manner. We shall compute, for the two epochs of
[3849′] 1750 and 1950, which embrace an interval of 200 Julian years, the values of $\frac{de}{dt}$, $\frac{d\varpi}{dt}$, $\frac{de'}{dt}$, $\frac{d\varpi'}{dt}$, $\frac{d\gamma}{dt}$, $\frac{d\Pi}{dt}$; and shall represent these
[3850] quantities, at the second of these epochs, by $\frac{de_{\prime}}{dt}$, $\frac{d\varpi_{\prime}}{dt}$, $\frac{de'_{\prime}}{dt}$, &c.; we shall then have, by supposing t to be expressed in Julian years,*

[3851]
$$\frac{de_{\prime}}{dt}=\frac{de}{dt}+200\cdot\frac{dde}{dt^2};$$

in which the differentials de, dde, in the second member, correspond to the epoch 1750. The value of $e_{\prime}$† for any time t, neglecting the cube

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* (2427) We have, as in [607, &c.],

[3850a]
$$u=U+t\cdot\left(\frac{dU}{dt}\right)+\tfrac{1}{2}t^2\cdot\left(\frac{d^2U}{dt^2}\right)+\&c.,$$

[3850b] u being a function of t, which becomes U, when $t=0$. Now putting $u=\frac{de_{\prime}}{dt}$, $U=\frac{de}{dt}$, as in [3850], we get, by retaining only the first power of t, $\frac{de_{\prime}}{dt}=\frac{de}{dt}+t\cdot\frac{dde}{dt^2}$, which, by putting $t=200$, the interval mentioned in [3849′], becomes as in [3851]. From this
[3850c] we get $\frac{dde}{dt^2}=\frac{1}{200}\cdot\left\{\frac{de_{\prime}}{dt}-\frac{de}{dt}\right\}$. The values of $\frac{de}{dt}$, $\frac{de_{\prime}}{dt}$, being computed, as in [4238, &c., 4330a, &c.], for the epochs 1750, 1950; we obtain, by substitution, in [3850c], the value of $\frac{dde}{dt^2}$, corresponding to the epoch 1750.

† (2428) Putting $U=e$, $u=e_{\prime}$, in [3850a], we get

[3852a]
$$e_{\prime}=e+t\cdot\frac{de}{dt}+\tfrac{1}{2}t^2\cdot\frac{dde}{dt^2}\qquad[3852];$$

in which we must substitute the values of e, $\frac{de}{dt}$, $\frac{dde}{dt^2}$ [3850, 3850c], for the epoch 1750.

of t and its higher powers, is

$$e+t\cdot\frac{de}{dt}+\frac{t^2}{2}\cdot\frac{dde}{dt^2}; \qquad [3852]$$

e, $\frac{de}{dt}$, $\frac{dde}{dt^2}$, being supposed to correspond to the year 1750; *this expression may be used for ten or twelve centuries before or after that epoch.** [3853]
In like manner, we may determine the values of ϖ, e', ϖ', γ, and Π; thence we may compute the values of P, corresponding to the three epochs 1750, 2250, and 2750. If we represent these values by P, $P_{,}$, $P_{,,}$, and the general expression of P by† [3853′]

$$P+t\cdot\frac{dP}{dt}+\frac{t^2}{2}\cdot\frac{ddP}{dt^2}; \qquad [3854]$$

we shall have, by putting successively, $t=500$, $t=1000$,

$$P_{,}=P+\ 500\cdot\frac{dP}{dt}+\ 250000\cdot\tfrac{1}{2}\cdot\frac{ddP}{dt^2}; \qquad [3855]$$

$$P_{,,}=P+1000\cdot\frac{dP}{dt}+1000000\cdot\tfrac{1}{2}\cdot\frac{ddP}{dt^2}; \qquad [3855']$$

hence we obtain‡

Values of dP, ddP.

$$\frac{dP}{dt}=\frac{4P_{,}-3P-P_{,,}}{1000}; \qquad \frac{ddP}{dt^2}=\frac{P_{,,}-2P_{,}+P}{250000}. \qquad [3856]$$

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* (2429) To give some idea of the rapidity with which the terms of the series [3852] decrease, we may take the value of e^{iv} [4407] for the case of $t=1000$, and we shall find $t\cdot\frac{de}{dt}=329^s$, $-\frac{1}{2}t^2\cdot\frac{dde}{dt^2}=8^s$; so that the second is about $\frac{1}{40}$ part of the first; and with the same rate of decrease, the third term $\frac{1}{6}t^3\cdot\frac{d^3e}{dt^3}$ will be insensible; similar remarks may be made relative to the other terms of [4407, &c.]. [3853a] [3853b]

† (2430) The expression [3854] is similar to [3850a], and by putting, successively, $t=500$, $t=1000$, we get $P_{,}$, $P_{,,}$ [3855, 3855′]. [3855a]

‡ (2431) Multiplying [3855] by 4, [3855′] by -1, adding the products, and then dividing by 1000, we get $\frac{dP}{dt}$ [3856]. Again, multiplying [3855] by -2, adding the product to [3855′], and then dividing by 250000, we get $\frac{ddP}{dt^2}$ [3856]. [3856a]

9. *The terms depending on the fifth powers of the excentricities may have*
[3856'] *a sensible influence on the great inequalities of Jupiter and Saturn; but the calculation is very troublesome on account of its excessive length.* The importance of the subject has, however, induced that very skilful astronomer Burckhardt, to undertake the computation. He has discussed, with scrupulous
[3857] attention, all the terms of this order depending on the angle $5n't-2nt$, neglecting merely those terms which depend on the products of the excentricities by the fourth power of the mutual inclinations of the orbits; which produce only insensible quantities. The expression of R [3742]
[3857'] corresponds to the action of m' upon m; and the part of the expression which has the most influence on this inequality, is the product of m' by the following factor,*

[3858] $$\frac{R}{m'}=-\frac{1}{\sqrt{r^2-2rr'.\cos.(v'-v)+r'^2}}+\frac{\frac{\gamma^2}{4}.rr'.\{\cos.(v'-v)-\cos.(v'+v)\}}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{3}{2}}}.$$

[3858'] *This factor is the same for both planets;*† by developing it, and noticing

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* (2432) If we proceed by a method similar to that used in [3829a, &c.], we may prove, as in [3829n, &c.], that the second and third terms of R [3742], namely,

[3858a] $$-\frac{m'.\gamma^2}{4}.\frac{r}{r'^2}.\{\cos.(v'-v)-\cos.(v'+v)\},$$

do not have any influence in producing terms of the order now under consideration, depending on the angle $5n't-2nt$, and by neglecting them, and also the first term of [3742], which is noticed in [3861, 3868], we obtain the value of $\frac{R}{m'}$ [3858].

† (2433) As γ enters into R [3858] only in the even powers, and the quantities
[3859a] multiplied by γ^4 are neglected [3857], the terms of R of the fifth order, must contain factors of the following forms,

[3859b] $$e'^5,\quad e'^4e,\quad e'^3e^2,\quad e'^2e^3,\quad e'e^4,\quad e^5;\qquad \gamma^2e'^3,\quad \gamma^2e'^2e,\quad \gamma^2e'e^2,\quad \gamma^2e^3;$$

of which the six first terms compose all the combinations of e, e', of the fifth dimension, and the remaining terms all the combinations of e, e', of the third dimension, multiplied by γ^2 of the second dimension. Now we see, as in [957$^{\text{viii}}$, 957$^{\text{ix}}$], that if R contain a series of terms of the form $m'.k.\cos.(5n't-2nt+A)$, the first term of the series
[3859c] will be of the order $i'-i=5-2=3$, or of the *third* order; the second term will be of the order $i'-i+2$, or of the *fifth* order; and by noticing only terms of the *fifth* order, the angles will become, respectively, of the forms [3859]. For in the elliptical
[3859d] motion the angle $nt+\varepsilon$ is always connected with $-\varpi$, $n't+\varepsilon'$ with $-\varpi'$ [669, 957$^{\text{xi}}$];

only the products of the excentricities and inclinations corresponding to the angle $5n't-2nt$, we shall have a function of this form, [3858'']

Forms of the terms in R of the fifth dimension in e, e', γ.

$$\left.\begin{array}{rl}\frac{R}{m'}= & N^{(0)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-4\varpi'+\varpi)\\ & +N^{(1)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-3\varpi')\\ & +N^{(2)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi'-\varpi)\\ & +N^{(3)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'-2\varpi)\\ & +N^{(4)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-3\varpi)\\ & +N^{(5)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon+\varpi'-4\varpi)\\ & +N^{(6)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi'+\varpi-2\Pi)\\ & +N^{(7)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'-2\Pi)\\ & +N^{(8)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-2\Pi)\\ & +N^{(9)}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon+\varpi'-2\varpi-2\Pi)\end{array}\right\};\quad (O) \qquad [3859]$$

and we find*

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and in the terms depending on γ^2, the angle $2n't+2\varepsilon'$ is connected with -2Π; so that if the coefficients of ϖ, ϖ', Π, be represented by g, g', g'', respectively, we shall always have, by noticing the signs $g+g'+g''=-3$; which is similar to [959], changing the signs of the coefficients. Moreover, the sum of the coefficients g, g', g'', considering them all as positive, *must not exceed* 5 [957$^{\text{ix}}$], because the present calculation is restricted to terms of the fifth order. Thus, for example, a term depending on the angle $5n't-2nt+5\varepsilon'-2\varepsilon-5\varpi'+2\varpi$, must be rejected, because the sum of the coefficients of ϖ', ϖ, taking them positively, is 7, corresponding to terms of the seventh order. Now a slight examination will show, that the values of g, g', g'', which satisfy the equation $g+g'+g''=-3$ [3859e], with the prescribed condition, are as in the following table; the corresponding numbers being placed in the same vertical lines. These numbers agree with [3859]; [3859e] [3859f] [3859g]

Values of g',	-4,	-3,	-2,	-1,	0,	1;	-2,	-1,	0,	1;
Values of g,	1,	0,	-1,	-2,	-3,	-4;	1,	0,	-1,	-2;
Values of g'',	0,	0,	0,	0,	0,	0;	-2,	-2,	-2,	-2.

[3859h]

* (2434) The signs of all these values of $a'N^{(0)}$, $a'N^{(1)}$, &c. [3860—3860$^{\text{ix}}$], have been changed from the original so as to correct the error mentioned by the author in [5974, &c.]. Before the discovery of this mistake, he had computed and used these erroneous values in ascertaining the inequalities of Jupiter and Saturn [4431, 4187]; hence it becomes necessary to apply the corrections of the mean longitudes, given in [5976, 5977, &c.]. We have given [3860—3860$^{\text{ix}}$] as they were printed by the author, [3860a]

[3860]
$$a'N^{(0)} = -\frac{e'^4 e}{768}\cdot\left\{\begin{array}{l} 3138\, b_{\frac{1}{2}}^{(1)} - 13\,\alpha\,.\,\dfrac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} - 1556\,\alpha^2.\,\dfrac{d^2\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} - 438\,\alpha^3.\,\dfrac{d^3\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^3} \\ - 38\,\alpha^4.\,\dfrac{d^4\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^4} - \alpha^5.\,\dfrac{d^5\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^5} \end{array}\right\};$$

Terms of the fifth dimension in e, e', γ.

$$a'N^{(1)} = -\frac{e'^3}{768}\cdot\left\{\begin{array}{l} -(20267\,e'^2 + 24896\,e^2)\,.\,b_{\frac{1}{2}}^{(2)} - (7223\,e'^2 + 8144\,e^2)\,.\,\alpha\,.\,\dfrac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} \\ + (1094\,e'^2 + 3692\,e^2)\,.\,\alpha^2.\,\dfrac{d^2 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} + (482\,e'^2 + 1436\,e^2)\,.\,\alpha^3.\,\dfrac{d^3 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^3} \\ + (41\,e'^2 + 140\,e^2)\,.\,\alpha^4.\,\dfrac{d^4\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^4} + (e'^2 + 4\,e^2)\,.\,\alpha^5.\,\dfrac{d^5 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^5} \end{array}\right\}$$

[3860']
$$+\frac{e'^3\gamma^2}{384}\cdot\left\{\begin{array}{l} 590\,\alpha\,.\,\left(b_{\frac{3}{2}}^{(1)} + b_{\frac{3}{2}}^{(3)}\right) + 255\,\alpha^2.\,\left(\dfrac{d\,b_{\frac{3}{2}}^{(1)}}{d\,\alpha} + \dfrac{d\,b_{\frac{3}{2}}^{(3)}}{d\,\alpha}\right) \\ + 30\,\alpha^3.\left(\dfrac{d^2 b_{\frac{3}{2}}^{(1)}}{d\,\alpha^2} + \dfrac{d^2 b_{\frac{3}{2}}^{(3)}}{d\,\alpha^2}\right) + \alpha^4.\left(\dfrac{d^3 b_{\frac{3}{2}}^{(1)}}{d\,\alpha^3} + \dfrac{d^3 b_{\frac{3}{2}}^{(3)}}{d\,\alpha^3}\right) \end{array}\right\};$$

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correcting the signs as above; but without pretending to verify more than one or two terms of each of the coefficients. The calculations of Burckhardt, on this subject, are given in the *Mémoires de l'Institut*, T. IX, 1808, p. 59, supp., but generally with wrong signs.

From what has been said in the preceding notes [3809*a*—3856*a*], concerning the terms
[3860*b*] of the third order, we may form some idea of the great labor of computing and reducing the terms of the fifth order [3860—3860^{iv}]. The series [3829*g*—*m*, 3834*b*] must be very
[3860*c*] much increased by the introduction of terms of the fourth and fifth orders; a table similar to [3835*a*] must be formed, containing terms of the fifth order, depending on the proposed angles and on the powers and products of α_0, α', α'', as far as the fifth order inclusively. Then we obtain, as in [3836*d*, 3837*c*, &c.], values of $N^{(0)}$, $N^{(1)}$, &c., depending on $A^{(i)}$
[3860*d*] and its differentials relatively to a, a'; which may be reduced to the differentials relative to a only, by extending the table [1003] to differentials of the fifth order; finally, by the
[3860*e*] substitution of the values $A^{(i)}$, $B^{(i)}$, and their differentials, in terms of $b_{\frac{1}{2}}^{(i)}$, $b_{\frac{3}{2}}^{(i)}$, and their differentials [996—1008], we get the required values of $N^{(0)}$, $N^{(1)}$, &c. This short sketch of the method of computing the terms of the fifth and higher orders, must suffice; more minuteness would be inconsistent with the prescribed limits to the notes on this work;
[3860*f*] in which we have proposed to point out and illustrate the methods of computing the various inequalities, by occasional examples, without attempting to verify the immense number of numerical calculations with which the work abounds.

$$a'N^{(2)} = \frac{e'^2 e}{768} \cdot \left\{ \begin{array}{l} -(109392\,e'^2+53064\,e^2).b_{\frac{1}{2}}^{(3)} - (42368\,e'^2+23436\,e^2).\alpha.\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} \\ +(1064\,e'^2+2088\,e^2).\alpha^2.\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} + (1572\,e'^2+1710\,e^2).\alpha^3.\frac{d^3 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^3} \\ +(152\,e'^2+192\,e^2).\alpha^4.\frac{d^4 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^4} + (4\,e'^2+6\,e^2).\alpha^5.\frac{d^5 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^5} \end{array} \right\}$$

$$-\frac{e'^2 e\gamma^2}{128} \cdot \left\{ \begin{array}{l} 595\,\alpha.\left(b_{\frac{3}{2}}^{(2)}+b_{\frac{3}{2}}^{(4)}\right) + 245\,\alpha^2.\left(\frac{d\,b_{\frac{3}{2}}^{(2)}}{d\,\alpha}+\frac{d\,b_{\frac{3}{2}}^{(4)}}{d\,\alpha}\right) \\ +29\,\alpha^3.\left(\frac{d^2 b_{\frac{3}{2}}^{(2)}}{d\,\alpha^2}+\frac{d^2 b_{\frac{3}{2}}^{(4)}}{d\,\alpha^2}\right) + \alpha^4.\left(\frac{d^3 b_{\frac{3}{2}}^{(2)}}{d\,\alpha^3}+\frac{d^3 b_{\frac{3}{2}}^{(4)}}{d\,\alpha^3}\right) \end{array} \right\};$$ [3860″]

$$a'N^{(3)} = -\frac{e' e^2}{768} \cdot \left\{ \begin{array}{l} -(42912\,e^2+199848\,e'^2).b_{\frac{1}{2}}^{(4)} - (21728\,e^2+82032\,e'^2).\alpha.\frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} \\ -(640\,e^2+2970\,e'^2).\alpha^2.\frac{d^2 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^2} + (864\,e^2+1854\,e'^2).\alpha^3.\frac{d^3 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^3} \\ +(116\,e^2+210\,e'^2).\alpha^4.\frac{d^4 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^4} + (4\,e^2+6\,e'^2).\alpha^5.\frac{d^5 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^5} \end{array} \right\}$$

Terms of the fifth dimension in e, e', γ.

$$+\frac{e' e^2\gamma^2}{128} \cdot \left\{ \begin{array}{l} 580\,\alpha.\left(b_{\frac{3}{2}}^{(3)}+b_{\frac{3}{2}}^{(5)}\right) + 234\,\alpha^2.\left(\frac{d\,b_{\frac{3}{2}}^{(3)}}{d\,\alpha}+\frac{d\,b_{\frac{3}{2}}^{(5)}}{d\,\alpha}\right) \\ +28\,\alpha^3.\left(\frac{d^2 b_{\frac{3}{2}}^{(3)}}{d\,\alpha^2}+\frac{d^2 b_{\frac{3}{2}}^{(5)}}{d\,\alpha^2}\right) + \alpha^4.\left(\frac{d^3 b_{\frac{3}{2}}^{(3)}}{d\,\alpha^3}+\frac{d^3 b_{\frac{3}{2}}^{(5)}}{d\,\alpha^3}\right) \end{array} \right\};$$ [3860‴]

$$a'N^{(4)} = \frac{e^3}{768} \cdot \left\{ \begin{array}{l} -(11840\,e^2+152000\,e'^2).b_{\frac{1}{2}}^{(5)} - (6560\,e^2+65168\,e'^2).\alpha.\frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} \\ -(592\,e^2+4720\,e'^2).\alpha^2.\frac{d^2 b_{\frac{1}{2}}^{(5)}}{d\,\alpha^2} + (152\,e^2+920\,e'^2).\alpha^3.\frac{d^3 b_{\frac{1}{2}}^{(5)}}{d\,\alpha^3} \\ +(26\,e^2+128\,e'^2).\alpha^4.\frac{d^4 b_{\frac{1}{2}}^{(5)}}{d\,\alpha^4} + (e^2+4\,e'^2).\alpha^5.\frac{d^5 b_{\frac{1}{2}}^{(5)}}{d\,\alpha^5} \end{array} \right\}$$

$$-\frac{e^3\gamma^2}{384} \cdot \left\{ \begin{array}{l} 554\,\alpha.\left(b_{\frac{3}{2}}^{(4)}+b_{\frac{3}{2}}^{(6)}\right) + 222\,\alpha^2.\left(\frac{d\,b_{\frac{3}{2}}^{(4)}}{d\,\alpha}+\frac{d\,b_{\frac{3}{2}}^{(6)}}{d\,\alpha}\right) \\ +27\,\alpha^3.\left(\frac{d^2 b_{\frac{3}{2}}^{(4)}}{d\,\alpha^2}+\frac{d^2 b_{\frac{3}{2}}^{(6)}}{d\,\alpha^2}\right) + \alpha^4.\left(\frac{d^3 b_{\frac{3}{2}}^{(4)}}{d\,\alpha^3}+\frac{d^3 b_{\frac{3}{2}}^{(6)}}{d\,\alpha^3}\right) \end{array} \right\};$$ [3860iv]

[3860v] $$a'N^{(5)}=\frac{e'e^4}{768}\cdot\left\{\begin{array}{l} 41448\,.\,b_{\frac{1}{2}}^{(6)}+18392\,\alpha\,.\,\frac{d\,b_{\frac{1}{2}}^{(6)}}{d\,\alpha}+1780\,\alpha^2.\,\frac{d^2b_{\frac{1}{2}}^{(6)}}{d\,\alpha^2} \\ -156\,\alpha^3.\,\frac{d^3b_{\frac{1}{2}}^{(6)}}{d\,\alpha^3}-29\,\alpha^4.\,\frac{d^4b_{\frac{1}{2}}^{(6)}}{d\,\alpha^4}-\alpha^5.\,\frac{d^5b_{\frac{1}{2}}^{(6)}}{d\,\alpha^5}\end{array}\right\};$$

[3860vi] $$a'N^{(6)}=\frac{e'^2e\gamma^2}{128}.\left\{-85\,\alpha\,.\,b_{\frac{3}{2}}^{(2)}+85\,\alpha^2.\frac{d\,b_{\frac{3}{2}}^{(2)}}{d\,\alpha}+21\,\alpha^3.\,\frac{d^2b_{\frac{3}{2}}^{(2)}}{d\,\alpha^2}+\alpha^4.\,\frac{d^3b_{\frac{3}{2}}^{(2)}}{d\,\alpha^3}\right\};$$

Terms of the fifth dimension in e, e', γ.

[3860vii] $$a'N^{(7)}=\frac{e'\gamma^2}{128}.\left\{\begin{array}{l}(56\,e^2+842\,e'^2)\,.\,\alpha\,.\,b_{\frac{3}{2}}^{(3)}+(4\,e^2+87\,e'^2)\,.\,\alpha^2.\,\frac{d\,b_{\frac{3}{2}}^{(3)}}{d\,\alpha} \\ -(16\,e^2+2\,e'^2)\,.\,\alpha^3.\,\frac{d^2b_{\frac{3}{2}}^{(3)}}{d\,\alpha^2}-(2\,e^2+e'^2)\,.\,\alpha^4.\,\frac{d^3b_{\frac{3}{2}}^{(3)}}{d\,\alpha^3}\end{array}\right\};$$

[3860viii] $$a'N^{(8)}=\frac{e\gamma^2}{128}.\left\{\begin{array}{l}-(174\,e^2+196\,e'^2)\,.\,\alpha\,.\,b_{\frac{3}{2}}^{(4)}+(50\,e^2+180\,e'^2)\,.\,\alpha^2.\,\frac{d\,b_{\frac{3}{2}}^{(4)}}{d\,\alpha} \\ +(14\,e'^2-e^2)\,.\,\alpha^3.\,\frac{d^2b_{\frac{3}{2}}^{(4)}}{d\,\alpha^2}+(2\,e'^2+e^2)\,.\,\alpha^4.\,\frac{d^3b_{\frac{3}{2}}^{(4)}}{d\,\alpha^3}\end{array}\right\};$$

[3860ix] $$a'N^{(9)}=\frac{e'e^2\gamma^2}{128}.\left\{580\,\alpha\,.\,b_{\frac{3}{2}}^{(5)}+86\,\alpha^2.\,\frac{d\,b_{\frac{3}{2}}^{(5)}}{d\,\alpha}-8\,\alpha^3.\,\frac{d^2b_{\frac{3}{2}}^{(5)}}{d\,\alpha^2}-\alpha^4.\,\frac{d^3b_{\frac{3}{2}}^{(5)}}{d\,\alpha^3}\right\}.$$

[3861] When we consider the action of m' upon m, we must augment $a'N^{(0)}$ [3860], by increasing $b_{\frac{1}{2}}^{(1)}$ with the term $-\frac{a}{a'}$, or $-\alpha$ [3743], which increases $a'N^{(0)}$ by $\frac{3125\,\alpha\,.\,e'^4e}{768}$.* When we consider the action of m upon m',

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* (2435) In [996], we have, generally, $\frac{1}{a'}\,.\,b_{\frac{1}{2}}^{(i)}=-A^{(i)}$; but in the particular case

[3861a] of $i=1$, this becomes, as in [997], $\frac{1}{a'}.b_{\frac{1}{2}}^{(1)}-\frac{a}{a'^2}=-A^{(1)}$. The part $\frac{a}{a'^2}$ being introduced by the term $\frac{a}{a'^2}\,.\cos.(n't-nt+\varepsilon'-\varepsilon)$ [954], which does not occur in the terms noticed in the value of R [3858], so that wherever the quantity $\frac{1}{a'}.b_{\frac{1}{2}}^{(1)}$ occurs,

[3861b] we ought to add $-\frac{a}{a'^2}$; or in other words, $b_{\frac{1}{2}}^{(1)}$ ought to be increased by the term $-\frac{a}{a'}$, or $-\alpha$. To notice this circumstance, we must apply a correction to the value

we must add to $b_{\frac{1}{2}}^{(1)}$ the term $-\frac{1}{\alpha^2}$; which increases $a'N^{(0)}$ by $\frac{500\,e'^4e}{768\,\alpha^2}$. [3862]

This being premised, we shall multiply the preceding values of $a'N^{(0)}$, $a'N^{(1)}$, &c. by m', and shall reduce each of the cosines by which they are multiplied in the function [3859], into sines and cosines of [3862′]
$5n't-2nt+5\varepsilon'-2\varepsilon$; which gives to this function the following form,*

Value of R.

$$\begin{aligned} a'R = {} & m'.a'P_{,}.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon) \\ & + m'.a'P'_{,}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon). \end{aligned}$$ [Action of m' on m]. [3863]

We shall likewise multiply by m the values of $a'N^{(0)}$, $a'N^{(1)}$, &c. relative to the action of m upon m'; and shall reduce the sines and cosines

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of $a'N^{(0)}$ [3860], which may be computed by supposing $b_{\frac{1}{2}}^{(1)}=-\alpha$, which gives $\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}=-1$, $\frac{ddb_{\frac{1}{2}}^{(1)}}{d\alpha}=0$, &c. Substituting these in [3860], it becomes

$$-\frac{e'^4e}{768}.\{-3138\,\alpha+13\,\alpha\}=\frac{3125\,\alpha\,.\,e'^4e}{768},$$ [3861c]

as in [3861]. When we are computing the action of m on m', the formula [3861a] becomes

$$\frac{1}{a'}.b_{\frac{1}{2}}^{(1)}-\frac{a'}{a^2}=-A^{(1)}, \qquad \text{or} \qquad -A^{(1)}=\frac{1}{a'}.\left\{b_{\frac{1}{2}}^{(1)}-\frac{a'^2}{a^2}\right\}=\frac{1}{a'}.\left\{b_{\frac{1}{2}}^{(1)}-\alpha^{-2}\right\};$$

so that the correction of $b_{\frac{1}{2}}^{(1)}$ is $-\alpha^{-2}$, and the correction of $a'N^{(0)}$ for this case, will be found by putting $b_{\frac{1}{2}}^{(1)}=-\alpha^{-2}$ in the expression [3860]. Now this value of $b_{\frac{1}{2}}^{(1)}$ gives

$$\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}=2\,\alpha^{-3}; \quad \frac{ddb_{\frac{1}{2}}^{(1)}}{d\alpha^2}=-6\,\alpha^{-4}; \quad \frac{d^3b_{\frac{1}{2}}^{(1)}}{d\alpha^3}=24\,\alpha^{-5}; \quad \frac{d^4b_{\frac{1}{2}}^{(1)}}{d\alpha^4}=-120\,\alpha^{-6};$$ [3861d]

substituting these in that expression of $a'N^{(0)}$, it becomes

$$-\frac{e'^4e}{768\,\alpha^2}.\{-3138-2\times13+6\times1556-24\times438+120\times38-720\}=\frac{500\,e'^4e}{768\,\alpha^2},$$

as in [3862].

* (2436) The reduction here used is the same as that in [3842b, &c.], by which the function [3835] is reduced to the form of [3842a′], and were it not for the terms [3861, 3862], the values of $P_{,}$, $P'_{,}$ [3863] would be identical with $P_{,,}$, $P'_{,,}$ [3865], respectively; for the factor [3858] is the same for both planets; and the reasoning made [3864a]
use of in [3846a—g] will serve to prove, in [3863, 3865], that $P_{,}$, $P'_{,}$ will be respectively equal to $P_{,,}$, $P'_{,,}$, if we neglect the terms [3861, 3862], and we shall show, in [3866b], [3864b]
that these terms do not affect the result.

[3864] of the function [3859] to sines and cosines of $5n't - 2nt + 5\varepsilon' - 2\varepsilon$; which will give to it the following form,

Value of R'.

[3865]
$$a'R' = m \cdot a'P_{\prime\prime} \cdot \sin. (5n't - 2nt + 5\varepsilon' - 2\varepsilon) + m \cdot a'P_{\prime\prime}' \cdot \cos. (5n't - 2nt + 5\varepsilon' - 2\varepsilon).$$

[Action of m on m'].

We shall then substitute these values successively, in the expressions of δv, $\delta v'$, of the preceding article [3844, 3846], neglecting their second
[3865'] differences, because of the smallness of these quantities; and in this way we shall obtain the parts of the inequalities of Jupiter and Saturn, corresponding to the angle $5n't - 2nt$, and depending on the powers and products of the excentricities and inclinations of the orbits of the fifth order.

We may here observe, that in consequence of the ratio which obtains
[3866] between the mean motions of Jupiter and Saturn, we have $3125\,\alpha^3 = 500$;*
[3867] for $\alpha^3 = \frac{n'^2}{n^2}$ and $5n'$ is very nearly equal to $2n$; consequently $\frac{n'^2}{n^2} = \frac{4}{25}$. Hence it follows, that the *value of $a'N^{(0)}$ is the same, whether we consider the action of m' upon m, or that of m upon m'.* Hence we may deduce the preceding part of $\delta v'$ from the corresponding part of δv, by multiplying
[3868] the latter by $-\frac{5m \cdot n'^2}{2m' \cdot n^2} \cdot \frac{a'}{a}$.†

[3866a] * (2437) We have nearly $1 = n^2a^3 = n'^2a'^3$ [3709']; hence $\frac{n'^2}{n^2} = \frac{a^3}{a'^3} = \alpha^3$ [3829b]; but by [3818d], we have nearly $5n' - 2n = 0$, or $\frac{n'}{n} = \frac{2}{5}$; therefore $\alpha^3 = \left(\frac{n'}{n}\right)^2 = \frac{4}{25}$, as in [3867], and $3125\,\alpha^3 = 500$, or $3125\,\alpha = \frac{500}{\alpha^2}$; substituting this in the increment
[3866b] of $a'N^{(0)}$ [3861], corresponding to the action of m' upon m, it changes into the expression [3862], representing the increment of $a'N^{(0)}$ in the action of m upon m', as we have remarked in [3864b].

† (2438) If we multiply the factor $-\frac{6m' \cdot n^2}{(5n' - 2n)^2}$, connected with the chief term
[3868a] of δv [3844], by the quantity $-\frac{5m \cdot n'^2}{2m' \cdot n^2} \cdot \frac{a'}{a}$ [3868], the product becomes

[3868b]
$$\frac{15m \cdot n'^2}{(5n' - 2n)^2} \cdot \frac{a'}{a} = \frac{15m \cdot n'^2}{(5n' - 2n)^2} \cdot \frac{1}{\alpha};$$

in which the part $\frac{15m \cdot n'^2}{(5n' - 2n)^2}$ is the same as the corresponding factor of the terms of $\delta v'$ [3846]; the other part, $\frac{a'}{a}$, being multiplied into the terms aP, aP', $a\,dP$, $a\,dP'$, &c. [3844],

10. *In the theory of Mercury disturbed by the Earth, we must notice the inequality depending on the angle* $nt-4n't$; because the mean motion [3869]
of Mercury is very nearly four times that of the Earth [4077a]. Supposing
m to be Mercury and m' the Earth, we shall obtain the proposed inequality
by putting $i=4$, in the expression of δv [3817]. Considering the [3870]
extreme minuteness of this inequality, we *may neglect all the terms*
depending on $\frac{dP}{dt}$, $\frac{dP'}{dt}$, *and retain only those having the divisor* $(n-4n')^2$. [3871]
Hence we shall get*

Inequality of a long period in Mercury.

$$\delta v=\frac{3m'.n^2}{(n-4n')^2}.\{aP'.\sin.(nt-4n't+\varepsilon-4\varepsilon')+aP.\cos.(nt-4n't+\varepsilon-4\varepsilon')\}. \qquad [3872]$$

We can easily determine P and P' in the following manner. We may calculate, by formula [3711], the value of $\frac{r\delta r}{a^2}$, corresponding to the angle $4n't-2nt$, by substituting in it $i=4$. Hence we obtain a [3873]
value of $\frac{r\delta r}{a^2}$ of the form,†

$$\begin{aligned}\frac{r\delta r}{a^2}= \;& L\;.\,e^2.\cos.(4n't-2nt+4\varepsilon'-2\varepsilon-2\varpi)\\ &+L^{(1)}.\,ee'.\cos.(4n't-2nt+4\varepsilon'-2\varepsilon-\varpi-\varpi')\\ &+L^{(2)}.\,e'^2.\cos.(4n't-2nt+4\varepsilon'-2\varepsilon-2\varpi')\\ &+L^{(3)}.\,\gamma^2.\cos.(4n't-2nt+4\varepsilon'-2\varepsilon-2\Pi).\end{aligned} \qquad [3874]$$

We shall then observe, that this value of $\frac{r\delta r}{a^2}$ results from the variations of the excentricity and perihelion, depending on $nt-4n't$, in the elliptical

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produces the corresponding expressions $a'P$, $a'P'$, $a'dP$, $a'dP'$, &c. [3846]; the values P, P' of $\delta v'$, having been proved in the two last notes to be respectively equal to those of P, P', in δv. [3868c]

* (2139) Neglecting dP, dP', ddP, ddP', and H, in [3817], and putting $i=4$, [3872a]
we obtain the expression [3872].

† (2140) The two first of the angles [3874], connected with e^2, ee', are explicitly contained in [3711]; the others, as well as these two, are included in the form

$$\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+K\}, \qquad [3873a]$$

which occurs in [3711], and is developed in [3745—3745'''].

[3875] expression of $\frac{1}{2}\cdot\frac{r^2}{a^2}$. This expression contains the term $-e.\cos.(nt+\varepsilon-\varpi)$, whose variation is*

[3876] $$\frac{r\,\delta r}{a^2}=-\delta e.\cos.(nt+\varepsilon-\varpi)-e\,\delta\varpi.\sin.(nt+\varepsilon-\varpi);$$

δe and $\delta\varpi$ being the variations of e and ϖ, depending on $nt-4n't$.

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* (2411) If we square the value of r [3701], and substitute

$$\cos.^2(nt+\varepsilon-\varpi)=\tfrac{1}{2}+\tfrac{1}{2}\cos.2.(nt+\varepsilon-\varpi),$$

we shall get

[3876a] $$r^2=a^2.\{1+\tfrac{3}{2}e^2-2e.\cos.(nt+\varepsilon-\varpi)-\tfrac{1}{2}e^2.\cos.2.(nt+\varepsilon-\varpi)+\&c.\}.$$

In the troubled orbit the elements $r,\ a,\ e,\ \varepsilon,\ \varpi,\ nt,$ are increased by the variations
[3876b] $\delta r,\ \delta a,\ \delta e,\ \delta\varpi,\ \delta v,$ respectively; and if we neglect the squares and products of these variations, the increment of the preceding expression will be found by taking its differential relatively to the characteristic δ; hence we get

[3876c] $$2r\,\delta r=2a\,\delta a.\{1+\tfrac{3}{2}e^2-\&c.\}$$
$$+a^2.\{3e\,\delta e-2\,\delta e.\cos.(nt+\varepsilon-\varpi)-2e\,\delta\varpi.\sin.(nt+\varepsilon-\varpi)-\&c.\}.$$

Dividing this by $2a^2$, it becomes of the form

[3876d] $$\frac{r\delta r}{a^2}=-\delta e.\cos.(nt+\varepsilon-\varpi)-e\,\delta\varpi.\sin.(nt+\varepsilon-\varpi)+\mathrm{X};$$

representing, for brevity, by the symbol X, all the terms of the second member, excepting the two parts explicitly retained by the author in [3876]. If we neglect X, and substitute
[3876e] in the remaining terms the values of δe, $e\,\delta\varpi$ [3877, 3878], we shall get the expression of $\frac{r\delta r}{a^2}$ [3879], which the author supposes to be identical with [3874], and thence by integration obtains δv [3882]. In the Memoirs of the Astronomical Society of London,
[3876f] Vol. II, page 358, &c., Mr. Plana has pointed out some defects in this method, and has shown, that the terms depending on X materially alter the result. To prove this, he has computed directly the terms of δv depending on the divisor $(n-4n')^2$, using formulas similar to those in [3835—3841]; which we shall give in [3881r—w']; after going over
[3876g] the calculation by the method of the author. From the comparison made in [3883u, v], it appears, that this method of La Place cannot be considered, in an analytical point of view, as a very near approximation to the truth; though he seems rather unwilling
[3876h] to admit the fact, in a note he published on the subject in the *Connaisance des Tems*, for 1829, page 249.

We shall have, by [1288, 1297],*

$$\delta e = \frac{m'.an}{n-4n'}.\left\{ \left(\frac{dP}{de}\right).\sin.(4n't-nt+4\varepsilon'-\varepsilon) + \left(\frac{dP'}{de}\right).\cos.(4n't-nt+4\varepsilon'-\varepsilon) \right\};$$ [3877]

$$e\delta\varpi = \frac{m'.an}{n-4n'}.\left\{ -\left(\frac{dP}{de}\right).\cos.(4n't-nt+4\varepsilon'-\varepsilon) + \left(\frac{dP'}{de}\right).\sin.(4n't-nt+4\varepsilon'-\varepsilon) \right\};$$ [3878]

hence the variation of $-e.\cos.(nt+\varepsilon-\varpi)$ becomes†

$$\frac{r\delta r}{a^2} = \frac{m'.an}{n-4n'}.\left\{ \left(\frac{dP}{de}\right).\sin.(2nt-4n't+2\varepsilon-4\varepsilon'-\varpi) - \left(\frac{dP'}{de}\right).\cos.(2nt-4n't+2\varepsilon-4\varepsilon'-\varpi) \right\}.$$ [3879]

This function is identical with the preceding expression of $\frac{r\delta r}{a^2}$ [3874]; therefore if we change, in both of them, $2nt+2\varepsilon$ into $nt+\varepsilon+\varpi+\frac{\pi}{2}$, [3880]
π being the semi-circumference, we shall obtain‡

$$\begin{aligned}
\frac{m'.an}{n-4n'}.\left\{ \left(\frac{dP}{de}\right).\cos.(nt-4n't+\varepsilon-4\varepsilon') + \left(\frac{dP'}{de}\right).\sin.(nt-4n't+\varepsilon-4\varepsilon') \right\} \\
= L\ .\ e^2.\sin.(4n't-nt+4\varepsilon'-\varepsilon-3\varpi) \\
+ L^{(1)}.ee'.\sin.(4n't-nt+4\varepsilon'-\varepsilon-\varpi'-2\varpi) \\
+ L^{(2)}.e'^2.\sin.(4n't-nt+4\varepsilon'-\varepsilon-2\varpi'-\varpi) \\
+ L^{(3)}.\gamma^2.\sin.(4n't-nt+4\varepsilon'-\varepsilon-\varpi-2\Pi).
\end{aligned}$$ [3881]

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* (2442) The expression of R [1287] is the same as in [3810]; so that P, P' have the same values in both formulas. Now putting $i'=4$, $i=1$, $\mu=1$ [3709], in the expression of $\delta\varpi$ [1297], and then multiplying it by e, we get the value of $e\delta\varpi$ [3878]. [3877a]
The variation δe [1288] becomes, by similar substitutions, of the same form as in [3877].

† (2443) Putting, for a moment, $4n't-nt+4\varepsilon'-\varepsilon=T_7$, $nt+\varepsilon-\varpi=W$; then multiplying [3877] by $-\cos.W$, also [3878] by $-\sin.W$, and adding the products, we get for the second member of [3876], or the value of $\frac{r\delta r}{a^2}$, the expression [3879b]; reducing this by means of [22, 24] Int., it becomes as in [3879c], which is equivalent to [3879]; [3879a]

$$\frac{r\delta r}{a^2} = \frac{m'.an}{n-4n'}.\left\{ \left(\frac{dP}{de}\right).(-\sin.T_7.\cos.W+\cos.T_7.\sin.W) - \left(\frac{dP'}{de}\right).(\cos.T_7.\cos.W+\sin.T_7.\sin.W) \right\}$$ [3879b]

$$= \frac{m'.an}{n-4n'}.\left\{ \left(\frac{dP}{de}\right).\sin.(W-T_7) - \left(\frac{dP'}{de}\right).\cos.(W-T_7) \right\}.$$ [3879c]

‡ (2444) We have two expressions of $\frac{r\delta r}{a^2}$ [3874, 3879], depending upon the angle $2nt-4n't$, and it is evident, that if it were not for the terms produced by the [3880a]

[3881'] If we integrate this equation relatively to e,* and then multiply it by $\frac{3n}{n-4n'}$, we shall obtain

[3882] $$\delta v = \frac{3n}{n-4n'} \cdot \left\{ \begin{array}{l} \tfrac{1}{3} L \;\;. e^3 . \sin.(4n't - nt + 4\varepsilon' - \varepsilon - 3\varpi) \\ + \tfrac{1}{2} L^{(1)}. e^2 e'. \sin.(4n't - nt + 4\varepsilon' - \varepsilon - \varpi' - 2\varpi) \\ + \;\; L^{(2)}. e e'^2. \sin.(4n't - nt + 4\varepsilon' - \varepsilon - 2\varpi' - \varpi) \\ + \;\; L^{(3)}. e \gamma^2. \sin.(4n't - nt + 4\varepsilon' - \varepsilon - \varpi - 2\Pi) \end{array} \right\}.$$

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function X [3876c], they would be identical; therefore they will still be equal to each other, if we change the angle $2nt+2\varepsilon$ into $nt+\varepsilon+\varpi+\frac{1}{2}\pi$. Now if we make this change

[3880b] in [3874], we shall find, that a term of the form $\cos.(4n't-2nt+4\varepsilon'-2\varepsilon+A)$, becomes

$$\cos.(4n't - nt + 4\varepsilon' - \varepsilon + A - \varpi - \tfrac{1}{2}\pi) = \sin.(4n't - nt + 4\varepsilon' - \varepsilon + A - \varpi);$$

and the second member of the expression [3874] changes into the second member of [3881]. In like manner, $\sin.(2nt - 4n't + 2\varepsilon - 4\varepsilon' - \varpi)$ becomes

[3880c] $$\sin.(nt - 4n't + \varepsilon - 4\varepsilon' + \tfrac{1}{2}\pi) = \cos.(nt - 4n't + \varepsilon - 4\varepsilon');$$

and $\cos.(2nt - 4n't + 2\varepsilon - 4\varepsilon' - \varpi)$ becomes

[3880d] $$\cos.(nt - 4n't + \varepsilon - 4\varepsilon' + \tfrac{1}{2}\pi) = -\sin.(nt - 4n't + \varepsilon - 4\varepsilon');$$

hence the second member of [3879] becomes as in the first member of [3881].

[3881a] * (2415) Multiplying the equation [3881] by de, and then integrating it relatively to e, in order to obtain the values of P, P', we get

[3881b] $$\begin{array}{l} \frac{m'.an}{n-4n'} \cdot \left\{ P.\cos.(nt - 4n't + \varepsilon - 4\varepsilon') + P'.\sin.(nt - 4n't + \varepsilon - 4\varepsilon') \right\} \\ = \;\; \tfrac{1}{3} L \;\;. e^3 \;. \sin.(4n't - nt + 4\varepsilon' - \varepsilon - 3\varpi) \\ \quad + \tfrac{1}{2} L^{(1)}. e^2 e'. \sin.(4n't - nt + 4\varepsilon' - \varepsilon - \varpi' - 2\varpi) \\ \quad + \;\; L^{(2)}. e e'^2. \sin.(4n't - nt + 4\varepsilon' - \varepsilon - 2\varpi' - \varpi) \\ \quad + \;\; L^{(3)}. e \gamma^2. \sin.(4n't - nt + 4\varepsilon' - \varepsilon - \varpi - 2\Pi). \end{array}$$

The first member of this expression being multiplied by $\frac{3n}{n-4n'}$, becomes equal to the value of δv [3872]; therefore δv will be obtained by multiplying the second member

[3881c] of [3881b] by $\frac{3n}{n-4n'}$; and in this way we obtain [3882]. In the integration relative to e [3881a, b], we may add terms depending on e'^3, and $e'\gamma^2$, which are considered as constant in the integrations; but the excentricity of the Earth's orbit e', being only

[3881d] about $\frac{1}{12}$ of e [4080], the term depending on e'^3, must be much smaller than the

In this integration, we neglect the terms of P and P' depending on [3882']

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others; and the same remark will apply to the term depending on $e'\gamma^2$. The author has neglected these terms, because they are so much less than those which are included in the expression [3882].

Having followed the author in this indirect method of computing the value of δv [3882], we shall now proceed to the direct investigation of the same inequality. For this purpose we must have an expression of R, similar to [3835], depending on the angle $4n't-nt$. This expression is evidently of the following form, [3881e]

$$\begin{aligned} R = \ & M^{(0)}.c'^3.\cos.(4n't-nt+4\varepsilon'-\varepsilon-3\varpi') \\ & +M^{(1)}.c'^2c.\cos.(4n't-nt+4\varepsilon'-\varepsilon-2\varpi'-\varpi) \\ & +M^{(2)}.c'c^2.\cos.(4n't-nt+4\varepsilon'-\varepsilon-\varpi'-2\varpi) \\ & +M^{(3)}.\,e^3\,.\cos.(4n't-nt+4\varepsilon'-\varepsilon-3\varpi) \\ & +M^{(4)}.c'\gamma^2.\cos.(4n't-nt+4\varepsilon'-\varepsilon-\varpi'-2\Pi) \\ & +M^{(5)}.c\gamma^2.\cos.(4n't-nt+4\varepsilon'-\varepsilon-\varpi-2\Pi)\,; \end{aligned} \quad [3881f]$$

but the factors $M^{(0)}$, $M^{(1)}$, &c. are different from those in [3836, &c.]; we shall give their values in [3881r—w']. If we suppose, for a moment, the preceding expression of R to be put under the form $R=\Sigma M.\cos.(4n't-nt+K)$, we shall have $\mathrm{d}R=n\,\Sigma M.\sin.(4n't-nt+K)$ [916']. Substituting this in the expression of the mean longitude ζ [3715l], we shall get the corresponding term, [3881g]

$$\delta v = 3ffan\,dt.\mathrm{d}R = -\frac{3an^2}{(4n'-n)^2}.\Sigma M.\sin.(4n't-nt+K)\,; \quad [3881h]$$

therefore the value of δv *may be easily derived from* R [3881f], *by multiplying it by* $-\frac{3an^2}{(4n'-n)^2}$, *and changing the cosines into sines.* The terms of R may be very easily obtained from the values of $M^{(0)}$, $M^{(1)}$, &c., computed in [3836d—3840o], by merely *decreasing the value of* i *by unity;* so as to change the angle $5n't-2nt$ into $4n't-nt$. In this way of computing $M^{(0)}$, we must use the decreased value $i=1$ [3836a], and then [3836d] becomes as in [3881r]. In computing $M^{(1)}$ from [3837c], we have the decreased value $i=2$ [3837a]; hence we get [3881s]. From [3838e, h], we get the decreased value $i=3$, and $M^{(2)}$ [3881t]. From [3839a, b], we get the decreased value $i=4$, and $M^{(3)}$ [3881u]. These expressions are reduced, in the first place, by means of the formulas [1003], and then by [996—1001]; so that we finally obtain the values [3881r', s', t', u']. Observing, that in computing $M^{(1)}$ [3881r'], we must notice the increments of $b_{\frac{1}{2}}^{(1)}$ and $\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\alpha}$, represented by $-\alpha$ and -1, respectively, as in [3861b—c], by which means we shall obtain the first term, $-\frac{m'}{48a'}.\{-256\alpha\}$, [3881i] [3881k] [3881l] [3881m] [3881n] [3881n'] [3881o]

[3883] e'^3 and $e'\gamma^2$; but as the excentricity of the orbit of the Earth is quite small,

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in the expression [3881r′], which is omitted by Mr. Plana by mistake. In like manner,
[3881o′] from [3840h] and the *decreased* value $i=3$ [3840g], we obtain $M^{(4)}$ [3881v]; also from
[3881p] [3840n] and the decreased value $i=4$ [3840m], we obtain $M^{(5)}$ [3881w]; which, by similar substitutions [1008, &c.], are reduced to the forms [3881v′, w′]. In making these
[3881q] successive reductions, we have used the abridged expression [3755a], $A_m^{(n)}=a^m.\left(\frac{d^m A^{(n)}}{da^m}\right)$.

[3881r] $$M^{(0)}=\frac{m'}{48}.\left\{64A^{(1)}-48a'.\left(\frac{dA^{(1)}}{da'}\right)+12a'^2.\left(\frac{ddA^{(1)}}{da'^2}\right)-a'^3.\left(\frac{d^3A^{(1)}}{da'^3}\right)\right\}$$

$$=\frac{m'}{48}.\left\{\begin{array}{l}64A^{(1)}+48.[A^{(1)}+A_1^{(1)}]+12.[2A^{(1)}+4A_1^{(1)}+A_2^{(1)}]\\+[6A^{(1)}+18A_1^{(1)}+9A_2^{(1)}+A_3^{(1)}]\end{array}\right\}$$

$$=\frac{m'}{48}.\{142A^{(1)}+114A_1^{(1)}+21A_2^{(1)}+A_3^{(1)}\}$$

[3881r′] $$=-\frac{m'}{48a'}.\left\{-256\alpha+142b_{\frac{1}{2}}^{(1)}+114\alpha.\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}+21\alpha^2.\frac{d^2b_{\frac{1}{2}}^{(1)}}{d\alpha^2}+\alpha^3.\frac{d^3b_{\frac{1}{2}}^{(1)}}{d\alpha^3}\right\};$$

[3881s] $$M^{(1)}=-\frac{m'}{16}.\left\{\begin{array}{l}104A^{(2)}+26a.\left(\frac{dA^{(2)}}{da}\right)-40a'.\left(\frac{dA^{(2)}}{da'}\right)-10a'a.\left(\frac{ddA^{(2)}}{da'da}\right)\\+4a'^2.\left(\frac{d^2A^{(2)}}{da'^2}\right)+a'^2a.\left(\frac{d^3A^{(2)}}{da'^2da}\right)\end{array}\right\}$$

$$=-\frac{m'}{16}.\left\{\begin{array}{l}104A^{(2)}+26A_1^{(2)}+40.[A^{(2)}+A_1^{(2)}]+10.[2A_1^{(2)}+A_2^{(2)}]\\+4.[2A^{(2)}+4A_1^{(2)}+A_2^{(2)}]+[6A_1^{(2)}+6A_2^{(2)}+A_3^{(2)}]\end{array}\right\}$$

$$=-\frac{m'}{16}.\{152A^{(2)}+108A_1^{(2)}+20A_2^{(2)}+A_3^{(3)}\}$$

[3881s′] $$=\frac{m'}{16a'}.\left\{152b_{\frac{1}{2}}^{(2)}+108\alpha.\frac{db_{\frac{1}{2}}^{(2)}}{d\alpha}+20\alpha^2.\frac{d^2b_{\frac{1}{2}}^{(2)}}{d\alpha^2}+\alpha^3.\frac{d^3b_{\frac{1}{2}}^{(2)}}{d\alpha^3}\right\};$$

[3881t] $$M^{(2)}=\frac{m'}{16}.\left\{\begin{array}{l}126A^{(3)}-21a'.\left(\frac{dA^{(3)}}{da'}\right)+60a.\left(\frac{dA^{(3)}}{da}\right)-10aa'.\left(\frac{d^2A^{(3)}}{dada'}\right)\\+6a^2.\left(\frac{d^2A^{(3)}}{da^2}\right)-a^2a'.\left(\frac{d^3A^{(3)}}{da^2da'}\right)\end{array}\right\}$$

$$=\frac{m'}{16}.\left\{\begin{array}{l}126A^{(3)}+21.[A^{(3)}+A_1^{(3)}]+60A_1^{(3)}+10.[2A_1^{(3)}+A_2^{(3)}]\\+6A_2^{(3)}+[3A_2^{(3)}+A_3^{(3)}]\end{array}\right\}$$

$$=\frac{m'}{16}.\{147A^{(3)}+101A_1^{(3)}+19A_2^{(3)}+A_3^{(3)}\}$$

[3881t′] $$=-\frac{m'}{16a'}.\left\{147b_{\frac{1}{2}}^{(3)}+101\alpha.\frac{db_{\frac{1}{2}}^{(3)}}{d\alpha}+19\alpha^2.\frac{d^2b_{\frac{1}{2}}^{(3)}}{d\alpha^2}+\alpha^3.\frac{d^3b_{\frac{1}{2}}^{(3)}}{d\alpha^3}\right\};$$

in comparison with that of Mercury, and the inequality in question is very [3883']

$$M^{(3)} = -\frac{m'}{48} \cdot \left\{ 136\, A^{(4)} + 93\, a \cdot \left(\frac{d\,A^{(4)}}{da}\right) + 18\, a^2 \cdot \left(\frac{d^2 . A^{(4)}}{da^2}\right) + a^3 \cdot \left(\frac{d^3 . A^{(4)}}{da^3}\right) \right\} \qquad [3881u]$$

$$= \frac{m'}{48\, a'} \cdot \left\{ 136\, b_{\frac{1}{2}}^{(4)} + 93\, \alpha \cdot \frac{d\, b_{\frac{1}{2}}^{(4)}}{d\alpha} + 18\, \alpha^2 \cdot \frac{d^2 b_{\frac{1}{2}}^{(4)}}{d\alpha^2} + \alpha^3 \cdot \frac{d^3 b_{\frac{1}{2}}^{(4)}}{d\alpha^3} \right\}; \qquad [3881u']$$

$$M^{(4)} = \frac{m'}{16} \cdot a'a \cdot \left\{ -5\, B^{(2)} + a' \cdot \left(\frac{d\,B^{(2)}}{da'}\right) \right\} = \frac{m'}{16} \cdot a'a \cdot \left\{ -5\, B^{(2)} + \left[-3\, B^{(2)} - a \cdot \left(\frac{d\,B^{(2)}}{da}\right) \right] \right\} \qquad [3881v]$$

$$= -\frac{m'}{16} \cdot a'a \cdot \left\{ 8\, B^{(2)} + a \cdot \left(\frac{d\,B^{(2)}}{da}\right) \right\} = -\frac{m'}{16} \cdot \frac{\alpha}{a'} \cdot \left\{ 8\, b_{\frac{3}{2}}^{(2)} + \alpha \cdot \frac{d\, b_{\frac{3}{2}}^{(2)}}{d\alpha} \right\}; \qquad [3881v']$$

$$M^{(5)} = \frac{m'}{16} \cdot a'a \cdot \left\{ 5\, B^{(3)} + a \cdot \left(\frac{d\,B^{(3)}}{da}\right) \right\} \qquad [3881w]$$

$$= \frac{m'}{16} \cdot \frac{\alpha}{a'} \cdot \left\{ 5\, b_{\frac{3}{2}}^{(3)} + \alpha \cdot \frac{d\, b_{\frac{3}{2}}^{(3)}}{d\alpha} \right\}. \qquad [3881w']$$

If we substitute in these the numerical values [4095—4102], also $\frac{d^3 b_{\frac{1}{2}}^{(1)}}{d\alpha^3} = 5{,}340815$, $\alpha \cdot \frac{d\, b_{\frac{3}{2}}^{(2)}}{d\alpha} = 1{,}96112$, given by Mr. Plana, in Vol. II, page 366, of the Memoirs of the Astronomical Society of London, we shall obtain, by supposing $a' = 1$, [3883a]

$$\begin{array}{lll} a' M^{(0)} = -\, m' . 0{,}3411\,; & a' M^{(1)} = \, m' . 3{,}3192\,; & a' M^{(2)} = -\, m' . 1{,}4808\,; \\ a' M^{(3)} = \, m' . 0{,}2181\,; & a' M^{(4)} = -\, m' . 0{,}1921\,; & a' M^{(5)} = \, m' . 0{,}0690. \end{array} \qquad [3883b]$$

The last four of these numbers agree nearly with those given by Mr. Plana; but he finds $a' M^{(0)} = -m' . 2{,}40567$, $a' M^{(1)} = m' . 2{,}9430$; so that he makes $M^{(0)}$ seven [3883c] times too great, and $M^{(1)}$ about a seventh part too small. The first of these mistakes arises from the omission of the term $-256\,\alpha$ [3881o]; the second is an error in the numerical calculations. We must observe, that the indices of M in La Place's notation, namely, 0, 1, 2, 3, 4, 5, correspond, respectively, to 3, 2, 1, 0, 5, 4, in [3883d] the notation used by Mr. Plana. In computing the value of δv, Mr. Plana uses the elements corresponding to the year 1800, namely,

$$\begin{array}{llll} e' = 0{,}0168532\,; & e = 0{,}2056163\,; & \gamma = \text{tang.}\, 7^d\, 0^m\, 6^s\,; & \varpi' = 99^d\, 30^m\, 5^s\,; \\ \varpi = 74^d\, 21^m\, 47^s\,; & \Pi = 45^d\, 57^m\, 31^s\,; & a' = 1\,; \quad a = 0{,}38709\,; & \text{and } n', n \text{ [4077]}; \end{array} \qquad [3883e]$$

he also reduces the mass m' from $\frac{1}{329630}$ [4061] to $\frac{1}{354936}$, which makes [3883f]

[3883''] small, we may neglect these terms without any sensible error [3881*d*].

$\mu''=-0{,}0713$ [4230']; then by the method [3881*i*], he finally obtains

[3883*g*] $$\delta v = 0^s{,}5596 \,.\, \sin.\,(4\,n' t - n\,t + 4\,\varepsilon' - \varepsilon - 16^d\,59^m\,20^s).$$

If we correct the errors mentioned in [3883*c*]; also another error, in his substitution of the value of $2\,\Pi$, which is taken too small by 40^d, in [3881*f*]; it will become

[3883*h*] $$\delta v = 0^s{,}61 \,.\, \sin.\,(4\,n' t - n\,t + 4\,\varepsilon' - \varepsilon - 21^d\,19^m).$$

This differs but very little from the computation of La Place in [4283], namely,

[3883*i*] $$\begin{aligned}\delta v &= (1+\mu'') \,.\, 0^s{,}69 \,.\, \sin.\,(4\,n'' t - n\,t + 4\,\varepsilon'' - \varepsilon - 19^d\,2^m\,13^s)\\ &= 0^s{,}64 \,.\, \sin.\,(4\,n'' t - n\,t + 4\,\varepsilon'' - \varepsilon - 19^d\,2^m\,13^s) \qquad [3883f].\end{aligned}$$

Notwithstanding this near agreement in the numerical results, the method of La Place is essentially defective, as may be seen by comparing the term depending on e^3 in the expression [3881*i*, *f*], namely,

[3883*k*] $$\delta v = -\frac{3\,a\,n^2}{(4\,n'-n)^2} \,.\, M^{(3)} \,.\, e^3 \,.\, \sin.\,(4\,n' t - n\,t + 4\,\varepsilon' - \varepsilon - 3\,\varpi),$$

with that given by La Place in [3882],

[3883*l*] $$\delta v = \frac{n}{n-4\,n'} \,.\, L \,.\, e^3 \,.\, \sin.\,(4\,n' t - n\,t + 4\,\varepsilon' - \varepsilon - 3\,\varpi).$$

[3883*l*'] To compute the value of L, we may observe, that $L\,.\,e^2.\cos.(4\,n't-2\,n\,t+4\,\varepsilon'-2\,\varepsilon-2\,\varpi)$ is the term of $\frac{r\,\delta r}{a^2}$, depending on e^2, in [3874]. Now the term of $\frac{r\,\delta r}{a^2}$ [3711],
[3883*m*] corresponding to $i=4$, and having the divisor $4\,n'-n$, is

[3883*n*] $$\frac{\frac{-4\,n}{4\,n'-2\,n} \,.\, a\,M + a^2.\left(\frac{d\,M}{d\,a}\right)}{(4\,n'-n)\,.\,(4\,n'-3\,n)} \,.\, n^2 \,.\, \cos.\,(4\,n' t - 2\,n\,t + 4\,\varepsilon' - 2\,\varepsilon - 2\,\varpi);$$

and as we retain here only the terms depending on e^2, we may put $M=M^{(0)}\,e^2$ [3703, 3745]; moreover, we have, in the present case, very nearly $4\,n'-2\,n=-n$, $4\,n'-3\,n=-2\,n$ [3869]; hence this term of $\frac{r\,\delta r}{a^2}$ becomes

[3883*o*] $$-\frac{\left\{4\,a\,M^{(0)} + a^2.\left(\frac{d\,M^{(0)}}{d\,a}\right)\right\}}{2\,.\,(4\,n'-n)} \,.\, n\,e^2 \,.\, \cos.\,(4\,n' t - 2\,n\,t + 4\,\varepsilon' - 2\,\varepsilon - 2\,\varpi).$$

Now we may obtain the expression of $M^{(0)}$ [3883*p*], by putting $i=4$ [3883*m*], in [3750]. The partial differential, relative to a, is as in [3883*q*]. Substituting these two values

11. It follows, from [1337′—1342], that the two terms of R [3835], represented by

$$R = \quad M^{(4)}.\,e'\gamma^2.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'-2\Pi)$$
$$+M^{(5)}.\,e\gamma^2.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-2\Pi),$$ [3884]

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in the first member of [3883r], and making the same reductions as in [999, &c.], we get [3883s], by putting $a'=1$,

$$M^{(0)} = \frac{m'}{8}.\left\{44A^{(4)}+14a.\left(\frac{dA^{(4)}}{da}\right)+a^2.\left(\frac{d^2A^{(4)}}{da^2}\right)\right\}$$ [3883p]

$$\left(\frac{dM^{(0)}}{da}\right) = \frac{m'}{8}.\left\{58.\left(\frac{dA^{(4)}}{da}\right)+16a.\left(\frac{d^2A^{(4)}}{da^2}\right)+a^2.\left(\frac{d^3A^{(4)}}{da^3}\right)\right\}$$ [3883q]

$$4aM^{(0)}+a^2.\left(\frac{dM^{(0)}}{da}\right) = \frac{m'a}{8}.\left\{176A^{(4)}+114a.\left(\frac{dA^{(4)}}{da}\right)+20a^2.\left(\frac{d^2A^{(4)}}{da^2}\right)+a^3.\left(\frac{d^3A^{(4)}}{da^3}\right)\right\}$$ [3883r]

$$=-\frac{m'a}{8}.\left\{176\,b_{\frac{1}{2}}^{(4)}+114\,\alpha.\frac{db_{\frac{1}{2}}^{(4)}}{d\alpha}+20\,\alpha^2.\frac{d^2b_{\frac{1}{2}}^{(4)}}{d\alpha^2}+\alpha^3.\frac{d^3b_{\frac{1}{2}}^{(4)}}{d\alpha^2}\right\}.$$ [3883s]

Substituting this in [3883o], and putting the result equal to

$$L.\,e^2.\cos.(4n't-2nt+4\varepsilon'-2\varepsilon-2\varpi) \qquad [3883l'],$$

we get

$$L=\frac{m'.an}{16.(4n'-n)}.\left\{176\,b_{\frac{1}{2}}^{(4)}+114\,\alpha.\frac{db_{\frac{1}{2}}^{(4)}}{d\alpha}+20\,\alpha^2.\frac{d^2b_{\frac{1}{2}}^{(4)}}{d\alpha^2}+\alpha^3.\frac{d^3b_{\frac{1}{2}}^{(4)}}{d\alpha^3}\right\};$$ [3883t]

consequently the part of δv [3883l], computed by La Place, is

$$\delta v=-\frac{m'.an^2.e^3}{16.(4n'-n)^2}.\left\{176\,b_{\frac{1}{2}}^{(4)}+114\alpha^2.\frac{db_{\frac{1}{2}}^{(4)}}{d\alpha}+20\,\alpha^2.\frac{d^2b_{\frac{1}{2}}^{(4)}}{d\alpha^2}+\alpha^3.\frac{d^3b_{\frac{1}{2}}^{(4)}}{d\alpha^3}\right\};$$ [3883u]

whereas the real value, obtained by the direct method [3881i, u′], is

$$\delta v=-\frac{m'.an^2.e^3}{16.(4n'-n)^2}.\left\{136\,b_{\frac{1}{2}}^{(4)}+93\,\alpha.\frac{db_{\frac{1}{2}}^{(4)}}{d\alpha}+18\,\alpha^2.\frac{d^2b_{\frac{1}{2}}^{(4)}}{d\alpha^2}+\alpha^3.\frac{d^3b_{\frac{1}{2}}^{(4)}}{d\alpha^3}\right\}.$$ [3883v]

If we substitute in these expressions the values given in [4095, &c.], we shall find, that the coefficient of $-\frac{m'.an^2.e^3}{16.(4n'-n)^2}$, in the first is 12, 54, and in the second 10, 50; so [3883w]
that La Place's method makes this term too great by about one *fifth* part; and the same [3883x]
discrepancy occurs in the coefficients of most of the terms of these two formulas.

produce in the value of s, or in the motion of m in latitude, the inequality,*

[3885] $$\delta s = -\frac{2an}{5n'-2n} \cdot \left\{ \begin{array}{l} M^{(4)} \cdot e'\gamma \cdot \sin.(5n't - 3nt + 5\varepsilon' - 3\varepsilon - \varpi' - \Pi) \\ + M^{(5)} \cdot e\gamma \cdot \sin.(5n't - 3nt + 5\varepsilon' - 3\varepsilon - \varpi - \Pi) \end{array} \right\}.$$

Moreover the same terms produce in the value of s', or in the motion of m' in latitude, the inequality†

[3886] $$\delta s' = \frac{2a'n'}{5n'-2n} \cdot \frac{m}{m'} \cdot \left\{ \begin{array}{l} M^{(4)} \cdot e'\gamma \cdot \sin.(4n't - 2nt + 4\varepsilon' - 2\varepsilon - \varpi' - \Pi) \\ + M^{(5)} \cdot e\gamma \cdot \sin.(4n't - 2nt + 4\varepsilon' - 2\varepsilon - \varpi - \Pi) \end{array} \right\};$$

There is a small inequality in the motion of the Earth, depending on the same angle $nt - 4n''t$, given by the author in [4311]. He seems to have computed it from

[3883y] the term for Mercury [4283], by means of the formula [1208], $\delta v'' = -\delta v \cdot \frac{m\sqrt{a}}{m''\sqrt{a''}}$, using $\delta v = -0^s{,}690412$ [4283], and the other elements [4061, 4079]. This method will answer, as the inequality is extremely small.

[3885a] * (2446) Putting, in the term of R [1337''], $\tang.\varphi_{\prime}' = \gamma$, it becomes

$$R = m'k \cdot \gamma^g \cdot \cos.(i'n't - int + A - g\theta_{\prime}');$$

[3885a'] comparing this with [3884], we get $g = 2$, $\theta_{\prime}' = \Pi$, $i' = 5$, $i = 2$; also in the first term, $m'k = M^{(4)} \cdot e'$, $A = 5\varepsilon' - 2\varepsilon - \varpi'$; and in the second term, $m'k = M^{(5)} \cdot e$,

[3885b] $A = 5\varepsilon' - 2\varepsilon - \varpi$. Substituting these in [1342], which is obtained from the integrals [1341a, 1341], we obtain in s, from the first term, the quantity

[3885c] $$-\frac{2an}{5n'-2n} \cdot M^{(4)} \cdot e'\gamma \cdot \sin.(5n't - 2nt - v + 5\varepsilon' - 2\varepsilon - \varpi' - \Pi);$$

and from the second term, the quantity

[3885d] $$-\frac{2an}{5n'-2n} \cdot M^{(5)} \cdot e\gamma \cdot \sin.(5n't - 2nt - v + 5\varepsilon' - 2\varepsilon - \varpi - \Pi);$$

observing, that $\mu = 1$ [3709]. Putting, in these, for v, its mean value $nt + \varepsilon$ [3834], and connecting the two preceding terms, they become as in [3885].

† (2447) The terms of R [3884], used in computing s [3885], are deduced from the function [3831], which is multiplied by the factor or mass m'. In computing the

[3886a] value of s', corresponding to the planet m', and to the same angles, we must use the factor m, instead of m'; therefore the value of R to be used in computing s', is equal to the function [3884], multiplied by $\frac{m}{m'}$; which amounts to the same thing as to change

[3886b] $M^{(4)}$, $M^{(5)}$, into $\frac{m}{m'} \cdot M^{(4)}$, and $\frac{m}{m'} \cdot M^{(5)}$, respectively.

Π *being, as in the preceding inequality of s, the longitude of the ascending node of the orbit of* m' *upon that of* m. These are the only sensible inequalities in latitude, in the planetary system, depending on the product of the excentricities and inclinations of the orbits. [3886']

We have seen, in [3800], that the value of δs produces in the motion of m, reduced to the fixed plane, the term $-\text{tang.}\,\varphi\,.\,\delta s\,.\,\cos.(v_{,}-\theta)$; [3887] by substituting the preceding inequality of s [3885] in this term, we shall obtain a term depending on $5n't-2nt$, which must be added to the

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If we now compare the value of s [3885] with the value of R [3884], we shall find, that s may be derived from R, by multiplying it by $\frac{-2an.dt}{\gamma}$; then integrating relatively [3886c] to t, as in [3885*b*, &c., 1311*a*], and after integration, decreasing the angles by the quantity $v-\Pi$ [3885*c*], or by its mean value $nt+\varepsilon-\Pi$. In like manner, we may derive s' [3886d] from R [3884], after multiplying it by the factor $\frac{m}{m'}$ [3886*b*]. This value of $\frac{m}{m'}\,.\,R$ is to be multiplied by $-\frac{2a'n'.dt}{\gamma}$, to correspond with [3886*c*], and it will become

$$-2a'n'.dt\,.\,\frac{m}{m'}\,.\left\{\begin{array}{l} M^{(4)}.\,e'\gamma\,.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'-2\Pi) \\ +M^{(5)}.\,e\gamma\,.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-2\Pi) \end{array}\right\};\qquad [3886e]$$

and then by integration, we get

$$-\frac{2a'n'}{5n'-2n}\,.\,\frac{m}{m'}\,.\left\{\begin{array}{l} M^{(4)}.\,e'\gamma\,.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'-2\Pi) \\ +M^{(5)}.\,e\gamma\,.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-2\Pi) \end{array}\right\}.\qquad [3886e']$$

The angles $5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-2\Pi$, &c., must now be decreased by $v'-\Pi'=n't+\varepsilon'-\Pi'$, corresponding to the planet m', as in [3886*d*]; the angle Π' being the longitude of the *ascending* node of the orbit of m upon that of m'; in the same [3886f] manner as Π [3746] is the *ascending* node of m' upon that of m; and it is evident, that $\Pi'=180^d+\Pi$; hence $v'-\Pi'=n't+\varepsilon'-\Pi-180^d$. Subtracting this from the angles which occur in [3886*e'*], it becomes

$$-\frac{2a'n'}{5n'-2n}\,.\,\frac{m}{m'}\,.\left\{\begin{array}{l} M^{(4)}.\,e'\gamma\,.\sin.(4n't-2nt+4\varepsilon'-2\varepsilon-\varpi'-\Pi+180^d) \\ +M^{(5)}.\,e\gamma\,.\sin.(4n't-2nt+4\varepsilon'-2\varepsilon-\varpi-\Pi+180^d) \end{array}\right\},\qquad [3886g]$$

which is easily reduced to the form [3886].

[3888] great inequality of the motion of m; but this term is insensible for Jupiter and Saturn.*

[3887a] * (2448) The functions δs, $\delta s'$ [3885, 3886], reduced to numbers in [4458, 4513], are of the order 3^s or 9^s; these are multiplied by $\tang. \varphi$ in [3887], and as this tangent is very small [4082], these terms may be neglected.

CHAPTER II.

INEQUALITIES DEPENDING ON THE SQUARE OF THE DISTURBING FORCE.

12. *The great inequalities which we have just investigated, produce other sensible ones, depending on the square of the disturbing force.* We have given the analytical expressions in [1213, 1214, 1306—1309]; and it follows, from [1197, 1213], that if we put

the great inequality of Jupiter $\zeta = \overline{H} . \sin. (5n't - 2nt + 5\varepsilon' - 2\varepsilon + \overline{A})$, [3889]

we shall have

Great inequalities of Jupiter.

$$\delta v = -\frac{\overline{H}^2}{8} . \frac{(2m'\sqrt{a'} + 5m\sqrt{a})}{m'\sqrt{a'}} . \sin. 2 . (5n't - 2nt + 5\varepsilon' - 2\varepsilon + \overline{A}), \qquad [3890]$$

for the corresponding inequality of Jupiter, depending on the square of the disturbing force. This inequality, like that from which it is derived, is to be added to the mean motion of Jupiter.* [3890']

In like manner, if we put

the great inequality of Saturn $\zeta' = -\overline{H}' . \sin. (5n't - 2nt + 5\varepsilon' - 2\varepsilon + \overline{A}')$, [3891]

we shall have

Great inequalities of Saturn.

$$\delta v' = \frac{\overline{H}'^2}{8} . \frac{(2m'\sqrt{a'} + 5m\sqrt{a})}{m\sqrt{a}} . \sin. 2 . (5n't - 2nt + 5\varepsilon' - 2\varepsilon + \overline{A}'), \qquad [3891']$$

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* (2449) The great inequality of Jupiter is found, by substituting, in ζ [1197], $\mu = 1$ [3709], also $i = 2$, $i' = 5$; and if we put [3890a]

$$A = 5\varepsilon' - 2\varepsilon + \overline{A}, \qquad T_5 = 5n't - 2nt + 5\varepsilon' - 2\varepsilon, \qquad \overline{H} = -\frac{6m' . a n^2 k}{(5n' - 2n)^2}, \qquad [3890b]$$

we get $\zeta = \overline{H} . \sin. (T_5 + \overline{A})$, as in [3889]. Making the same substitutions in the terms of the second order [1213], it becomes as in [3890]. [3890c]

[3891''] *for the corresponding inequality of Saturn,* which must be added to the mean motion of Saturn.*

The variations of the excentricities and perihelion may introduce similar inequalities in the mean motions of the two planets. To determine them,
[3891'''] we shall observe, that if we notice only the cubes and products of three dimensions, of the excentricities and inclinations of the orbits, we shall have†

[3892] $$3a\,.\int\!\!\int n\,dt\,.\,\mathrm{d}R = -6\,a\,m'.\int\!\!\int n^2\,dt^2.\left\{\begin{array}{l} P.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon) \\ -P'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\}.$$

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* (2450) Substituting ζ [3890c] in [1208], we get

[3891a] $$\text{the great inequality of Saturn } \zeta' = -\frac{m\sqrt{a}}{m'\sqrt{a'}}\,.\,\overline{H}\,.\sin.(T_5+\overline{A})\,;$$

putting this equal to the assumed value [3891], we obtain

[3891b] $$\overline{H}' = \frac{m\sqrt{a}}{m'\sqrt{a'}}\,.\,\overline{H}, \quad \text{and} \quad \overline{A} = \overline{A}'.$$

Now by comparing the two formulas [1213, 1214], we find, that the part of the great inequality of Saturn, depending on the square of the disturbing force, is equal to the
[3891c] corresponding part of the great inequality of Jupiter, multiplied by $-\frac{m\sqrt{a}}{m'\sqrt{a'}}$; and by using the expression of this inequality of Jupiter [3890], that of Saturn becomes

[3891d] $$\tfrac{1}{8}\overline{H}^2.\frac{m\sqrt{a}}{m'\sqrt{a'}}\,.\,\frac{(2m'\sqrt{a'}+5m\sqrt{a})}{m'\sqrt{a'}}.\sin.2.(T_5+\overline{A}) = \tfrac{1}{8}\overline{H}'^2.\frac{(2m'\sqrt{a'}+5m\sqrt{a})}{m\sqrt{a}}.\sin.2.(T_5+\overline{A})\,;$$

the second of these formulas being deduced from the first, by the substitution of $\overline{H}$ [3891b]. This last expression agrees with that in [3891'], except that $\overline{A}$ is changed into $\overline{A}'$, so as to make both the expressions [3891, 3891'] depend on the same argument; observing, that these quantities are very nearly equal to each other, since, in the year 1750, we have
[3891e] $\overline{A} = 4^d\,22^m\,21^s$ [4434], and $\overline{A}' = 4^d\,21^m\,20^s$ [4492].

† (2451) The part of R depending on the angle $5n't-2nt$, and terms of the
[3892a] third degree in e, e', γ, &c., is given in [3842a']. Its differential, relatively to the characteristic d [916'], is

[3892b] $$\mathrm{d}R = -2\,m'.\,n\,dt\,.\{P.\cos.\,T_5 - P'.\sin.\,T_5\}.$$

Multiplying this by $3a.ndt$, and prefixing the double sign of integration, we get [3892], which represents the part of δv [3715b], depending on $\mathrm{d}R$, the divisor $\sqrt{(1-e^2)}$ being
[3892c] neglected, as in [3718']. The quantities P, P', which occur in this expression, are given in [3842, 3843], in terms of the elements of the orbits of m, m'.

which gives, in $3a.\int\int n\,dt\,.\,\mathrm{d}R$, the quantity*

$$-6am'.\int\int n^2dt^2.\left\{\begin{array}{l}
\delta e.\left\{\left(\frac{dP}{de}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{de}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\right\} \\
+\delta\varpi.\left\{\left(\frac{dP}{d\varpi}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{d\varpi}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\right\} \\
+\delta e'.\left\{\left(\frac{dP}{de'}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{de'}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\right\} \\
+\delta\varpi'.\left\{\left(\frac{dP}{d\varpi'}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{d\varpi'}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\right\} \\
+\delta\gamma.\left\{\left(\frac{dP}{d\gamma}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{d\gamma}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\right\} \\
+\delta\Pi.\left\{\left(\frac{dP}{d\Pi}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{d\Pi}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\right\}
\end{array}\right\}\ (X);$$ [3893]

δe, $\delta\varpi$, $\delta e'$, $\delta\varpi'$, $\delta\gamma$, $\delta\Pi$, being the parts of e, ϖ, e', ϖ', γ, Π, respectively, depending upon the angle $5n't-2nt$. We have, by means of [3842c],† [3893']

$$\left(\frac{dP}{d\varpi}\right)=e.\left(\frac{dP'}{de}\right); \qquad \left(\frac{dP'}{d\varpi}\right)=-e.\left(\frac{dP}{de}\right);$$ [3894]

$$\left(\frac{dP}{d\varpi'}\right)=e'.\left(\frac{dP'}{de'}\right); \qquad \left(\frac{dP'}{d\varpi'}\right)=-e'.\left(\frac{dP}{de'}\right);$$ [3894']

$$\left(\frac{dP}{d\Pi}\right)=\gamma.\left(\frac{dP'}{d\gamma}\right); \qquad \left(\frac{dP'}{d\Pi}\right)=-\gamma.\left(\frac{dP}{d\gamma}\right).$$ [3894'']

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* (2452) We have already noticed the effect of the *secular* variations of P, P', in the terms of $3a.\int\int n\,dt.\mathrm{d}R$ [3812, 3812f], depending on $\sin.T_5$, $\cos.T_5$; using, for brevity, T_5 [3890b]. The object of the present investigation is to ascertain whether the *periodical* variations of e, e', ϖ, ϖ', Γ, Π, depending on the angle T_5, which are computed in [1288, 1297, &c.], produce, in the function $3a.\int\int n\,dt\,.\,\mathrm{d}R$, any *secular* or *periodical* inequalities. Now if we suppose the elements e, e', ϖ, ϖ', γ, Π, to be increased by the variations δe, $\delta e'$, $\delta\varpi$, $\delta\varpi'$, $\delta\gamma$, $\delta\Pi$, respectively, the corresponding increments of P, P', will be obtained, by means of [607—612], in the following forms, [3893a] [3893b]

$$\delta P=\left(\frac{dP}{de}\right).\delta e+\left(\frac{dP}{d\varpi}\right).\delta\varpi+\left(\frac{dP}{de'}\right).\delta e'+\left(\frac{dP}{d\varpi'}\right).\delta\varpi'+\left(\frac{dP}{d\gamma}\right).\delta\gamma+\left(\frac{dP}{d\Pi}\right).\delta\Pi;$$ [3893c]

$$dP'=\left(\frac{dP'}{de}\right).\delta e+\left(\frac{dP}{d\varpi}\right).\delta\varpi+\left(\frac{dP'}{de'}\right).\delta e'+\left(\frac{dP'}{d\varpi'}\right).\delta\varpi'+\left(\frac{dP'}{d\gamma}\right).\delta\gamma+\left(\frac{dP'}{d\Pi}\right).\delta\Pi;$$ [3893d]

these parts of the general values of P, P', being substituted in [3892], produce the expression [3893].

† (2453) The equations [3894—3894''], are easily deduced from the general values

Moreover we have, as in [1297, 1288],*

[3895] $$\left(\frac{dP}{de}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{de}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)=\frac{(5n'-2n)}{m'.an}.e\delta\varpi;$$

[3895'] $$\left(\frac{dP'}{de}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)+\left(\frac{dP}{de}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)=-\frac{(5n'-2n)}{m'.an}.\delta e;$$

we likewise have†

[3896] $$\left(\frac{dP}{de'}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{de'}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)=\frac{(5n'-2n)}{m.a'n'}.e'\delta\varpi';$$

[3896'] $$\left(\frac{dP'}{de'}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)+\left(\frac{dP}{de'}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)=-\frac{(5n'-2n)}{m.a'n'}.\delta e'.$$

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of P, P' [3842c], which give

[3894a] $$\left(\frac{dP}{d\varpi}\right)=\Sigma\, b\,.\,M'.\,e'^{b'}.\,e^{b}.\,\gamma^{2c}.\cos.\,(b'\varpi'+b\,\varpi+2\,c\,\Pi);$$

[3894b] $$\left(\frac{dP'}{de}\right)=\Sigma\, b\,.\,M'.\,e'^{b'}.\,e^{b-1}.\,\gamma^{2c}.\cos.\,(b'\varpi'+b\,\varpi+2\,c\,\Pi).$$

These expressions satisfy the first of the equations [3891]; and in like manner, we may prove the others to be accurate, by the substitution of the partial differentials of P, P' [3842c].

* (2454) The value of R [3842a'], is the same as that assumed in [1287],
[3895a] supposing $\mu=1$, $i'=5$, $i=2$, as in [3890a]. Making the same substitutions in δe, $\delta\varpi$ [1288, 1297], we get, by using the abridged symbols [3846b, d], the following expressions, which are easily reduced to the forms [3895', 3895];

[3895b] $$\delta e=-\frac{m'.an}{5n'-2n}.\left\{\left(\frac{dP'}{de}\right).\cos.\,T_5+\left(\frac{dP}{de}\right).\sin.\,T_5\right\};$$

[3895c] $$\delta\varpi=\frac{m'.an}{(5n'-2n).e}.\left\{\left(\frac{dP}{de}\right).\cos.\,T_5-\left(\frac{dP'}{de}\right).\sin.\,T_5\right\}.$$

† (2455) The values $\delta e'$, $e'\delta\varpi'$, depending on the angle T_5, noticing only terms of
[3895d] the third order in e, e', γ [3891'''], are easily deduced from those of δe, $e\delta\varpi$ [3895, 3895'], by a process similar to that employed in [3846a—g]; using also the same abridged symbols
[3895e] T_5, T_6, P_0, P_0', &c. For if we substitute, in [1288], the values $i'=-2$, $i=-5$, we get the following term of δe, which may be added to [3895b], to obtain a symmetrical form of δe, similar to [3846b, &c.],

[3895f] $$\delta e=-\frac{m'.an}{5n-2n'}.\left\{\left(\frac{dP_0'}{de}\right).\cos.\,T_6+\left(\frac{dP_0}{de}\right).\sin.\,T_6\right\}.$$

This last term may, however, be neglected in computing the value of δe; because it has not the small divisor $5n'-2n$. Now changing the elements m, a, n, e, &c. into
[3895g] m', a', n', e', &c., and the contrary, as in [3846a, d], we find, that the part of $\delta e'$, arising

To obtain the values of $\delta\gamma$ and $\delta\Pi$, we shall observe, that the latitude of m, [3896″]
above the *primitive* orbit of m', is $s=-\gamma\,.\,\sin.(v-\Pi)$,* which gives [3897]

$$\delta s=-\delta\gamma\,.\,\sin.(v-\Pi)+\gamma\,.\,\delta\Pi\,.\,\cos.(v-\Pi). \qquad [3898]$$

Now we have, in [1342],†

$$\delta s=\frac{m'.an}{5n'-2n}.\left\{\begin{array}{l}\left(\frac{dP}{d\gamma}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-v+\Pi)\\ -\left(\frac{dP'}{d\gamma}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-v+\Pi)\end{array}\right\}. \qquad [3899]$$

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from [3895b], has the divisor $5n-2n'$, which is large; therefore this part is small, and may be neglected. The other part, derived from [3895f], becomes

$$\delta e'=-\frac{m.a'n'}{5n'-2n}.\left\{\left(\frac{dP'}{de'}\right).\cos.T_5+\left(\frac{dP}{de'}\right).\sin.T_5\right\}; \qquad [3895h]$$

which is easily reduced to the form [3896′]. In the same manner, we may derive $\delta\varpi'$ [3895i]
[3896] from $\delta\varpi$ [3895c].

* (2456) It may not be amiss to remark, that the object of the calculation in [3896″—3902], is to ascertain the parts of $\delta\gamma$, $\gamma\delta\Pi$ [3900, 3901], arising from the [3897a]
perturbation of m in latitude, by the action of m'; *supposing the fixed plane to be the* [3897b]
primitive orbit of m' [3897]; these parts are denoted by $\delta_{\prime\prime}\gamma$, $\gamma\,\delta_{\prime\prime}\Pi$, respectively, in [3899′]. In like manner, the action of m upon m' affects the values of $\delta\gamma$, $\gamma\delta\Pi$, [3897c]
by terms which are represented by $\delta_{\prime}\gamma$, $\gamma\,\delta_{\prime}\Pi$, respectively, [3904]. The sum of these two parts of $\delta\gamma$ gives the complete value of $\delta\gamma$, as in the first equation [3905]; and the sum of the two parts of $\delta\Pi$ gives the complete value of $\delta\Pi$, as in the second of the [3897d]
equations [3905]. Having made these preliminary observations, we shall now remark, that the expression [3897] is similar to [679], changing $v_{\prime}$ into v, tang.φ into γ [669″, 3739]; [3897e]
and θ into $\Pi+180^d$ [669″, 3746]; observing, that as Π [3746 or 3902] is the longitude of the ascending node of m' upon the orbit of m, we shall have $\Pi+180^d$, for that of the [3897f]
ascending node of m upon the orbit of m', taken for the fixed plane [3896″]. Hence [679] becomes $s=\gamma.\sin.(v-\Pi-180^d)=-\gamma.\sin.(v-\Pi)$, as in [3897]. Supposing [3897g]
now γ, Π to vary; the corresponding variation of s will be as in [3898].

† (2457) Using the values [3895a], also $g=2$, tang.$\varphi_{\prime}'=\gamma$, $\theta_{\prime}'=\Pi$ [3902, 1337′]; also, for brevity

$$T_5=5n't-2nt+5\varepsilon'-2\varepsilon, \qquad T_8=5n't-2nt+A-2\Pi; \qquad [3899a]$$

the expressions of R [1337″], and s or δs [1342], become

$$R=m'k.\gamma^2.\cos.T_8, \qquad \delta s=\frac{-2m'k.an}{5n'-2n}.\gamma.\sin.(T_8-v+\Pi). \qquad [3899b]$$

Comparing this expression with the preceding [3898], we shall obtain,
[3899'] *for the parts of* $\delta\gamma$, $\gamma\,\delta\Pi$, *depending upon the action of* m' *upon* m, which
$\delta_{\prime\prime}$. we shall represent by $\delta_{\prime\prime}\gamma$, $\gamma\,\delta_{\prime\prime}\Pi$,

[3900] $$\delta_{\prime\prime}\gamma=-\frac{m'.an}{5n'-2n}\cdot\left\{\left(\frac{dP}{d\gamma}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)+\left(\frac{dP'}{d\gamma}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\right\};$$

[3901] $$\gamma\,\delta_{\prime\prime}\Pi=\frac{m'.an}{5n'-2n}\cdot\left\{\left(\frac{dP}{d\gamma}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{d\gamma}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\right\};$$

γ, Π. *in which* γ *is the mutual inclination of the two orbits to each other, and* Π *the*
[3902] *longitude of the ascending node of* m' *upon the orbit of* m [3746]. These
[3903] quantities also vary by the action of m upon m'; so that if we put these
$\delta_{\prime}$. last variations equal to $\delta_{\prime}\gamma$, $\delta_{\prime}\Pi$; the whole variations being $\delta\gamma$, $\delta\Pi$;
[3904] we shall have*

[3905] $$\delta\gamma=\delta_{\prime}\gamma+\delta_{\prime\prime}\gamma\,; \qquad \delta\Pi=\delta_{\prime}\Pi+\delta_{\prime\prime}\Pi\,;$$

[3906] $$\delta_{\prime}\gamma=\frac{m.a'n'}{m'.an}\cdot\delta_{\prime\prime}\gamma=\frac{m\sqrt{a}}{m'\sqrt{a'}}\cdot\delta_{\prime\prime}\gamma\,; \qquad \delta_{\prime}\Pi=\frac{m.a'n'}{m'.an}\cdot\delta_{\prime\prime}\Pi=\frac{m\sqrt{a}}{m'\sqrt{a'}}\cdot\delta_{\prime\prime}\Pi.$$

[3899c] If we compare this value of δs with that of R, we shall find, that $\delta s=\frac{an}{5n'-2n}\cdot\left(\frac{dR}{d\gamma}\right)$,
provided we increase the angle $5n't-2nt$ by the quantity $90^d-v+\Pi$, by which
[3899d] means $\cos.T_8$ will change into $\cos.(T_8+90^d-v+\Pi)=-\sin.(T_8-v+\Pi)$;
and if we use R [3842a'], the expression of δs [3899c], becomes as in [3899f, g, or 3899].
[3899e] Now if we put, for brevity, $v-\Pi=v_{\prime}$, and develop the terms of [3899g], by means
of [22, 24] Int., it becomes, as in [3899h],

[3899f] $$\delta s=\frac{an}{5n'-2n}\cdot m'\cdot\left\{\left(\frac{dP}{d\gamma}\right).\sin.(T_5+90^d-v+\Pi)+\left(\frac{dP'}{d\gamma}\right).\cos.(T_5+90^d-v+\Pi)\right\}$$

[3899g] $$=\frac{an}{5n'-2n}\cdot m'\cdot\left\{\left(\frac{dP}{d\gamma}\right).\cos.(T_5-v_{\prime})-\left(\frac{dP'}{d\gamma}\right).\sin.(T_5-v_{\prime})\right\}$$

[3899h] $$=\frac{m'.an}{5n'-2n}\cdot\left\{\left[\left(\frac{dP}{d\gamma}\right).\sin.T_5+\left(\frac{dP'}{d\gamma}\right).\cos.T_5\right].\sin.v_{\prime}+\left[\left(\frac{dP}{d\gamma}\right).\cos.T_5-\left(\frac{dP'}{d\gamma}\right).\sin.T_5\right].\cos.v_{\prime}\right\}.$$

[3899i] Comparing this with $\delta s=-\delta\gamma\,.\sin.v_{\prime}+\gamma\,\delta\Pi\,.\cos.v_{\prime}$ [3898], and putting the coefficients
of $\sin.v_{\prime}$, $\cos.v_{\prime}$, separately equal to each other in both expressions, we get [3900, 3901].
If we compare the value of $\gamma\,\delta_{\prime\prime}\Pi$ [3901] with that of R [3842a'], we easily perceive
[3899k] that it may be put under the form $\gamma\,\delta_{\prime\prime}\Pi=-an\,.\int dt\,.\left(\frac{dR}{d\gamma}\right)$; and having found $\gamma\,\delta_{\prime\prime}\Pi$
[3899l] by this formula, we get from it the value of $\delta_{\prime\prime}\gamma$, by changing the angle T_5 into T_5+90^d,
as is evident by comparing the two expressions [3901, 3900].

* (2458) From the expression of $\gamma\,\delta_{\prime\prime}\Pi$ [3899k], we may obtain the value of $\gamma\,\delta_{\prime}\Pi$,
[3906a] corresponding to the action of m upon m'; by observing that the values of P, P', which

This being premised, if we substitute these different quantities in the function [3893], we shall find that it vanishes.* Therefore *the variations of the excentricities, of the perihelia, of the nodes and of the inclinations of the orbits, corresponding to the two great inequalities of Jupiter and Saturn, do not introduce into the mean motion of Jupiter, or into the greater axis of its*

The inequality of the mean motion, arising [3906'] from the terms here [3906''] treated of, vanishes.

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occur in R [3842a'], are the same in both cases, as is remarked in [3832 or 3846f, &c.]; so that it is only necessary to change R [3831] into $\frac{m}{m'}.R$, and an into $a'n'$, to obtain from [3899k], the expression $\gamma\,\delta_{\prime}\Pi = -\frac{m}{m'}.a'n'.\int dt.\left(\frac{dR}{d\gamma}\right)$. Dividing this [3906$b$] by $\gamma\delta_{\prime\prime}\Pi$ [3899k], we get the first form of $\delta_{\prime}\Pi$ [3906]; and by applying the principle [3906c] of derivation [3899l] to this value of $\gamma\,\delta_{\prime}\Pi$, we obtain that of $\delta_{\prime}\gamma$ [3906]. The second forms [3906] are derived from the first, by putting $an=a^{-\frac{1}{2}}$, $a'n'=a'^{-\frac{1}{2}}$ [3709']. [3906d] Substituting the values [3906] in [3905], we get

$$\delta\gamma = \frac{m'.an+m.a'n'}{m'.an}.\delta_{\prime\prime}\gamma; \qquad \gamma\,\delta\,\Pi = \frac{m'.an+m.a'n'}{m'.an}.\gamma\,\delta_{\prime\prime}\Pi; \qquad [3906e]$$

in which we must substitute for $\delta_{\prime\prime}\gamma$, $\gamma\,\delta_{\prime\prime}\Pi$, their values [3900, 3901]. Therefore, to obtain the complete values of $\delta\gamma$, $\gamma\,\delta\,\Pi$, we must change the factor $m'.an$ into [3906f] $m'.an+m.a'n'$, in the formulas [3900, 3901].

* (2459) If we substitute the values [3894—3901] in [3893], we shall find, that the terms of this expression mutually destroy each other. In proving this, we shall neglect the factor $-6\,a\,m'.\iint n^2\,dt^2$, which affects all the terms; and shall use the symbol T_5 [3890b], also, for brevity,

$$M_1=\frac{5n'-2n}{m'.an}, \qquad M_2=\frac{5n'-2n}{m.a'n'}, \qquad M_3=\frac{5n'-2n}{m'.an+m.a'n'}. \qquad [3907a]$$

Then the expressions [3895, 3895'] may be put under the following forms [3907b]; the similar values [3896, 3896'] become as in [3907c]; and if we change, in [3900, 3901], the factor $m'.an$ into $m'.an+m.a'n'$, in order to obtain the complete values of $\delta\gamma$, $\gamma\,\delta\,\Pi$ [3906f], they will become as in [3907d];

$$\left(\frac{dP'}{de}\right).\cos.T_5+\left(\frac{dP}{de}\right).\sin.T_5=-M_1.\delta e; \quad \left(\frac{dP}{de}\right).\cos.T_5-\left(\frac{dP'}{de}\right).\sin.T_5=M_1.e\,\delta\varpi; \quad [3907b]$$

$$\left(\frac{dP'}{de'}\right).\cos.T_5+\left(\frac{dP}{de'}\right).\sin.T_5=-M_2.\delta e'; \quad \left(\frac{dP}{de'}\right).\cos.T_5-\left(\frac{dP'}{de'}\right).\sin.T_5=M_2.e'\delta\varpi'; \quad [3907c]$$

$$\left(\frac{dP'}{d\gamma}\right).\cos.T_5+\left(\frac{dP}{d\gamma}\right).\sin.T_5=-M_3.\delta\gamma; \quad \left(\frac{dP}{d\gamma}\right).\cos.T_5-\left(\frac{dP'}{d\gamma}\right).\sin.T_5=M_3.\gamma\,\delta\,\Pi. \quad [3907d]$$

orbit, considered as a variable ellipsis, any sensible inequality, depending on
[3907] *the square of the disturbing force; and it is evident, that the same result holds good in the mean motion of Saturn and in the greater axis of its orbit.*

Substituting these values in the first members of the following equations [3907e—g], then reducing, by the neglect of the terms which mutually destroy each other and putting $\sin.^2 T_5 + \cos.^2 T_5 = 1$, we get

[3907e] $$-M_1.\delta e.\cos.T_5 - M_1.e\delta\varpi.\sin.T_5 = \left(\frac{dP'}{de}\right); \quad M_1.\delta e.\sin.T_5 - M_1.e\delta\varpi.\cos.T_5 = -\left(\frac{dP}{de}\right);$$

[3907f] $$-M_2.\delta e'.\cos.T_5 - M_2.e'\delta\varpi'.\sin.T_5 = \left(\frac{dP'}{de'}\right); \quad M_2.\delta e'.\sin.T_5 - M_2.e'\delta\varpi'.\cos.T_5 = -\left(\frac{dP}{de'}\right);$$

[3907g] $$-M_3.\delta\gamma.\cos.T_5 - M_3.\gamma\delta\Pi.\sin.T_5 = \left(\frac{dP'}{d\gamma}\right); \quad M_3.\delta\gamma.\sin.T_5 - M_3.\gamma\delta\Pi.\cos.T_5 = -\left(\frac{dP}{d\gamma}\right);$$

Now the first line of [3893] becomes, by the substitution of $M_1.e\delta\varpi$ [3907b] equal
[3907h] to $\delta e.(M_1.e\delta\varpi) = M_1.e\delta e.\delta\varpi$. The second line of [3893] becomes, by the
[3907i] substitution of [3894], equal to $e\delta\varpi.\left\{\left(\frac{dP'}{de}\right).\cos.T_5 + \left(\frac{dP}{de}\right).\sin.T_5\right\}$, and by using $-M_1.\delta e$ [3907b], it is reduced to $e\delta\varpi.(-M_1.\delta e) = -M_1.e\delta e.\delta\varpi$; adding this to the first line [3907h], the sum becomes zero. In like manner, the third line of [3893], by the substitution of $M_2.e'\delta\varpi'$ [3907c], is equal to $\delta e'.(M_2.e'\delta\varpi') = M_2.e'\delta e'.d\varpi'$; and the fourth line, by the successive substitutions of [3894'] and $-M_2.\delta e'$ [3907c],
[3907k] is $e'\delta\varpi'.(-M_2.\delta e') = -M_2.e'\delta e.\delta\varpi'$; the sum of these two lines is therefore equal to zero. Substituting $M_3.\gamma\delta\Pi$ [3907d] in the fifth line of [3893], it becomes $\delta\gamma.(M_3.\gamma\delta\Pi) = M_3.\gamma\delta\gamma.\delta\Pi$; and by successively using the equations [3894''],
[3907l] also the value of $-M_3\delta\gamma$ [3907d], we shall find, that the sixth line of [3893] is $\gamma\delta\Pi.(-M_3.\delta\gamma) = -M_3.\gamma\delta\gamma.\delta\Pi$; therefore the sum of the fifth and sixth lines
[3907m] is equal to zero. Hence we see that all the terms of [3893], included between the braces, mutually destroy each other, as is observed in [3906']; consequently the values of
[3907n] δe, $\delta\varpi$, $\delta e'$, $\delta\varpi'$, $\delta\gamma$, $\delta\Pi$ [3895—3901], do not produce in $3a.\int\int n\,dt.\mathrm{d}R$ [3892 or 3715b] any term of the order of the square of the disturbing forces. The function $3a.\int\int n\,dt.\mathrm{d}R$, represents the mean motion of the planet m [1183]; therefore
[3907o] the variation of the mean motion, arising from these values of δe, $\delta\varpi$, $\delta e'$, &c. is nothing. Again, from [3709'], we have $2a = 2n^{-\frac{2}{3}}$, and as the mean motion nt or n, is
[3907p] not affected by these values of δe, $\delta\varpi$, &c., it follows, that the transverse axis of the ellipsis $2a$ is not affected by the variations δe, $\delta\varpi$, &c. now under consideration, as is observed in [3906'']. The same result holds good when we notice the variations of the
[3907q] motions of the body m', disturbed by m, as in [3907].

13. *We shall now consider the variations of the excentricities and of the perihelia.* We have given, in [1287—1309], the expressions of the increments of $\frac{de}{dt}$, $\frac{d\varpi}{dt}$, $\frac{de'}{dt}$, $\frac{d\varpi'}{dt}$,* depending on the two great inequalities of Jupiter and Saturn, and we have observed, in [1309″, &c.], that the variations of e, ϖ, e', ϖ', relative to the angle $5n't-2nt$,† [3908]

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* (2460) The expression de [1284], is integrated in [1286], and put under another form in [1288]. Now as this last expression is used in this article, we shall take its differential relatively to t, and then change the angles $n't$, nt into ζ', ζ, respectively, as in [1194‴]; for the purpose of noticing the inequalities of the mean motion. If we put $\mu=1$, $i'=5$, $i=2$, as in [3895a], we shall get from [1288] the following value of de; and in like manner, from [1297], we get $d\varpi$ [3908d]; [3908a] [3908b]

$$de=-m'.andt.\left\{\left(\frac{dP}{de}\right).\cos.(5\zeta'-2\zeta+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{de}\right).\sin.(5\zeta'-2\zeta+5\varepsilon'-2\varepsilon)\right\}; \quad [3908c]$$

$$d\varpi=-m'.andt.\left\{\frac{1}{e}.\left(\frac{dP}{de}\right).\sin.(5\zeta'-2\zeta+5\varepsilon'-2\varepsilon)+\frac{1}{e}.\left(\frac{dP'}{de}\right).\cos.(5\zeta'-2\zeta+5\varepsilon'-2\varepsilon)\right\}. \quad [3908d]$$

† (2461) If we put the values of ζ, ζ', under the forms $\zeta=nt+N$, $\zeta'=n't+N'$, we shall find, by comparison with [1304, 1305], and using the symbols [3890a, b], [3909a]

$$N=\frac{6m'.an^2}{(5n'-2n)^2}.\{P.\cos.T_5-P'.\sin.T_5\}; \quad [3909b]$$

$$N'=-\frac{6m'.an^2}{(5n'-2n)^2}.\frac{m\sqrt{a}}{m'\sqrt{a'}}.\{P.\cos.T_5-P'.\sin.T_5\}. \quad [3909b']$$

Substituting the values [3909a] in the first member of the following expression, we get

$$5\zeta'-2\zeta+5\varepsilon'-2\varepsilon=5n't-2nt+5\varepsilon'-2\varepsilon+(5N'-2N)=T_5+(5N'-2N), \quad [3909c]$$

and by neglecting the square and higher powers of $5N'-2N$, using also [60, 61] Int., we obtain

$$\sin.(5\zeta'-2\zeta+5\varepsilon'-2\varepsilon)=\sin.T_5+(5N'-2N).\cos.T_5;$$
$$\cos.(5\zeta'-2\zeta+5\varepsilon'-2\varepsilon)=\cos.T_5-(5N'-2N).\sin.T_5. \quad [3909d]$$

Substituting these in the value of de [3908c], or, as it may be called, $d\delta e$, we get

$$d\delta e= m'.andt.\left\{-\left(\frac{dP}{de}\right).\cos.T_5+\left(\frac{dP'}{de}\right).\sin.T_5\right\} \quad [3909e]$$
$$+m'.andt.(5N'-2N).\left\{\left(\frac{dP'}{de}\right).\cos.T_5+\left(\frac{dP}{de}\right).\sin.T_5\right\}.$$

may introduce in these expressions some variations similar to those produced

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The part of this expression, depending on the factor $5\mathcal{N}'-2\mathcal{N}$, is of the order m'^2; and as the other terms are of the order m', we must notice, in them, the variations of $\left(\frac{dP}{de}\right)$, $\left(\frac{dP'}{de}\right)$, arising from the variations of δe, $\delta\varpi$, &c. The additional

[3909f] terms of the values of $\left(\frac{dP}{de}\right)$, $\left(\frac{dP'}{de}\right)$, from this source, may be found by changing P, P' into $\left(\frac{dP}{de}\right)$, $\left(\frac{dP'}{de}\right)$, respectively, in [3893c, d]; and as the former quantity

[3909g] is multiplied by $-m'.a n d t.\cos.T_5$, in [3909e], and the latter by $m'.a n d t.\sin.T_5$, the complete expression of $d\delta e$ will be

$$d\delta e = m'.andt.\left\{-\left(\frac{dP}{de}\right).\cos.T_5+\left(\frac{dP'}{de}\right).\sin.T_5\right\}$$

$$+m'.andt.(5\mathcal{N}'-2\mathcal{N}).\left\{\left(\frac{dP'}{de}\right).\cos.T_5+\left(\frac{dP}{de}\right).\sin.T_5\right\}$$

[3909h]
$$-m'.andt.\cos.T_5.\left\{\begin{array}{l}+\left(\frac{ddP}{de^2}\right).\delta e+\left(\frac{ddP}{de\,d\varpi}\right).\delta\varpi+\left(\frac{ddP}{de\,de'}\right).\delta e' \\ +\left(\frac{ddP}{de\,d\varpi'}\right).\delta\varpi'+\left(\frac{ddP}{de\,d\gamma}\right).\delta\gamma+\left(\frac{ddP}{de\,d\Pi}\right).\delta\Pi\end{array}\right\}$$

$$+m'.andt.\sin.T_5.\left\{\begin{array}{l}+\left(\frac{ddP'}{de^2}\right).\delta e+\left(\frac{ddP'}{de\,d\varpi}\right).\delta\varpi+\left(\frac{ddP'}{de\,de'}\right).\delta e' \\ +\left(\frac{ddP'}{de\,d\varpi'}\right).\delta\varpi'+\left(\frac{ddP'}{de\,d\gamma}\right).\delta\gamma+\left(\frac{ddP'}{de\,d\Pi}\right).\delta\Pi\end{array}\right\}.$$

Now if we take the partial differentials of [3894—3894''], relatively to e, we get

[3909i]
$$\left(\frac{ddP}{de\,d\varpi}\right)=\left(\frac{dP'}{de}\right)+e.\left(\frac{ddP'}{de^2}\right); \qquad \left(\frac{ddP'}{de\,d\varpi}\right)=-\left(\frac{dP}{de}\right)-e.\left(\frac{ddP}{de^2}\right);$$

$$\left(\frac{ddP}{de\,d\varpi'}\right)=e'.\left(\frac{ddP'}{de\,de'}\right); \qquad \left(\frac{ddP'}{de\,d\varpi'}\right)=-e'.\left(\frac{ddP}{de\,de'}\right);$$

$$\left(\frac{ddP}{de\,d\Pi}\right)=\gamma.\left(\frac{ddP'}{de\,d\gamma}\right); \qquad \left(\frac{ddP'}{de\,d\Pi}\right)=-\gamma.\left(\frac{ddP}{de\,d\gamma}\right).$$

Substituting these in [3909h], and retaining only the terms of the order m'^2; or in other words, neglecting those terms of the first line of [3909h], which are independent

by the two great inequalities. If we apply this method to the elements of

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of the factor $5\mathcal{N}''-2\mathcal{N}$ and the second differentials ddP, ddP', we get

$$d\delta e = m'.andt.(5\mathcal{N}''-2\mathcal{N}).\left\{\left(\tfrac{dP'}{de}\right).\cos.T_5+\left(\tfrac{dP}{de}\right).\sin.T_5\right\}$$

$$-m'.andt.\cos.T_5.\left\{\begin{array}{l}\left(\tfrac{ddP}{de^2}\right).\delta e+\left(\tfrac{dP'}{de}\right).\delta\varpi+\left(\tfrac{ddP'}{de^2}\right).e\delta\varpi+\left(\tfrac{ddP}{dede'}\right).\delta e'\\ +\left(\tfrac{ddP'}{dede'}\right).e'\delta\varpi'+\left(\tfrac{ddP}{ded\gamma}\right).\delta\gamma+\left(\tfrac{ddP'}{ded\gamma}\right).\gamma\delta\Pi\end{array}\right\}$$

$$+m'.andt.\sin.T_5.\left\{\begin{array}{l}\left(\tfrac{ddP'}{de^2}\right).\delta e-\left(\tfrac{dP}{de}\right).\delta\varpi-\left(\tfrac{ddP}{de^2}\right).e\delta\varpi+\left(\tfrac{ddP'}{dede'}\right).\delta e'\\ -\left(\tfrac{ddP}{dede'}\right).e'\delta\varpi'+\left(\tfrac{ddP'}{ded\gamma}\right).\delta\gamma-\left(\tfrac{ddP}{ded\gamma}\right).\gamma\delta\Pi\end{array}\right\}. \quad [3909k]$$

We must substitute in this the values [3895—3896', 3906f], and then by integration, we shall obtain δe [3910], as will appear by the following calculations, using the abridged symbols

$$\mathcal{N}_1=\frac{3m'^2.a^2n^3}{(5n'-2n)^2}.\frac{(5m\surd a+2m'\surd a')}{m'\surd a'}, \qquad \mathcal{N}_2=\frac{m'^2.a^2n^2}{5n'-2n}, \qquad \mathcal{N}_3=\frac{mm'.aa'nn'}{5n'-2n}, \quad [3909l]$$

to denote the factors of the three different groups of terms which occur in [3910]. If we compare these expressions with those in [3907a], we shall obtain the following values of $m'.an$, which will be used hereafter; these equations are easily proved to be identical, by the substitution of [3907a, 3909l] and reducing. $m'.an=M_1\mathcal{N}_2=M_2\mathcal{N}_3=M_3.(\mathcal{N}_2+\mathcal{N}_3)$. [3909m]

First. We have, by means of [3909b, b'],

$$\begin{aligned}m'.andt.(5\mathcal{N}''-2\mathcal{N})&=-\frac{6m'^2.a^2n^3}{(5n'-2n)^2}.\frac{(5m\surd a+2m'\surd a')}{m'\surd a'}.\{P.\cos.T_5-P'.\sin.T_5\}.dt\\&=-2\mathcal{N}_1.\{P.\cos.T_5-P'.\sin.T_5\}.dt.\end{aligned} \quad [3909n]$$

Multiplying this by $\left(\frac{dP'}{de}\right).\cos.T_5+\left(\frac{dP}{de}\right).\sin.T_5$, we obtain the value of the first line of [3909k], as in the first member of the following expression, which, by means of [1, 6, 31] Int., is reduced to the form [3909o];

$$-2N_1.dt.\{P.\cos.T_5-P'.\sin.T_5\}.\left\{\left(\tfrac{dP'}{de}\right).\cos.T_5+\left(\tfrac{dP}{de}\right).\sin.T_5\right\}$$

$$=-2\mathcal{N}_1.dt.\left\{P.\left(\tfrac{dP'}{de}\right).\cos.^2T_5-P'.\left(\tfrac{dP}{de}\right).\sin.^2T_5-\left[P'.\left(\tfrac{dP'}{de}\right)-P.\left(\tfrac{dP}{de}\right)\right].\sin.T_5.\cos.T_5\right\}$$

$$=-\mathcal{N}_1.dt.\left\{\begin{array}{l}P.\left(\tfrac{dP'}{de}\right)-P'.\left(\tfrac{dP}{de}\right)+\left[P.\left(\tfrac{dP'}{de}\right)+P'.\left(\tfrac{dP}{de}\right)\right].\cos.2T_5\\ \qquad\qquad -\left[P'.\left(\tfrac{dP'}{de}\right)-P.\left(\tfrac{dP}{de}\right)\right].\sin.2T_5\end{array}\right\}. \quad [3909o]$$

Its integral gives the terms of δe [3910], depending on the factor $(5m\surd a+2m'\surd a')$.

[3909] the orbits of Jupiter and Saturn, and *put* δe, $\delta \varpi$, *for the variations arising*

Second. The term of [3909k], connected with the factor $\left(\frac{ddP}{de^2}\right).dt$, is as in the first member of [3909p]; which, by the successive substitutions of [3909m, 3907e], becomes as in [3909q], whose integral gives the corresponding term in the fourth line of δe [3910];

[3909p] $m'.an.\{-\delta e.\cos.T_5-e\delta\varpi.\sin.T_5\}=M_1N'_2.\{-\delta e.\cos.T_5-e\delta\varpi.\sin.T_5\}$

[3909q] $$=N'_2.\{-M_1.\delta e.\cos.T_5-M_1.e\delta\varpi.\sin.T_5\}=N'_2.\left(\frac{dP'}{de}\right).$$

Third. The term of [3909k], connected with $\left(\frac{ddP'}{de^2}\right).dt$, is as in [3909r], and by reduction, using [3909m, 3907e], it becomes as in [3909s]; whose integral gives the corresponding term of the fourth line of [3910];

[3909r] $m'.an.\{\delta e.\sin.T_5-e\delta\varpi.\cos.T_5\}=M_1N'_2.\{\delta e.\sin.T_5-e\delta\varpi.\cos.T_5\}$

[3909s] $$=N'_2.\{M_1.\delta e.\sin.T_5-M_1.e\delta\varpi.\cos.T_5\}=-N'_2.\left(\frac{dP}{de}\right).$$

Fourth. We may proceed in the same manner with the terms of [3909k] connected with the factors $\left(\frac{ddP}{de\,de'}\right).dt$, $\left(\frac{ddP'}{de\,de'}\right)dt$; which will be found to be represented,

[3909t] respectively, by the first members of [3909p, r], accenting the symbols e, δe, $\delta\varpi$. If we also put $m'.an=M_2N'_3$ [3909m], and reduce the formulas as in [3909q, s] by

[3909u] using the expressions [3907f], they will become, respectively, $N'_3.\left(\frac{dP'}{de'}\right)$, $-N'_3.\left(\frac{dP}{de'}\right)$. Multiplying these by the factors [3909t], and integrating relatively to t, they become as in the last line of the expression [3910].

[3909v] *Fifth.* In like manner, the terms of [3909k], connected with the factors $\left(\frac{ddP}{de\,d\gamma}\right).dt$, $\left(\frac{ddP'}{de\,d\gamma}\right).dt$, will be represented by the first members of [3909p, r], changing e, δe, $\delta\varpi$, into γ, $\delta\gamma$, $\delta\Pi$, respectively. Then substituting $m'.an=M_3.(N'_2+N'_3)$ [3909m], and reducing the formulas, as in [3909q, s], using [3907g], they become respectively,

[3909w] $(N'_2+N'_3).\left(\frac{dP'}{d\gamma}\right)$, $-(N'_2+N'_3).\left(\frac{dP}{d\gamma}\right)$. Multiplying these by the factors [3909v], and integrating relatively to t, we get the corresponding terms of δe [3910]; the terms depending on N'_2 being in the fourth line, and those on N'_3 in the last line of [3910].

Sixth. The two remaining terms of [3909k] are as in the first member of [3909x]; which is reduced to the form in the second member, by the substitution of $m'.an=M_1N'_2$ [3909m], and $M_1.\delta\varpi$ [3907b]. Reducing the products by means of [31, 32] Int.,

from the square of the disturbing force, we shall find

$$\delta e = -\frac{3m'^2.a^2n^3}{(5n'-2n)^2}\cdot\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m'\sqrt{a'}}\cdot\left\{\begin{array}{l} \left\{P.\left(\frac{dP'}{de}\right)-P'.\left(\frac{dP}{de}\right)\right\}.t \\ +\frac{\left\{P.\left(\frac{dP'}{de}\right)+P'.\left(\frac{dP}{de}\right)\right\}}{2.(5n'-2n)}.\sin.2.(5n't-2nt+5\varepsilon'-2\varepsilon) \\ +\frac{\left\{P'.\left(\frac{dP'}{de}\right)-P.\left(\frac{dP}{de}\right)\right\}}{2.(5n'-2n)}.\cos.2.(5n't-2nt+5\varepsilon'-2\varepsilon) \end{array}\right\}$$

Inequality of the excentricity of Jupiter.

$$+\frac{m'^2.a^2n^2}{5n'-2n}\cdot\left\{\begin{array}{l} \left\{\left(\frac{dP'}{de}\right).\left(\frac{ddP}{de^2}\right)-\left(\frac{dP}{de}\right).\left(\frac{ddP'}{de^2}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP}{de\,d\gamma}\right)-\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP'}{de\,d\gamma}\right)\right\}.t \\ +\frac{\left\{\left(\frac{dP}{de}\right)^2-\left(\frac{dP'}{de}\right)^2\right\}}{4.(5n'-2n).e}.\cos.2.(5n't-2nt+5\varepsilon'-2\varepsilon) \\ -\frac{\left(\frac{dP}{de}\right).\left(\frac{dP'}{de}\right)}{2.(5n'-2n).e}.\sin.2.(5n't-2nt+5\varepsilon'-2\varepsilon) \end{array}\right\} \qquad [3910]$$

$$+\frac{mm'.aa'.nn'.t}{5n'-2n}.\left\{\left(\frac{dP'}{de'}\right).\left(\frac{ddP}{de\,de'}\right)-\left(\frac{dP}{de'}\right).\left(\frac{ddP'}{de\,de'}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP}{de\,d\gamma}\right)-\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP'}{de\,d\gamma}\right)\right\};*$$

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it becomes as in [3909y]; then integrating relatively to t, it produces the terms depending on $\cos.2T$, $\sin.2T$, in the fifth or sixth lines of [3910];

$$m'.and\,t.\left\{-\left(\frac{dP}{de}\right).\sin.T_5-\left(\frac{dP'}{de}\right).\cos.T_5\right\}.\delta\varpi$$

$$= \frac{N_2}{e}.dt.\left\{-\left(\frac{dP}{de}\right).\sin.T_5-\left(\frac{dP'}{de}\right).\cos.T_5\right\}.\left\{\left(\frac{dP}{de}\right).\cos.T_5-\left(\frac{dP'}{de}\right).\sin.T_5\right\} \qquad [3909x]$$

$$=-\frac{N_2}{e}.dt.\left\{\left(\frac{dP}{de}\right)^2-\left(\frac{dP'}{de}\right)^2\right\}.\sin.T_5.\cos.T_5$$

$$-\frac{N_2}{e}.dt.\left(\frac{dP}{de}\right).\left(\frac{dP'}{de}\right).\{\cos.^2T_5-\sin.^2T_5\}$$

$$=-\frac{N_2}{2e}.dt.\left\{\left(\frac{dP}{de}\right)^2-\left(\frac{dP'}{de}\right)^2\right\}.\sin.2T_5-\frac{N_2}{e}.dt.\left\{\left(\frac{dP}{de}\right).\left(\frac{dP'}{de}\right).\cos.2T_5\right\}. \qquad [3909y]$$

* (2462) If we compare the expressions of de, $d\varpi$ [3908c, d], we shall find, that $d\varpi$ may be derived from de, by subtracting 90^d from the angle $5\zeta'-2\zeta+5\varepsilon'-2\varepsilon$, [3910a]
and connecting the factor $\frac{1}{e}$ with each of the quantities $\left(\frac{dP}{de}\right)$, $\left(\frac{dP'}{de}\right)$; by this means [3910b]
the angle T_5 is also changed into T_5-90^d, in all the terms of [3909e, h, k], in which

Inequality of the perigee of Jupiter.

[3911]

$$\delta\varpi = \frac{3m'^2.a^2n^3}{(5n'-2n)^2.e}.\frac{(5m\surd a+2m'\surd a')}{m'\surd a'}.\left\{\begin{array}{l} \left\{P.\left(\frac{dP}{de}\right)+P'.\left(\frac{dP'}{de}\right)\right\}.t \\ +\frac{\left\{P.\left(\frac{dP}{de}\right)-P'.\left(\frac{dP'}{de}\right)\right\}}{2.(5n'-2n)}.\sin.2.(5n't-2nt+5\varepsilon'-2\varepsilon) \\ +\frac{\left\{P'.\left(\frac{dP}{de}\right)+P.\left(\frac{dP'}{de}\right)\right\}}{2.(5n'-2n)}.\cos.2.(5n't-2nt+5\varepsilon'-2\varepsilon) \end{array}\right\}$$

$$+\frac{m'^2.a^2n^2}{(5n'-2n).e}.\left\{\begin{array}{l} \left\{\left(\frac{dP}{de}\right).\left(\frac{ddP}{de^2}\right)+\left(\frac{dP'}{de}\right).\left(\frac{ddP'}{de^2}\right)+\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP}{de\,d\gamma}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP'}{de\,d\gamma}\right)\right\}.t \\ +\frac{\left\{\left(\frac{dP}{de}\right)^2-\left(\frac{dP'}{de}\right)^2\right\}}{2.(5n'-2n).e}.\sin.2.(5n't-2nt+5\varepsilon'-2\varepsilon) \\ +\frac{\left(\frac{dP}{de}\right).\left(\frac{dP'}{de}\right)}{(5n'-2n).e}.\cos.2.(5n't-2nt+5\varepsilon'-2\varepsilon) \end{array}\right\}$$

$$+\frac{mm'.aa'nn'.t}{(5n'-2n).e}.\left\{\left(\frac{dP}{de'}\right).\left(\frac{ddP}{de\,de'}\right)+\left(\frac{dP'}{de'}\right).\left(\frac{ddP'}{de\,de'}\right)+\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP}{de\,d\gamma}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP'}{de\,d\gamma}\right)\right\}.$$

[3910c] T_5 *explicitly* occurs; observing that no change must be made in the factor $5N'-2N$. Hence it appears, that if we change in [3909h] the angle T_5 into T_5-90°, without altering $5N'-2N$, and then multiply the resulting expression by $\frac{1}{e}$, we shall obtain

[3910d] all the terms of $d\,\delta\varpi$, except those arising from the variation of the factor $\frac{1}{e}$, connected with the quantities $\left(\frac{dP}{de}\right)$, $\left(\frac{dP'}{de}\right)$ [3910b]. These last quantities depend upon the two following terms of $d\,\delta\varpi$, namely,

[3910e] $$m'.a\,n\,d\,t.\frac{1}{e}.\left\{-\left(\frac{dP}{de}\right).\sin.T_5-\left(\frac{dP'}{de}\right).\cos.T_5\right\},$$

corresponding to the two first terms of [3909e]; and as the variation of $\frac{1}{e}$ is

[3910f] $$-\frac{\delta e}{e^2}=\frac{1}{M_1.e^2}.\left\{\left(\frac{dP}{de}\right).\sin.T_5+\left(\frac{dP'}{de}\right).\cos.T_5\right\}\quad [3907b];$$

also $m'.a\,n=M_1\,N_2$ [3909m], this part of $d\,\delta\varpi$ will be represented by

[3910g] $$\frac{N_2}{e^2}.d\,t.\left\{-\left(\frac{dP}{de}\right).\sin.T_5-\left(\frac{dP'}{de}\right).\cos.T_5\right\}.\left\{\left(\frac{dP}{de}\right).\sin.T_5+\left(\frac{dP'}{de}\right).\cos.T_5\right\}$$
$$=\frac{N_2}{e^2}.d\,t.\left\{-\left(\frac{dP}{de}\right)^2.\sin.^2T_5-\left(\frac{dP'}{de}\right)^2.\cos.^2T_5-2.\left(\frac{dP}{de}\right).\left(\frac{dP'}{de}\right).\sin.T_5.\cos.T_5\right\}.$$

The parts of these expressions, proportional to the time t, give the secular variations of the excentricity and of the perihelion, depending on the square of the disturbing forces. *To obtain the periodical terms of v depending on this square*, we shall consider the term $2e.\sin.(nt+\varepsilon-\varpi)$ [3748], in the elliptical expression of the true longitude. If we put δe, $\delta\varpi$, for the variations of e, ϖ, depending upon the angle $5n't-2nt+5\varepsilon'-2\varepsilon$, [3912] [3912'] [3912'']

This is to be connected with the terms mentioned in [3910d], to obtain the complete value of $d\delta\varpi$; and then by integration, we shall get $\delta\varpi$ [3911], as will appear by the following investigation, taking the terms in the same order as in the preceding note [3909n—y]. [3910h]

In the first place, the terms depending on $5\mathcal{N}'-2\mathcal{N}$, are multiplied by the factor $\left(\frac{dP'}{de}\right).\cos.T_5+\left(\frac{dP}{de}\right).\sin.T_5$, in the expression of $d\delta e$ [3909h], which becomes $\frac{1}{e}.\left(\frac{dP'}{de}\right).\sin.T_5-\frac{1}{e}.\left(\frac{dP}{de}\right).\cos.T_5$, in $d\delta\varpi$ [3910d]. Now it is evident, by inspection, that this last expression may be derived from the first, by changing $\left(\frac{dP}{de}\right)$ into $\frac{1}{e}.\left(\frac{dP'}{de}\right)$, and $\left(\frac{dP'}{de}\right)$ into $-\frac{1}{e}.\left(\frac{dP}{de}\right)$, without varying the angle T_5, or the factor $5\mathcal{N}'-2\mathcal{N}$; therefore we may use the same process of derivation in obtaining the part of $d\delta\varpi$, depending on $5\mathcal{N}'-2\mathcal{N}$, from the similar part of $d\delta e$ [3909k]; or in other words, the part of $\delta\varpi$ [3911], connected with the factor $5m\sqrt{a}+2m'\sqrt{a'}$, from the similar part of δe [3910]. [3910i] [3910k] [3910l]

We shall now apply the principle of derivation mentioned in [3910d], to the terms [3909p—w], and we shall find, that the factor of $\frac{1}{e}.\left(\frac{ddP}{de^2}\right).dt$, in $d\delta\varpi$, deduced from [3909q], is $\mathcal{N}_2.\{-M_1.\delta e.\sin.T_5+M_1.e\delta\varpi.\cos.T_5\}=\mathcal{N}_2.\left(\frac{dP}{de}\right)$ [3907e], producing the term $\frac{\mathcal{N}_2}{e}.\left(\frac{dP}{de}\right).\left(\frac{ddP}{de^2}\right).dt$. in $d\delta\varpi$, whose integral is as in the first term of the fourth line of $\delta\varpi$ [3911]. The term [3909s], by similar reductions, gives $\frac{\mathcal{N}_2}{e}.\left(\frac{dP'}{de}\right).\left(\frac{ddP'}{de^2}\right).t$; the terms [3909$t$] give [3910$m$] [3910$m'$] [3910$n$]

$$\frac{\mathcal{N}_3}{e}.\left(\frac{dP}{de'}\right).\left(\frac{ddP}{de\,de'}\right).t; \qquad \frac{\mathcal{N}_3}{e}.\left(\frac{dP'}{de'}\right).\left(\frac{ddP'}{de\,de'}\right).t; \qquad [3910o]$$

the terms [3909w] give

$$(\mathcal{N}_2+\mathcal{N}_3).\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP}{de\,d\gamma}\right).t; \qquad (\mathcal{N}_2+\mathcal{N}_3).\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP'}{de\,d\gamma}\right).t; \qquad [3910o']$$

as in the fourth and seventh lines of $\delta\varpi$ [3911].

and upon the first power of the disturbing force,* also $\delta' e$, $\delta' \varpi$, for the
[3913] preceding variations of e, ϖ, depending upon the double of this angle; †
[3913'] moreover, if we denote by $\delta\varepsilon$ the sum of the two inequalities of m, the

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Lastly, the terms of $d\,\delta\varpi$, deduced from those of $d\,\delta e$, in the first member of [3909x], by the principle of derivation [3910d], are

[3910p] $$m'.a\,n\,d\,t.\left\{-\frac{1}{e}.\left(\frac{d\,P'}{d\,e}\right).\sin.T_5+\frac{1}{e}.\left(\frac{d\,P}{d\,e}\right).\cos.T_5\right\}.\delta\varpi\,;$$

which, by the substitution of $m'.a\,n=M_1\,N_2$ [3909m], and $\delta\varpi$ [3907b], becomes

[3910q] $$\frac{N_2}{e^2}.d\,t.\left\{\left(\frac{d\,P}{d\,e}\right).\cos.T_5-\left(\frac{d\,P'}{d\,e}\right).\sin.T_5\right\}.\left\{\left(\frac{d\,P}{d\,e}\right).\cos.T_5-\left(\frac{d\,P'}{d\,e}\right).\sin.T_5\right\}$$
$$=\frac{N_2}{e^2}.d\,t.\left\{\left(\frac{d\,P}{d\,e}\right)^2.\cos.^2T_5+\left(\frac{d\,P'}{d\,e}\right)^2.\sin.^2T_5-2.\left(\frac{d\,P}{d\,e}\right).\left(\frac{d\,P'}{d\,e}\right).\sin.T_5.\cos.T_5\right\}.$$

Adding these terms to those in [3910g], and putting $\cos.^2T_5-\sin.^2T_5=\cos.2T_5$, $2\sin.T_5.\cos.T_5=\sin.2T_5$, we get

[3910r] $$\frac{N_2}{e^2}.dt.\left\{\left(\frac{d\,P}{d\,e}\right)^2.\cos.2T_5-\left(\frac{d\,P'}{d\,e}\right)^2.\cos.2T_5-2.\left(\frac{d\,P}{d\,e}\right).\left(\frac{d\,P'}{d\,e}\right).\sin.2T_5\right\};$$

and by integration, it produces the terms of $\delta\varpi$, depending on $\sin.2T_5$, $\cos.2T_5$, in the fifth and sixth lines of [3911].

[3912a] * (2163) These values of δe, $\delta\varpi$, are given by the formulas [3907b].

† (2164) The formulas [3910—3912'] give, by using T_5 [3890b],

[3913a]
$$\delta' e=-\frac{3m'^2.a^2n^3}{2.(5n'-2n)^3}.\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m'\sqrt{a'}}.\left\{\begin{array}{l}\left[P.\left(\frac{d\,P'}{d\,e}\right)+P'.\left(\frac{d\,P}{d\,e}\right)\right].\sin.2T_5\\+\left[P'.\left(\frac{d\,P'}{d\,e}\right)-P.\left(\frac{d\,P}{d\,e}\right)\right].\cos.2T_5\end{array}\right\}$$
$$+\frac{m'^2.a^2n^2}{4.(5n'-2n)^2.e}.\left\{\left[\left(\frac{d\,P}{d\,e}\right)^2-\left(\frac{d\,P'}{d\,e}\right)^2\right].\cos.2T_5-2.\left(\frac{d\,P}{d\,e}\right).\left(\frac{d\,P'}{d\,e}\right).\sin.2T_5\right\}$$

[3913b]
$$\delta'\varpi=\frac{3m'^2.a^2n^3}{2.(5n'-2n)^3.e}.\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m'\sqrt{a'}}.\left\{\begin{array}{l}\left[P.\left(\frac{d\,P}{d\,e}\right)-P'.\left(\frac{d\,P'}{d\,e}\right)\right].\sin.2T_5\\+\left[P'.\left(\frac{d\,P}{d\,e}\right)+P.\left(\frac{d\,P'}{d\,e}\right)\right].\cos.2T_5\end{array}\right\}$$
$$+\frac{m'^2.a^2n^2}{2.(5n'-2n)^2.e^2}.\left\{\left[\left(\frac{d\,P}{d\,e}\right)^2-\left(\frac{d\,P'}{d\,e}\right)^2\right].\sin.2T_5+2.\left(\frac{d\,P}{d\,e}\right).\left(\frac{d\,P'}{d\,e}\right).\cos.2T_5\right\}.$$

one depending on the angle $5n't-2nt+5\varepsilon'-2\varepsilon$, the other upon the double of this angle,* the term $2e.\sin.(nt+\varepsilon-\varpi)$, will become [3913'']

$$(2e+2\delta e+2\delta' e).\sin.(nt+\varepsilon+\delta\varepsilon-\varpi-\delta\varpi-\delta'\varpi). \qquad [3914]$$

If we neglect the cube of the disturbing force, the preceding expression may be developed in the following form,† [3914']

$$\begin{aligned}&2e.\sin.(nt+\varepsilon+\delta\varepsilon-\varpi)\\&+2\delta e.\sin.(nt+\varepsilon-\varpi)-2e\delta\varpi.\cos.(nt+\varepsilon-\varpi)\\&+\{2\delta' e+2e\delta\varpi.\delta\varepsilon-e.(\delta\varpi)^2\}.\sin.(nt+\varepsilon-\varpi)\\&-\{2e\delta'\varpi+2\delta e.\delta\varpi-2\delta\varepsilon.\delta e\}.\cos.(nt+\varepsilon-\varpi).\end{aligned} \qquad [3915]$$

The term $2e.\sin.(nt+\varepsilon+\delta\varepsilon-\varpi)$ is that obtained by increasing the [3915']

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* (2465) The great inequalities [1197, 1213, &c.], are to be applied to the mean motion of the planet [1070'']. If we notice only the chief terms of $\delta\varepsilon$, having the divisor $(5n'-2n)^2$, they will become, by putting $i=5$ in [3817], and using T_5 [3890b]; [3914a]

$$\delta\varepsilon=\frac{6m'.an^2}{(5n'-2n)^2}.\{P.\cos.T_5-P'.\sin.T_5\}. \qquad [3914b]$$

We may remark, that the terms of v [3748], depending on e^2, e^3, &c., are here neglected by the author, on account of their smallness; they are, however, noticed by him in the fourth volume [9062, &c.]. [3914c]

† (2466) Putting $a=nt+\varepsilon+\delta\varepsilon-\varpi$, $b=\delta\varpi+\delta\varpi'$, in [22] Int., we get the second member of [3915b], which is successively reduced to the form [3915c], by using [43, 44] Int., neglecting terms of the order m'^3, and finally putting $\cos.a=\cos.(nt+\varepsilon-\varpi)-\delta\varepsilon.\sin.(nt+\varepsilon-\varpi)$ in the term multiplied by $\delta\varpi$; [3915a] [3915a']

$$\sin.(nt+\varepsilon+\delta\varepsilon-\varpi-\delta\varpi-\delta'\varpi)=\sin.a.\cos.(\delta\varpi+\delta'\varpi)-\cos.a.\sin.(\delta\varpi+\delta'\varpi) \qquad [3915b]$$

$$\begin{aligned}&=\{1-\tfrac{1}{2}.(\delta\varpi)^2\}.\sin.a-(\delta\varpi+\delta'\varpi).\cos.a\\&=\sin.a-\tfrac{1}{2}.(\delta\varpi)^2.\sin.(nt+\varepsilon-\varpi)-(\delta\varpi+\delta'\varpi).\cos.(nt+\varepsilon-\varpi)\\&\qquad+\delta\varpi.\delta\varepsilon.\sin.(nt+\varepsilon-\varpi).\end{aligned} \qquad [3915c]$$

Multiplying this by $2e+2\delta e+2\delta' e$, and neglecting terms of the order m'^3, it becomes as in [3915]; observing, that in the term multiplied by $2\delta e$, we may put

$$\sin.a=\sin.(nt+\varepsilon-\varpi)+\delta\varepsilon.\cos.(nt+\varepsilon-\varpi). \qquad [3915d]$$

[3915''] mean motion nt, by $\delta\varepsilon$, in the elliptical part, according to the directions in [1070'']. The two terms

[3916] $$2\,\delta e\,.\,\sin.\,(nt+\varepsilon-\varpi)-2\,e\,\delta\varpi\,.\,\cos.\,(nt+\varepsilon-\varpi),$$

[3916'] form the inequality depending on the angle $3nt-5n't+3\varepsilon-5\varepsilon'$, given by the formula [3718].* If we then substitute in the other terms, the

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* (2467) If we put $i=5$ in [3814, 3825], where only the terms having the divisor $5n'-2n$ are retained [3818', 3824], we get

[3916a] $$\frac{r\delta r}{a^2}=H.\cos.(5n't-3nt+5\varepsilon'-3\varepsilon+A);\qquad \delta v=2H.\sin.(5n't-3nt+5\varepsilon'-3\varepsilon+A);$$

and we may observe, that this value of δv is easily obtained from that of $r\delta r$, by means

[3916b] of the formula [3718]; retaining only its first term $\delta v=\frac{2d.(r\delta r)}{a^2.ndt}$, which contains the small divisor $5n'-2n$ [3814, &c.]. If we substitute, in this last expression of δv, the value of $r\delta r$ [3876d], neglecting the small terms depending on X, it becomes

[3916c] $$\delta v=2\,\delta e\,.\,\sin.\,(nt+\varepsilon-\varpi)-2\,e\,\delta\varpi\,.\cos.\,(nt+\varepsilon-\varpi).$$

Comparing these two values of δv [3916a, c], we find, that the two terms in the second line of [3915], depend on the angle $5n't-3nt+5\varepsilon'-3\varepsilon$, or $3nt-5n't+3\varepsilon-5\varepsilon'$, as in [3916']. The same result may be obtained by the substitution of the values

[3916d] of δe, $e\delta\varpi$ [3907b] in [3916], and using the symbols $T_5=5n't-2nt+5\varepsilon'-2\varepsilon$, $W=nt+\varepsilon-\varpi$; since it becomes, by successive reductions, as in [3916g]; being of the form mentioned in [3916'];

[3916e] $$2\delta e.\sin.W-2e\delta\varpi.\cos.W=-\frac{2}{M_1}.\left\{\left(\frac{dP'}{de}\right).\cos.T_5+\left(\frac{dP}{de}\right).\sin.T_5\right\}.\sin.W$$
$$-\frac{2}{M_1}.\left\{\left(\frac{dP}{de}\right).\cos.T_5-\left(\frac{dP'}{de}\right).\sin.T_5\right\}.\cos.W$$

[3916e'] $$=-\frac{2}{M_1}.\left(\frac{dP}{de}\right).\{\cos.T_5.\cos.W+\sin.T_5.\sin.W\}$$
$$+\frac{2}{M_1}.\left(\frac{dP'}{de}\right).\{\sin.T_5.\cos.W-\cos.T_5.\sin.W\}$$

[3916f] $$=-\frac{2}{M_1}.\left(\frac{dP}{de}\right).\cos.(T_5-W)+\frac{2}{M_1}.\left(\frac{dP'}{de}\right).\sin.(T_5-W)$$

[3916g] $$=-\frac{2}{M_1}.\left(\frac{dP}{de}\right).\cos.(5n't-3nt+5\varepsilon'-3\varepsilon+\varpi)$$
$$+\frac{2}{M_1}.\left(\frac{dP'}{de}\right).\sin.(5n't-3nt+5\varepsilon'-3\varepsilon+\varpi).$$

values of δe, $\delta\varpi$ [3907b], and for $\delta' e$, $\delta'\varpi$, their preceding values [3913a, b]; the sum will give, by neglecting terms depending on the sine and cosine of $nt+\varepsilon$, because they are comprised in the equation of the centre,* [3917]

$$-\frac{3m'^2.a^2n^3}{(5n'-2n)^3}.\frac{(5m\sqrt{a}+4m'\sqrt{a'})}{m'\sqrt{a'}}.\left\{P.\left(\frac{dP'}{de}\right)+P'.\left(\frac{dP}{de}\right)\right\}.\cos.(5nt-10n't+5\varepsilon-10\varepsilon'-\varpi) \qquad [3918]$$

$$-\frac{3m'^2.a^2n^3}{(5n'-2n)^3}.\frac{(5m\sqrt{a}+4m'\sqrt{a'})}{m'\sqrt{a'}}.\left\{P'.\left(\frac{dP'}{de}\right)-P.\left(\frac{dP}{de}\right)\right\}.\sin.(5nt-10n't+5\varepsilon-10\varepsilon'-\varpi). \qquad [3918]$$

If we put, in [3916g], $\left(\frac{dP}{de}\right)=-M_1H.\sin.(A-\varpi)$, $\left(\frac{dP'}{de}\right)=M_1H.\cos.(A-\varpi)$, [3916$h$]
and reduce the result by means of [21] Int., it becomes equal to

$$2H.\sin.(5n't-3nt+5\varepsilon'-3\varepsilon+A). \qquad [3916i]$$

This is of the same form as [3825], which represents the most important term of this form and order, having the small divisor $5n'-2n$ [3821]. The factor H is of the second dimension in e, e' [3814b], being of the same order as the quantities $\left(\frac{dP}{de}\right)$, $\left(\frac{dP'}{de}\right)$. [3916$i$]
For the values of P, P' [1287], which correspond to the angle T_5, are of the third dimension in e, e', &c. [957$^{\text{viii}}$, &c.], and their differential coefficients, which occur in [3916g], are of a lower order by one degree. [3916k]

* (2468) The first and second lines of the expression [3915] are accounted for in [3915'', 3916]; the remaining terms become, by using the abridged symbols W, T_5 [3916d],

$$\{2\delta'e+2e\delta\varpi.\delta\varepsilon-e.(\delta\varpi)^2\}.\sin.W+\{-2e.\delta'\varpi-2\delta e.\delta\varpi+2\delta\varepsilon.\delta e\}.\cos.W; \qquad [3917a]$$

in which we must substitute the values of δe, $\delta\varpi$ [3907b], $\delta' e$, $\delta'\varpi$ [3913a, b], $\delta\varepsilon$ [3914b]. In making these substitutions, the terms $\delta\varpi.\delta\varepsilon$, $(\delta\varpi)^2$, $\delta e.\delta\varpi$, $\delta\varepsilon.\delta e$, will produce factors of the forms $A.\cos.^2T_5$, $A'.\sin.^2T_5$, $A''.\sin.T_5.\cos.T_5$, [3917b]
or $\frac{1}{2}A+\frac{1}{2}A.\cos.2T_5$, $\frac{1}{2}A'-\frac{1}{2}A'.\cos.2T_5$, $\frac{1}{2}A.\sin.2T_5$ [1, 6, 31] Int. Substituting these in [3917a], we find that the parts $\frac{1}{2}A$, $\frac{1}{2}A'$, independent of $2T_5$, produce terms depending on $\sin.W$, $\cos.W$, of the form $a.\sin.W+b.\sin.W$; [3917c]
which, by putting $a=k.\sin.\beta$, $b=k.\cos.\beta$, and reducing by [21] Int., becomes $k.\sin.(W+\beta)=k.\sin.(nt+\varepsilon-\varpi+\beta)$. This may be connected with the equation of the centre [3915'], as is observed in [3917]; therefore these terms may be neglected, and we may substitute in [3917a] the following values, [3917c]

$$\cos.^2T_5=\tfrac{1}{2}\cos.2T_5;\quad \sin.^2T_5=-\tfrac{1}{2}\cos.2T_5;\quad \sin.T_5.\cos.T_5=\tfrac{1}{2}\sin.2T_5. \qquad [3917d]$$

Substituting these in the square of $\delta\varpi$, multiplied by $-e$, deduced from [3907b, a], we get

$$-e.(\delta\varpi)^2=-\frac{m'^2.a^2n^2}{2.(5n'-2n)^2.e}.\left\{\left[\left(\frac{dP}{de}\right)^2-\left(\frac{dP'}{de}\right)^2\right].\cos.2T_5-2.\left(\frac{dP}{de}\right).\left(\frac{dP'}{de}\right).\sin.2T_5\right\}. \qquad [3917e]$$

This inequality may be put under the form [3921]; for if we represent by

[3919] $$\delta v = K \cdot \sin. (5n't - 3nt + 5\varepsilon' - 3\varepsilon + B),$$

This term is destroyed by the corresponding terms of $2\delta' e$, deduced from the third line of [3913a], so that the sum becomes

[3917f] $$2\delta' e - e \cdot (\delta\varpi)^2 = -\frac{3m'^2 \cdot a^2 n^3}{(5n'-2n)^3} \cdot \frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m'\sqrt{a'}} \cdot \left\{ \begin{array}{c} \left[P \cdot \left(\frac{dP'}{de}\right) + P' \cdot \left(\frac{dP}{de}\right)\right] \cdot \sin. 2T_5 \\ + \left[P' \cdot \left(\frac{dP'}{de}\right) - P \cdot \left(\frac{dP}{de}\right)\right] \cdot \cos. 2T_5 \end{array} \right\}.$$

Multiplying the value of $e\,\delta\varpi$ [3907b, a], by $\delta\varepsilon$ [3914a], and reducing the product by means of the expressions [3917d], we get, by putting the factor 6, in this last expression, under the form $3 \cdot \frac{2m'\sqrt{a'}}{m'\sqrt{a'}}$,

[3917g] $$2e\,\delta\varpi \cdot \delta\varepsilon = -\frac{3m'^2 \cdot a^2 n^3}{(5n'-2n)^3} \cdot \frac{2m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ \begin{array}{c} \left[P \cdot \left(\frac{dP'}{de}\right) + P' \cdot \left(\frac{dP}{de}\right)\right] \cdot \sin. 2T_5 \\ + \left[P' \cdot \left(\frac{dP'}{de}\right) - P \cdot \left(\frac{dP}{de}\right)\right] \cdot \cos. 2T_5 \end{array} \right\}.$$

[3917h] Adding this to [3917f], and putting, for brevity, $M_4 = -\frac{3m'^2 \cdot a^2 n^3}{(5n'-2n)^3} \cdot \frac{(5m\sqrt{a}+4m'\sqrt{a'})}{m'\sqrt{a'}}$, we get

[3917i] $$2\delta' e + 2e\,\delta\varpi \cdot \delta\varepsilon - e \cdot (\delta\varpi)^2 = \left\{ \begin{array}{c} M_4 \cdot \left[P \cdot \left(\frac{dP'}{de}\right) + P' \cdot \left(\frac{dP}{de}\right)\right] \cdot \sin. 2T_5 \\ + M_4 \cdot \left[P' \cdot \left(\frac{dP'}{de}\right) - P \cdot \left(\frac{dP}{de}\right)\right] \cdot \cos. 2T_5 \end{array} \right\}.$$

Again, multiplying together the two equations [3907b], and dividing by $\frac{1}{2} M_1^2 \cdot e$ [3907a], we get, by substituting the values [3917d],

[3917k] $$-2\delta e \cdot \delta\varpi = \frac{m'^2 \cdot a^2 n^2}{(5n'-2n)^2 \cdot e} \cdot \left\{ \begin{array}{c} \left[\left(\frac{dP}{de}\right)^2 - \left(\frac{dP'}{de}\right)^2\right] \cdot \sin. 2T_5 \\ + 2 \cdot \left[\left(\frac{dP}{de}\right) \cdot \left(\frac{dP'}{de}\right)\right] \cdot \cos. 2T_5 \end{array} \right\}.$$

Adding this to the expression $\delta'\varpi$ [3913b], multiplied by $-2e$, it is destroyed by the term depending on the third line of [3913b], and the sum becomes

[3917l] $$-2e\,\delta'\varpi - 2\delta e \cdot \delta\varpi = -\frac{3m'^2 \cdot a^2 n^3}{(5n'-2n)^3} \cdot \frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m'\sqrt{a'}} \cdot \left\{ \begin{array}{c} + \left[P \cdot \left(\frac{dP'}{de}\right) + P' \cdot \left(\frac{dP}{de}\right)\right] \cdot \cos. 2T_5 \\ - \left[P' \cdot \left(\frac{dP'}{de}\right) - P \cdot \left(\frac{dP}{de}\right)\right] \cdot \sin. 2T_5 \end{array} \right\}.$$

the inequality of m, depending on $3nt-5n't+3\varepsilon-5\varepsilon'$;* and as in [3889],

the great inequality $\zeta=\overline{H}.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon+\bar{A})$, [3920]

Multiplying $-M_1.\delta e$ [3907b] by $-\frac{2}{M_1}$, also by $\delta\varepsilon$ [3914b], and then reducing by means of [3907a, 3917d], we get

$$2\delta\varepsilon.\delta e=-\frac{3m'^2.a^2n^3}{(5n'-2n)^3}.\frac{2m'\sqrt{a'}}{m'\sqrt{a'}}.\left\{\begin{array}{l}+\left[P.\left(\frac{dP'}{de}\right)+P'.\left(\frac{dP}{de}\right)\right].\cos.2T_5\\ -\left[P'.\left(\frac{dP'}{de}\right)-P.\left(\frac{dP}{de}\right)\right].\sin.2T_5\end{array}\right\}. \qquad [3917m]$$

The sum of [3917l, m], using M_4 [3917h], is

$$\{-2e\delta'\varpi-2\delta e.\delta\varpi+2\delta\varepsilon.\delta e\}=\left\{\begin{array}{l}+M_4.\left[P.\left(\frac{dP'}{de}\right)+P'.\left(\frac{dP}{de}\right)\right].\cos.2T_5\\ -M_4.\left[P'\left(\frac{dP'}{de}\right)-P.\left(\frac{dP}{de}\right)\right].\sin.2T_5\end{array}\right\}. \qquad [3917n]$$

Multiplying [3917i] by $\sin.W$, and [3917n] by $\cos.W$, then adding the products, we find that the first member is equal to the expression [3917a]; and the second member, by the substitution of $\sin.2T_5.\sin.W+\cos.2T_5.\cos.W=\cos.(W-2T_5)$, $\cos.2T_5.\sin.W-\sin.2T_5.\cos.W=\sin.(W-2T_5)$, becomes [3917o]

$$M_4.\left\{P.\left(\frac{dP'}{de}\right)+P'.\left(\frac{dP}{de}\right)\right\}.\cos.(W-2T_5)+M_4.\left\{P'.\left(\frac{dP'}{de}\right)-P.\left(\frac{dP}{de}\right)\right\}.\sin.(W-2T_5); \qquad [3917p]$$

and by resubstituting the values of M_4, T_5, W [3917h, 3916d], it becomes as in [3918, 3918'].

* (2469) The expression [3919] is of the same form as that assumed in [3826], or that computed in [3916g], assuming $i=5$; moreover [3920] is the same as [3889]. [3920a]
Hence if we put, for brevity, $T_5=5n't-2nt+5\varepsilon'-2\varepsilon$, $W_2=nt+\varepsilon$, and [3920a']
then make the two expressions [3919, 3916g] equal to each other; also [3920, 3909b, a], using M [3907a]; we shall obtain the two following equations;

$$K.\sin.(T_5-W_2+B)=\frac{2m'.an}{(5n'-2n)}.\left\{-\left(\frac{dP}{de}\right).\cos.(T_5-W_2+\varpi)+\left(\frac{dP'}{de}\right).\sin.(T_5-W_2+\varpi)\right\}; \qquad [3920b]$$

$$\overline{H}.\sin.(T_5+\bar{A})=\frac{6m'.an^2}{(5n'-2n)^2}.\{P.\cos.T_5-P'.\sin.T_5\}. \qquad [3920c]$$

the preceding inequality will be, by §69, of the second book,*

[3921] $$\delta v = \tfrac{1}{4}\cdot\frac{(5m\sqrt{a}+4m'\sqrt{a'})}{m'\sqrt{a'}}\cdot \overline{H} K \cdot \sin.(5nt-10n't+5\varepsilon-10\varepsilon'-B-\overline{A}).$$

In like manner, we shall find, by noticing only the secular variations,†

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* (2470) Multiplying together the equations [3920b, c], and reducing the products by [17—20] Int., we find that the first member becomes equal to

[3921a] $$\tfrac{1}{2}\,\overline{H}K\cdot\cos.(W_2+\overline{A}-B)-\tfrac{1}{2}\,\overline{H}K\cdot\cos.(W_2-2\,T_5-B-\overline{A});$$

and the product, in the second member, depends on similar angles W_2, $W_2-2\,T_5$. Now as these expressions must be equal to each other, whatever be the value of t, we may put the terms depending on the angle $W_2-2\,T_5$ in both members, separately equal to each other, and we shall get

[3921b] $$-\tfrac{1}{2}\overline{H}K\cdot\cos.(W_2-2T_5-B-\overline{A})=-\frac{6m'^2.a^2n^3}{(5n'-2n)^3}\cdot\left\{\begin{array}{l}\left[P.\left(\frac{dP'}{de}\right)+P'.\left(\frac{dP}{de}\right)\right].\sin.(W_2-2T_5-\varpi)\\ -\left[P'.\left(\frac{dP'}{de}\right)-P.\left(\frac{dP}{de}\right)\right].\cos.(W_2-2T_5-\varpi)\end{array}\right\}.$$

This equation being identical, we may change $W_2-2\,T_5$, into $W_2-2\,T_5+90^d$; by which means the expressions $\cos.(W_2-2\,T_5-B-\overline{A})$, $\sin.(W_2-2\,T_5-\varpi)$,
[3921c] $\cos.(W_2-2\,T_5-\varpi)$, become, respectively, $-\sin.(W_2-2\,T_5-B-\overline{A})$, $\cos.(W_2-2\,T_5-\varpi)$, $-\sin.(W_2-2\,T_5-\varpi)$; substituting these in [3921b], and multiplying the result by $\frac{5m\sqrt{a}+4m'\sqrt{a'}}{2m'\sqrt{a'}}$, the first member of the product becomes as in the second member of [3921]; and the second member of this product includes the terms
[3921d] [3918, 3918′]; observing, that $W_2-2\,T_5=5nt-10n't+5\varepsilon-10\varepsilon'$ [3920a']; therefore the inequality [3921] is equal to the sum of the two expressions [3918, 3918′].

† (2471) Using the abridged symbols P_0, P_0', T_0, &c. [3816b—d]; also
[3922a] $Z=5\varpi'-2\varpi+5\varepsilon'-2\varepsilon$, $Z_1=5\varpi-2\varpi'+5\varepsilon-2\varepsilon'$; we find, that the expression of de [3908c] may be rendered symmetrical by the introduction of the two terms depending on the angle Z_0, or T_6, in the value of R [3846c]; so that we may put

[3922b] $$de=-m'.andt.\left\{\left(\frac{dP}{de}\right).\cos.Z-\left(\frac{dP'}{de}\right).\sin.Z+\left(\frac{dP_0}{de}\right).\cos.Z_0-\left(\frac{dP'_0}{de}\right).\sin.Z_0\right\}.$$

In computing δe from this expression, it is not necessary to notice the angle Z_0, because
[3922c] it does not produce terms which are so essentially increased by the small divisor $5n'-2n$, as has been already observed in [3846d''']. From this expression of de, we may derive

depending on the square of the disturbing force,

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$$\delta e' = -\frac{3m^2.a^3n^3.t}{(5n'-2n)^2.a'}\cdot\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m\sqrt{a}}\cdot\left\{P.\left(\frac{dP'}{de'}\right)-P'.\left(\frac{dP}{de'}\right)\right\}$$
$$+\frac{m^2.a'^2n'^2.t}{5n'-2n}\cdot\left\{\left(\frac{dP'}{de'}\right).\left(\frac{ddP}{de'^2}\right)-\left(\frac{dP}{de'}\right).\left(\frac{ddP'}{de'^2}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP}{de'd\gamma}\right)-\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP'}{de'd\gamma}\right)\right\} \quad [3922]$$
$$+\frac{mm'.aa'.nn'.t}{5n'-2n}\cdot\left\{\left(\frac{dP'}{de}\right).\left(\frac{ddP}{dede'}\right)-\left(\frac{dP}{de}\right).\left(\frac{ddP'}{dede'}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP}{de'd\gamma}\right)-\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP'}{de'd\gamma}\right)\right\}.$$

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that of de', by changing the elements of the body m into those of m', and the contrary; by which means P changes into P_0 [3846d, &c.], P' into P'_0, Z into Z_0, a into a', [3922d]
e into e', &c.; hence we have

$$de' = -m.a'n'dt.\left\{\left(\frac{dP_0}{de'}\right).\cos.Z_0-\left(\frac{dP'_0}{de'}\right).\sin.Z_0+\left(\frac{dP}{de'}\right).\cos.Z-\left(\frac{dP'}{de'}\right).\sin.Z\right\}. \quad [3922e]$$

Neglecting the terms of this expression depending on the angle Z_0, because they do not produce by integration the small divisor $5n'-2n$; then substituting the values of $\sin.Z$, $\cos.Z$ [3909d, 3922a], we get the following value of de', or as it may be written $d\delta e'$, being similar to [3909e], [3922e']

$$d\delta e' = \quad m.a'n'dt.\left\{-\left(\frac{dP}{de'}\right).\cos.T_5+\left(\frac{dP'}{de'}\right).\sin.T_5\right\}$$
$$+m.a'n'dt.(5\mathcal{N}'-2\mathcal{N}).\left\{\left(\frac{dP'}{de'}\right).\cos.T_5+\left(\frac{dP}{de'}\right).\sin.T_5\right\}. \quad [3922f]$$

The part of this expression depending on $5\mathcal{N}'-2\mathcal{N}$, is easily deduced from that in the first line of [3909k], or from its development in [3909o]; by multiplying it by $\frac{m.a'n'}{m'.an}$, [3922g]
and changing the partial differentials of P, P', relative to e, into those relative to e'. Hence we obtain the following expression of the part of $d\delta e'$, depending on the factor $(5\mathcal{N}'-2\mathcal{N})$ [3922f],

$$-\mathcal{N}_1.\frac{m.a'n'}{m'.an}.dt.\left\{\begin{array}{r} P.\left(\frac{dP'}{de'}\right)-P'.\left(\frac{dP}{de'}\right)+\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\cos.2T_5 \\ -\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\sin.2T_5 \end{array}\right\}. \quad [3922h]$$

Now by successive reductions, using $an=a^{-\frac{1}{2}}$ [3709'], $a'n'=a'^{-\frac{1}{2}}$ we get

$$\frac{m.a'n'}{m'.an}=\frac{m.a^{\frac{1}{2}}}{m'.a'^{\frac{1}{2}}}=\frac{m.a}{m'.a'^{\frac{1}{2}}a^{\frac{1}{2}}}; \quad [3922h']$$

hence from [3909l], we obtain

$$-\mathcal{N}_1.\frac{m.a'n'}{m'.an}=-\frac{3m'^2.a^2n^3}{(5n'-2n)^2}.\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m'.a'^{\frac{1}{2}}}.\frac{m.a}{m'.a'^{\frac{1}{2}}a^{\frac{1}{2}}}=-\frac{3m^2.a^3n^3}{(5n'-2n)^2.a'}.\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m\sqrt{a}}. \quad [3922i]$$

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[3923]
$$\delta\varpi' = \frac{3m^2.a^3n^3.t}{(5n'-2n)^2.a'e'}.\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m\sqrt{a}}.\left\{P.\left(\frac{dP}{de'}\right)+P'.\left(\frac{dP'}{de'}\right)\right\}$$
$$+\frac{m^2.a'^2n^2.t}{(5n'-2n).e'}.\left\{\left(\frac{dP}{de'}\right).\left(\frac{ddP}{de'^2}\right)+\left(\frac{dP'}{de'}\right).\left(\frac{ddP'}{de'^2}\right)+\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP}{de'd\gamma}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP'}{de'd\gamma}\right)\right\}$$
$$+\frac{mm'.aa'.nn'.t}{(5n'-2n).e'}.\left\{\left(\frac{dP}{de}\right).\left(\frac{ddP}{dede'}\right)+\left(\frac{dP'}{de}\right).\left(\frac{ddP'}{dede'}\right)+\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP}{de'd\gamma}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP'}{de'd\gamma}\right)\right\}.*$$

Substituting this in [3922h], and integrating, we find, that the terms multiplied by t, become as in the first line of [3922]; the other part depending on $2T_5$, produce in $\delta e'$ the terms

[3922k]
$$-\frac{3m^2.a^3n^3}{2.(5n'-2n)^3.a'}.\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m\sqrt{a}}.\left\{\begin{array}{l}\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\sin.2T_5 \\ +\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\cos.2T_5\end{array}\right\}.$$

If we compare the terms of $d\delta e$, which are independent of $(5N'-2N)$ [3909e], with those of $d\delta e'$ [3922f], we find, that the latter may be derived from the former
[3922l] *by changing the elements* m, a, n, e, ϖ, &c. *into* m', a', n', e', ϖ', &c., *respectively, without altering* P, P', T_5; and as the divisor $5n'-2n$ is introduced merely by
[3922m] the integration of terms depending on the sine or cosine of the angle T_5 and its multiples, *this divisor will also be unchanged.* Now making these changes in the secular terms, in the fourth and seventh lines of δe [3910], we obtain the similar terms in the second and third lines of $\delta e'$ [3922]; moreover the periodical terms, depending on $2T_5$, in the fifth and sixth lines of δe [3910], produce the following terms of $\delta e'$,

[3922n]
$$\frac{m^2.a'^2n^2}{4.(5n'-2n)^2.e'}.\left\{\left[\left(\frac{dP}{de'}\right)^2-\left(\frac{dP'}{de'}\right)^2\right].\cos.2T_5-2.\left(\frac{dP}{de'}\right).\left(\frac{dP'}{de'}\right).\sin.2T_5\right\}.$$

[3922o] The *sum of the expressions* [3922k, n] *may be represented by* $\delta' e'$, to conform to the notation in [3913], the characteristic δ' being used to include the terms depending on
[3922p] the angle $2T_5$. These terms are used in [3924e].

* (2472) In the same manner as we have deduced the expressions [3922b, e, f] from [3908c], we may obtain the following expressions of $d\varpi$, $d\varpi'$, $d\delta\varpi'$ from [3908d];

[3923a]
$$d\varpi = -m'.an\,dt.\left\{\frac{1}{e}.\left(\frac{dP}{de}\right).\sin.Z+\frac{1}{e}.\left(\frac{dP'}{de}\right).\cos.Z+\frac{1}{e}.\left(\frac{dP_0}{de}\right).\sin.Z_0+\frac{1}{e}.\left(\frac{dP'_0}{de}\right).\cos.Z_0\right\};$$

[3923b]
$$d\varpi' = -m.a'n'dt.\left\{\frac{1}{e'}.\left(\frac{dP_0}{de'}\right).\sin.Z_0+\frac{1}{e'}.\left(\frac{dP'_0}{de'}\right).\cos.Z_0+\frac{1}{e'}.\left(\frac{dP}{de'}\right).\sin.Z+\frac{1}{e'}.\left(\frac{dP'}{de'}\right).\cos.Z\right\};$$

[3923c]
$$d\delta\varpi' = m.a'n'dt.\left\{-\frac{1}{e'}.\left(\frac{dP}{de'}\right).\sin.T_5-\frac{1}{e'}.\left(\frac{dP'}{de'}\right).\cos.T_5\right\}$$
$$+m.a'n'dt.(5N'-2N).\left\{-\frac{1}{e'}.\left(\frac{dP}{de'}\right).\cos.T_5+\frac{1}{e'}.\left(\frac{dP'}{de'}\right).\sin.T_5\right\}.$$

We also find, that the motion of m' in longitude, is affected with the inequality*

$$-\frac{3m^2.a^3n^3}{(5n'-2n)^3.a'}.\frac{(3m\sqrt{a}+2m'\sqrt{a'})}{m\sqrt{a}}.\left\{\begin{array}{l}\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\cos.(4nt-9n't+4\varepsilon-9\varepsilon'-\varpi')\\+\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\sin.(4nt-9n't+4\varepsilon-9\varepsilon'-\varpi')\end{array}\right\}. \quad [3924]$$

This last expression being developed, as in [3922g, &c.], and integrated, gives this part of $\delta\varpi'$. It may also be derived from $d\delta e'$ [3922f], in the following manner. We perceive, by inspection, that the part of [3923c], depending on the factor $5N'-2N$, can be derived from the corresponding terms of $d\delta e'$ [3922f], by changing $\left(\frac{dP}{de'}\right)$ into $\frac{1}{e'}.\left(\frac{dP'}{de'}\right)$, and $\left(\frac{dP'}{de'}\right)$ into $-\frac{1}{e'}.\left(\frac{dP}{de'}\right)$. If we make the same changes in the first line of $\delta e'$ [3922], which was derived from the factor $5N'-2N$, [3922i, &c.], we get the first line of the expression of $\delta\varpi'$ [3923]; and the periodical terms of $e'\delta\varpi'$, corresponding to [3922k], become equal to the following function, which is used in [3924n]; [3923d] [3923e]

$$\frac{3m^2.a^3n^3}{2.(5n'-2n)^3.a'}.\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m\sqrt{a}}.\left\{\begin{array}{l}\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\cos.2T_5\\-\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\sin.2T_5\end{array}\right\}. \quad [3923f]$$

The part of $d\delta\varpi'$ [3923c], which is independent of $5N'-2N$, may be derived from the corresponding part of $d\delta\varpi$ [3908d, 3910a—e, or 3911], by the principle of derivation mentioned in [3922l, &c.]; that is, *by changing* m, a, n, e, ϖ, &c. *into* m', a', n', e', ϖ', &c., *respectively, without altering* P, P', T_5, *or the divisor* $5n'-2n$. In this way, we find that the fourth and seventh lines of [3911] give the second and third lines of [3923]; and the periodical terms, corresponding to the fifth and sixth lines of [3911], produce in $e'\delta\varpi'$ the following terms, [3923g]

$$\frac{m^2.a'^2n'^2}{2.(5n'-2n)^2.e'}.\left\{\left[\left(\frac{dP}{de'}\right)^2-\left(\frac{dP'}{de'}\right)^2\right].\sin.2T_5+2.\left(\frac{dP}{de'}\right).\left(\frac{dP'}{de'}\right).\cos.2T_5\right\}. \quad [3923h]$$

The *sum of the expressions* [3923f, h] *depending on the angle* $2T_5$, represents the value of $e'\delta'\varpi'$, [3913]; which is used in the next note. [3923i]

* (2473) The expression [3924] represents, for the planet m', the terms similar to those in [3918, 3918'], which correspond to the planet m, and are derived from the function [3917a]. The similar function, relative to the planet m', using the symbols $T_5=5n't-2nt+5\varepsilon'-2\varepsilon$, $W'=n't+\varepsilon'-\varpi'$, is [3924a]

$$\{2\delta' e'+2e'\delta\varpi'.\delta\varepsilon'-e'.(\delta\varpi')^2\}.\sin.W'-\{2e'\delta'\varpi'+2\delta e'.\delta\varpi'-2\delta\varepsilon'.\delta e'\}.\cos.W'. \quad [3924b]$$

If we denote the inequality of m', depending on the angle $2nt-4n't+2\varepsilon-4\varepsilon'$, by

[3925] $$\delta v' = K'.\sin.(4n't-2nt+4\varepsilon'-2\varepsilon+B'),$$

By the inspection of [3907*b, c, a*], we perceive, that δe, $\delta\varpi$, become equal to $\delta e'$, $\delta\varpi'$, respectively, by *changing the elements* m, a, e, &c. *into* m', a', e', &c., *without altering* P, P', T_5, *or the divisor* $5n'-2n$; upon the principles of derivation used in [3923*g*]. By this method of derivation, we may obtain $-e'.(\delta\varpi')^2$ from [3917*e*], [3924*c*] and we find, that it is equal to, and of an opposite sign to the part of $2\delta' e'$ [3922*n*]; so that these terms destroy each other, in the value of $2\delta' e'-e'.(\delta\varpi')^2$; and then the other part of $2\delta' e'$ [3922*k*], spoken of in [3922*o*], produces the following expression;

[3924*d*] $$2\delta'e'-e'.(\delta\varpi')^2=-\frac{3m^2.a^3n^3}{(5n'-2n)^3.a'}.\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m\sqrt{a}}.\left\{\begin{array}{l}\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\sin.2T_5\\+\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\cos.2T_5\end{array}\right\}.$$

Now if we represent, as in [3913'], by $\delta\varepsilon'$, the part of $\delta v'$ [3846, &c.], depending on the angles T_5, $2T_5$, and notice, as in [3914*a*, &c.], only the chief terms of $\delta\varepsilon'$ depending on T_5, we shall get the following value, which is similar to [3914*b*],

[3924*e*] $$\delta\varepsilon'=\frac{15m.a'n'^2}{(5n'-2n)^2}.\{-P.\cos.T_5+P'.\sin.T_5\}.$$

Multiplying this by $2e'\delta\varpi'$ [3907*c, a*], and substituting the values [3917*d*], we get

[3924*f*] $$2e'\delta\varpi'.\delta\varepsilon'=\frac{15m^2.a'^2n'^3}{(5n'-2n)^3}.\left\{\begin{array}{l}\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\sin.2T_5\\+\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\cos.2T_5\end{array}\right\}.$$

[3924*g*] We have very nearly $5n'=2n$ [3818*d*], and $n'^2a'^3=n^2a^3$ [3709']; multiplying these two equations together, and the product by $3m^2$, we get $15m^2.a'^3n'^3=6m^2.a^3n^3$; substituting this in the first factor of the second member of [3924*f*], it becomes

[3924*h*] $$\frac{15m^2.a'^2n'^3}{(5n'-2n)^3}=\frac{3m^2.a^3n^3}{(5n'-2n)^3.a'}.\frac{2m\sqrt{a}}{m\sqrt{a}};$$

and then the sum of [3924*d, f*] becomes, by writing, for brevity,

[3924*i*] $$M_5=-\frac{3m^2.a^3n^3}{(5n'-2n)^3.a'}.\frac{(3m\sqrt{a}+2m'\sqrt{a'})}{m\sqrt{a}};$$

[3924*k*] $$2\delta'e'+2e'\delta\varpi'.\delta\varepsilon'-e'.(\delta\varpi')^2=\left\{\begin{array}{l}M_5.\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\sin.2T_5\\+M_5.\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\cos.2T_5\end{array}\right\}.$$

and the great inequality of m' [3891] by*

$$\zeta' = -\overline{H}'.\sin.(5n't - 2nt + 5\varepsilon' - 2\varepsilon + \overline{A}'),$$ [3926]

Again, if we multiply together the two equations [3907c], and divide the product by $\frac{1}{2}M^2_2 . e'$ [3907a], we shall get an expression of $-2\delta e'.\delta\varpi'$, similar to [3917k], m', a, n, e, being changed into m, a', n', e', respectively, without altering the divisor $5n' - 2n$. Adding this to the part of $-2e'\delta'\varpi'$, deduced from [3923h], we find that the sum becomes nothing; and the term of $e'\delta'\varpi'$ [3923f] produces the following expression, [3924l]

$$-2e'\delta'\varpi' - 2\delta e'.\delta\varpi' = -\frac{3m^2.a^3n^3}{(5n'-2n)^3.a'}.\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m\sqrt{a}}.\left\{\begin{array}{l}\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\cos.2T_5 \\ -\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\sin.2T_5\end{array}\right\}.$$ [3924m]

Multiplying $-M_2.\delta e'$ [3907c] by $-\frac{2}{M_2}$, and by $\delta\varepsilon'$ [3924e], and reducing, using [3907a, 3917d], we get

$$2\delta\varepsilon'.\delta e' = \frac{15m^2.a'^2n'^3}{(5n'-2n)^3}.\left\{\begin{array}{l}\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\cos.2T_5 \\ -\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\sin.2T_5\end{array}\right\};$$ [3924n]

in which we must substitute the factor [3924h]; then the resulting expression being added to [3924m], using M_5 [3924i], the sum becomes

$$-\{2e'\delta'\varpi' + 2\delta e'.\delta\varpi' - 2\delta\varepsilon'.\delta e'\} = \left\{\begin{array}{l}M_5.\left[P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right].\cos.2T_5 \\ -M_5.\left[P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right].\sin.2T_5\end{array}\right\}.$$ [3924o]

Multiplying the equation [3924k] by $\sin.W'$, and [3924o] by $\cos.W'$, then adding the products, we find that the first member of the sum is equal to the function [3924b]; the second member, reduced by formulas similar to [3917o], is [3924p]

$$M_5.\left\{P.\left(\frac{dP'}{de'}\right)+P'.\left(\frac{dP}{de'}\right)\right\}.\cos.(W'-2T_5)+M_5.\left\{P'.\left(\frac{dP'}{de'}\right)-P.\left(\frac{dP}{de'}\right)\right\}.\sin.(W'-2T_5);$$ [3924q]

which, by resubstituting the values [3924i, a], becomes as in [3924].

* (2474) If we interchange the elements of the bodies m, m', in [3826], and suppose B to become B', and $i=-2$, we shall obtain an inequality of the body m', of the form [3925]. Substituting $T_5 = 5n't - 2nt + 5\varepsilon' - 2\varepsilon$, $W_3 = n't + \varepsilon'$, $W' = n't + \varepsilon' - \varpi'$, we find that the expressions [3925, 3926] become, respectively, [3926a]

$$\delta v' = K'.\sin.(T_5 - W_3 + B'); \qquad \zeta' = -\overline{H}'.\sin.(T_5 + \overline{A}').$$ [3926b]

we shall find, that the inequality of m', depending on the angle $4nt - 9n't + 4\varepsilon - 9\varepsilon'$, is represented by

[3927] $$\delta v' = \tfrac{1}{4} \cdot \frac{(3m\sqrt{a} + 2m'\sqrt{a'})}{m\sqrt{a}} \cdot \overline{H}' K' . \sin. (4nt - 9n't + 4\varepsilon - 9\varepsilon' - B' - \overline{A}').$$

These may be reduced to forms similar to [3920*b*, *c*], respectively, by observing, that the term $2e'.\sin.(n't + \varepsilon' - \varpi')$, in the motion of m', similar to that of m [3913''], may be developed as in [3915], and will contain the terms $2\,\delta e'.\sin.W' - 2e'\,\delta\varpi'.\cos.W'$, which may be reduced, as in [3916*f*], to the form

[3926*c*] $$-\frac{2}{M_2}\cdot\left(\frac{dP}{de'}\right).\cos.(T_5 - W') + \frac{2}{M_2}\cdot\left(\frac{dP'}{de'}\right).\sin.(T_5 - W');$$

[3926*d*] and by the usual process, as in [3916*h*, *i*], it may be reduced to the form $K'.\sin.(T_5 - W' + B_1)$. Now if we put $B_1 = B' - \varpi'$, and $W' = W_3 - \varpi'$ [3926*a*], it becomes, as in [3926*b*], $K'.\sin.(T_5 - W_3 + B')$; so that by substituting the value of M_2 [3907*a*], we shall have identically, in like manner as in [3920*b*],

[3926*e*] $$K'.\sin.(T_5 - W_3 + B') = \frac{2m.a'n'}{(5n' - 2n)}\cdot\left\{-\left(\frac{dP}{de'}\right).\cos.(T_5 - W_3 + \varpi') + \left(\frac{dP'}{de'}\right).\sin.(T_5 - W_3 + \varpi')\right\}.$$

Putting the two expressions of the chief terms of the great inequality [3924*e*, 3926*b*] equal to each other, we get, by changing the signs,

[3926*f*] $$\overline{H}'.\sin.(T_5 + \overline{A}') = \frac{15m.a'n'^2}{(5n' - 2n)^2}\cdot\{P.\cos.T_5 - P'.\sin.T_5\}.$$

The identical equations [3926*e*, *f*] are similar to [3920*b*, *c*], and may be derived from them [3926*g*] by changing m', a, n, e, ϖ, $\overline{A}$, B, K, $\overline{H}$, W_2, into m, a', n', e', ϖ', $\overline{A}'$, B', K', $\overline{H}'$, W_3, respectively; also multiplying the second member of [3920*c*] by $\frac{15}{6}$, without altering the angle T_5, or the divisor $(5n' - 2n)$. Making the same changes in the product of these two equations, and in [3921*b*], we get from this last the following equation;

[3926*h*] $$-\tfrac{1}{2}\overline{H}'K'.\cos.(W_3 - 2T_5 - B' - \overline{A}') = -\frac{15m^2.a'^2n'^3}{(5n' - 2n)^3}\cdot\left\{\begin{array}{l} \left[P.\left(\frac{dP'}{de'}\right) + P'.\left(\frac{dP}{de'}\right)\right].\sin.(W_3 - 2T_5 - \varpi') \\ -\left[P'.\left(\frac{dP'}{de'}\right) - P.\left(\frac{dP}{de'}\right)\right].\cos.(W_3 - 2T_5 - \varpi') \end{array}\right\}.$$

This equation being identical, we may change $W_3 - 2T_5$, into $W_3 - 2T_5 + 90^d$; then multiplying by $\frac{(3m\sqrt{a} + 2m'\sqrt{a'})}{2m\sqrt{a}}$, we find, that the second member of the product becomes equal to the expression [3924]; and the first member becomes equal to [3927];
[3926*i*] observing that $W_3 - 2T_5 = 4nt - 9n't + 4\varepsilon - 9\varepsilon'$ and $15m^2.a'^3n'^3 = 6m^2.a^3n^3$ [3924*g*]; therefore the expression [3927] is equivalent to [3924].

14. *The nodes and inclinations of the orbits of Jupiter and Saturn are subjected to variations analogous to the preceding. To determine them, we shall observe, that* φ, φ', *being the inclinations of the orbits to a fixed plane, and* θ, θ' *the longitude of their ascending nodes*, we shall have, as in [1338], [3928]
by reason of the smallness of φ, φ',*

$$\gamma \,.\, \sin. \Pi = \varphi' . \sin. \theta' - \varphi \,.\, \sin. \theta \,; \qquad [3929]$$

$$\gamma \,.\, \cos. \Pi = \varphi' . \cos. \theta' - \varphi \,.\, \cos. \theta . \qquad [3929']$$

Moreover, from [3906], we have†

$$\delta \,.\, (\varphi' . \sin. \theta') = - \frac{m\sqrt{a}}{m'\sqrt{a'}} \,.\, \delta \,.\, (\varphi \,.\, \sin. \theta) \,; \qquad [3930]$$

$$\delta \,.\, (\varphi' . \cos. \theta') = - \frac{m\sqrt{a}}{m'\sqrt{a'}} \,.\, \delta \,.\, (\varphi \,.\, \cos. \theta) . \qquad [3930']$$

The subject of the small inequalities, treated of in this article, is resumed by the author [3926k]
in the fourth volume [9062, &c.]; where he notices terms of the order $m'^2 . e^2$, &c., which are omitted in [3914e]. His object in using the indirect methods, adopted in this [3926l]
article, is to avoid the great labor of a direct calculation; assuming as a principle, that these very small inequalities may be determined in this manner to a sufficient degree of exactness, for all the purposes of practical astronomy; as will appear from the minute examination [3926m]
of the terms of this kind in [9041—9114].

* (2475) Comparing the notation in [1337′, 3902], we get $\theta'_{\prime} = \Pi$; $\text{tang.}\, \varphi'_{\prime} = \text{tang.}\, \gamma = \gamma$ [3929a]
nearly; hence the equations [1338] become $p' - p = \gamma \,.\, \sin. \Pi$, $q' - q = \gamma \,.\, \cos. \Pi$. [3929b]
Now on account of the smallness of φ, we have very nearly $p = \varphi \,.\, \sin. \theta$, $q = \varphi \,.\, \cos. \theta$ [3929c]
[1334]; and in like manner, for the orbit of m', $p' = \varphi' . \sin. \theta'$, $q' = \varphi' . \cos. \theta'$. Substituting these in [3929b], we get [3929, 3929′].

† (2476) The variation of the second member of [3929], arising from the action of the body m' upon m, is represented by $- \delta \,.\, (\varphi \,.\, \sin. \theta)$, because φ', θ', do not [3930a]
vary by the action of m'. The variation of the first member of the same equation, using the characteristics $\delta_{\prime}$, $\delta_{\prime\prime}$, as in [3899′, 3904], is $\delta_{\prime\prime} . (\gamma \,.\, \sin. \Pi)$; hence by development, we have

$$- \delta \,.\, (\varphi \,.\, \sin. \theta) = \delta_{\prime\prime} \gamma \,.\, \sin. \Pi + \gamma \,.\, \delta_{\prime\prime} \Pi \,.\, \cos. \Pi . \qquad [3930b]$$

In like manner, the variation of the second member of [3929], relative to the action of the body m, which does not affect φ, θ, is $\delta \,.\, (\varphi' . \sin. \theta')$; and that of the first member is

From these four equations, we deduce the following,*

[3931] $$\delta\varphi = -\frac{m'\sqrt{a'}}{m\sqrt{a}+m'\sqrt{a'}}.\{\delta\gamma.\cos.(\Pi-\theta)-\gamma.\delta\Pi.\sin.(\Pi-\theta)\};$$

[3931'] $$\varphi\,\delta\theta = -\frac{m'\sqrt{a'}}{m\sqrt{a}+m'\sqrt{a'}}.\{\delta\gamma.\sin.(\Pi-\theta)+\gamma.\delta\Pi.\cos.(\Pi-\theta)\};$$

[3932] $$\delta\varphi' = \frac{m\sqrt{a}}{m\sqrt{a}+m'\sqrt{a'}}.\{\delta\gamma.\cos.(\Pi-\theta')-\gamma.\delta\Pi.\sin.(\Pi-\theta')\};\dagger$$

[3932'] $$\varphi'\,\delta\theta' = \frac{m\sqrt{a}}{m\sqrt{a}+m'\sqrt{a'}}.\{\delta\gamma.\sin.(\Pi-\theta')+\gamma.\delta\Pi.\cos.(\Pi-\theta')\}.$$

[3930c] $\delta_{,}.(\gamma\sin.\Pi)$; hence we get [3930d]. Substituting successively in this the values [3906, 3930b], we finally obtain [3930f], as in [3930],

[3930d] $$\delta.(\varphi'.\sin.\theta') = \delta_{,}\gamma.\sin.\Pi+\gamma.\delta_{,}\Pi.\cos.\Pi$$

[3930e] $$= \frac{m\sqrt{a}}{m'\sqrt{a'}}.\{\delta_{,,}\gamma.\sin.\Pi+\gamma.\delta_{,,}\Pi.\cos.\Pi\} \quad [3906];$$

[3930f] $$= -\frac{m\sqrt{a}}{m'\sqrt{a'}}.\delta.(\varphi.\sin.\theta) \quad [3930b].$$

[3930g] In the same way, we may deduce [3930'] from [3929'].

[3931a] * (2477) We shall put, for brevity, $M_6 = \frac{m\sqrt{a}}{m\sqrt{a}+m'\sqrt{a'}}$, $M_7 = \frac{m'\sqrt{a'}}{m\sqrt{a}+m'\sqrt{a'}}$: then taking the variation of [3929], relative to the characteristic δ, we get, by the substitution of [3930], the following equation,

[3931b] $$\delta.(\gamma.\sin.\Pi) = \delta.(\varphi'.\sin.\theta')-\delta.(\varphi.\sin.\theta)$$
$$= -\frac{m\sqrt{a}}{m'\sqrt{a'}}.\delta.(\varphi.\sin.\theta)-\delta.(\varphi.\sin.\theta) = -\frac{1}{M_7}.\delta.(\varphi.\sin.\theta).$$

or

[3931b'] $$\delta.(\varphi.\sin.\theta) = -M_7.\delta.(\gamma.\sin.\Pi).$$

[3931b''] In like manner, from [3929', 3930'], we get $\delta.(\varphi.\cos.\theta) = -M_7.\delta.(\gamma.\cos.\Pi)$. Developing these two equations, we obtain

[3931c] $$\delta\varphi.\sin.\theta+\varphi\,\delta\theta.\cos.\theta = -M_7.(\delta\gamma.\sin.\Pi+\gamma.\delta\Pi.\cos.\Pi);$$

[3931d] $$\delta\varphi.\cos.\theta-\varphi\,\delta\theta.\sin.\theta = -M_7.(\delta\gamma.\cos.\Pi-\gamma.\delta\Pi.\sin.\Pi).$$

[3931e] Multiplying [3931c, d] by $\sin.\theta$, $\cos.\theta$, respectively; adding the products, and substituting $\sin.^2\theta+\cos.^2\theta=1$, $\sin.\Pi.\sin.\theta+\cos.\Pi.\cos.\theta=\cos.(\Pi-\theta)$, $\cos.\Pi.\sin.\theta-\sin.\Pi.\cos.\theta=-\sin.(\Pi-\theta)$, we get [3931]. Again, multiplying [3931c, d] by $\cos.\theta$, $-\sin.\theta$, respectively; adding the products, and making similar substitutions, we get [3931'].

† (2478) We may compute the equations [3932, 3932'] from [3929—3930'], in like
[3932a] manner as in the last note; or more simply by derivation, in the following manner.

Therefore the variations of φ, θ, φ', θ', depend on the variations of γ and Π. We have, by § 12,*

$$\frac{d\gamma}{dt}=-\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\,.\,m'\,.\,a\,n\,.\left\{\begin{array}{l}\left(\frac{dP}{d\gamma}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\\ -\left(\frac{dP'}{d\gamma}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\}; \qquad [3933]$$

$$\frac{\gamma.d\Pi}{dt}=-\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\,.\,m'\,.\,a\,n\,.\left\{\begin{array}{l}\left(\frac{dP}{d\gamma}\right).\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\\ +\left(\frac{dP'}{d\gamma}\right).\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\}. \qquad [3933']$$

If we change m, a, φ, θ, γ, into m', a', φ', θ', $-\gamma$, and the contrary respectively, in [3932b]
the equations [3929—3930'], they will remain unaltered, as will be evident by changing the signs of the two first of these equations, and multiplying those which are derived from the two last by the factor $-\frac{m\sqrt{a}}{m'\sqrt{a'}}$. Making the changes [3932b] in [3931, 3931'], which [3932c]
are deduced from [3929—3930'], we get [3932, 3932'].

* (2479) Substituting the values $\delta_{\prime\prime}\gamma$, $\delta_{\prime\prime}\Pi$ [3900, 3901], in [3906c], and using, for brevity, the symbols T_5 [3890b], also $a\,n=a^{-\frac{1}{2}}$, $a'\,n'=a'^{-\frac{1}{2}}$,

$$M_8=\frac{m'.an+m.a'n'}{m'.an}=\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}, \qquad M_9=M_8\,.\,m'.\,a\,n=\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\,.\,m'.\,a\,n, \qquad [3933a]$$

we get

$$\delta\gamma=-M_8\,.\,\frac{m'.\,a\,n}{5n'-2n}\,.\left\{\left(\frac{dP}{d\gamma}\right).\sin.\,T_5+\left(\frac{dP'}{d\gamma}\right).\cos.\,T_5\right\}; \qquad [3933b]$$

$$\delta\Pi=\quad M_8\,.\,\frac{m'.\,a\,n}{(5n'-2n).\gamma}\,.\left\{\left(\frac{dP}{d\gamma}\right).\cos.\,T_5-\left(\frac{dP'}{d\gamma}\right).\sin.\,T_5\right\}. \qquad [3933c]$$

The divisor $5n'-2n$ is introduced in δs, &c. [1342, 3899—3901], by the integration relative to t, spoken of in [1341a, &c.], in finding p, q, s [1341, 1342]; where the angle T_5 is considered as the only variable quantity; the very small terms, of a different [3933e]
form or order, depending on the variations of the elements, which enter into the second members of [1312, &c., 3933b, c], being neglected. If we again resume the differentials of the expressions [3933b, c], upon the same principles, we shall get

$$\frac{d.(\delta\gamma)}{dt}=-M_8\,.\,m'.\,a\,n\,.\left\{\left(\frac{dP}{d\gamma}\right).\cos.\,T_5-\left(\frac{dP'}{d\gamma}\right).\sin.\,T_5\right\}; \qquad [3933d]$$

$$\frac{d.(\delta\Pi)}{dt}=-M_8\,.\,\frac{m'.\,a\,n}{\gamma}\,.\left\{\left(\frac{dP}{d\gamma}\right).\sin.\,T_5+\left(\frac{dP'}{d\gamma}\right).\cos.\,T_5\right\}. \qquad [3933e]$$

These equations are equivalent to [3933, 3933'], omitting the characteristic δ, which [3933f]
merely signifies, that the calculation is restricted to terms depending on the angle T_5 [3893'].

[3933″] Hence we deduce, by *neglecting periodical quantities* whose effect is insensible*, and observing that†

[3934] $$\left(\frac{dP'}{d\gamma}\right)\cdot\left(\frac{ddP}{d\gamma^2}\right)-\left(\frac{dP}{d\gamma}\right)\cdot\left(\frac{ddP'}{d\gamma^2}\right)=0\,;‡$$

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* (2179*a*) If we compare the expressions [3842, 4401] with the numerical values
[3933g] of e^{iv}, e^{v}, γ, or tang. γ [4080, 4409], we shall easily perceive, that the terms of P [3842],
[3933h] depending on γ, are not a thirtieth part so great as some of the terms depending on e^{iv}, e^{v}; therefore the periodical inequalities depending on the variation of γ, will evidently be
[3933i] much less than those arising from the variations of e^{iv}, e^{v}. Now from the computation made in [4438, 4496], it appears, that these last inequalities are nearly 4^s and 9^s; hence it is evident, that we may neglect the periodical quantities spoken of in [3933″].

† (2180) Dividing [3842] by a', and taking the partial differentials relatively to γ, we get

[3934a] $$m'.\left(\frac{dP}{d\gamma}\right)=2\,M^{(4)}\,.\,e'\gamma\,.\,\text{sin.}\,(2\,\Pi+\varpi')+2\,M^{(5)}\,.\,e\gamma\,.\,\text{sin.}\,(2\,\Pi+\varpi)\,;$$

[3934b] $$m'.\left(\frac{ddP}{d\gamma^2}\right)=2\,M^{(4)}\,.\,e'\,.\,\text{sin.}\,(2\,\Pi+\varpi')+2\,M^{(5)}\,.\,e\,.\,\text{sin.}\,(2\,\Pi+\varpi).$$

Multiplying the second of these equations by γ, it becomes equal to the first; hence we get, by dividing by m', $\gamma\,.\left(\frac{ddP}{d\gamma^2}\right)=\left(\frac{dP}{d\gamma}\right)$. In like manner, from the values of
[3934c] $m'.a'P'$ [3843], we obtain $\gamma\,.\left(\frac{ddP'}{d\gamma^2}\right)=\left(\frac{dP'}{d\gamma}\right)$; dividing the first of these expressions by the second, we get an equation, which is easily reduced to the form [3934].

‡ (2181) To obtain the effect of the variations of P, P', ξ, ξ', in $d\gamma$ [3933], we may proceed in the same manner as we have done in notes 2461, 2462 [3909*a*, &c.], in finding the variations of de, $d\varpi$. In the first place, we must substitute, as in [3908*a*],
[3935a] ξ, ξ' for nt, $n't$, in [3933], and use the symbols [3933*a*]; hence we get

[3935b] $$d\gamma=-M_8.m'.andt.\left\{\left(\frac{dP}{d\gamma}\right).\text{cos.}\,(5\xi'-2\xi+5\varepsilon'-2\varepsilon)-\left(\frac{dP'}{d\gamma}\right).\text{sin.}\,(5\xi'-2\xi+5\varepsilon'-2\varepsilon)\right\}.$$

[3935b′] Substituting in this the values [3909*d*], we get the following expression, which is nearly similar to [3909*e*], changing e into γ, &c., and writing, as usual, $d\delta\gamma$ for $\delta\gamma$,

[3935c] $$\begin{aligned} d\delta\gamma= {} & M_8\,.\,m'\,.\,a\,n\,d\,t\,.\left\{-\left(\frac{dP}{d\gamma}\right).\,\text{cos.}\,T_5+\left(\frac{dP'}{d\gamma}\right).\,\text{sin.}\,T_5\right\} \\ & +M_8\,.\,m'\,.\,a\,n\,d\,t\,.\,(5N'-2N)\,.\left\{\left(\frac{dP'}{d\gamma}\right).\,\text{cos.}\,T_5+\left(\frac{dP}{d\gamma}\right).\,\text{sin.}\,T_5\right\}. \end{aligned}$$

The variation of this expression, arising from δe, $\delta\varpi$, $\delta e'$, $\delta\varpi'$, $\delta\gamma$, $\delta\Pi$, in the two first terms, may be found as in [3909*e*—*k*]; or more simply by derivation, in the following

Inequality in the inclination of the orbits of Jupiter and Saturn.

$$\delta\gamma = -\frac{3\,m'^2.\,a^2\,n^3}{(5\,n'-2\,n)^2}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\cdot\frac{(5\,m\sqrt{a}+2\,m'\sqrt{a'})}{m'\sqrt{a'}}\,.\,t\,.\left\{P.\left(\frac{d\,P'}{d\,\gamma}\right)-P'.\left(\frac{d\,P}{d\,\gamma}\right)\right\}$$

$$+\frac{m'^2.\,a^2\,n^2}{5\,n'-2\,n}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\,.\,t\,.\left\{\left(\frac{d\,P'}{d\,e}\right).\left(\frac{d\,d\,P}{d\,e\,d\gamma}\right)-\left(\frac{d\,P}{d\,e}\right).\left(\frac{d\,d\,P'}{d\,e\,d\,\gamma}\right)\right\}$$

$$+\frac{m\,m'.\,a\,a'.\,n\,n'}{5\,n'-2\,n}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\,.\,t\,.\left\{\left(\frac{d\,P'}{d\,e'}\right).\left(\frac{d\,d\,P}{d\,e'\,d\,\gamma}\right)-\left(\frac{d\,P}{d\,e'}\right).\left(\frac{d\,d\,P'}{d\,e'\,d\,\gamma}\right)\right\};$$ [3935]

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manner. If we change, in $d\,\delta\,e$ [3909e], e into γ, ϖ into Π, and the contrary; also [3935d]
m' into $M_8.\,m'$, without altering the values of P, P', N, N', T_5, e', ϖ', &c.; we shall find, that this expression of $d\,\delta\,e$ becomes equal to that of $d\,\delta\,\gamma$ [3935c]; and by making the same changes in the other expressions of $d\,\delta\,e$ [3909h, k], we shall get the similar values of $d\,\delta\,\gamma$. After making these changes in [3909k], and putting, for brevity,
$M_9 = M_8\,.\,m'.\,a\,n$ [3933a], we may alter the arrangement of the quantities, so that the [3935e]
terms depending on the same differential coefficient may be connected together, and we shall get

$$d\,\delta\gamma = \quad M_9\,.\,dt\,.\,(5\,N'-2\,N)\,.\left\{\left(\frac{d\,P'}{d\,\gamma}\right).\cos.\,T_5+\left(\frac{d\,P}{d\,\gamma}\right).\sin.\,T_5\right\}$$

$$+M_9\,.\,dt\,.\left(\frac{d\,d\,P}{d\,e\,d\,\gamma}\right).(-\delta e.\cos.\,T_5-e\,\delta\varpi.\sin.\,T_5)+M_9.dt.\left(\frac{d\,d\,P'}{d\,e\,d\,\gamma}\right).(\delta e.\sin.\,T_5-e\delta\varpi.\cos.\,T_5)$$

$$+M_9\,.\,dt\,.\left(\frac{d\,d\,P}{d\,e'\,d\,\gamma}\right).(-\delta e'.\cos.\,T_5-e'\delta\varpi'.\sin.\,T_5)+M_9.dt.\left(\frac{d\,d\,P'}{d\,e'\,d\,\gamma}\right).(\delta e'.\sin.\,T_5-e'\delta\varpi'.\cos.\,T_5)$$ [3935f]

$$+M_9\,.\,dt\,.\left(\frac{d\,d\,P}{d\,\gamma^2}\right).(-\delta\gamma.\cos.\,T_5-\gamma\delta\,\Pi.\sin.\,T_5)+M_9.dt.\left(\frac{d\,d\,P'}{d\,\gamma^2}\right).(\delta\gamma.\sin.\,T_5-\gamma\delta\Pi.\cos.\,T_5)$$

$$-M_9\,.\,dt\,.\,\delta\,\Pi\,.\left\{\left(\frac{d\,P}{d\,\gamma}\right).\sin.\,T_5+\left(\frac{d\,P'}{d\,\gamma}\right).\cos.\,T_5\right\}.$$

We may neglect the fourth and fifth lines of this expression. For if we substitute the values [3907g] in the fourth line, it becomes equal to $\frac{M_9}{M_3}$, multiplied by the terms in [3935g]
the first member of [3934], and is therefore equal to nothing. Moreover, by using the value of $\delta\,\Pi$ [3907d], we find that the lower line of the expression [3935f] becomes of a similar form to that in the second member of [3909x]; the partial differentials of P, P' being taken relative to γ, instead of e. Hence we find, as in [3909y], that this line
of [3935f] depends upon the *periodical* quantities $\sin.\,2\,T_5$, $\cos.\,2\,T_5$, which are [3935h]
neglected in the present calculation [3933″]. The three remaining lines of the expression [3935f] being reduced, and integrated relatively to t, produce respectively the three lines of the expresson of $\delta\,\gamma$ [3935]. For if we compare the first line of [3909k], multiplied
by $M_8 = \frac{M_9}{m'.\,a\,n}$ [3933a], with the first line of [3935f], we shall find that they become [3935i]
identical, by changing the partial differentials relative to e into those relative to γ; hence

$$\delta\Pi = \frac{3m'^2.a^2n^3}{(5n'-2n)^2.\gamma}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\cdot\frac{(5m\sqrt{a}+2m'\sqrt{a'})}{m'\sqrt{a'}}.t.\left\{P.\left(\frac{dP}{d\gamma}\right)+P'.\left(\frac{dP'}{d\gamma}\right)\right\}^{*}$$

Inequality in the place of the node.

$$+\frac{m'^2.a^2n^2}{(5n'-2n).\gamma}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}.t.\left\{\begin{array}{l}\left(\frac{dP}{de}\right).\left(\frac{ddP}{de\,d\gamma}\right)+\left(\frac{dP'}{de}\right).\left(\frac{ddP'}{de\,d\gamma}\right)\\+\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP}{d\gamma^2}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP'}{d\gamma^2}\right)\end{array}\right\}$$

[3936]

$$+\frac{m\,m'\,a\,a'\,n\,n'}{(5n'-2n).\gamma}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}.t.\left\{\begin{array}{l}\left(\frac{dP}{de'}\right).\left(\frac{ddP}{de'd\gamma}\right)+\left(\frac{dP'}{de'}\right).\left(\frac{ddP'}{de'd\gamma}\right)\\+\left(\frac{dP}{d\gamma}\right).\left(\frac{ddP}{d\gamma^2}\right)+\left(\frac{dP'}{d\gamma}\right).\left(\frac{ddP'}{d\gamma^2}\right)\end{array}\right\}.$$

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we obtain the coefficient of t, in the term of $\delta\gamma$, depending on the first line of [3935f], by

[3935k] multiplying the first line of [3910], which is derived from the first of [3909k], by M_8 [3935i], and changing the differential divisor de into $d\gamma$, as in the first line of [3935]. Again, substituting the values [3907e] in the second line of [3935f], and using

[3935l] $$\frac{M_0}{M_1}=\frac{m'^2.a^2n^2}{5n'-2n}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\quad[3933a,\ 3907a],$$

we get the second line of [3935]. Lastly, substituting [3907f], and

[3935m] $$\frac{M_0}{M_2}=\frac{m\,m'.a\,a'\,n\,n'}{5n'-2n}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\quad[3933a,\ 3907a],$$

in the third line of [3935f], we get the third line of [3935].

* (2482) We may compute $\delta\Pi$ from [3933e], in the same manner as we have found $\delta\gamma$ [3935] from [3933d] in the last note; or we may use the principle of derivation;

[3936a] observing that the expressions of $d\gamma$, $\gamma\,d\Pi$ [3933d, e] have a relation to each other, which is similar to that of de, $e\,d\varpi$ [3908c, d]. Moreover the former values may

[3936b] be derived from the latter, by changing e, ϖ, &c., into γ, Π, &c., respectively, as in [3935d]; therefore we may derive the expression of $\delta\Pi$ from that of $\delta\gamma$, in the same manner as we have derived $\delta\varpi$ from δe, in note 2462 [3910a, &c.]. Proceeding now as in that note, we shall find, by changing e into γ, &c. in the terms [3910p, q], and

[3936c] reducing as in [3910r], that these terms depend on the periodical quantities $\sin.2T_5$, $\cos.2T_5$, which are neglected in [3933″] and in [3935h]. In the terms depending on the factor $5N'-2N$, we find, by proceeding as in [3910k], that we must change

[3936d] $\left(\frac{dP}{d\gamma}\right)$ into $\frac{1}{\gamma}.\left(\frac{dP'}{d\gamma}\right)$, and $\left(\frac{dP'}{d\gamma}\right)$ into $-\frac{1}{\gamma}.\left(\frac{dP}{d\gamma}\right)$; and by making these changes in the first line of $\delta\gamma$ [3935], we get the corresponding terms of $\delta\Pi$ in the first line of [3936]. The remaining terms corresponding to those which are computed in

[3936e] [3910m—o], depend on the second differentials ddP, ddP', and may be computed from the second, third, and fourth lines of [3935f]; changing T_5 into T_5-90^d, as

15. If we wish to determine, for any time whatever, the elements of the planetary orbits, we must integrate the differential equations [1089, 1132], by the method explained in [1096, &c.]; but in our present ignorance of the exact values of the masses of several of the planets, this calculation would be of no practical use in astronomy; and it becomes indispensable to notice the secular variations, depending on the square of the disturbing force, which we have just determined; since they are very sensible in the orbits of Jupiter and Saturn. These variations increase the values of $\frac{dh^{iv}}{dt}$, $\frac{dl^{iv}}{dt}$, $\frac{dp^{iv}}{dt}$, $\frac{dq^{iv}}{dt}$, $\frac{dh^{v}}{dt}$, &c., relative to these two planets, by the quantities* $\frac{h^{iv}.\delta e^{iv}}{e^{iv}t}+\frac{l^{iv}.\delta\varpi^{iv}}{t}$; $\frac{l^{iv}.\delta e^{iv}}{e^{iv}t}-\frac{h^{iv}.\delta\varpi^{iv}}{t}$; $\frac{p^{iv}.\delta\varphi^{iv}}{\varphi^{iv}t}+\frac{q^{iv}.\delta\theta^{iv}}{t}$; &c., [3937] [3938]

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in [3910*a*—*d*], and substituting the values [3907*c*—*g*]; by this means we shall obtain the corresponding terms, which are to be multiplied by $\frac{dt}{\gamma}$ in $d\delta\Pi$; or by $\frac{t}{\gamma}$ in $\delta\Pi$, namely,

$$\frac{M_9}{M_1}\cdot\left\{\left(\frac{dP}{de}\right)\cdot\left(\frac{ddP}{de\,d\gamma}\right)+\left(\frac{dP'}{de}\right)\cdot\left(\frac{ddP'}{de\,d\gamma}\right)\right\}+\frac{M_9}{M_2}\cdot\left\{\left(\frac{dP}{de'}\right)\cdot\left(\frac{ddP}{de'd\gamma}\right)+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{ddP'}{de'd\gamma}\right)\right\}$$
$$+\frac{M_9}{M_3}\cdot\left\{\left(\frac{dP}{d\gamma}\right)\cdot\left(\frac{ddP}{d\gamma^2}\right)+\left(\frac{dP'}{d\gamma}\right)\cdot\left(\frac{ddP'}{d\gamma^2}\right)\right\}. \qquad [3936f]$$

Substituting in this the values [3935*l*, *m*], also

$$\frac{M_9}{M_3}=\left\{\frac{m'^2.a^2n^2}{5n'-2n}+\frac{m'm.aa'nn'}{5n'-2n}\right\}.\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}} \quad [3933a, 3907a], \qquad [3936g]$$

we get, by a slight reduction, the second and third lines of [3936].

* (2183) The equations [1022], corresponding to Jupiter and Saturn, are

$$h^{iv}=e^{iv}.\sin.\varpi^{iv};\qquad l^{iv}=e^{iv}.\cos.\varpi^{iv};\qquad h^{v}=e^{v}.\sin.\varpi^{v};\qquad l^{v}=e^{v}.\cos.\varpi^{v}. \qquad [3938a]$$

Taking the variations of these quantities, relatively to the characteristic δ, used as in [3938'], and then substituting the values of $\sin.\varpi^{iv}$, $\cos.\varpi^{iv}$, &c., deduced from [3938*a*], we get

$$\delta h^{iv}=\delta e^{iv}.\sin.\varpi^{iv}+e^{iv}.\delta\varpi^{iv}.\cos.\varpi^{iv}=\delta e^{iv}.\frac{h^{iv}}{e^{iv}}+e^{iv}.\delta\varpi^{iv}.\frac{l^{iv}}{e^{iv}}; \qquad [3938b]$$

$$\delta l^{iv}=\delta e^{iv}.\cos.\varpi-e^{iv}.\delta\varpi^{iv}.\sin.\varpi^{iv}=\delta e^{iv}.\frac{l^{iv}}{e^{iv}}-e^{iv}.\delta\varpi^{iv}.\frac{h^{iv}}{e^{iv}},\ \&c. \qquad [3938c]$$

The secular part of any one of the quantities δe^{iv}, $\delta\varpi^{iv}$, δe^{v}, $\delta\varpi^{v}$ [3910, 3911, 3922, 3923], may be put under the form $\delta e^{iv}=At$; A being a function of the elements of the orbits, of the order m'^2. Its differential, divided by dt, gives $\frac{d\delta e^{iv}}{dt}=A=\frac{\delta e^{iv}}{t}$; observing, [3938d] that the variations of A may be neglected, because they are of the order m', and are

considering only in δe^{iv}, $\delta \varpi^{\text{iv}}$, *the quantities proportional to the time t,*
[3938′] *determined in the preceding articles.* We must substitute, in these last quantities, the values of e^{iv}, $\sin.\varpi^{\text{iv}}$, $\cos.\varpi^{\text{iv}}$, &c., expressed in terms of h^{iv}, l^{iv}, &c.* The differential equations [1089] will then cease to
[3939] be linear; but it will be easy to integrate them by known methods of approximation, when, after the lapse of many centuries, the exact values of the planetary masses shall be known. In the present state of astronomy, it is sufficiently accurate to have the secular variations of the elements of the
[3940] orbits, expressed in a series ascending according to the powers of the time, carrying on the approximation no farther than to include the second power.

We have seen, in [1114″, 1139‴], that the state of the planetary system
[3940′] is stable, or in other words, that the excentricities of the orbits are small, and their planes but little inclined to each other. We have deduced this important result of the system of the world from the equation [1153],†

[3941]
$$\text{constant} = (e^2+\varphi^2)\,.\,m\sqrt{a} + (e'^2+\varphi'^2)\,.\,m'\sqrt{a'} + \&c.;$$

for the second member of this equation being small in the present state of the system, it must always remain so; consequently the excentricities
[3942] and inclinations of the orbits will always be quite small.‡ We shall now prove that the differential of the preceding equation [3941],

[3943]
$$(e\,d\,e + \varphi\,d\,\varphi)\,.\,m\sqrt{a} + (e'\,d\,e' + \varphi'\,d\,\varphi')\,.\,m'\sqrt{a'} + \&c. = 0,$$

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multiplied by δe^{iv}, which is of the order m'^2, producing terms of the order m'^3. For a similar reason, we may neglect the variations of $\frac{h^{\text{iv}}}{e^{\text{iv}}}$, $\frac{l^{\text{iv}}}{e^{\text{iv}}}$, &c. in finding the differentials of [3938b, &c.]. Hence the differential of the last expression in [3938b], divided by dt, is
[3938e] $\frac{d\,\delta h^{\text{iv}}}{dt} = \frac{d\,\delta e^{\text{iv}}}{dt}\cdot\frac{h^{\text{iv}}}{e^{\text{iv}}} + e^{\text{iv}}\cdot\frac{d\,\delta\varpi^{\text{iv}}}{dt}\cdot\frac{l^{\text{iv}}}{e^{\text{iv}}} = \frac{\delta e^{\text{iv}}}{t}\cdot\frac{h^{\text{iv}}}{e^{\text{iv}}} + e^{\text{iv}}\cdot\frac{\delta\varpi^{\text{iv}}}{t}\cdot\frac{l^{\text{iv}}}{e^{\text{iv}}}$, as in [3938], omitting the characteristic δ in the first member. In a similar way, we may obtain the other values
[3938f] [3938] from [3938e, &c.]; also the variations of $\frac{dp^{\text{iv}}}{dt}$, $\frac{dq^{\text{iv}}}{dt}$, &c. from [1132, 1032].

* (2484) The equations [3938a] give $e^{\text{iv}} = \sqrt{(h^{\text{iv}2} + l^{\text{iv}2})}$, $e^{\text{v}} = \sqrt{(h^{\text{v}2} + l^{\text{v}2})}$, as
[3939a] in [1108]; which are to be substituted in [3938]; and when the resulting quantities are added, respectively, to the second members of [1089, 1132], they cease to be linear in h^{iv}, l^{iv}, &c., as is observed in [3939].

[3940a] † (2485) Neglecting terms of the order φ^4, we may put $\tan g.^2\varphi = \varphi^2$, and then [1153] becomes as in [3941].

[3941a] ‡ (2486) This must be understood with the restrictions mentioned in note 762 [1114a, &c.].

obtains even when we notice the secular variations of the elements of the orbits determined in the preceding articles [3910, 3922, 3935, &c.]. *Hence it* [3943'] *will follow, that these variations do not affect the stability of the planetary system.* To render this evident, it is only necessary to prove, that if we represent the mass of Jupiter by m, that of Saturn by m', and put [3943''] δe, $\delta e'$, $\delta\varphi$, $\delta\varphi'$, respectively, for the secular variations of e, e', φ, φ', which were found by the preceding calculations, we shall have

$$(e\,\delta e+\varphi\,\delta\varphi)\,.\,m\sqrt{a}+(e'\,\delta e'+\varphi'\,\delta\varphi')\,.\,m'\sqrt{a'}=0. \qquad [3944]$$

If we substitute, in the function $\varphi\,\delta\varphi\,.\,m\sqrt{a}+\varphi'\,\delta\varphi'.\,m'\sqrt{a'}$, the values [3944'] of φ, $\delta\varphi$, φ', $\delta\varphi'$, given in the preceding article, it becomes*

$$\frac{m\,m'\sqrt{a\,a'}}{m\sqrt{a}+m'\sqrt{a'}}\,.\,\gamma\,\delta\gamma\,; \qquad [3945]$$

which changes the equation [3944] into

$$e\,\delta e\,.\,m\sqrt{a}+e'\,\delta e'.\,m'\sqrt{a'}+\frac{m\,m'\sqrt{a\,a'}}{m\sqrt{a}+m'\sqrt{a'}}\,.\,\gamma\,\delta\gamma=0. \qquad [3946]$$

We shall now commence with the consideration of the first line of the [3946'] expression of δe [3910], which becomes, by the substitution of $a^3\,n^2=1$ [3709'],†

$$\delta e=\frac{-3m'.(5m\sqrt{a}+2m'\sqrt{a'})}{(5n'-2n)^2.a\sqrt{a'}}\,.\,nt\,.\left\{P.\left(\frac{dP'}{de}\right)-P'.\left(\frac{dP}{de}\right)\right\}. \qquad [3947]$$

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* (2487) Multiplying [3931, 3932] by $\varphi.m\sqrt{a}$, $\varphi'.m'\sqrt{a'}$, respectively, and adding the products, we get

$$\varphi\delta\varphi\; m\sqrt{a}+\varphi'\delta\varphi'\; m'\sqrt{a'}=\frac{m\,m'\sqrt{a\,a'}}{m\sqrt{a}+m'\sqrt{a'}}\,.\left\{\begin{array}{l}\delta\gamma\,.\,\{-\varphi\,.\cos.(\Pi-\theta)+\varphi'.\cos.(\Pi-\theta')\}\\ +\gamma\,\delta\Pi\,.\,\{\varphi\,.\sin.(\Pi-\theta)-\varphi'.\sin.(\Pi-\theta')\}\end{array}\right\}. \qquad [3944a]$$

Now multiplying [3929, 3929'] by $\sin.\Pi$, $\cos.\Pi$, respectively, adding the products, and putting $\sin.^2\Pi+\cos.^2\Pi=1$, $\sin.\Pi.\sin.\theta'+\cos.\Pi.\cos.\theta'=\cos.(\Pi-\theta')$, &c. [24] Int., [3944b] we get [3944c]. In like manner, multiplying [3929] by $-\cos.\Pi$, and [3929'] by $\sin.\Pi$, and reducing the sum of the products, it becomes as in [3944d];

$$\varphi'.\cos.(\Pi-\theta')-\varphi\,.\cos.(\Pi-\theta)=\gamma\,; \qquad [3944c]$$

$$\varphi'.\sin.(\Pi-\theta')-\varphi\,.\sin.(\Pi-\theta)=0. \qquad [3944d]$$

Substituting these in [3944a], it becomes as in [3945]; and by this means [3944] changes into [3946].

† (2488) Substituting $a^2\,n^3=\frac{n}{a}$ [3946'] in the first line of δe [3910], it becomes as in [3917]. Again, substituting $a^3\,n^3=n$ [3946'], in the first line of $\delta e'$ [3922], [3946a] we get [3948]; in like manner, the first line of [3935] becomes as in [3949].

In the second place, we shall consider the first line of the expression of $\delta e'$ [3922],

[3948] $$\delta e' = -\frac{3m.(5m\sqrt{a}+2m'\sqrt{a'})}{(5n'-2n)^2.a'\sqrt{a}}.nt.\left\{P.\left(\frac{dP'}{de'}\right)-P'.\left(\frac{dP}{de'}\right)\right\}.$$

Lastly, we shall notice the first line of the expression of $\delta\gamma$ [3935],

[3949] $$\delta\gamma = -\frac{3m'.(5m\sqrt{a}+2m'\sqrt{a'})}{(5n'-2n)^2.a\sqrt{a'}}.\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}.nt.\left\{P.\left(\frac{dP'}{d\gamma}\right)-P'.\left(\frac{dP}{d\gamma}\right)\right\}.$$

If we notice only these terms, we shall find*

[3950] $$e\,\delta e.m\sqrt{a}+e'\delta e'.m'\sqrt{a'}+\frac{mm'\sqrt{aa'}}{m\sqrt{a}+m'\sqrt{a'}}.\gamma\,\delta\gamma$$
$$= -\frac{3mm'.(5m\sqrt{a}+2m'\sqrt{a'})}{(5n'-2n)^2.\sqrt{aa'}}.nt.\left\{\begin{array}{l} P.\left[e.\left(\frac{dP'}{de}\right)+e'.\left(\frac{dP'}{de'}\right)+\gamma.\left(\frac{dP'}{d\gamma}\right)\right] \\ -P'.\left[e.\left(\frac{dP}{de}\right)+e'.\left(\frac{dP}{de'}\right)+\gamma.\left(\frac{dP}{d\gamma}\right)\right]\end{array}\right\}.$$

[3950'] Now P, P', being homogeneous functions of e, e', γ, of the third dimension, we shall have†

[3951] $$e.\left(\frac{dP}{de}\right)+e'.\left(\frac{dP}{de'}\right)+\gamma.\left(\frac{dP}{d\gamma}\right)=3P;$$

[3951'] $$e.\left(\frac{dP'}{de}\right)+e'.\left(\frac{dP'}{de'}\right)+\gamma.\left(\frac{dP'}{d\gamma}\right)=3P';$$

therefore the equation [3950] will become

[3952] $$e\,\delta e.m\sqrt{a}+e'\,\delta e'.m'\sqrt{a'}+\frac{mm'\sqrt{aa'}}{m\sqrt{a}+m'\sqrt{a'}}.\gamma\,\delta\gamma=0.$$

[3949a] * (2489) Substituting the terms of δe, $\delta e'$, $\delta\gamma$ [3947, 3948, 3949], in the first member of the expression [3946], it becomes as in the second member of [3950].

† (2490) The expressions of P, P' [3842, 3843], are evidently homogeneous in e, e', γ, and of the third dimension. Now the theorem in homogeneous functions
[3950a] [1001a], by putting $n=3$, $a=e$, $a'=e'$, $a''=\gamma$, $A^{(i)}=P$, becomes as in [3951]; and if we put $A^{(i)}=P'$, we get [3951']. Substituting these in [3950], we get [3952].

We shall, in the next place, consider the following terms in the fourth line of δe [3910],*

$$\delta e=\frac{m'^2.t}{(5n'-2n).a}\cdot\left\{\left(\frac{dP'}{de}\right)\cdot\left(\frac{ddP}{de^2}\right)-\left(\frac{dP}{de}\right)\cdot\left(\frac{ddP'}{de^2}\right)+\left(\frac{dP'}{d\gamma}\right)\cdot\left(\frac{ddP}{de\,d\gamma}\right)-\left(\frac{dP}{d\gamma}\right)\cdot\left(\frac{ddP'}{de\,d\gamma}\right)\right\};$$ [3953]

and the terms in the third line of $\delta e'$ [3922],

$$\delta e'=\frac{m\,m'.t}{(5n'-2n).\sqrt{a\,a'}}\cdot\left\{\left(\frac{dP'}{de}\right)\cdot\left(\frac{ddP}{de\,de'}\right)-\left(\frac{dP}{de}\right)\cdot\left(\frac{ddP'}{de\,de'}\right)+\left(\frac{dP'}{d\gamma}\right)\cdot\left(\frac{ddP}{de'\,d\gamma}\right)-\left(\frac{dP}{d\gamma}\right)\cdot\left(\frac{ddP'}{de'\,d\gamma}\right)\right\};$$ [3954]

also the terms in the second line of $\delta\gamma$ [3935],

$$\delta\gamma=\frac{m'^2.t}{(5n'-2n).a}\cdot\frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}}\cdot\left\{\left(\frac{dP'}{de}\right)\cdot\left(\frac{ddP}{de\,d\gamma}\right)-\left(\frac{dP}{de}\right)\cdot\left(\frac{ddP'}{de\,d\gamma}\right)\right\};$$ [3955]

we shall have, by noticing these terms only, and observing that we have, as in [3934],

$$\left(\frac{dP'}{d\gamma}\right)\cdot\left(\frac{ddP}{d\gamma^2}\right)-\left(\frac{dP}{d\gamma}\right)\cdot\left(\frac{ddP'}{d\gamma^2}\right)=0;$$ [3956]

$$e\,\delta e\,.\,m\sqrt{a}+e'\,\delta e'.\,m'\sqrt{a'}+\frac{m\,m'\sqrt{a\,a'}}{m\sqrt{a}+m'\sqrt{a'}}\cdot\gamma\,\delta\gamma$$

$$=\frac{m'^2.\,m\,t}{(5n'-2n).\sqrt{a}}\cdot\left\{\begin{array}{l}\left(\frac{dP'}{de}\right)\cdot\left\{e\cdot\left(\frac{ddP}{de^2}\right)+e'\cdot\left(\frac{ddP}{de\,de'}\right)+\gamma\cdot\left(\frac{ddP}{de\,d\gamma}\right)\right\}\\ -\left(\frac{dP}{de}\right)\cdot\left\{e\cdot\left(\frac{ddP'}{de^2}\right)+e'\cdot\left(\frac{ddP'}{de\,de'}\right)+\gamma\cdot\left(\frac{ddP'}{de\,d\gamma}\right)\right\}\\ +\left(\frac{dP'}{d\gamma}\right)\cdot\left\{e\cdot\left(\frac{ddP}{de\,d\gamma}\right)+e'\cdot\left(\frac{ddP}{de'\,d\gamma}\right)+\gamma\cdot\left(\frac{ddP}{d\gamma^2}\right)\right\}\\ -\left(\frac{dP}{d\gamma}\right)\cdot\left\{e\cdot\left(\frac{ddP'}{de\,d\gamma}\right)+e'\cdot\left(\frac{ddP'}{de'\,d\gamma}\right)+\gamma\cdot\left(\frac{ddP'}{d\gamma^2}\right)\right\}\end{array}\right\};\dagger$$ [3957]

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* (2491) The part of δe in the fourth line of [3910], by the substitution of $a^2n^2=\frac{1}{a}$ [3946'], becomes as in [3953]. Again, we have $an=\frac{1}{\sqrt{a}}$, $a'n'=\frac{1}{\sqrt{a'}}$ [3946'], $a\,a'.\,n\,n'=\frac{1}{\sqrt{a\,a'}}$; substituting this in the third line $\delta e'$ [3922], it becomes as in [3954]. [3952a]
Lastly, substituting $a^2\,n^2=\frac{1}{a}$ [3746'], in the second line of $\delta\gamma$ [3935], it becomes as in [3955]. [3952b]

† (2492) Adding the two terms [3956] to the *two terms between the braces*, in the last factor of the expression of $\delta\gamma$ [3955]; it becomes of a symmetrical form with the values of δe, $\delta e'$ [3953, 3954]. Substituting these values of δe, $\delta e'$, $\delta\gamma$, in the first member of [3957], and connecting together the terms depending on the same factors of the first order, it becomes as in the second member of [3957]. [3957a] [3957b]

[3957'] $\left(\frac{dP}{de}\right)$ and $\left(\frac{dP'}{de}\right)$ are homogeneous in e, e', γ, and of the second dimension; hence we have*

[3958] $$e.\left(\frac{ddP}{de^2}\right)+e'.\left(\frac{ddP}{dede'}\right)+\gamma.\left(\frac{ddP}{ded\gamma}\right)=2.\left(\frac{dP}{de}\right);$$

[3958'] $$e.\left(\frac{ddP'}{de^2}\right)+e'.\left(\frac{ddP'}{dede'}\right)+\gamma.\left(\frac{ddP'}{ded\gamma}\right)=2.\left(\frac{dP'}{de}\right).$$

[3958''] Moreover $\left(\frac{dP}{d\gamma}\right)$, $\left(\frac{dP'}{d\gamma}\right)$ are homogeneous in e, e', γ, of the second dimension; therefore we have

[3959] $$e.\left(\frac{ddP}{ded\gamma}\right)+e'.\left(\frac{ddP}{de'd\gamma}\right)+\gamma.\left(\frac{ddP}{d\gamma^2}\right)=2.\left(\frac{dP}{d\gamma}\right);$$

[3959'] $$e.\left(\frac{ddP'}{ded\gamma}\right)+e'.\left(\frac{ddP'}{de'd\gamma}\right)+\gamma.\left(\frac{ddP'}{d\gamma^2}\right)=2.\left(\frac{dP'}{d\gamma}\right);$$

hence we find, by noticing these terms only,†

[3960] $$e\,\delta e\,.\,m\sqrt{a}+e'\,\delta e'.\,m'\sqrt{a'}+\frac{mm'\sqrt{aa'}}{m\sqrt{a}+m'\sqrt{a'}}.\,\gamma\,\delta\gamma=0.$$

Lastly, we shall consider the following terms of δe,‡ included in

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* (2493) It evidently appears from the values of P, P' [3842, 3843], that

[3958a] $\left(\frac{dP}{de}\right)$, $\left(\frac{dP'}{de}\right)$, $\left(\frac{dP}{d\gamma}\right)$, $\left(\frac{dP'}{d\gamma}\right)$ are homogeneous functions in e, e', γ, of the *second* degree, corresponding to the formula [1001a,], supposing $a=e$, $a'=e$, $a''=\gamma$, $m=2$. If we put, in this formula, $A^{(i)}=\left(\frac{dP}{de}\right)$, we get [3958]; and $A^{(i)}=\left(\frac{dP'}{de}\right)$ gives

[3958b] [3958']. In like manner, by putting successively, $A^{(i)}=\left(\frac{dP}{d\gamma}\right)$, $A^{(i)}=\left(\frac{dP'}{d\gamma}\right)$ [1001a], we get [3959, 3959'].

† (2494) Substituting the values [3958, 3958'] in the first and second lines of the

[3960a] second member of [3957], we find that these terms mutually destroy each other. In like

[3960b] manner, the terms in the third and fourth lines of [3957], are destroyed by the substitution of [3959, 3959']; and the whole expression becomes as in [3960].

‡ (2495) Substituting $aa'.nn'=\frac{1}{\sqrt{aa'}}$ [3952a], in the last lines of the values

[3961a] of δe, $\delta\gamma$ [3910, 3935], we get [3961, 3963], respectively. Putting $a'^2n'^2=\frac{1}{a'}$ [3952a], in the second line of $\delta e'$ [3922], we get [3962].

the seventh line of [3910],

$$\delta e = \frac{m m'.t}{(5n'-2n).\sqrt{a a'}} \cdot \left\{ \left(\frac{dP'}{de'}\right) \cdot \left(\frac{ddP}{de\,de'}\right) - \left(\frac{dP}{de'}\right) \cdot \left(\frac{ddP'}{de\,de'}\right) + \left(\frac{dP'}{d\gamma}\right) \cdot \left(\frac{ddP}{de\,d\gamma}\right) - \left(\frac{dP}{d\gamma}\right) \cdot \left(\frac{ddP'}{de\,d\gamma}\right) \right\}; \qquad [3961]$$

and the terms of $\delta e'$, in the second line of [3922], namely,

$$\delta e' = \frac{m^2.t}{(5n'-2n).a'} \cdot \left\{ \left(\frac{dP'}{de'}\right) \cdot \left(\frac{ddP}{de'^2}\right) - \left(\frac{dP}{de'}\right) \cdot \left(\frac{ddP'}{de'^2}\right) + \left(\frac{dP'}{d\gamma}\right) \cdot \left(\frac{ddP}{de'd\gamma}\right) - \left(\frac{dP}{d\gamma}\right) \cdot \left(\frac{ddP'}{de'd\gamma}\right) \right\}; \qquad [3962]$$

also those terms of $\delta\gamma$, in the third line of [3935],

$$\delta\gamma = \frac{m m'}{(5n'-2n).\sqrt{a a'}} \cdot \frac{(m\sqrt{a}+m'\sqrt{a'})}{m'\sqrt{a'}} \cdot t \cdot \left\{ \left(\frac{dP'}{de'}\right) \cdot \left(\frac{ddP}{de'd\gamma}\right) - \left(\frac{dP}{de'}\right) \cdot \left(\frac{ddP'}{de'd\gamma}\right) \right\}. \qquad [3963]$$

Hence we shall have, by noticing these terms only,*

The stability of the orbit of a planet is not disturbed by

$$e\,\delta e\,.\,m\sqrt{a} + e'\,\delta e'.\,m'\sqrt{a'} + \frac{m m'\sqrt{a a'}}{m\sqrt{a}+m'\sqrt{a'}} \cdot \gamma\,\delta\gamma = 0. \qquad [3964]$$

terms of the order of the [3964'] square of the disturbing forces.

Therefore the equations [3946, 3941] *hold good, even when we notice the terms depending on the square of the disturbing force* [3910, 3922, 3935].

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* (2496) Substituting the values of δe, $\delta e'$, $\delta\gamma$ [3961—3963], in the first member of [3964], and reducing, as in the preceding notes, by means of formulas similar to [3958—3959'], we shall find, that the terms mutually destroy each other. But without taking the trouble of writing down these formulas at full length, we may abridge the calculation, by the principle of derivation, in the following manner. If we multiply the values of δe, $\delta e'$, $\delta\gamma$ [3953, 3954, 3955], by the factor $\frac{m\sqrt{a}}{m'\sqrt{a'}}$, and in the terms which are connected with the two differential coefficients $\left(\frac{dP}{de}\right)$, $\left(\frac{dP}{de'}\right)$, change the partial differentials of P, P', of the *first order relative to* de, into those relative to de'; and in the differentials of the *second order*, de^2 into $de\,de'$, $de\,de'$ into de'^2, $de\,d\gamma$ into $de'd\gamma$, *the other differentials being unchanged*; we shall obtain the three expressions [3961, 3962, 3963], respectively. The same changes in the partial differentials may be made in [3958 3958']; as is evident by putting, in [1001a], $a=e$, $a'=e'$, $a''=\gamma$; and then $A^{(i)} = \left(\frac{dP}{de'}\right)$, to obtain the equation corresponding to [3958]; also $A^{(i)} = \left(\frac{dP'}{de'}\right)$, to obtain the equation corresponding to [3958']. To render the expression [3963] symmetrical, we may, as in [3957a], add the two terms [3956] to those between the braces in [3963]. Hence it is evident, that if we substitute these values of δe, $\delta e'$, $\delta\gamma$ [3961, 3962, 3963, 3964f], in the first member of [3957], the result will be equal to the second member of [3957], multiplied by the factor [3964b], changing also the partial

[3964a] [3964b] [3964c] [3964d] [3964e] [3964f] [3964g]

The *determination of the invariable plane*, given in §62, Book II, is founded on the three equations,*

[3965] $$c = m\sqrt{a.(1-e^2)}\,.\cos.\varphi + m'\sqrt{a'.(1-e'^2)}\,.\cos.\varphi' + \&c.;$$

[3965'] $$c' = m\sqrt{a.(1-e^2)}\,.\sin.\varphi\,.\cos.\theta + m'\sqrt{a'.(1-e'^2)}\,.\sin.\varphi'.\cos.\theta' + \&c.;$$

[3965''] $$c'' = m\sqrt{a.(1-e^2)}\,.\sin.\varphi\,.\sin.\theta + m'\sqrt{a'.(1-e'^2)}\,.\sin.\varphi'.\sin.\theta' + \&c.;$$

a and a' being constant, having regard even to the terms [3906'—3907],
[3965'''] depending on the square of the disturbing force. The first of these equations gives, by neglecting the products of four dimensions in e, e', &c., φ, φ', &c.,†

[3966] $$\text{constant} = (e^2+\varphi^2)\,.m\sqrt{a} + (e'^2+\varphi'^2)\,.m'\sqrt{a'} + \&c.;$$

and we have just seen, in [3964'], that the terms depending on the square
[3966'] of the disturbing force, do not affect the accuracy of this equation. The

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[3964h] differentials, as in [3964c]. Now the third and fourth lines of the terms between the braces, in the second member of [3957], remain unchanged [3961d]; they must therefore vanish, as in [3960^{i}], by the substitution of the expressions [3959, 3959']. In like
[3964i] manner, the first and second lines vanish, as in [3960a], by the substitution of the two equations found in [3964e], corresponding to [3958, 3958']. Hence the second member wholly vanishes, and the result becomes as in [3964]. We may remark, that this
[3964k] demonstration is restricted to terms having the small divisor $(5n'-2n)$; but it is extended to other terms in [5935, &c.].

* (2197) Substituting $(1+\text{tang.}^2\varphi)^{-\frac{1}{2}} = \cos.\varphi$; $(1+\text{tang.}^2\varphi')^{-\frac{1}{2}} = \cos.\varphi'$, &c.
[3965a] in [1151], it becomes as in [3965]. Making the same substitutions in c', c'' [1158, 1159], and putting also, as in [1156],

[3965b] $$p\,.\cos.\varphi = \sin.\varphi\,.\sin.\theta; \quad q\,.\cos.\varphi = \sin.\varphi\,.\cos.\theta; \quad p'.\cos.\varphi' = \sin.\varphi'.\sin.\theta', \quad \&c.,$$

we get [3965', 3965''] It may be remarked, that the quantities c', c'', are in the original
[3965c] work called c'', c', respectively; they are here altered so as to conform to the notation in [1158, 1159].

† (2198) If we neglect terms of the order e^4, φ^4, we shall have

[3966a] $$\sqrt{a.(1-e^2)} = (1-\tfrac{1}{2}e^2)\,.\sqrt{a}, \qquad \cos.\varphi = 1-\tfrac{1}{2}\varphi^2 \quad [41]\ \text{Int.};$$

hence $m\sqrt{a.(1-e^2)}\,.\cos.\varphi = m\sqrt{a} - \frac{1}{2}\,.(e^2+\varphi^2)\,.m\sqrt{a}$; substituting this and the similar terms of a', e', φ', &c., in [3965], it becomes

[3966b] $$c = m\sqrt{a} + m'\sqrt{a'} + \&c. - \tfrac{1}{2}\,.\{(e^2+\varphi^2)\,.m\sqrt{a} + (e'^2+\varphi'^2)\,.m'\sqrt{a'} + \&c.\}.$$

Multiplying this by -2, and transposing the constant terms $-2m\sqrt{a}$, $-2m'\sqrt{a'}-$&c. to the first member, we get [3966].

equation [3965″] gives, by neglecting the products of three dimensions in e, e', &c., φ, φ', &c.,*

$$\delta \,.\, (\varphi \,.\, \sin.\, \theta) \,.\, m\sqrt{a} + \delta \,.\, (\varphi' .\, \sin.\, \theta') \,.\, m'\sqrt{a'} + \&c. = 0. \qquad [3967]$$

Now if we notice only the terms depending on the square of the disturbing force [3931—3936],† this equation will hold good; therefore the expression [3967′]

$$c'' = m\sqrt{a.(1-e^2)} \,.\, \sin.\, \varphi \,.\, \sin.\, \theta + m'\sqrt{a'.(1-e'^2)} \,.\, \sin.\, \varphi' .\, \sin.\, \theta' + \&c. \qquad [3968]$$

[3965″], will not be affected by these terms. In like manner, we find, that a similar result is obtained from the equation [3965′], [3968′]

$$c' = m\sqrt{a.(1-e^2)} \,.\, \sin.\, \varphi \,.\, \cos.\, \theta + m'\sqrt{a'.(1-e'^2)} \,.\, \sin.\, \varphi' .\, \cos.\, \theta' + \&c. \qquad [3969]$$

Hence the invariable plane, determined in § 62, of the second book [1162, 1162′], *remains unchanged, even when we notice these terms depending on the square of the disturbing force.* [3969′]

16. *The terms depending on the square of the disturbing force, have a sensible influence on the two great inequalities of Jupiter and Saturn;*‡ *we* [3969″]

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* (2499) Neglecting terms of the order φ^3, φ'^3, we may put $\sin.\varphi = \varphi$; $\sin.\varphi' = \varphi'$, &c. [43] Int. If we also neglect terms of the order $e^2\varphi$, $e'^2\varphi'$, &c., the equation [3965″] may be put under the form $c'' = (\varphi \,.\, \sin.\, \theta) \,.\, m\sqrt{a} + (\varphi' .\, \sin.\, \theta') \,.\, m'\sqrt{a'} + \&c.$; and [3967a] if we take the variation relatively to the characteristic δ, it becomes as in [3967].

† (2500) The terms here referred to, are those mentioned in [3943′], and computed for two planets in [3929—3933′]. The equations [3930, 3930′] may be put under the following forms, [3968a]

$$\delta \,.\, (\varphi \,.\, \sin.\, \theta) \,.\, m\sqrt{a} + \delta \,.\, (\varphi' .\, \sin.\, \theta') \,.\, m'\sqrt{a'} = 0\,;$$
$$\delta \,.\, (\varphi \,.\cos.\, \theta) \,.\, m\sqrt{a} + \delta \,.\, (\varphi' .\, \cos.\, \theta') \,.\, m'\sqrt{a'} = 0. \qquad [3968b]$$

In the same manner, other planets produce similar expressions, and the sum of all the equations, corresponding to the first, forms the equation [3967]; a similar equation may [3968c] also be obtained from the sum of the equations of the second form.

‡ (2501) Substituting the expressions [3756b, c, e], in δR [3764], it becomes as in [3970]; observing, that the coefficients of $h^2 + l^2$, $h'^2 + l'^2$ [3764], are equal to each other, as appears by multiplying [3752d] by -4.

shall proceed to determine the most considerable of these terms. We have seen, in [3764], that the expression of R or δR contains the function

$$\delta R = \frac{m'}{8}\cdot(e^2+e'^2)\cdot\left\{2a\cdot\left(\frac{dA^{(0)}}{da}\right)+a^2\cdot\left(\frac{ddA^{(0)}}{da^2}\right)\right\}$$

[3970] $$+\frac{m'}{4}\cdot ee'\cdot\cos.(\varpi'-\varpi)\cdot\left\{4A^{(1)}+2a\cdot\left(\frac{dA^{(1)}}{da}\right)+2a'\cdot\left(\frac{dA^{(1)}}{da'}\right)+aa'\cdot\left(\frac{ddA^{(1)}}{da\,da'}\right)\right\}$$

$$+\frac{m'}{8}\cdot aa'\cdot B^{(1)}\cdot\gamma^2.$$

[3970′] If we increase the quantities e, e', ϖ, ϖ', γ, in this expression, by their variations, depending on the angle $5n't-2nt$,* we shall obtain in R some terms depending on the same angle; and it would seem, on account of the divisor $5n'-2n$, connected with these variations, that these terms might become sensible. But we must observe, that this divisor
[3970″] disappears in $\mathrm{d}R$, because the differential characteristic d, refers only to the co-ordinates of m, or to the variations of e, ϖ [916′]; so that it introduces the multiplicator $5n'-2n$. Now we have seen, that the great
[3970‴] inequality of m depends chiefly on the term $3a\int\int n\,dt\,.\,\mathrm{d}R$ [1070″]. The inequalities of the radius vector and the longitude, which depend on the variations of the excentricities and perihelion, relative to the angle
[3971] $5n't-2nt$, have therefore very little influence on the two great inequalities of Jupiter and Saturn.

We shall see hereafter [4392, &c., 4466, &c.], that the most sensible inequalities of these two planets, depending on the simple excentricities

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* (2502) The variation of e, e', ϖ, &c., here referred to, are those represented
[3970a] by δe, $\delta e'$, $\delta\varpi$, &c. [3907b, c, d]; all of which have the divisor $5n'-2n$ [3907a]; but the divisor is destroyed in finding their differentials de, $d\varpi$, &c., as is evident from [3908c, &c.]. Hence it follows, that the differential of the expression [3970] gives,
[3970b] in $\mathrm{d}\delta R$ or $\mathrm{d}R$, terms depending on $e\,de$, $ee'\,d\varpi$, &c., which do not contain this divisor; and if we substitute them in the chief term of the great inequality [3970‴], they will produce terms which are of the order m'^2, or of the order m', in comparison
[3970c] with the chief terms computed in [3844, 4418, 4174]; but as these terms of the order m'^2,
[3970d] have the same divisor $(5n-2n)^2$, as the chief term, it seems proper to examine carefully into their exact values, instead of neglecting them, as the author has done. We
[3970e] shall also see, in [4006t, &c., 4431f], that several terms, omitted by the author, similar to those treated of in this article, are quite as important as those which he has retained.

of the orbits, are relative to the angle $n t - 2 n' t$. We shall put*

$$\frac{\delta r}{a} = F . \cos . (n t - 2 n' t + \varepsilon - 2 \varepsilon' + A), \qquad [3972]$$

for the term of $\frac{\delta r}{a}$, depending on this angle; and

$$\delta v = E . \sin . (n t - 2 n' t + \varepsilon - 2 \varepsilon' + B), \qquad [3973]$$

for the term of δv, depending on the same angle; also for the corresponding terms of $\frac{\delta r'}{a'}$ and $\delta v'$,

$$\frac{\delta r'}{a'} = F' . \cos . (n t - 2 n' t + \varepsilon - 2 \varepsilon' + A'); \qquad [3974]$$

$$\delta v' = E' . \sin . (n t - 2 n' t + \varepsilon - 2 \varepsilon' + B'). \qquad [3974']$$

If we suppose that R *corresponds to Saturn, disturbed by Jupiter*, and [3974'']
then develop it relatively to the squares and products of the excentricities
and inclinations of the orbits, noticing only the angle $3 n' t - n t$, we shall [3974''']
obtain, as in [3745, &c.], a function of this form,†

$$\begin{aligned} R = \; & M^{(0)} . e'^2 . \cos . (3 n' t - n t + 3 \varepsilon' - \varepsilon - 2 \varpi') \\ & + M^{(1)} . e e' . \cos . (3 n' t - n t + 3 \varepsilon' - \varepsilon - \varpi - \varpi') \\ & + M^{(2)} . e^2 . \cos . (3 n' t - n t + 3 \varepsilon' - \varepsilon - 2 \varpi) \\ & + M^{(3)} . \gamma^2 . \cos . (3 n' t - n t + 3 \varepsilon' - \varepsilon - 2 \Pi). \end{aligned} \qquad [3975]$$

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* (2503) The terms of δv [4392], depending on the angle $n t - 2 n' t$, or rather on $2 n^{\rm v} t - n^{\rm iv} t$, are of the order 138'' or 56'', and may be reduced to the form [3973]; [3973a]
those of $\delta v'$ [4466] are of the order 182'', 417'', and may be reduced to the form [3974'];
they are the largest terms of the expressions [4392, 4666]. In like manner, the parts
of $\frac{\delta r}{a}$, $\frac{\delta r'}{a'}$ [4393, 4467], may be reduced to the forms [3972, 3974]; the last of [3973b]
which is the greatest term of [4467].

† (2504) This value of R is similar to that assumed in [3745—3745'''], changing
reciprocally the elements of m' into those of m; also $M^{(2)}$ into $M^{(0)}$, $M^{(0)}$ into $M^{(2)}$; [3975a]
and afterwards putting $i = -1$. This form of the angles in the value of R, is selected
because it produces, in connexion with the variations [3972—3974'], terms in dR, $d'R$, [3975b]
of the order m^2, depending on the same angle $5 n' t - 2 n t$, as the great inequality, as is
seen, in [3979, 3982, 3985, 3989, 3991]. We may remark incidentally, that in this article

[3976] The quantity $M^{(0)}.e'^2.\cos.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi')$ arises from the development of the term of R, denoted by $A^{(1)}.\cos.(v'-v)$;* in which
[3976'] we must increase r by δr, r' by $\delta r'$, v by δv, v' by $\delta v'$. This is the same as to increase, in the development of this term, a by δr,
[3977] a' by $\delta r'$, and $n't-nt$ by $\delta v'-\delta v$; by which means it produces the following expression,†

[3978]
$$\begin{aligned}R = &-M^{(0)}.e'^2.(\delta v'-\delta v).\sin.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi')\\ &+a.\left(\frac{dM^{(0)}}{da}\right).e'^2.\frac{\delta r}{a}.\cos.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi')\\ &+a'.\left(\frac{dM^{(0)}}{da'}\right).e'^2.\frac{\delta r'}{a'}.\cos.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi').\end{aligned}$$

[3975c] the values R, $R_{,}$, differ from those in other parts of the work; since R, $R_{,}$ [3974'', 4005'] take the place of R', R [1199'], respectively; m being the mass of Jupiter, m' that of Saturn. The object of the author, in making this change in the value of R, is to
[3975d] obtain express formulas for the *direct computation* of the inequalities of Saturn, which are much *larger* than those of Jupiter; and then to deduce the corresponding *smaller* ones
[3975e] of Jupiter, by means of the formula [1208]; it being evident, that this method of deduction, *in the cases where it can be applied*, must be more accurate in finding the *small* inequalities of Jupiter from the *large* ones of Saturn, than in an inverse process.

* (2505) The part of R, independent of γ^2, *corresponding to the action of Jupiter*
[3976a] *upon Saturn*, is found by changing, in [3742], m', r, r', v, v', into m, r', r, v', v,
[3976b] respectively; and if we suppose, that when a, a', $nt+\varepsilon$, $n't+\varepsilon'$, are changed into r', r, v', v, respectively, the quantity $A^{(i)}$ [3743] becomes $A_1^{(i)}$, we shall get, from [3742, 3743], *for this part of* R, the following expression,

[3976c] $$R=\frac{mr'}{r^2}.\cos.(v'-v)-\frac{m}{\sqrt{\{r^2-2rr'.\cos.(v'-v)+r'^2\}}}=\frac{m}{2}.\Sigma.A_1^{(i)}.\cos.i.(v'-v).$$

Substituting in this the values of r, r', v, v' [952, 953], we obtain an expression of R,
[3976d] of the same form as [957], and possessing the properties mentioned in [957—963]; moreover, the term multiplied by the factor e'^2, being represented by

[3976e] $$M^{(0)}.e'^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2n't+2\varepsilon'-2\varpi'\}\quad[957—959'],$$

becomes of the form [3976], by putting $i=1$; then the corresponding term of R [3976c] is of the same form as in [3976'].

† (2506) The term $M^{(0)}.e'^2.\cos.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi')$ [3975], is produced in the function R, by a development similar to that which is used in [957], that is, by
[3977a] the substitution of the *elliptical* values of $u_{,}$, $v_{,}$, &c., without noticing the perturbations [3972—3974']. If we wish also to notice these terms, we may suppose a, a', r, r', to be

This produces in R, the terms*

$$R = -\tfrac{1}{2} M^{(0)}.\, E'.\, e'^2.\cos.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - 2\varpi' - B')$$
$$+\tfrac{1}{2} M^{(0)}.\, E.\, e'^2.\cos.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - 2\varpi' - B)$$
$$+\tfrac{1}{2} a'.\left(\frac{d.M^{(0)}}{da'}\right).\, F'.\, e'^2.\cos.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - 2\varpi' - A')$$
$$+\tfrac{1}{2} a.\left(\frac{d.M^{(0)}}{da}\right).\, F.\, e'^2.\cos.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - 2\varpi' - A). \qquad [3979]$$

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increased, respectively, by δr, $\delta r'$, δv, $\delta v'$; by which means $A^{(i)}.\cos.i.(v'-v)$ will be augmented by the three terms in the second member of the following expression, in which we have retained the factor $i=1$, for the purpose of more easy derivation hereafter; [3977a']

$$\delta.\{A^{(i)}.\cos.i.(v'-v)\} = -A^{(i)}.i.(\delta v' - \delta v).\sin.i.(v'-v)$$
$$+a.\left(\frac{d.A^{(i)}}{da}\right).\frac{\delta r}{a}.\cos.i.(v'-v) + a'.\left(\frac{d.A^{(i)}}{da'}\right).\frac{\delta r'}{a'}.\cos.i.(v'-v); \qquad [3977b]$$

and in the same manner as we have derived from $A^{(i)}.\cos.i.(v'-v)$ the term

$$M^{(0)}.e'^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2n't+2\varepsilon'-2\varpi'\} \qquad [3976e], \qquad [3977c]$$

we may derive the three terms [3978] from those in [3977b]. Thus the first term of the second member of [3977b] is the variation of $A^{(i)}.\cos.i.(v'-v)$ or of $A^{(1)}.\cos.(v'-v)$, supposing the angle $i.(v'-v)$ to increase by $i.(\delta v'-\delta v)$; in like manner, the first line of [3978] is the variation of the term [3977d]

$$M^{(0)}.c'^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2n't+2\varepsilon'-2\varpi'\}, \qquad [3977\varepsilon]$$

supposing the angle $i.(n't-nt+\varepsilon'-\varepsilon)$, corresponding to $i.(v'-v)$, to increase by the same quantity $\delta v'-\delta v$. The second line of [3978] is deduced from the second term in the second member of [3977b], by supposing a to be increased by δr in $A^{(1)}$ and $M^{(0)}$. Lastly, the third line of [3978] is derived from the third term of the second member of [3977b], by supposing a' to be increased by $\delta r'$, in $A^{(1)}$ and $M^{(0)}$. [3977e] [3977f]

* (2507) The expression [3979] is deduced from [3978] by the substitution of [3972—3974], and reducing by [17—20] Int., retaining only the angles which are similar to that of the great inequality, depending on [3979a]

$$5n't - 2nt = (3n't - nt) - (nt - 2n't); \qquad [3979b]$$

or the difference between the angles contained in [3978] and those in [3972—3974].

d' R. [3980] *We shall put* $\mathrm{d}'R$ *for the differential of* R, *supposing the co-ordinates of* m' *to be the only variable quantities.* In the terms multiplied by E'
[3981] and F', the part $5n't-nt$, of the angle $5n't-2nt$,* is relative to these co-ordinates. In the terms multiplied by E and F, the part $3n't$ of the same angle $5n't-2nt$, is relative to the same co-ordinates; therefore we shall have, by noticing only the preceding terms of R [3979],

[3982]
$$\begin{aligned} a'\mathrm{d}'R = {} & \tfrac{1}{2}.(5n'-n).dt.a'M^{(0)}.E'.e'^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi'-B') \\ & -\tfrac{1}{2}.(5n'-n).dt.a'^2.\left(\frac{dM^{(0)}}{da'}\right).F'.e'^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi'-A') \\ & -\tfrac{3}{2}.n'dt.a'M^{(0)}.E.e'^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi'-B) \\ & -\tfrac{3}{2}.n'dt.aa'.\left(\frac{dM^{(0)}}{da}\right).F.e'^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi'-A). \end{aligned}$$

[3983] The term $M^{(1)}.ee'.\cos.(3n't-nt+3\varepsilon'-\varepsilon-\varpi-\varpi')$ [3975], results from the development of $A^{(2)}.\cos.2.(v'-v)$, in the expression

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* (2508) The differential relative to d' [3980], does not affect nt in the angle
[3982a] $3n't-nt$, which occurs explicitly in [3975], so that $\mathrm{d}'.(3n't-nt)=3n'dt$; but
[3982b] this characteristic d' affects the *whole* of the values of $\frac{\delta r'}{a'}$, $\delta v'$ [3974, 3974'], connected
[3982b'] with F', E', consequently the whole of the angle $nt-2n't$, which occurs in these values, must be considered as variable, and its differential is $ndt-2n'dt$. The difference of these two expressions gives

[3982c]
$$\mathrm{d}'.(5n't-2nt)=\mathrm{d}'.(3n't-nt)-\mathrm{d}'.(nt-2n't)=(5n'-n).dt;$$

which must be taken for the differential of the angle $5n't-2nt$ [3979b], depending on E', F', in the first and third lines of [3979]; hence we obtain the first and second
[3982d] lines of [3982]. In like manner, the differential relative to d' does not affect the
[3982e] expressions of $\frac{\delta r}{a}$, δv [3972, 3973], connected with the factors F, E; or in other
[3982f] words, the differential of the angle $nt-2n't$, connected with these factors, must vanish; and we shall have $\mathrm{d}'.(nt-2n't)=0$; subtracting this from [3982a], we get, in this case, for the differential of [3979b],

[3982g]
$$\mathrm{d}'.(5n't-2nt)=\mathrm{d}'.(3n't-nt)-\mathrm{d}'.(nt-2n't)=3n'dt.$$

[3982h] Substituting this in the differential of the second and fourth lines of [3979], we get, respectively, the third and fourth lines of [3982].

of R.* Therefore we must vary, in this term, a by δr, a' by $\delta r'$, [3983']
also $2n't-2nt$ by $2\delta v'-2\delta v$; and by this means we obtain the following terms of R,

$$R=-2M^{(1)}.ee'.(\delta v'-\delta v).\sin.(3n't-nt+3\varepsilon'-\varepsilon-\varpi-\varpi')$$
$$+a.\left(\frac{dM^{(1)}}{da}\right).ee'.\frac{\delta r}{a}.\cos.(3n't-nt+3\varepsilon'-\varepsilon-\varpi-\varpi') \qquad [3984]$$
$$+a'.\left(\frac{dM^{(1)}}{da'}\right).ee'.\frac{\delta r'}{a'}.\cos.(3n't-nt+3\varepsilon'-\varepsilon-\varpi-\varpi').$$

Hence the part of $a'\mathrm{d}'R$, relative to this expression, is

$$a'\mathrm{d}'R=\quad(5n'-n).dt.a'M^{(1)}.E'.ee'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-\varpi'-B')$$
$$-\tfrac{1}{2}.(5n'-n).dt.a'^2.\left(\frac{dM^{(1)}}{da'}\right).F'.ee'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-\varpi'-A')$$
$$-3n'dt.a'M^{(1)}.E.ee'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-\varpi'-B) \qquad [3985]$$
$$-\tfrac{3}{2}n'dt.aa'.\left(\frac{dM^{(1)}}{da}\right).F.ee'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi-\varpi'-A).$$

The term $M^{(2)}.e^2.\cos.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi)$ [3975], arises [3986]
from the development of $A^{(3)}.\cos.(3v'-3v)$, in the expression

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* (2509) Proceeding with the term depending on $M^{(1)}$. [3975], in the same manner as we have done with that multiplied by $M^{(0)}$, in the three preceding notes, we find, that it may be put under the form

$$M^{(1)}.ee'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+n't+nt+\varepsilon'+\varepsilon-\varpi'-\varpi\}, \qquad [3984a]$$

supposing $i=2$; by which means it becomes as in the second line of [3975], and the corresponding term of [3976c], is of the form

$$\tfrac{1}{2}m.A_{,}^{(i)}.\cos.i.(v'-v)=A^{(2)}.\cos.2.(v'-v). \qquad [3984b]$$

The variations of this term, depending on δr, $\delta r'$, δv, $\delta v'$, are as in [3977b], supposing $i=2$; and from these we may deduce the functions [3984, 3985], by a computation similar to that used in finding [3978, 3982]. We may, however, obtain the former by derivation in a more simple manner; for if we change $M^{(0)}$, e'^2, $-2\varpi'$, into [3984c]
$M^{(1)}$, ee', $-\varpi-\varpi'$, respectively, we shall find, that the first term of [3975] becomes like the second; and the doubling the values of $\delta v'$, δv, in [3977b], on account of the [3984d]
factor $i=2$, make it necessary that we should double the values of E, E' [3973, 3974']. Making these changes in [3978, 3982], they become, respectively, as in [3984, 3985].

of R.* Therefore we must vary, in this term, a by δr, a' by $\delta r'$, and
[3987] $3n't-3nt$ by $3\delta v'-3\delta v$; hence we get the following terms of R,

[3988]
$$\begin{aligned} R = & -3M^{(2)}.e^2.(\delta v'-\delta v).\sin.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi) \\ & + a.\left(\frac{dM^{(2)}}{da}\right).e^2.\frac{\delta r}{a}.\cos.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi) \\ & + a'.\left(\frac{dM^{(2)}}{da'}\right).e^2.\frac{\delta r'}{a'}.\cos.(3n't-nt+3\varepsilon'-\varepsilon-2\varpi). \end{aligned}$$

Therefore the part of $a'\text{d}'R$, relative to this expression, is

[3989]
$$\begin{aligned} a'\text{d}'R = & \tfrac{3}{2}.(5n'-n).dt.a'M^{(2)}.E'.e^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi-B') \\ & -\tfrac{1}{2}.(5n'-n).dt.a'^2.\left(\frac{dM^{(2)}}{da'}\right).F'.e^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi-A') \\ & -\tfrac{9}{2}.n'dt.a'M^{(2)}.E.e^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi-B) \\ & -\tfrac{3}{2}.n'dt.aa'.\left(\frac{dM^{(2)}}{da}\right).F.e^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\varpi-A). \end{aligned}$$

[3989′] Lastly, the term $M^{(3)}.\gamma^2.\cos.(3n't-nt+3\varepsilon'-\varepsilon-2\,\Pi)$ [3975].
[3989″] arises from the term multiplied by $\gamma^2.\cos.(3v'-v)$, in the expression of R;†

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* (2510) Proceeding as in the last note, we may put the term [3975], depending on $M^{(2)}$, under the form

[3988a]
$$M^{(2)}.e^2.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-2\varpi\},$$

supposing $i=3$; and then the corresponding term of [3976c] is of the form

[3988b]
$$\tfrac{1}{2}m'.A_{,}^{(i)}.\cos.i.(v'-v)=A^{(3)}.\cos.3.(v'-v).$$

The variations of this term are as in [3977b], supposing $i=3$; from which we may get [3988, 3989], in the same manner as [3978, 3982] were found. The same result may be obtained more easily by derivation, as in the last note; by changing, in [3975, &c.],
[3988c] $M^{(0)}$, e'^2, $A^{(0)}$, $2\varpi'$, into $M^{(2)}$, e^2, $A^{(3)}$, 2ϖ, respectively; by which means the first term of [3975], changes into the third; and the trebling of the values of $\delta v'$, δv, in
[3988d] [3977b], on account of the factor $i=3$, makes it necessary to change E, E' [3973, 3974′] into $3E$, $3E'$, respectively. Making these changes in [3978, 3982], they become as in [3988, 3989], respectively.

† (2511) We must now compute the terms arising from the introduction of the increments
[3990a] δr, $\delta r'$, δv, $\delta v'$, in the expressions of r, r', v, v', connected with the factor γ^2, in the value of R [3742]; which were neglected in [3976a]. These terms of R may be

we must therefore vary a by δr, a' by $\delta r'$, $3n't$ by $3\delta v'$, and nt by δv; hence we obtain the following terms,

$$\begin{aligned} R = & - M^{(3)}.\gamma^2.(3\delta v' - \delta v).\sin.(3n't - nt + 3\varepsilon' - \varepsilon - 2\Pi) \\ & + a.\left(\frac{dM^{(3)}}{da}\right).\gamma^2.\frac{\delta r}{a}.\cos.(3n't - nt + 3\varepsilon' - \varepsilon - 2\Pi) \\ & + a'.\left(\frac{dM^{(3)}}{da'}\right).\gamma^2.\frac{\delta r'}{a'}.\cos.(3n't - nt + 3\varepsilon' - \varepsilon - 2\Pi). \end{aligned}$$ [3990]

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deduced from those depending on γ^2, in [3742], by changing the elements as in [3976a]. These four terms of R [3742] are already multiplied by the factor γ^2, of the second dimension, and as none of a higher order are noticed in [3975], we may substitute in these terms, $r=a$, $r'=a'$, $v=nt+\varepsilon-\Pi$, $v'=n't+\varepsilon'-\Pi$; and retain only [3990b] angles of the form $3n't-nt$, assumed in [3975]. Now it is evident, that the two first of these terms of R [3742], depending on the angles $\cos.(v'-v)$, $\cos.(v'+v)$, [3990c] produce the angles $n't-nt$, $n't+nt$, which are not included in the proposed form. The third of these terms [3742] contains $v'-v$ in its numerator and denominator, [3990d] and when the denominator is developed, as in [3744], the whole term will depend on quantities of the form $\cos.i.(v'-v)$ or $\cos.i.(n't-nt)$, which are not comprised in the form $3n't-nt$, now under consideration; so that we need only retain the last [3990d'] term of [3742], which, by making the changes indicated in [3976a], may be put under the form $R = -\frac{m\gamma^2}{4}.\frac{rr'.\cos.(v'+v)}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{3}{2}}}$. Now if in the formula [3744], [3990e] we change a, a', $nt+\varepsilon$, $n't+\varepsilon'$, $B^{(i)}$, into r, r', v, v', $B_{\prime}^{(i)}$, we shall get

$$\{r^2 - 2rr'.\cos.(v'-v) + r'^2\}^{-\frac{3}{2}} = \tfrac{1}{2}\Sigma.B_{\prime}^{(i)}.\cos.i.(v'-v). \qquad [3990f]$$

Substituting this in R [3990e], and reducing by means of formula [3749], it becomes

$$R = -\tfrac{1}{4}m.\gamma^2.rr'.\tfrac{1}{2}\Sigma.B_{\prime}^{(i)}.\cos.\{i.(v'-v)+v'+v\}. \qquad [3990f']$$

If we change i into $i-1$, and put $-\frac{1}{8}m.rr'.B_{\prime}^{(i-1)} = M^{(i)}$, we get

$$R = \gamma^2.\Sigma.M^{(i)}.\cos.\{i.(v'-v)+2v\}; \qquad [3990g]$$

which in the case of $i=3$, produces a term of the form $R = M^{(3)}.\gamma^2.\cos.(3v'-v)$. [3990h] Taking the variations of this term, as in [3977a', &c.], we get the following expression, similar to [3977b],

$$\begin{aligned} \delta.\{M^{(3)}.\gamma^2.\cos.(3v'-v)\} = & -M^{(3)}.\gamma^2.(3\delta v'-\delta v).\sin.(3v'-v) \\ & + a.\left(\frac{dM^{(3)}}{da}\right).\gamma^2.\frac{\delta r}{a}.\cos.(3v'-v) + a'.\left(\frac{dM^{(3)}}{da'}\right).\gamma^2.\frac{\delta r'}{a'}.\cos.(3v'-v). \end{aligned}$$ [3990i]

Substituting in this the values [3990b], we obtain [3990].

Hence we obtain in $a'\,\mathrm{d}'R$, the following terms,*

[3991]
$$\begin{aligned}a'\mathrm{d}'R = {} & \tfrac{3}{2}.(5n'-n).dt.a'M^{(3)}.E'.\gamma^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\Pi-B') \\ & -\tfrac{1}{2}.(5n'-n).dt.a'^2.\left(\frac{dM^{(3)}}{da'}\right).F'.\gamma^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\Pi-A') \\ & -\tfrac{3}{2}n'dt.a'M^{(3)}.E.\gamma^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\Pi-B) \\ & -\tfrac{3}{2}n'dt.aa'.\left(\frac{dM^{(3)}}{da}\right).F.\gamma^2.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-2\Pi-A).\end{aligned}$$

The most sensible inequalities, arising from the squares and products of the
[3991'] excentricities and inclinations of the orbits, which neither have $5n'-2n$†
for a divisor, nor depend upon the variations of the elements relative to the

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* (2512) The expression [3991] is deduced from [3990], in the same manner as [3982] is from [3978]; or more easily by the principle of derivation. For if we change
[3991a] $M^{(0)}$, e'^2, $\delta v'$, $-2\varpi'$, into $M^{(3)}$, γ^2, $3\delta v'$, -2Π, respectively, the function [3978] will become as in [3990]; consequently E' [3974'] must be changed, as in [3984d],
[3991b] into $3E'$. Making the same changes in [3982], which was deduced from [3978], we get [3991].

† (2513) The divisors in [3714, 3715], which may be small, in the theory of the perturbations of Jupiter and Saturn, are $in'+(3-i).n$, $in'+(1-i).n$, $in'+(2-i).n$;
[3992a] and since $n'=\frac{2}{5}n$ nearly [3818d], they become $(3-\frac{3}{5}i).n$, $(1-\frac{3}{5}i).n$, $(2-\frac{3}{5}i).n$. If we put $i=5$, the first divisor becomes 0, the others being large. If $i=4$, the
[3992b] last divisor becomes $-\frac{2}{5}n$, and the others are larger. If $i=3$, the last divisor becomes $\frac{1}{5}n$, and the others are greater then this quantity; and it is evident, that next to $i=5$, this value of i gives the least value to the divisors [3992a]; therefore the terms of $r\delta r$, δv [3714, 3715], of the second order, relative to the quantities e, e', γ, and depending on the angle $3n't-nt$, may be increased by this divisor, so as to become
[3992c] greater than other terms of the same order, relative to e, e', γ, which have not a small divisor. This reasoning is confirmed *a posteriori* by the inspection of the numerical values of δr^{iv}, δr^{v}, δv^{iv}, δv^{v} [4397, 4470, 4394, 4468], in which the terms depending on the angle $3n't-nt$, are generally greater than any of those that are noticed in [3991'],
[3992d] excepting $4n't-2nt$. This last angle is here neglected, because the terms δr, δv, &c., depending upon it, do not produce in [3995], functions of the form [3998], depending on the angle $5n't-2nt$, which are the only ones under consideration at the present moment. Now if we notice only the terms depending on the angle $3n't-nt$, in
[3992e] [3714, 3715], we shall obtain for $\frac{\delta r}{a}$, δv, quantities of the forms [3992, 3993], and in like manner, in $\frac{\delta r'}{a'}$, $\delta v'$, terms of the forms [3994, 3994'].

angle $5n't - 2nt$, are those corresponding to the angle $3n't - nt$. We shall put

$$\frac{\delta r}{a} = G \cdot \cos. (3n't - nt + 3\varepsilon' - \varepsilon + C), \qquad [3992]$$

for the part of $\frac{\delta r}{a}$, depending on this angle; also

$$\delta v = H \cdot \sin. (3n't - nt + 3\varepsilon' - \varepsilon + D), \qquad [3993]$$

for the part of δv, depending on the same angle; in like manner,

$$\frac{\delta r'}{a'} = G' \cdot \cos. (3n't - nt + 3\varepsilon' - \varepsilon + C'), \qquad [3994]$$

$$\delta v' = H' \cdot \sin. (3n't - nt + 3\varepsilon' - \varepsilon + D'), \qquad [3994']$$

for the parts of $\frac{\delta r'}{a'}$, $\delta v'$, depending on the same angle. The expression of R, developed relative to the first power of the excentricities, contains the two following terms,*

$$\begin{aligned} R = \; & N^{(0)} \cdot e \cdot \cos. (nt - 2n't + \varepsilon - 2\varepsilon' + \varpi) \\ & + N^{(1)} \cdot e' \cdot \cos. (nt - 2n't + \varepsilon - 2\varepsilon' + \varpi'). \end{aligned} \qquad [3995]$$

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* (2514) In the same manner as we have deduced, from R [3976c], the three terms [3976e, 3981a, 3988a], of the second order in e, e', we may obtain two of the first order in e, e', of the following forms, [3995a]

$$\begin{aligned} R = \; & \mathcal{N}^{(0)} \cdot e \cdot \cos. \{ i \cdot (n't - nt + \varepsilon' - \varepsilon) + nt + \varepsilon - \varpi \} \\ & + \mathcal{N}^{(1)} \cdot e' \cdot \cos. \{ i \cdot (n't - nt + \varepsilon' - \varepsilon) + n't + \varepsilon' - \varpi' \}. \end{aligned} \qquad [3995b]$$

If we put $i = 2$, in the first of these terms, it becomes of the same form as the first term of [3995]; and by proceeding in like manner as in note 2506, we easily perceive that this term arises from the development of $\mathcal{A}^{(i)} \cdot \cos. i \cdot (v' - v)$, supposing $i = 2$, as in [3995c]. Moreover the second term of R [3995b], becomes of the same form as the second term of [3995], by putting $i = 1$; and then the term $\mathcal{A}^{(i)} \cdot \cos. i \cdot (v' - v)$, upon which it depends, becomes $\mathcal{A}^{(1)} \cdot \cos. (v' - v)$, as in [3998']. [3995c] [3995d] [3995e] [3995f]

We have already computed, in the case of $i = 2$, the effect of the substitution of the variations δr, $\delta r'$, δv, $\delta v'$, in the development of $\mathcal{A}^{(2)} \cdot \cos. 2 \cdot (v' - v)$ [3984b], and we have found that this substitution, in [3984b], produces the function [3984]. A similar method may be followed with the first line of R [3995b]; but it is more simple to derive it from [3984a, 3984]. This is done by changing, in [3984a], the factor $\mathcal{M}^{(1)} \cdot e e'$ into $\mathcal{N}^{(0)} \cdot e$, and decreasing the angle, which is contained under the sign *cos.*, by the [3995g] [3995h] [3995i]

[3995′] The first of these terms arises from the development of $A^{(2)}.\cos.(2v'-2v)$, in the expression of R; and in this development we must increase a by δr,
[3995″] a' by $\delta r'$, $2n't-2nt$ by $2\delta v'-2\delta v$; from which we obtain the following expression,

[3996]
$$\begin{aligned} R = \ & 2N^{(0)}.e.(\delta v'-\delta v).\sin.(nt-2n't+\varepsilon-2\varepsilon'+\varpi) \\ & +a.\left(\frac{dN^{(0)}}{da}\right).e.\frac{\delta r}{a}.\cos.(nt-2n't+\varepsilon-2\varepsilon'+\varpi) \\ & +a'.\left(\frac{dN^{(0)}}{da'}\right).e.\frac{\delta r'}{a'}.\cos.(nt-2n't+\varepsilon-2\varepsilon'+\varpi). \end{aligned}$$

Hence we get in R, the following terms,*

[3997]
$$\begin{aligned} R = \ & N^{(0)}.H'.e.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+D') \\ & -N^{(0)}.H.e.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+D) \\ & +\tfrac{1}{2}a'.\left(\frac{dN^{(0)}}{da'}\right).G'.e.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+C') \\ & +\tfrac{1}{2}a.\left(\frac{dN^{(0)}}{da}\right).G.e.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+C). \end{aligned}$$

To obtain the corresponding part of $\mathrm{d}'R$, we must vary the angle
[3997′] $5n't-nt$, in the terms multiplied by H' and G';† but in the terms

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[3995k] quantity $n't+\varepsilon'-\varpi'$; by which means it becomes as in the first line of [3995b]; then putting $i=2$, it becomes as in the first term of [3995]. The same changes being made in [3984], which was derived from [3984a], it becomes as in [3996]; observing that when the angle $3n't-nt+3\varepsilon'-\varepsilon-\varpi-\varpi'$ [3984] is decreased by the quantity $n't+\varepsilon'-\varpi'$ [3995k], its sine becomes

[3995l]
$$\sin.(2n't-nt+2\varepsilon'-\varepsilon-\varpi)=-\sin.(nt-2n't+\varepsilon-2\varepsilon'+\varpi),$$

as in the first line of [3996], and its cosine is as in the second and third lines of the same expression.

[3997a] * (2515) Substituting, in [3996], the values of δr, δr, $\delta r'$, $\delta v'$ [3992–3994′], reducing the products by [17–20] Int., and retaining only the terms depending on the angle $5n't-2nt$, it becomes as in [3997].

[3998a] † (2516) The characteristic d' [3980] affects only the angle $2n't$, in [3995], so that in these terms we shall have $\mathrm{d}'.(nt-2n't)=-2n'dt$; but d' affects the the whole values of $\frac{\delta r'}{a'}$, $\delta v'$, consequently also the whole of the angle $3n't-nt$,

multiplied by H and G, we must only vary $2n't$; hence we obtain [3997'']

$$a'\mathrm{d}'R = -(5n'-n).dt.a'N^{(0)}.H'.e.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+D')$$

$$-\tfrac{1}{2}.(5n'-n).dt.a'^2.\left(\frac{dN^{(0)}}{da'}\right).G'.e.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+C')$$

$$+2n'dt.a'N^{(0)}.H.e.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+D)$$

$$-n'dt.aa'.\left(\frac{dN^{(0)}}{da}\right).G.e.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi+C). \qquad [3998]$$

The term $N^{(1)}.e'.\cos.(nt-2n't+\varepsilon-2\varepsilon'+\varpi')$, arises from the development of the term of R, represented by $A^{(1)}.\cos.(v'-v)$* [3995f]; [3998']

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which occurs in the terms [3994, 3994'], which are multiplied by G', H'; so that in these terms we shall have $\mathrm{d}'.(3n't-nt)=3n'dt-ndt$. Subtracting [3998a] from this, we get [3998b]

$$\mathrm{d}'.(5n't-2nt)=\mathrm{d}'.(3n't-nt)-\mathrm{d}'.(nt-2n't)=(5n'-n).dt, \qquad [3998c]$$

for the differential of the angle $5n't-2nt$, which occurs in the terms of R [3997], depending on G', H'; it being evident, that the angle $5n't-2nt$ is produced in these terms by combining the angles $3n't-nt$, $nt-2n't$, as in [3998c]. Substituting this in the differential of the first and third lines of [3997], taken relatively to d', we get the first and second lines of [3998], containing the factors G', H', as in [3997']. [3998d]
Again, the characteristic d' [3980] does not affect $\frac{\delta r}{a}$, δv, so that in their values [3992, 3993], which contain the factors G, H, we have $\mathrm{d}'.(3n't-nt)=0$; subtracting from this the expression [3998a], we get [3998e]

$$\mathrm{d}'.(5n't-2nt)=\mathrm{d}'.(3n't-nt)-\mathrm{d}'.(nt-2n't)=2n'dt; \qquad [3998e']$$

which is to be substituted in the differential of the second and fourth lines of [3997], taken relatively to d', to obtain the third and fourth lines of [3998], containing the factors G, H, as in [3997'']. The whole value of $\mathrm{d}'R$ is to be multiplied by a', to obtain $a'\mathrm{d}R$ [3998]. [3998f]

* (2517) We have seen, in [3995f], that the second term of [3995],

$$N^{(1)}.e'.\cos.(nt-2n't+\varepsilon-2\varepsilon'+\varpi'), \qquad [3999a]$$

is derived from a term of R, of the form $A^{(1)}.\cos(v'-v)$, corresponding to $i=1$; being of the same form as [3977d]. Now the effect of the substitution of the variations of δr, $\delta r'$, δv, dv', in the development of this quantity, having been computed in [3978], we may deduce from it the terms of R [3999], corresponding to the present case, by a similar method of derivation to that made use of in [3995h—l]. Thus, instead of the [3999b]

[3998''] we must therefore vary, in this term, a by δr, a' by $\delta r'$, $n't-nt$ by $\delta v' - \delta v$, and we get the following expression,

[3999]
$$\begin{aligned} R = \; & N^{(1)}.e'.(\delta v' - \delta v).\sin.(nt - 2n't + \varepsilon - 2\varepsilon' + \varpi') \\ & + a.\left(\frac{dN^{(1)}}{da}\right).e'.\frac{\delta r}{a}.\cos.(nt - 2n't + \varepsilon - 2\varepsilon' + \varpi') \\ & + a'.\left(\frac{dN^{(1)}}{da'}\right).e'.\frac{\delta r'}{a'}.\cos.(nt - 2n't + \varepsilon - 2\varepsilon' + \varpi'). \end{aligned}$$

Therefore the part of $a'\mathrm{d}'R$, relative to these terms, is*

[4000]
$$\begin{aligned} a'\mathrm{d}'R = & -\tfrac{1}{2}.(5n'-n).dt.a'N^{(1)}.H'.e'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'+D') \\ & -\tfrac{1}{2}.(5n'-n).dt.a'^2.\left(\frac{dN^{(1)}}{da'}\right).G'.e'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'+C') \\ & + n'dt.a'N^{(1)}.H.e'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'+D) \\ & - n'dt.aa'.\left(\frac{dN^{(1)}}{da}\right).G.e'.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-\varpi'+C). \end{aligned}$$

[4000'] The values of $M^{(0)}$, $M^{(1)}$, $M^{(2)}$, $M^{(3)}$, are determined in the formulas of § 4, by changing the quantities relative to m into those relative to m', and the contrary [3975a, b].† The values of $N^{(0)}$ and $N^{(1)}$ are determined

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operations mentioned in [3995i], we must, in the present case, change the factor $M^{(0)}.e'^2$ [3977e] into $N^{(1)}.e'$; and decrease the angle which is contained under the sign *cos.*,
[3999c] by $n't + \varepsilon' - \varpi'$; by which means [3977$e$] becomes as in the second line of [3995b],
[3999d] or the second line of [3995], supposing $i = 1$. Now making the same changes in [3978], which is derived from [3977e], it becomes as in [3999]; observing that when the
[3999e] angle $3n't - nt + 3\varepsilon' - \varepsilon - 2\varpi'$ [3978], is decreased by $n't + \varepsilon' - \varpi'$ [3999c], it becomes $2n't - nt + 2\varepsilon' - \varepsilon - \varpi' = -(nt - 2n't + \varepsilon - 2\varepsilon' + \varpi')$.

* (2518) The function [4000] may be deduced from [3999], by the method we have used in computing [3997] from [3996]. It may, however, be deduced more easily from
[3999f] [3996, 3997]; by changing $N^{(0)}$, e, ϖ, δv, $\delta v'$, into $N^{(1)}$, e', ϖ', $\frac{1}{2}\delta v$, $\frac{1}{2}\delta v'$, respectively. For by this means, [3996] changes into [3999]; and H, H' [3993, 3994'] become
[3999g] $\frac{1}{2}H$, $\frac{1}{2}H'$, respectively. These changes being made in [3998], it becomes as in [4000].

† (2519) If we put $i = -1$, in the terms of R [1011], depending on e, e', and
[4000a] retain only these two terms, putting also $A^{(-i)} = A^{(i)}$ [954''], we get, for this part of R, relative to the action of Saturn on Jupiter,

[4000b]
$$\begin{aligned} R = & -\frac{m'}{2}.\left\{a.\left(\frac{dA^{(1)}}{da}\right) - 2A^{(1)}\right\}.e.\cos.(2nt - n't + 2\varepsilon - \varepsilon' - \varpi) \\ & -\frac{m'}{2}.\left\{a'.\left(\frac{dA^{(2)}}{da'}\right) + 4A^{(2)}\right\}.e'.\cos.(2nt - n't + 2\varepsilon - \varepsilon' - \varpi'). \end{aligned}$$

by the equations,

$$a'N^{(0)}=-2m.a'A^{(2)}-\tfrac{1}{2}m.aa'.\left(\frac{dA^{(2)}}{da}\right);$$ [4001]

$$a'N^{(1)}=\quad m.a'.A^{(1)}-\tfrac{1}{2}m.a'^2.\left(\frac{dA^{(1)}}{da'}\right).$$ [4001']

Connecting together all these partial expressions of $a'\text{d}'R$, we obtain a term of this form,*

$$a'\text{d}'R=mn'.I.dt.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-O).$$ [4002]

Hence the term $3a'ffn'dt.\text{d}'R$, of the expression of $\delta v'$, gives† [4002']

$$\delta v'=-\frac{3n'^2.I.m}{(5n'-2n)^2}.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon-O).$$ [4003]

This is the most sensible term of the great inequality of Saturn, depending on the square of the disturbing force.

Changing, reciprocally, the elements of m' into those of m, we get the corresponding part of R, relative to the action of Jupiter on Saturn. Comparing this with the assumed form [3995], after having changed the signs of all the terms contained under the sign *cos.*, in [3995], we get the expressions of $N^{(0)}$, $N^{(1)}$ [4001, 4001']. [4000c]

* (2520) Adding together the parts of $a'\text{d}'R$ [3982, 3985, 3989, 3991, 3998, 4000], and putting, for brevity, $T_5=5n't-2nt+5\varepsilon'-2\varepsilon$, we get a series of terms of the first form [4002c]; I' being used for brevity, for the coefficients, and O' for the quantity connected with T_5. Developing this by [23] Int., we get the second form [4002c or 4002d]; in which we may substitute [4002a]

$$\Sigma.I'.\cos.O'=mn'.I.\cos.O,\qquad \Sigma.I'.\sin.O'=-mn'.I.\sin.O,$$ [4002b]

and we obtain the first form [4002e], which by means of [22] Int., becomes as in the second form of [4002e], agreeing with [4002],

$$a'\text{d}'R=dt.\Sigma.I'.\sin.(T_5+O')=dt.\Sigma.I'.\{\sin.T_5.\cos.O'+\cos.T_5.\sin.O'\}$$ [4002c]

$$=dt.\sin.T_5.\Sigma.I'.\cos.O'+dt.\cos.T_5.\Sigma.I'.\sin.O'$$ [4002d]

$$=mn'.I.dt.\{\sin.T_5.\cos.O-\cos.T_5.\sin O\}=mn'.I.dt.\sin.(T_5-O).$$ [4002e]

† (2521) Multiplying [4002] by $3n'dt$, and then integrating it twice, relatively to t, we get, for $3a'ffn'dt.\text{d}'R$, the expression [4003]; and this quantity is evidently the most important one in the value of $\delta v'$, depending on the term now under consideration, included in the expression [3715m]. [4003a]

If the expression of R, divided by the disturbing mass, be the same for Jupiter and Saturn, we shall have, as in [1208], the corresponding inequality of Jupiter δv, by substituting the preceding value $\delta v'$ [4003] in the formula

[4003']
$$\delta v = -\frac{m'\sqrt{a'}}{m\sqrt{a}} \cdot \delta v' ;$$

but the value of $A^{(1)}$ [3775c] is not the same for the two planets, therefore the terms*

[4004]
$$M^{(0)}. e'^2. \cos. (3n't - nt + 3\varepsilon' - \varepsilon - 2\varpi') ;$$
$$N^{(1)}. e'. \cos. (nt - 2n't + \varepsilon - 2\varepsilon' + \varpi') ;$$

divided by the disturbing mass, are different for each of them. But it follows, from [1202], that by noticing only the terms having the divisor $(5n' - 2n)^2$, we shall have in this case,†

[4005]
$$m . \int \mathrm{d} R_{,} + m'. \int \mathrm{d}' R = 0 ;$$

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* (2522) The terms mentioned in [4004] are derived from $A^{(1)}. \cos. (v' - v)$, as
[4004a] it appears in [3976', 3998']; but the value of $A^{(1)}$ is not the same, in computing the action
[4004b] of m upon m'; as it is in computing the action of m' upon m [3775c]. Now we have already remarked, in Vol. I, page 651, that the method of finding the inequality of Jupiter from that of Saturn, by means of the formula [1208 or 4003'], is not applicable, without some restriction, to the computation of terms of the order of the square of the disturbing force. This is evident from the consideration, that in the equation

[4004c]
$$m . \int \mathrm{d} R + m'. \int \mathrm{d}' R' = 0 \qquad [1202],$$

from which the formula [1208] is derived, terms of the third order in m, m' are neglected,
[4004d] which is equivalent to the neglect of terms of the *second* order in R, R'; being of the same order as the terms computed in [3982—4002].

† (2523) This formula is corrected for a typographical mistake in the original work,
[4005a] and is the same as in [4004c]; terms of the third order in m, m' being neglected. We have already spoken of the different meanings of the symbol R, and it may not be amiss again to repeat, that m is the mass of Jupiter, m' that of Saturn; also in formula
[4005b] [4004c], the value of R corresponds to the action of m' on m [913], and R' to the action of m on m' [1199']. These are changed in the present article to $R_{,}$ [4005'] and R [3974''], respectively.

$R_{\prime}$ being what R becomes relatively to the action of Saturn on Jupiter, and the differential characteristic d referring to the co-ordinates of Jupiter.* [4005']

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* (2524) Substituting $\delta v'$ [4003] in the formula [4003'], we get the corresponding inequality of δv [4006]. This method of deriving δv from $\delta v'$, would be sufficiently accurate, were it not for the terms of the third order in m, m', omitted in [4004c, 4003']. These neglected terms make it necessary either to correct the result obtained in [4006], or *to compute, in a direct manner, the value of* δv from the formula $\delta v = 3\,a f\!f n\,dt\,.\,\mathrm{d}R$ [3715*l*]. Thus, for the terms of $R_{\prime}$, which are similar to those of R [3978, 3984, 3988, 3990, 3996, 3999], we must compute the corresponding values of $a\,\mathrm{d}R_{\prime}$, similar to [3982, 3985, &c.—4000], and by combining all of them together, we get the value of $a\,\mathrm{d}R_{\prime}$, corresponding to [4002]. This is to be substituted in [4005c], to obtain the required inequality δv, which is to be used instead of [4006]. It will not, however, be necessary to repeat the whole of these calculations, since we shall soon show that the terms of R, of the form and order in the development [3742], combined with those of a similar development of $R_{\prime}$, satisfy the equation [4005], when we except the terms depending on $A^{(1)}$, and notice only such quantities as have been under consideration in this article, namely, those which are of the order of the square of the disturbing force, and depend on the angle $5n't - 2nt$. For if we put [4005b'] [4005b''] [4005c] [4005d] [4005e] [4005f] [4005g]

$$A = \cos.(v'-v) - \tfrac{1}{4}\gamma^2.\cos.(v'-v) + \tfrac{1}{4}\gamma^2.\cos.(v'+v); \qquad [4005h]$$

$$\begin{aligned} B = &-\{r^2 - 2rr'.\cos.(v'-v) + r'^2\}^{-\frac{1}{2}} \\ &+ \tfrac{1}{4}\gamma^2.\{\cos.(v'-v) - \cos.(v'+v)\}.\{r^2 - 2rr'.\cos.(v'-v) + r'^2\}^{-\frac{3}{2}}; \end{aligned} \qquad [4005i]$$

we shall obtain the value of R [4005*l*], corresponding, as in [3974''], to the disturbing force of Jupiter upon Saturn; the expression is derived from [3742], by changing m, r, v into m', r', v', and the contrary. Moreover $R_{\prime}$ [4005*l*', 4005'] corresponds to the action of Saturn upon Jupiter, being the same as in [3742], [4005k]

$$R = mA.\frac{r'}{r^2} + mB; \quad \text{[Action of Jupiter on Saturn.]} \qquad [4005l]$$

$$R_{\prime} = m'A.\frac{r}{r'^2} + m'B; \quad \text{[Action of Saturn on Jupiter.]} \qquad [4005l']$$

If we neglect, for a moment, the term A, we shall have $R = mB$, $R_{\prime} = m'B$; whence $R_{\prime} = \frac{m'}{m}.R$; so that the terms of $R_{\prime}$, corresponding to R [3975], may be found by changing $M^{(0)}$, $M^{(1)}$, $M^{(2)}$, $M^{(3)}$, into $\frac{m'}{m}.M^{(0)}$, $\frac{m'}{m}.M^{(1)}$, $\frac{m'}{m}.M^{(2)}$, $\frac{m'}{m}.M^{(3)}$, respectively, in [3975—3991]; also $N^{(0)}$, $N^{(1)}$, into $\frac{m'}{m}.N^{(0)}$, $\frac{m'}{m}.N^{(1)}$ [3995—4001']; or in other words, we may compute the parts of $R_{\prime}$, depending on B, by multiplying the [4005m] [4005n]

Hence it follows, that the inequality of Jupiter, corresponding to the

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corresponding terms of R [3978, 3984, &c.] by $\frac{m'}{m}$. In finding the differentials relative to d, we shall proceed in the same order as we have done in finding those relative to d'

[4005o] [3982a, &c.], observing that d does not affect $3n't$, in the angle $3n't-nt$, which

[4005p] occurs explicitly in [3975]. Hence we shall have $\mathrm{d}.(3n't-nt)=-n\,dt$, similar to [3982a]; moreover, as the sign d does not affect the values of $\frac{\delta r'}{a'}$, $\delta v'$, the differential of the angle $nt-2n't$, which occurs in these values, or in the terms connected with

[4005q] E', F' [3974', 3974], is $\mathrm{d}.(nt-2n't)=0$. The difference of these two expressions, corresponding to the equation [3982c], is

[4005r] $$\mathrm{d}.(5n't-2nt)=\mathrm{d}.(3n't-nt)-\mathrm{d}.(nt-2n't)=-n\,dt;$$

[4005r'] now we have very nearly $5n'-2n=0$ [3818d]; and the inequalities δv, $\delta v'$, under consideration, are very small, as we shall see in [4431f]; therefore we may put $-n=-(5n'-n)$, and the preceding expression becomes

[4005s] $$\mathrm{d}.(5n't-2nt)=-(5n'-n).dt;$$

which is equal to that of $\mathrm{d}'.(5n't-2nt)$ [3982c], but has a different sign. Hence, by noticing only the part of R, depending on B, and connected with the factors E', F', we have $\mathrm{d}R=-\mathrm{d}'R$; substituting this in the differential of $R_{,}$ [4005m], taken

[4005t] relatively to d, we get $\mathrm{d}R_{,}=\frac{m'}{m}.\mathrm{d}R=-\frac{m'}{m}.\mathrm{d}'R$; which is easily reduced to the

[4005u] form $m.\mathrm{d}R_{,}+m'.\mathrm{d}'R=0$ [4005]. In like manner, the differential d affects the whole of the values $\frac{\delta r}{a}$, δv [3972, 3973], depending on the factors E, F; so that the differential d, of the angle $nt-2n't$, connected with these terms, is

[4005v] $$\mathrm{d}.(nt-2n't)=n\,dt-2n'dt.$$

Subtracting this from [4005p], we get

[4005w] $$\mathrm{d}.(5n't-2nt)=\mathrm{d}.(3n't-nt)-\mathrm{d}.(nt-2n't)=2n'dt-2n\,dt;$$

and by substituting $2n'-2n=-3n'$ [4005r'], it becomes

[4005w'] $$\mathrm{d}.(5n't-2nt)=-3n'dt=-\mathrm{d}'.(5n't-2nt) \quad [3982g];$$

[4005x] hence, for these terms, we also get, as in [4005t], $\mathrm{d}R=-\mathrm{d}'R$ and $m.\mathrm{d}R_{,}+m'.\mathrm{d}'R=0$. The same result holds good when the terms of R, instead of depending on the angle

[4005y] $3n't-nt$ [3975], have other forms, as for example, $nt-2n't$ [3995]; which are to be combined with the corresponding terms of δr, δv, $\delta r'$, $\delta v'$, so as to produce the angle $5n't-2nt$. Thus, if instead of the particular values of R, $\frac{\delta r'}{a'}$ [3975, 3974], we assume the following general values,

[4005z] $$R=M.\cos.(i'_1n't-i_1nt+A_1), \qquad \frac{\delta r'}{a'}=F'.\cos.(i_2nt-i'_2n't+A_2);$$

preceding expression [4003], is

$$\delta v = \frac{3m'.n'^2.\sqrt{\frac{a'}{a}}}{(5n'-2n)^2} . I . \sin.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - O). \qquad [4006]$$

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in which $i'_1 + i'_2 = 5$; $i_1 + i_2 = 2$; we shall find that the products of these two expressions, contained in a function similar to [3978], will produce a term depending on the angle $5n't - 2nt$, as in [3979]. In this case, the equations [3982c, 4005r] become, respectively, by substituting $i'_1 + i'_2 = 5$ [4005z'], [4005z'] [4006a]

$$\begin{aligned} d'.(5n't - 2nt) &= d'.(i'_1 n't - i_1 nt) - d'.(i_2 nt - i'_2 n't) \\ &= i'_1 n' dt - (i_2 n dt - i'_2 n' dt) = 5 n' dt - i_2 n dt ; \end{aligned} \qquad [4006b]$$

$$d.(5n't - 2nt) = d.(i'_1 n't - i_1 nt) - d.(i_2 nt - i'_2 n't) = - i_1 n dt. \qquad [4006c]$$

The sum of these two equations, substituting $i_1 + i_2 = 2$; $5n' - 2n = 0$ [4005z', r'], is

$$d'.(5n't - 2nt) + d.(5n't - 2nt) = 5n'dt - 2ndt = 0, \quad \text{or} \quad d'R + dR = 0, \qquad [4006d]$$

as in [4005t]; and from this we get, generally, as in [4005x, 4005], $m.dR_{,} + m'.d'R = 0$. [4006e]
Hence it follows, that if we put δv_1, δv_2, for the parts of δv, of this form and order, depending on A, B, respectively; also $\delta v'_1$, $\delta v'_2$, for the similar parts of $\delta v'$, we shall have [4006e']

$$\delta v = \delta v_1 + \delta v_2 ; \qquad \delta v' = \delta v'_1 + \delta v'_2 ; \qquad [4006f]$$

and the formula [4006e] gives, as in [1202, &c.], the following expression, similar to [4003'],

$$\delta v_2 = - \delta v'_2 . \frac{m'\sqrt{a'}}{m\sqrt{a}} . \qquad [4006g]$$

From this formula we may compute δv_2, after having found $\delta v'_2$, by a direct process similar to that used in [3975—4003].

In computing the terms of δv_1, $\delta v'_1$, depending on A [4005h], we may neglect the two terms containing γ^2, for the same reasons as in [3990a—c]. Then we shall have simply $A = \cos.(v' - v)$; hence the corresponding parts of R, $R_{,}$ [4005l, l'], become [4006h]

$$R = m . \frac{r'}{r^2} . \cos.(v' - v) ; \qquad R_{,} = m' . \frac{r}{r'^2} . \cos..(v' - v). \qquad [4006i]$$

These quantities evidently depend on the term connected with the coefficient $A^{(1)}$, in the development of $\frac{R}{m'}$ [954, 957], as is evident by the substitution of the values [952, 953]. Hence we have, by noticing only this part of $A^{(1)}$,

$A^{(1)} = m . \frac{a'}{a^2}$; in computing $\delta v'_1$, arising from the action of Jupiter on Saturn; [4006k]

$A^{(1)} = m' . \frac{a}{a'^2}$; in computing $\delta v'$, arising from the action of Saturn on Jupiter. [4006l]

Now $A^{(1)}$ occurs only in the development of the term $A^{(1)}.\cos.(v' - v)$; and it is [4006m]

17. In the inequalities of Jupiter and Saturn, in which the coefficient
[4006′] of t is neither $5n'-2n$, nor differs from it by the quantity n, in

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[4006n] therefore found in $M^{(0)}$ [3976, 3976′], also in $N^{(0)}$ [4001′]; but not in $M^{(1)}$, $M^{(2)}$, $M^{(3)}$,
[4006o] $N^{(0)}$ [3983, 3986, 3989″, 3995′]; so that in these last terms we shall have $\delta v_1=0$,
[4006p] $\delta v'_1=0$, $\delta v_2=\delta v$, $\delta v'_2=\delta v'$; consequently the value of δv may be correctly obtained from $\delta v'$, in these cases, by means of the formula [4003′]. A different process
[4006q] must be used with the terms depending on $M^{(0)}$, $N^{(1)}$, which contain $A^{(1)}$. For we must compute $\delta v'_1$ in a direct manner, by means of the value of $A^{(1)}$ [4006k]; also δv_1, from
[4006r] [4006l]; by a process similar to that used in computing $\delta v'$ or $\delta v'_2$, in [3982, 4002′].
[4006s] Having thus obtained δv_1, $\delta v'_1$, $\delta v'_2$, we get δv_2, by means of the formula [4006g], and then by substitution in [4006f], we obtain the values of δv, $\delta v'$, corresponding to
[4006s′] these terms. These remarks are not restricted to the two forms of R, treated of by the author in [3975, 3995], but apply generally to others of a similar nature, contained in the general table, which we shall give in [4006u].

In addition to the terms of R, depending on the angles $3n't-nt$, $nt-2n't$;
[4006t] treated of by the author in [3975, 3995]; there is an infinite number of a similar nature; some of which are deserving of peculiar notice, on account of their magnitudes; and one of them is of nearly the same order as those we have already noticed. The 20 forms of R, δr, δv, $\delta r'$, $\delta v'$, &c., producing the angle $5n't-2nt$, are contained in the annexed table. Thus the form which is marked with the number 6, includes the terms of R, depending on the angle $3n't-nt$, as in the first form assumed by the author in [3975]; and when this is combined with δr, δv, &c., of the form $2n't-nt$, it produces terms depending on $5n't-2nt$, as in [3979]. We may also take these angles in an inverse order, corresponding to the accented numbers, supposing, as in the number 6′, that R depends on the
[4006u] angle $2n't-nt$, corresponding to the second form of the author, in [3995], and δr, δv, &c. depend on the angle $3n't-nt$. The numerical values of these terms of δr, $\delta v'$, are given inaccurately in [4132, 4488]; as was first observed by Mr. Plana, in the second volume of the Memoirs of the Astronomical Society of London; in which he has given the calculations of the

No.	Coefficients of t in the terms of R.	Coefficients of t in the terms of δr, δv, $\delta r'$, $\delta v'$.	
1	0	$5n'-2n$	1′
2	n'	$4n'-2n$	2′
3	$2n'$	$3n'-2n$	3′
4	$3n'$	$2n'-2n$	4′
5	$n'-n$	$4n'-n$	5′
6	$3n'-n$	$2n'-n$	6′
	$\nu=n'-n$; $i=$ any positive integer.		
7	$5n'-2n+i\nu$	$i\nu$	7′
8	$5n'-3n+i\nu$	$i\nu-n$	8′
9	$5n'-4n+i\nu$	$i\nu-2n$	9′
10	$5n'-5n+i\nu$	$i\nu-3n$	10′
	Coefficients of t in the terms of δr, δv, $\delta r'$, $\delta v'$.	Coefficients of t in the terms of R.	No.

[4006v] separate terms at full length; and has also noticed the terms of R, of the forms 5′, 3, 4; observing, however, that they have hardly any sensible effect in the complete values of δv, $\delta v'$. The final values of δv, $\delta v'$, computed by Mr. Plana, by a direct process, and independently of each other, did not satisfy the equation [4003′]; and this numerical result, he considered as a demonstration *a posteriori*, that this formula could not be applied
[4006w] to *all* these terms of the order of the square of the disturbing masses. In consequence

Jupiter, or n' in Saturn; we must increase nt and $n't$ by their great inequalities depending on $5n't-2nt$. For we have seen [1070''], [4006'']

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of these remarks, La Place resumed the subject in a memoir published in the *Connaissance des Tems* for the year 1829; in which he tacitly admits the inaccuracy of the application of the formula [4003'] to *all* these terms of the order of the square of the disturbing forces; and gives a new formula [4008x], expressing the relation between the complete values of the terms of δv, $\delta v'$, like those computed in this article, and others of a similar form and order, [4006x]
calculated by Mr. Plana [4006r]. This new formula has been called the *last gift of La Place to astronomy.* Upon applying the numerical values of δv, $\delta v'$, given by Mr. Plana, to this formula, it was not satisfied; whence La Place inferred, that these numerical calculations of Mr. Plana were incomplete or inaccurate. Some strictures having been made on this [4006y]
formula by Mr. Plana, in the *Memorie della Reale Accademia delle Scienze di Torino,* Tom. XXXI; it was followed by two other demonstrations of this new formula; the first by Mr. Poisson in a memoir published in the *Connaissance des Tems* for 1831; the second by Mr. Pontécoulant, in the same work, for 1833. In the memoir of Mr. Poisson, he notices the term of the form 1, in the table [4006u], and shows, that it is of sufficient importance to be introduced into the calculation. Under these circumstances, he recommends a [4006z]
revision of the whole calculation, by taking into consideration all the forms comprised in the table [4006u], which produce terms of δv, $\delta v'$, of any sensible magnitude. This extremely laborious task has been performed by Mr. Pontécoulant, who has given the [4007a]
abridged results of his investigation in the *Connaissance des Tems* for the year 1833, from which we shall make some extracts, in the notes upon the twelfth and thirteenth chapters of this book, in treating of the orbits of Jupiter and Saturn. These results, so far as they relate to terms of the forms 6, 6' [4006u], computed in this article, differ but very little [4007b]
from those of La Place [4432, 4488], except in the signs; and upon referring to the original manuscript, in which these last calculations were made, a mistake in the signs was discovered. Finally, Mr. Pontécoulant suggested to Mr. Plana, some corrections [4007c]
which were necessary in his work; and upon the revision of his calculation, it was found, that the results were almost identical with those of Mr. Pontécoulant; these corrected [4007d]
values, combined with the other terms of this kind computed by Mr. Pontécoulant, are found to satisfy very nearly the new formula of La Place [4008x]. We shall now give the demonstration of this formula.

For this purpose, we shall use the same notation as in [1198], in which M represents the sun's mass, m the mass of Jupiter, m' the mass of Saturn; x, y, z, the rectangular [4007e]
co-ordinates of Jupiter, referred to the sun's centre; r its radius vector, &c.; and the same letters accented correspond to the orbit of Saturn. Then putting, for brevity, [4007f]

$$w=\frac{xx'+yy'+zz'}{r^3};\qquad w'=\frac{xx'+yy'+zz'}{r'^3};\qquad \lambda=-\frac{1}{\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}}, \qquad [4007g]$$

[4006'''] that these great inequalities must be added to the mean motion, in the

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[4007h] we get, as in [949, 1200], by observing that $r^2=x^2+y^2+z^2$, $r'^2=x'^2+y'^2+z'^2$ [914'],

[4007i] $$R=m'.(w'+\lambda);\qquad \text{[For the action of Saturn upon Jupiter.]}$$

[4007k] $$R'=m.(w+\lambda);\qquad \text{[For the action of Jupiter upon Saturn.]}$$

Now if we multiply the formula [1198] by $M+m+m'$, it will become of the form [4007o]; for the two first terms of the second member of the product, or those in the first line of [1198], may be put under the form,

[4007l]
$$\begin{aligned}(M+m').m.\frac{(dx^2+dy^2+dz^2)}{dt^2}&+(M+m).m'.\frac{(dx'^2+dy'^2+dz'^2)}{dt^2}\\&+m^2.\frac{(dx^2+dy^2+dz^2)}{dt^2}+m'^2.\frac{(dx'^2+dy'^2+dz'^2)}{dt^2};\end{aligned}$$

of which the first line is the same as in the first line of [4007o]. Connecting the terms in the second line of [4007l] with those produced by the second line of [1198], namely,

[4007m] $$-\frac{(m\,dx+m'\,dx')^2}{dt^2}-\frac{(m\,dy+m'\,dy')^2}{dt^2}-\frac{(m\,dz+m'\,dz')^2}{dt^2},$$

it produces the second line of [4007o]; observing, that

$$m^2\,dx^2+m'^2\,dx'^2-(m\,dx+m'\,dx')^2=-2\,m\,m'.\,dx\,dx',\quad \&c.$$

The first and second terms of the third line of [1198] produce, without any reduction, the
[4007n] third line of [4007o], and the last term of [1198] gives the last of [4007o], using λ [4007g]; hence we have

[4007o]
$$\begin{aligned}\text{constant}=(M+m').m.\frac{(dx^2+dy^2+dz^2)}{dt^2}&+(M+m).m'.\frac{(dx'^2+dy'^2+dz'^2)}{dt^2}\\&-2\,m\,m'.\left\{\frac{dx\,dx'}{dt^2}+\frac{dy\,dy'}{dt^2}+\frac{dz\,dz'}{dt^2}\right\}\\&-2.(M+m+m').\left(\frac{Mm}{r}+\frac{Mm'}{r'}\right)\\&+2.(M+m+m').m\,m'.\lambda.\end{aligned}$$

If we multiply the values of $\frac{dx^2+dy^2+dz^2}{dt^2}$, $\frac{dx'^2+dy'^2+dz'^2}{dt^2}$ [1199, 1200], by $(M+m').m$, $(M+m).m'$, respectively; and add the products, we shall get, for the first line of the second member of [4007o], the following expression,

[4007p] $$(M+m').m.\left\{\frac{2.(M+m)}{r}-2\textstyle\int\mathrm{d}R\right\}+(M+m).m'.\left\{\frac{2.(M+m')}{r'}-2\textstyle\int\mathrm{d}'R'\right\}.$$

formulas of the elliptical motion; they must therefore be added to the same

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If we substitute this in [4007o], we shall find, that the term having the divisor r, is

$$\frac{2m}{r}\cdot\{(M+m')\cdot(M+m)-(M+m+m')\cdot M\},$$ [4007p']

which, by reduction, is $\frac{2m^2m'}{r}$; and in like manner, the term depending on r', is $\frac{2mm'^2}{r'}$; so that if after this substitution is made, we divide the whole expression by 2, and transpose the terms depending on $\mathrm{d}R$, $\mathrm{d}'R'$, we shall obtain the following equation, *in which nothing is omitted*, the constant quantity being included in the signs $\int$,

$$\begin{aligned}(M+m')\cdot m\cdot\int\mathrm{d}R+(M+m)\cdot m'\cdot\int\mathrm{d}'R'= \quad & mm'\cdot\left(\frac{m}{r}+\frac{m'}{r'}\right)\\ & -mm'\cdot\left(\frac{dx\,dx'+dy\,dy'+dz\,dz'}{dt^2}\right)\\ & +(M+m+m')\cdot mm'\cdot\lambda.\end{aligned}$$ [4007q]

We must now consider the terms of this equation affected with the small divisor $5n'-2n$, and having $5n't-2nt$ for the argument; these terms being the only ones which can acquire the divisor $(5n'-2n)^2$ by another integration in $\int\int\mathrm{d}R$, $\int\int\mathrm{d}'R'$, or in [4007r] the expression of the longitudes of the two planets [3715l, m]; and in making this investigation, we shall reject all terms of the order m^4. In the first place, we shall observe, that the expression in the second line of the second member of [4007q] does not contain such terms of the order m^2, as is evident from the reasoning in note 819 [1201'], [4007s] where it is shown, that these terms of the order m^2, arise from the substitution of the elliptical values of x, x', y, y', &c.; and to obtain terms of the order m^3, we must augment these elliptical values of x, x', &c. by the terms depending on the perturbations. These terms may be easily obtained by considering the orbits as variable ellipses, in which we may suppose x, x', to be of the forms,

$$x=A_1+B_1\cdot\cos.(nt+C_1)+\&c.;$$ [4007t]

$$x'=A_2+B_2\cdot\cos.(n't+C_2)+\&c.;$$ [4007u]

A_1, B_1, C_1, &c., A_2, B_2, C_2, &c. being functions of the elements of the orbits. These elements for the planet Jupiter are; the mean longitude of this planet $nt+\varepsilon$; [4007u'] ε the mean longitude of the epoch; a the semi-transverse axis of the ellipsis; e the excentricity; ϖ the longitude of the perihelion; γ the inclination of the ellipsis to a fixed plane; and θ the longitude of the ascending node. The same letters being accented, [4007u''] represent the corresponding elements of the orbit of Saturn. In the values of all these elements, the secular inequalities are supposed to be included. The differential of the expression [4007t, u], being found as in [1168'], become

$$dx=-B_1\cdot n\,dt\cdot\sin.(nt+C_1)-\&c.;$$ [4007v]

$$dx'=-B_2\cdot n'dt\cdot\sin.(n't+C_2)-\&c.$$ [4007w]

quantities in the development of R. Let

[4007] $$R = H \,.\, \cos.\, (i'n't - int + A),$$

The product $dx\,dx'$, will therefore contain only periodical quantities of the form,

[4007x] $$H \,.\, \cos.\, (i'n't - int + E)\,;$$

H, E, being functions of the elements of the orbits; and i', i, integral numbers, positive or negative; moreover $n't$, nt, in the planetary system, are incommensurable quantities [1197″]. Now if we consider the elements as variable, their variations, corresponding to the great inequalities of Jupiter and Saturn, will have the same argument as these inequalities,
[4007y] namely, $5n't - 2nt$, and they have $5n' - 2n$ for a divisor, as is evident from what we have seen in [1197, 1286, 1294, 1311, 1345′], or more completely in the appendix to this volume [5872—5879]. Substituting these variations in [4007r], and reducing by [17—20] Int., we shall obtain terms having this divisor; but it is evident, that they will
[4007z] not have the same argument, except $i' = 10$ and $i = 4$; in which case H will be of the order e^6 [957viii, &c.], which is neglected, because we notice only terms of the third order relative to the excentricities e, e', and of the same order relative to the masses m, m'.
[4008a] The same remarks may be made with regard to the products $dy\,dy'$, $dz\,dz'$; hence we conclude, that the function included in the second line of [4007q] does not contain terms of the order m^2 or m^3, which has for its argument $5n't - 2nt$, and for divisor $5n' - 2n$; so that we may substitute, in [4007q], the following expression,

[4008b] $$-mm' \,.\, \left(\frac{dx\,dx' + dy\,dy' + dz\,dz'}{dt^2}\right) = 0\,.$$

[4008b′] In the function comprised in the third line of [4007q], namely, $(M + m + m') \,.\, mm' \,.\, \lambda$, we may change the factor $M + m + m'$ into M; it being evident, that the neglected
[4008c] quantities do not comprise terms of the order m^3, having the argument $5n't - 2nt$ and the divisor $5n' - 2n$. Then substituting, in λ [4007g], the elliptical values of x, x' [4007t, u], and the similar values of y, y', z, z'; it becomes, by development, of the form,

[4008d] $$\lambda = A + K \,.\, \cos.\, (5n't - 2nt + I) + Q\,;$$

in which A *represents the part depending on the argument zero, and* Q *all the terms*
[4008e] *depending on angles of the form* $i'n't + int$, i', i, *being integral numbers, positive or negative, excluding those producing the argument* $5n't - 2nt$, *which is connected with* K, *and the argument zero connected with* A; hence we have

[4008f] $$(M + m + m') \,.\, mm' \,.\, \lambda = M \,.\, mm' \,.\, \{A + K \,.\, \cos.\, (5n't - 2nt + I) + Q\}.$$

The quantity $mm' \,.\, \frac{m}{r}$ [4007q], is of the third order in m, m', and as the value of r
[4008g] contains no term having the divisor $5n' - 2n$, except it be of the order m', we may neglect this term, because it produces nothing except of the order m^4; and the same is to

be any term of this development; and

$$\delta v = L \,.\, \sin.\,(i'n't - int + B), \qquad [4008]$$

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be observed relatively to $m\,m'.\frac{m'}{r}$. Substituting these and [4008b,f] in [4007q], we get

$$M.\{mf\mathrm{d}R + m'f\mathrm{d}'R'\} + m\,m'.\{f\mathrm{d}R + f\mathrm{d}'R'\} = M.m\,m'.\{A + K.\cos.(5n't - 2nt + I) + Q\}. \qquad [4008h]$$

We shall represent by (R), (R'), *the parts of* R, R', *respectively, of the order* m; [4008i]
then using the characteristic δ of variations, we shall put δR, $\delta R'$, *for the remaining parts of the same quantities of the order* m^2, &c., and we shall have

$$R = (R) + \delta R, \qquad R' = (R') + \delta R'. \qquad [4008k]$$

If we also put $[(A) + (K).\cos.(5n't - 2nt + I)]$ for the part of $A + K.\cos.(5n't - 2nt + I)$, [4008l]
which is independent of m, m'; and prefix the sign δ before the same quantity, to denote the remaining part, we shall have

$$\begin{aligned} A + K.\cos.(5n't - 2nt + I) + Q = {} & [(A) + (K).\cos.(5n't - 2nt + I)] \\ & + \delta.\{A + K.\cos.(5n't - 2nt + I\} + Q. \end{aligned} \qquad [4008m]$$

Substituting [4008k, m] in [4008h], and neglecting the terms $m\,m'.f\mathrm{d}\,\delta R$, $m\,m'.f\mathrm{d}'\,\delta R'$, which are of the order m^4; also the terms $M.m\,m'.Q$, because the integration does not introduce the divisor $5n' - 2n$, we get

$$\begin{aligned} & M.\{m.f\mathrm{d}(R) + m'.f\mathrm{d}'(R')\} + m\,m'.\{f\mathrm{d}(R) + f\mathrm{d}'(R')\} + M.\{m.f\mathrm{d}\delta R + m'.f\mathrm{d}'\delta R'\} \\ & = M.m\,m'.[(A) + (K).\cos.(5n't - 2nt + I)] + M.m\,m'.\delta.\{A + K.\cos.(5n't - 2nt + I)\}. \end{aligned} \qquad [4008n]$$

Now equating separately the parts of this equation, which are of the order m^2, and those of the order m^3; putting also $M = 1$ [3709], in terms of the order m^3, we get [4008o]

$$M.\{m.f\mathrm{d}(R) + m'.f\mathrm{d}'(R')\} = M.m\,m'.[(A) + (K).\cos.(5n't - 2nt + I)]; \qquad [4008p]$$

$$m\,m'.\{f\mathrm{d}(R) + f\mathrm{d}'(R')\} + m.f\mathrm{d}\delta R + m'.f\mathrm{d}'\delta R' = m\,m'.\delta.\{A + K.\cos.(5n't - 2nt + I)\}. \qquad [4008q]$$

If we neglect the terms of the second member of [4008q], *or in other words, the terms of the elliptical value of* λ, *depending on the two arguments zero and* $5n't - 2nt$, we shall have the following expression [4008s], which includes all the arguments except these two; and is accurate both as it regards terms of the third order of the masses m, m', and of the third order relative to the excentricities and inclinations, [4008r]

$$m\,m'.\{f\mathrm{d}(R) + f\mathrm{d}'(R')\} + m.f\mathrm{d}\delta R + m'.f\mathrm{d}'\delta R' = 0. \qquad [4008s]$$

Substituting $M = 1$ [4008o] in the product of [4008p], by the quantity m', we get, by neglecting terms of the two forms 0 and $5n't - 2nt$ [4008r], $m\,m'.f\mathrm{d}R + m'^2.f\mathrm{d}'R' = 0$. [4008t]
Subtracting this from [4008s], we obtain

$$m.f\mathrm{d}\delta R + m'.f\mathrm{d}'\delta R' + (m - m').m'.f\mathrm{d}'R' = 0. \qquad [4008u]$$

the corresponding inequality of Jupiter.* If we increase $n\,t$, $n't$, by their great inequalities in the expression [4007], there will result in R a term of the form,†

[4009] $$R = \pm q\,H.\cos.\{i'n't - i\,n\,t \pm (5\,n't - 2\,n\,t) + A \pm E\}.$$

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and since $a^{\frac{3}{2}}n = a'^{\frac{3}{2}}n' = 1$ [3866a], neglecting terms of the order m', this may be put under the following form, terms of the order m'^4 being neglected,

[4008v] $$m\,a^{\frac{3}{2}}n.\int\mathrm{d}\,\delta R + m'.a'^{\frac{3}{2}}n'.\int\mathrm{d}'\,\delta R' + (m-m').m'.a'^{\frac{3}{2}}n'.\int\mathrm{d}'R' = 0.$$

Now if we put ζ, ζ', for the great inequalities of Jupiter and Saturn; $\delta_{,}\zeta$, $\delta_{,}\zeta'$, for the
[4008v'] parts of ζ, ζ', depending on $\mathrm{d}\,\delta R$, $\mathrm{d}'\,\delta R'$; or in other words, those which depend on the combinations [4006u], *excluding the angles zero and* $5\,n't - 2\,n\,t$, we shall have, as in [3715l, m],

[4008w] $$\delta_{,}\zeta = 3\,a\,n.\iint dt.\mathrm{d}\,\delta R;\quad \delta_{,}\zeta' = 3\,a'n'.\iint dt.\mathrm{d}'\delta R';\quad \zeta' = 3\,a'n'.\iint dt.\mathrm{d}'R'.$$

La Place's last formula, which [4008x] includes some terms of the order m^2.

Now multiplying [4008v], by $3\,dt$, integrating and substituting [4008w], we get

$$m\sqrt{a}.\delta_{,}\zeta + m'\sqrt{a'}.\delta_{,}\zeta' + (m-m').m'.\sqrt{a'}.\zeta' = 0;$$

which is the last formula of La Place, proposed to be demonstrated in [4007d]; and the complete values of $\delta_{,}\zeta$, $\delta_{,}\zeta'$ ought to satisfy it; so that if one of these quantities be
[4008y] accurately computed, the other may be deduced from it; but the usefulness of the theorem is restricted by the circumstance, that it can only be applied to the results obtained from *all*
[4008z] the sensible terms of this kind, taken collectively; or to all the terms corresponding to each of the six factors e^3, e^2e', $e\,e'^2$, e'^3, $e\gamma^2$, $e'\gamma^2$.

* (2525) The relation between R and δv is expressed by the equation [3715b]. A particular case of this formula is considered in [3703, 3715], in which

[4009a] $$R = M.\cos.(m_{,}t + K)\quad [3703, 3711d];$$

and we find, by mere inspection, that the third and fourth terms of δv [3715b] have, as
[4009b] in [3715h], the divisors $m_{,}^2$, $m_{,}$; also by comparing [3702, 3711c], we find, that the terms of δv [3715b], depending on δr, have the divisor $m_{,}^2 - n^2$, or $m_{,} \pm n$. It is
[4009c] easy to generalize this result, as in [4010], where $m_{,} = i'n' - i\,n$.

† (2526) If we increase $n't$ by the great inequality of Saturn [3891], and $n\,t$ by that of Jupiter [3889], the angle $i'n't - i\,n\,t$, which occurs in [4007, 4008], will be increased by a quantity, which we shall represent by p; then putting, for brevity,

[4012a] $$T_5 = 5\,n't - 2\,n\,t + 5\,\varepsilon' - 2\,\varepsilon;\qquad -i'\overline{H}'.\cos.\bar{A}' - i\,\overline{H}.\cos.\bar{A} = 2\,q.\cos.c;$$

$$-i'\,\overline{H}'.\sin.\bar{A}' - i\,\overline{H}.\sin.\bar{A} = 2\,q.\sin.c;\quad 5\,\varepsilon' - 2\,\varepsilon + c = E,$$

Now the series of operations, which connects H and L, gives to the parts of H the divisors $(i'n'-in)^2$, $i'n'-in$, $i'n'-in\pm n$ [4009b, c]; [4010]
and the same series of operations gives to the inequalities corresponding to the parts of R [4009], the divisors* $\{i'n'-in\pm(5n'-2n)\}^2$, [4011]
$i'n'-in\pm(5n'-2n)$, $i'n'-in\pm(5n'-2n)\pm n$. If $i'n'-in$ [4012]
or $i'n'-in\pm n$ be not small quantities of the order $5n'-2n$, we may neglect $5n'-2n$ in these divisors,† and then the inequality, corresponding to [4012']

$$R=\pm qH.\cos.\{i'n't-int\pm(5n't-2nt)+A\pm E\},\qquad [4013]$$

will be

$$\delta v=\pm qL.\sin.\{i'n't-int\pm(5n't-2nt)+B\pm E\};\qquad [4014]$$

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we get, successively,

$$p=-i'\overline{H}'.\sin.(T_5+\overline{A}')-i\overline{H}.\sin.(T_5+\overline{A})\qquad [4012b]$$

$$=-i'\overline{H}'.\{\sin.T_5.\cos.\overline{A}'+\cos.T_5.\sin.\overline{A}'\}-i\overline{H}.\{\sin.T_5.\cos.\overline{A}+\cos.T_5.\sin.\overline{A}\}$$

$$=\quad 2q.\{\sin.T_5.\cos.c+\cos.T_5.\sin.c\}=2q.\sin.(T_5+c)=2q.\sin.(5n't-2nt+E).\qquad [4012c]$$

If we increase the angle $i'n't-int+A$ [4007] by the quantity p; then develop the expression by means of [61] Int., we shall obtain an additional term of the order p, and represented by $-pH.\sin.(i'n't-int+A)$. Substituting in this the value of [4012d] p [4012c], and then reducing by [17] Int., it becomes, as in [4009],

$$qH.\cos.\{i'n't-int+(5n't-2nt)+A+E\}-qH.\cos.\{i'n't-int-(5n't-2nt)+A-E\}.\qquad [4012e]$$

* (2527) The coefficient of t, in [4007], is $i'n'-in$, and from this arise the divisors [4010]; but in the term [4009], this coefficient is augmented by the quantity $\pm(5n'-2n)$; which requires a corresponding increase in the resulting divisors [4010]; [4014a]
by this means the divisors [4010], depending upon the term [4007], change into those given in [4012]. If we suppose $5n'-2n$ to be very small, in comparison with [4014b]
$i'n'-in$ or $i'n'-in\pm n$, we may neglect it; and then the chain of operations connecting H, L [4007, 4008], will have the same divisors as that connecting qH, qL [4014c]
[4013, 4014]. Now [4007] is changed into [4013], by multiplying by $\pm q$, and augmenting the angle $i'n't-int$ by $\pm(5n't-2nt)\pm E$. Applying the same [4014d]
process of derivation to [4008], we get the corresponding inequality of Jupiter, as in [4014].

† (2528) In restricting the formula [4014] to the terms mentioned in [4006'], we may consider the part which is neglected in [4012'], as of an order $\frac{5n'-2n}{n}$, or $\frac{1}{74}$ of [4015a]
that retained [3818d]; so that the error of the terms δv [4014] is of the order $\frac{1}{74}qL$;

[4015] which is the same as to increase $n t$, $n' t$, by the great inequalities in the term of δv [4008].*

We must also increase, in the terms depending on the first power of the excentricities, the quantities e, e', ϖ, ϖ', by their variations, depending
[4016] upon the angle $5 n' t - 2 n t$; but it is evident, that this will not produce any sensible inequalities.†

Manner of noticing the effect of the secular variations of the elements.

18. *The coefficients of the inequalities of the planets vary on account of the secular variations of the elements of their orbits; we may notice this in the following manner.* We must first put the inequality relative to any angle $i' n' t - i n t$, under the form‡

[4017] $$P . \sin. (i' n' t - i n t + i' \varepsilon' - i \varepsilon) + P' . \cos. (i' n' t - i n t + i' \varepsilon' - i \varepsilon).$$

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and as q is of the order $\frac{1}{2} p$ [4012c], it becomes of the order $\frac{1}{148} p L$. Now the great
[4015b] inequalities of Jupiter and Saturn being nearly 1265^s,—2957^s, [4434, 4474], the quantity p [4012a] becomes $-5 \times 2957^s - 3 \times 1265^s = -18580^s$, or about $\frac{1}{12}$ of the radius;
[4015c] consequently the quantity $\frac{1}{148} p L$ is less than $\frac{1}{148} \times \frac{1}{12} L$, or less than $\frac{1}{1700} L$; and the error of this computation of δv [4014], arising from this source, will generally be less than $\frac{1}{1700}$ of the inequality [4008], which is under consideration.

* (2528a) If we increase $n' t$, $n t$, by the great inequalities, using p [4012b], the expression δv [4008] will become $\delta v = L . \sin. (i' n' t - i n t + B + p)$. Developing
[4015d] this as in [60] Int., we get $\delta v = L . \sin. (i' n' t - i n t + B) + p L . \cos. (i' n' t - i n t + B)$. Substituting p [4012c], and reducing by [19] Int., it becomes equal to the sum of the two expressions [4008, 4014].

† (2529) The smallness of these terms may be seen, by a rough examination of the increment of the value of R [1011], arising from the introduction of the part of e or δe
[4016a] [1286], when we put $i' = 5$, $i = 2$, $a = 1$, $\frac{n}{5 n' - 2 n} = 74$ [3818d], $m' = \frac{1}{3512}$, $e = 0{,}05$ [4061d, 4080]; observing that as $i' - i = 3$, k [1281'], may be considered as of the order e^3, and $\left(\frac{d k}{d e}\right)$ of the order e^2; so that δe [1286] may be considered as of the order $74 m' . e^2 . \cos. (5 n' t - 2 n t + A)$, or $\frac{1}{900} e . \cos. (5 n' t - 2 n t + A)$ nearly.
[4016b] Consequently this increment of e produces terms of the order $\frac{1}{900}$, in comparison with those depending on e, in [4392], none of which amount to 200^s; hence it is evident, that these terms are insensible.

[4017a] ‡ (2530) The form assumed in [4017] has been frequently used, as, for example, in [3711i].

We must determine the values of P, P', for the epoch 1750, and then put

$$\text{tang.}\, A = \frac{P'}{P}\,; \qquad L = \sqrt{P^2 + P'^2}\,; \qquad [4018]$$

the sign of $\sin. A$ is the same as that of P', and its cosine is the same [4018′]
sign as that of P [4019d]; then the proposed inequality will be *

$$L \,.\, \sin.\, (i'n't - int + i'\varepsilon' - i\varepsilon + A). \qquad [4019]$$

We must determine the values of P, P', for 1950, noticing the secular variations of the elements of the orbits; and we shall have for this inequality, in 1950,

$$(L + \delta L)\,.\, \sin.\, (i'n't - int + i'\varepsilon' - i\varepsilon + A + \delta A). \qquad [4020]$$

If we denote by t the number of Julian years elapsed since 1750, the preceding inequality relative to the time t will assume the following form,†

$$\left(L + \frac{t\,.\,\delta L}{200}\right) .\, \sin.\, \left\{ i'n't - int + i'\varepsilon' - i\varepsilon + A + \frac{t\,.\,\delta A}{200} \right\}. \qquad [4021]$$

Under this form it may be used for several centuries before and after 1750. But this calculation is not necessary except with those inequalities which are quite large.

In the two great inequalities of Jupiter and Saturn, it will be useful to
continue the approximation as far as the square of the time, in the part [4021′]

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* (2531) Using, for brevity, $i'n't - int + i'\varepsilon' - i\varepsilon = T_9$; then developing [4019] [4019a]
by means of [21] Int., and putting the expressions [4017, 4019] equal to each other, we get, identically,

$$P.\sin. T_9 + P'.\cos. T_9 = L.\sin.(T_9 + A) = L.\cos.A.\sin.T_9 + L.\sin.A.\cos.T_9. \qquad [4019b]$$

Comparing the coefficients of $\sin. T_9$, $\cos. T_9$, separately, in both members, we get
$P = L.\cos.A$, $P' = L.\sin.A$. Dividing the second by the first, also taking the sum [4019c]
of their squares, we get [4018]. The quantity L being considered as positive, we [4019d]
get, from [4019c], the signs of *sin.A, cos.A*, as in [4018].

† (2532) If δL, δA, represent the variations of L, A, in 200 years, between
1750 and 1950; then their variations in t years will be represented by $\frac{t.\delta L}{200}$, $\frac{t.\delta A}{200}$, [4021a]
respectively. Substituting these in [4020], it becomes as in [4021].

which has the divisor $(5n'-2n)^2$. This part of the expression of δv is as in [3844],

[4022]
$$\delta v=-\frac{6m'.n^2}{(5n'-2n)^2}\cdot\left\{\begin{array}{l}\left\{aP'+\frac{2a.dP}{(5n'-2n).dt}-\frac{3a.ddP'}{(5n'-2n)^2.dt^2}\right\}.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\\ -\left\{aP-\frac{2a.dP'}{(5n'-2n).dt}-\frac{3a.ddP}{(5n'-2n)^2.dt^2}\right\}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\};$$

the values of P, P', and of their differentials, being relative to any time whatever t. By developing them in series, ascending according to the powers of the time, and retaining only the second power, and the first and
[4022'] second differentials of P, P', the preceding quantity will become*

Great inequality of Jupiter, reduced to a tabular form.

[4023]
$$\delta v=-\frac{6m'.n^2}{(5n'-2n)^2}\cdot\left\{\begin{array}{l}\left\{\begin{array}{l}aP'+\frac{2a.dP}{(5n'-2n).dt}-\frac{3a.ddP'}{(5n'-2n)^2.dt^2}\\ +t.\left\{a.\frac{dP'}{dt}+\frac{2a.ddP}{(5n'-2n).dt^2}\right\}+\frac{1}{2}t^2.a.\frac{ddP'}{dt^2}\end{array}\right\}.\sin.(5n't-2nt+5\varepsilon'-2\varepsilon)\\ -\left\{\begin{array}{l}aP-\frac{2a.dP'}{(5n'-2n).dt}-\frac{3a.ddP}{(5n'-2n)^2.dt^2}\\ +t.\left\{a.\frac{dP}{dt}-\frac{2a.ddP'}{(5n'-2n).dt^2}\right\}+\frac{1}{2}t^2.a.\frac{ddP}{dt^2}\end{array}\right\}.\cos.(5n't-2nt+5\varepsilon'-2\varepsilon)\end{array}\right\}.$$

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* (2533) The values of P, P', and their differentials [4022], must be computed for the particular time t, for which the value of δv is wanted; but this is an inconvenient method; therefore the functions by which $\sin.T_5$, $\cos.T_5$ [3842a], are multiplied in [4022],
[4022a] are developed in [4023] in series, ascending according to the powers of t. This is done by means of the formula [3850a], neglecting t^3, and the higher powers of t. Thus, if we put the factor of $\sin.T_5$, included between the braces in the first line of [4022],
[4022b] equal to u, and take its first and second differentials, neglecting the differentials of the third and higher orders; we shall get the following values of U, and its differentials; in which the terms in the second members correspond to the epoch $t=0$;

[4022c]
$$U=aP'+\frac{2a.dP}{(5n'-2n).dt}-\frac{3a.ddP'}{(5n'-2n)^2.dt^2};$$
$$\left(\frac{dU}{dt}\right)=a.\frac{dP'}{dt}+\frac{2a.ddP}{(5n'-2n).dt^2};\qquad\left(\frac{ddU}{dt^2}\right)=\frac{a.ddP'}{dt^2}.$$

Substituting these in [3850a], we get for u, the same expression as the factor of $\sin.T_5$,
[4022d] in the first and second lines of [4023]. In the same manner, the factor of $\cos.T_5$, in the second line of [4022], produces the corresponding factor, in the third and fourth lines of [4023].

The values of P, P', *and their differentials, correspond to the epoch of* 1750, *and are determined by the method in* [3850, &c.]; the other parts of the great inequality of m being rather small, it will be sufficient, by what has already been shown, to notice the first power of the time. This great inequality will then have the following form, [4023']

$$\begin{aligned}\delta v = &\ (A+B\,t+C\,t^2)\,.\,\sin.\,(5\,n't-2\,n\,t+5\,\varepsilon'-2\,\varepsilon)\\ &+(A'+B'\,t+C'\,t^2)\,.\,\cos.\,(5\,n't-2\,n\,t+5\,\varepsilon'-2\,\varepsilon).\end{aligned} \qquad [4024]$$

We may also put the great inequality of m' under the same form, by which means it will be easy to reduce these inequalities into tables.

If we wish to reduce the preceding inequality to one term, we must calculate it for the three epochs 1750, 2250, 2750. Let

$$\beta\,.\,\sin.\,(5\,n't-2\,n\,t+5\,\varepsilon'-2\,\varepsilon+\mathrm{A}) \qquad [4025]$$

be this inequality in the year 1750; and $\beta_{,}$, $\mathrm{A}_{,}$; $\beta_{,,}$, $\mathrm{A}_{,,}$, the values of β, A at the epochs 2250, 2750; then the inequality corresponding to any time whatever t, will be* [4025']

Great inequality of Jupiter, reduced to one term.

$$\delta v=\left(\beta+t\,.\frac{d\beta}{dt}+\tfrac{1}{2}t^2.\frac{dd\beta}{dt^2}\right).\sin.\left\{5\,n't-2\,n\,t+5\,\varepsilon'-2\,\varepsilon+\mathrm{A}+t\,.\frac{d\mathrm{A}}{dt}+\tfrac{1}{2}t^2.\frac{dd\mathrm{A}}{dt^2}\right\}; \qquad [4026]$$

the differentials β and A correspond to the epoch in 1750; and we shall have, by [3854—3856],†

$$\frac{d\beta}{dt}=\frac{4\,\beta_{,}-3\,\beta-\beta_{,,}}{1000}; \qquad \frac{dd\beta}{dt^2}=\frac{\beta_{,,}-2\,\beta_{,}+\beta}{250000}; \qquad [4027]$$

$$\frac{d\mathrm{A}}{dt}=\frac{4\,\mathrm{A}_{,}-3\,\mathrm{A}-\mathrm{A}_{,,}}{1000}; \qquad \frac{dd\mathrm{A}}{dt^2}=\frac{\mathrm{A}_{,,}-2\,\mathrm{A}_{,}+\mathrm{A}}{250000}. \qquad [4027']$$

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* (2534) β and A being functions of t, we shall have, as in [3850a],

$$\beta+t\,.\frac{d\beta}{dt}+\tfrac{1}{2}t^2.\frac{dd\beta}{dt^2}, \qquad \text{and} \qquad \mathrm{A}+t\,.\frac{d\mathrm{A}}{dt}+\tfrac{1}{2}t^2.\frac{dd\mathrm{A}}{dt^2}, \qquad [4025a]$$

for their values; using for β, A, and their differentials, the values corresponding to the epoch in 1750. Substituting these in [4025], it becomes as in [4026].

† (2535) If in the general formulas [3854—3856], we change P, $P_{,}$, $P_{,,}$, into β, $\beta_{,}$, $\beta_{,,}$, the expression [3854] will become like the first of the functions [4025a]; [4027a]
and by making the same changes in [3856], we shall get the values of $\frac{d\beta}{dt}$, $\frac{dd\beta}{dt^2}$

In conformity to the remark we have made in [3720], these two great
[4027''] inequalities of Jupiter and Saturn must be applied respectively to their mean motions.

[4027b] [4027]. In like manner, by changing, in [3851—3856], P, $P_{,}$, $P_{,,}$, into Λ, $\Lambda_{,}$, $\Lambda_{,,}$, the formula [3851] will become as in the second of the functions [4025a], and [3856]
[4027c] will give the values of $\frac{d\Lambda}{dt}$, $\frac{dd\Lambda}{dt^2}$ [4027'].

CHAPTER III.

PERTURBATIONS DEPENDING ON THE ELLIPTICITY OF THE SUN.

18'. Since the sun is endowed with a rotatory motion, its figure will not be perfectly spherical. We shall now investigate the effect of its ellipticity on the motions of the planets; putting

$\rho =$ the ellipticity of the sun, expressed in parts of its radius; Symbols.
$q =$ the ratio of the centrifugal force to the gravity at the sun's equator;
$\mu =$ the sine of the planet's declination relative to the sun's equator; [4028]
$D =$ the sun's semi-diameter;
$1 =$ the sun's mass, usually called M;

then it will follow, from [1812], that the sun's ellipticity adds to the function R [913], the quantity* Value of R, depending on the ellipticity.

$$R = (\rho - \tfrac{1}{2} q) \cdot \frac{D^2}{r^3} \cdot (\mu^2 - \tfrac{1}{3}). \qquad [4029]$$

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* (2536) We shall suppose m', m'', m''', &c. to represent the particles of the sun's mass; considering it as being composed of concentrical elliptical strata of variable densities, [4029a] symmetrically arranged about its centre of gravity, taken as the origin of the co-ordinates of these particles x', y', z'; x'', y'', z'', &c. The co-ordinates of the attracted planet m being represented by x, y, z, and its distance from the sun $r = \sqrt{(x^2 + y^2 + z^2)}$. In this case, the expression of R [913] will be reduced to its last term $R = -\frac{\lambda}{m}$; because [4029b] any term of the form $\frac{m'.(xx'+yy'+zz')}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}$, depending on the particle m', whose co-ordinates are x', y', z', is destroyed by a similar term, depending on an equal particle m', whose co-ordinates are $-x'$, $-y'$, $-z'$. Substituting, in [4029b], the value of λ [914], [4029c]

[4030] If we notice only this part of R, and put $\int dR = g + R$; g being a constant quantity; we shall find, that the differential equation in $r\delta r$ [926, 928'] becomes, by neglecting* the square of μ,

[4031] $$0 = \frac{d^2.(r\delta r)}{dt^2} + \frac{n^2 a^3 . r\delta r}{r^3} + 2g + \frac{(\rho - \frac{1}{2}q).D^2}{3r^3}.\dagger$$

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neglecting terms of the order $m'm''$, and using the sign $\int$ to represent the sum of the

[4029d] terms depending on all the particles, we get $R = -\int \frac{m'}{\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}}$. This expression of R corresponds to that of $-V$ in [1385'', 1386], m' being the attracting particle, and $\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}$ its distance from the attracted planet; hence $R = -V$; and by substituting the value of V [1812], we get

[4029e] $$R = -\frac{M}{r} - \frac{(\frac{1}{2}\alpha\varphi - \alpha h).D^2}{r^3}.M.(\mu^2 - \tfrac{1}{3}).$$

The last term being multiplied by D^2, to render it homogeneous with the first, because in [1812, 1795''], the semi-diameter of the body M is put equal to unity, and here it is

[4029f] supposed to be D. Again, by comparing [1670', 4028], we get $\alpha\varphi = q$; also by comparing [1801, &c., 4028], we get $\alpha h = \rho$. Substituting these in [4029e], we obtain

[4029g] $$R = -\frac{M}{r} - \frac{(\frac{1}{2}q - \rho).D^2}{r^3}.M.(\mu^2 - \tfrac{1}{3}).$$

Now if the sun were of a spherical form, with no rotatory motion, we should have

[4029h] $\rho = 0$, $q = 0$, and then $R = -\frac{M}{r}$ [4209g]. Subtracting this from the general value of R [4029g], we get the part of it depending on the sun's ellipticity, namely,

[4029i] $$R = -\frac{(\frac{1}{2}q - \rho).D^2}{r^3}.M.(\mu^2 - \tfrac{1}{3}),$$

and by putting, as in [4028], the sun's mass $M = 1$, it becomes as in [4029].

[4030a] * (2537) The inclination of the sun's equator to the ecliptic is less than 8^d, and its sine is nearly $\frac{1}{8}$, so that μ^2 must be less than $(\frac{1}{8})^2$, or $\frac{1}{64}$; which may be neglected in

[4030b] comparison with $\frac{1}{3}$; and then [4029] becomes $R = -\frac{1}{3}.(\rho - \frac{1}{2}q).\frac{D^2}{r^3}$.

[4031a] † (2538) Substituting, in [926], the value of $rR' = r.\left(\frac{dR}{dr}\right)$ [928'], also $\mu = n^2 a^3$ [3700], we get

[4031b] $$0 = \frac{d^2.(r\delta r)}{dt^2} + \frac{n^2 a^3 . r\delta r}{r^3} + 2\int dR + r.\left(\frac{dR}{dr}\right).$$

Now the value of R [4030b], depending on the sun's ellipticity, gives

[4031c] $$\int dR = -\tfrac{1}{3}.(\rho - \tfrac{1}{2}q).D^2.\int d.\frac{1}{r^3} = -\tfrac{1}{3}.(\rho - \tfrac{1}{2}q).\frac{D^2}{r^3} + g; \quad r.\left(\frac{dR}{dr}\right) = (\rho - \tfrac{1}{2}q).\frac{D^2}{r^3};$$

To determine the constant quantity g, we shall observe, that the formula [931] gives, in δv, the quantity*

$$3a \cdot ngt + (\rho - \tfrac{1}{2}q) \cdot \frac{D^2}{a^2} \cdot nt; \qquad [4032]$$

nt denoting the mean motion of the planet; this quantity must be equal to zero; therefore we have [4032′]

$$g = -\frac{(\rho - \frac{1}{2}q) \cdot D^2}{3a^3}. \qquad [4033]$$

Hence the differential equation in $r\delta r$ becomes, by neglecting the square of e, and observing that $n^2 a^3 = 1$ [3709′],† [4033′]

$$\frac{d^2 \cdot (r\delta r)}{dt^2} + n^2 \cdot r\delta r \cdot \{1 + 3e \cdot \cos.(nt + \varepsilon - \varpi)\} - \frac{2 \cdot (\rho - \frac{1}{2}q)}{3} \cdot n^2 \cdot D^2$$
$$+ \frac{(\rho - \frac{1}{2}q)}{3} \cdot n^2 \cdot D^2 \cdot \{1 + 3e \cdot \cos.(nt + \varepsilon - \varpi)\}. \qquad [4034]$$

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substituting these in [4031b], we get [4031]. We may observe, that the symbol μ [4031a] is entirely different from that in [4028].

* (2539) The constant quantity g is to be found, as in note 699, Vol. I, page 550, by putting the terms of [931], multiplied by t, or rather by $\frac{t}{\mu \cdot \sqrt{(1-e^2)}}$, equal to nothing. These terms are evidently produced by the two last terms of [931],

$$3a\textstyle\int n\,dt \cdot \int \mathrm{d}R + 2a\int n\,dt \cdot r \cdot \left(\frac{dR}{dr}\right); \qquad [4032a]$$

but from [4031c], we get

$$3a\textstyle\int \mathrm{d}R + 2ar \cdot \left(\frac{dR}{dr}\right) = 3ag + (\rho - \frac{1}{2}q) \cdot \frac{D^2 a}{r^3} = 3ag + (\rho - \frac{1}{2}q) \cdot \frac{D^2}{a^2}, \qquad [4032b]$$

noticing merely the term a of the value of r, which is evidently the only part which affects the coefficient of t, now under consideration. Multiplying this last expression by $n\,dt$, and integrating, it becomes as in [4032], which represents the part of δv, connected with the factor t. Putting this equal to nothing, we get [4033]. [4032c]

† (2540) We have $r = a \cdot \{1 - e \cdot \cos.(nt + \varepsilon - \varpi)\}$ [3747], neglecting e^2; hence we get, by using [4033′], [4034a]

$$\frac{1}{r^3} = \frac{1}{a^3} \cdot \{1 + 3e \cdot \cos.(nt + \varepsilon - \varpi)\} = n^2 \cdot \{1 + 3e \cdot \cos.(nt + \varepsilon - \varpi)\}; \qquad [4034b]$$

substituting this, and g [4033], in [4031], we get [4034].

This gives, by integration,*

[4035] $$\frac{r\,\delta r}{a^2}=\tfrac{1}{3}\,.\,(\rho-\tfrac{1}{2}\,q)\,.\,\frac{D^2}{a^2}\,.\,\{1-3\,e\,.\,n\,t\,.\,\sin.\,(n\,t+\varepsilon-\varpi)\}.$$

The elliptical part of $\frac{r^2}{a^2}$ is $1-2\,e\,.\cos.\,(n\,t+\varepsilon-\varpi)$ [3876a]; and if we suppose ϖ to vary by $\delta\varpi$, we shall have [3876d],†

[4036] $$\frac{r\,\delta r}{a^2}=-\,e\,\delta\varpi\,.\,\sin.\,(n\,t+\varepsilon-\varpi).$$

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* (2541) This integration is made as in [865—871″], putting $r\,\delta r=y'$; hence [4034] becomes, by connecting together the terms depending on e,

[4035a] $$0=\frac{d^2y'}{dt^2}+n^2.y'-\tfrac{1}{3}.(\rho-\tfrac{1}{2}q).n^2.D^2+\{n^2.y'+\tfrac{1}{3}.(\rho-\tfrac{1}{2}q).n^2.D^2\}.3e.\cos.(nt+\varepsilon-\varpi).$$

[4035b] Putting $y'=y+\frac{1}{3}.(\rho-\frac{1}{2}q).D^2$, and neglecting the term of the order $y\,e$, or e^2, we get

[4035c] $$0=\frac{d^2y}{dt^2}+n^2.\,y+2\,.\,(\rho-\tfrac{1}{2}q)\,.\,n^2.\,D^2.\,e\,.\cos.\,(n\,t+\varepsilon-\varpi);$$

[4035d] which is of the same form as [865a, 870′, 871′], changing a or m into n, ε into $\varepsilon-\varpi$, αK into $2\,.\,(\rho-\frac{1}{2}q)\,n^2.\,D^2.\,e$, and then [871″] becomes

$$y=-\frac{\alpha K t}{2n}.\sin.\,(n\,t+\varepsilon-\varpi)=-(\rho-\tfrac{1}{2}q)\,.\,n\,t\,.\,D^2.\,e\,.\sin.\,(n\,t+\varepsilon-\varpi)\,;$$

substituting this in y' or $r\,\delta r$ [4035b], we get

[4035e] $$r\,\delta r=\tfrac{1}{3}\,.\,(\rho-\tfrac{1}{2}\,q)\,.\,D^2-(\rho-\tfrac{1}{2}q)\,.\,n\,t\,.\,D^2.\,e\,.\sin.\,(n\,t+\varepsilon-\varpi)\,;$$

dividing this by a^2, we obtain [4035]. We may remark, that the term of the form $\alpha K.\cos.\,(n\,t+\varepsilon-\varpi)$ [871′] is included in the elliptical motion, and it is not necessary to notice this term in the present calculation.

† (2542) Comparing together the expressions of $\frac{r\,\delta r}{a^2}$ [3876 l, 4035], we find, that if the coefficients of $\sin.\,(n\,t+\varepsilon-\varpi)$ be put equal to each other, we shall get

[4036a] $$-\,e\,\delta\varpi=\tfrac{1}{3}.(\rho-\tfrac{1}{2}\,q).\frac{D^2}{a^2}.(-3\,e\,.\,n\,t)\,;$$

whence we obtain $\delta\varpi$, as in the first equation [4037]. The second expression [4037] is
[4036b] deduced from the first by the substitution of $n=a^{-\frac{3}{2}}$ [3709′]. Again, since the formula [4035] does not contain a term depending on $n\,t\,.\cos.\,(n\,t+\varepsilon-\varpi)$, and
[4036c] in [3876] this cosine is connected with the factor δe, we shall have $\delta e=0$. The

If we compare this expression of $\frac{r\,\delta r}{a^2}$ with the preceding, we shall obtain

Motion of the perihelion, arising

$$\delta\varpi = (\rho - \tfrac{1}{2}q)\cdot\frac{D^2}{a^2}\cdot nt = (\rho - \tfrac{1}{2}q)\cdot\frac{D^2.t}{a^{\frac{7}{2}}} \quad [4036a,\, b]\,;$$ [4037]

from the oblateness of the sun, is insensible.

therefore the most sensible effect of the ellipticity of the sun, upon the motion of a planet in its orbit, is a direct motion in its perihelion; but this motion [4037′] *being in the inverse ratio of the square root of the seventh power of the greater axis of the planetary ellipsis, we see that it cannot be sensible except* [4038] *in Mercury* [4036f].

To find the effect of the sun's ellipticity upon the position of the orbit, we shall resume the third of the equations [915]. This equation may be put under the following form,*

$$0 = \frac{d\,dz}{dt^2} + \frac{n^2\,a^3.\,z}{r^3} + \left(\frac{dR}{dz}\right).$$ [4039]

We shall take the solar equator for the fixed plane, which gives $\mu^2 = \frac{z^2}{r^2}$ [4039′] [4040a]; then by observing that $r^2 = x^2 + y^2 + z^2$, we shall have†

$$\left(\frac{dR}{dz}\right) = 3\,.\,(\rho - \tfrac{1}{2}q)\,.\,\frac{n^2.\,D^2}{a^2}\,.\,z\,;$$ [4040]

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constant part of $\frac{r\,\delta r}{a^2}$, which is nearly equal to that of $\frac{\delta r}{a}$, is represented in the present case by the first term of the second member of [4035]; so that we shall have

$$\frac{\delta r}{a} = \tfrac{1}{3}\cdot(\rho - \tfrac{1}{2}q)\cdot\frac{D^2}{a^2},$$ [4036d]

as in [4042]. Now we shall see, in [4262—4265′], that if the sun be homogeneous, we shall have, for the orbit of the planet Mercury, $\delta\varpi = (\rho - \tfrac{1}{2}q)\cdot\frac{D^2}{a^{\frac{7}{2}}}.t = 0'',012.t$ nearly [4036e] [4265]; and this expression is much smaller for the other planets, on account of the divisor $a^{\frac{7}{2}}$; so that it produces only 12″ in a thousand years for Mercury, and is much less for the other [4036f] planets. The quantity δr [4036d, 4260—4263] is evidently insensible.

* (2543) Substituting $\mu = n^2\,a^3$ [3700] in the third equation [915], it becomes [4039a] as in [4039].

† (2544) In [4028], μ is put for the sine of the planet's declination above the plane [4039b] of the sun's equator, its perpendicular distance above this plane being z, and its distance

hence the preceding differential equation becomes*

[4041] $$0=\frac{d\,d\,z}{d\,t^2}+n^2 z\,.\left\{1-\frac{3\,\delta\,r}{a}+3\,.\,(\rho-\tfrac{1}{2}\,q)\,.\,\frac{D^2}{a^2}\right\};$$

now by what precedes [4036d], we have

[4042] $$\frac{\delta\,r}{a}=\tfrac{1}{3}\,.\,(\rho-\tfrac{1}{2}\,q)\,.\,\frac{D^2}{a^2};$$

hence we obtain

[4043] $$0=\frac{d\,d\,z}{d\,t^2}+n^2 z\,.\left\{1+2\,.\,(\rho-\tfrac{1}{2}\,q)\,.\,\frac{D^2}{a^2}\right\}.$$

This gives, by integration,†

[4044] $$z=\varphi\,.\,\sin.\left\{n\,t\,.\left(1+(\rho-\tfrac{1}{2}\,q)\,.\,\frac{D^2}{a^2}\right)-\theta\right\};$$

[4045] $\frac{\varphi}{a}$ being the inclination of the orbit to the solar equator,‡ and θ an arbitrary

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[4040a] from the sun's centre r; hence we evidently have $\mu=\frac{z}{r}$; also $r=\sqrt{(x^2+y^2+z^2)}$ [914′]. Substituting this value of μ in [4029], we get

[4040b] $$R=(\rho-\tfrac{1}{2}\,q)\,.\,D^2.\left\{\frac{z^2}{r^5}-\frac{1}{3\,r^3}\right\}.$$

Taking its partial differential relatively to z, neglecting z^3, and observing that $\left(\frac{d\,r}{d\,z}\right)=\frac{z}{r}$, we get

[4040c] $$\left(\frac{d\,R}{d\,z}\right)=(\rho-\tfrac{1}{2}\,q)\,.\,D^2.\left\{\frac{2\,z}{r^5}+\frac{z}{r^5}\right\}=3\,.\,(\rho-\tfrac{1}{2}\,q)\,.\,\frac{D^2}{r^5}\,.\,z.$$

Retaining only the constant part of r, we may put $\frac{1}{r^5}=\frac{1}{a^5}=\frac{n^2}{a^2}$ [3709′], and then the preceding expression [4040c] becomes as in [4040].

* (2545) Noticing only the terms of r, depending on the sun's ellipticity, we may put,

[4041a] as in [4036d], $r=a+\delta\,r$, whence $\frac{1}{r^3}=\frac{1}{a^3}\cdot\left(1-\frac{3\,\delta\,r}{a}\right)$. Substituting this and [4040] in [4039], we get [4041]; and if we use [4036d], it becomes as in [4043].

† (2546) Comparing [865′, 4043], we get $y=z$, $a=n\,.\left\{1+(\rho-\tfrac{1}{2}\,q)\,.\frac{D^2}{a^2}\right\}$,

[4042a] by neglecting $(\rho-\tfrac{1}{2}\,q)^2$. Substituting these in the first value of y [864a]; changing also b into φ, and φ into $-\theta$, we get [4044].

‡ (2547) The sine of the declination is equal to $\frac{z}{r}$ [4040a], and its greatest value

[4045a] is equal to $\frac{\varphi}{r}$ [4044] or $\frac{\varphi}{a}$ nearly; which evidently represents the sine of the inclination of the orbit to the solar equator.

constant quantity. *Thus the nodes of the orbit on this equator have a retrograde motion equal to the direct motion of the perihelion, and which cannot therefore be sensible, except in the orbit of Mercury.** *At the same time we see that the sun's ellipticity has no influence on the excentricity of the planet's orbit* [4046c], *or on the inclination of this orbit to the solar equator; it cannot therefore alter the stability of the planetary system.* [4045′] [4046]

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* (2518) It is evident from the form of the angle, which occurs in [4044], that the retrograde motion of the node in the time t is represented by $nt.(\rho-\frac{1}{2}\varphi).\frac{D^2}{a^2}$, [4046a] because the body is in the node when $z=0$, and it completes its revolution, to the same node, while the angle $nt+nt.(\rho-\frac{1}{2}\varphi).\frac{D^2}{a^2}$ increases by 360^d; the mean [4046b] periodical revolution being performed in the time t, which makes $nt=360^d$ [4032′]. Hence it is evident, that the *retrograde motion of the node* in the time t is nearly equal to the difference of these quantities, as in [4046a], being the same as the *direct motion of the perihelion* [4037]. As $\delta e=0$ [4036c], the excentricity is not affected by the sun's [4046c] ellipticity, neither does it affect the inclination $\frac{\varphi}{a}$ of the planet's orbit to the sun's equator [4045a], which is constant, because φ is one of the constant quantities obtained by integration. The results found in this chapter agree with those found by Mr. Plana in the Memoirs of the Royal Society of London, Vol. II, page 344, &c., noticing the term neglected by [4046d] La Place in [4030]; making also the computation directly from the formulas [5788—5791], and carrying on the approximation to a rather greater degree of accuracy.

CHAPTER IV.

PERTURBATIONS OF THE MOTIONS OF THE PLANETS, ARISING FROM THE ACTION OF THEIR SATELLITES.

19. The theorems of § 10, Book II [442″, &c.], afford a simple and accurate method of ascertaining the perturbations of the planets from the action of their satellites. We have seen, in [451′, &c.], that the *common centre of gravity* of the planet and its satellites, describes very nearly an elliptical
[4047] orbit about the sun. If we consider this *common orbit* as the ellipsis of the planet; the relative position of the satellites, compared with each other and with the sun, will give the position of the planet, *relative to this common centre of gravity*, consequently also the perturbations which the planet suffers from its satellites. Let

$M =$ the mass of the planet;

Symbols. $R =$ the radius vector of the *common orbit*, or the orbit of the centre of gravity of the planet and satellites, the origin being the sun's centre;

$U =$ the angle formed by the radius R, and the invariable line, taken in the *common orbit*, as the origin of the longitudes;

m, m', &c. the masses of the satellites;

[4048] r, r', &c. the radii vectores of the satellites, the origin being the *common centre* of gravity of the planet and its satellites;

v, v', &c. the longitudes of the satellites, referred to this *common centre;*

s, s', &c. the latitudes of the satellites above the *common orbit*, and viewed from the *common centre;*

X, Y, Z the rectangular co-ordinates of the planet; taking the *common centre of gravity* of the planet and its satellites for their origin; the radius R for the axis of X; and for the axis of Z the line perpendicular to the plane of the *common orbit.*

We shall have very nearly, from the properties of the centre of gravity, and by observing that the masses of the satellites are very small, in comparison, with that of the planet,* [4049]

$$0 = MX + mr \,.\, \cos.\,(v - U) + m'r'.\,\cos.\,(v' - U) + \&c.\,;$$
$$0 = MY + mr \,.\, \sin.\,(v - U) + m'r'.\,\sin.\,(v' - U) + \&c.\,; \qquad [4050]$$
$$0 = MZ + m \,.\, rs + m'.\,r's' + \&c.$$

The perturbation of the radius vector is nearly equal to X; consequently it is equal to

Perturbations.

$$-\frac{m}{M}.r.\cos.(v-U) - \frac{m'}{M}.r'.\cos.(v'-U) - \&c. = \text{Perturbation of radius vector.} \qquad [4051]$$

The perturbation of the motion of the planet in longitude, as seen from the sun, is very nearly $\frac{Y}{R}$; therefore it is equal to

$$-\frac{m}{M}.\frac{r}{R}.\sin.(v-U) - \frac{m'}{M}.\frac{r'}{R}.\sin.(v'-U) - \&c. = \text{Perturbation in longitude.} \qquad [4052]$$

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* (2549) If we let fall from the points where the bodies M, m, m', &c. are situated, perpendiculars upon the axes of X, Y, Z, the distances of these perpendiculars from the common centre of gravity of the planet and its satellites, taken as the origin, will be, respectively, as follows; [4050a]

On the axis of X;	X;	$r \,.\, \cos.(v-U)$;	$r'.\cos.(v'-U)$, &c.;	[4050b]
On the axis of Y;	Y;	$r \,.\, \sin.\,(v-U)$;	$r'.\sin.\,(v'-U)$, &c.;	[4050c]
On the axis of Z;	Z;	rs;	$r's'$, &c. nearly.	[4050d]

Multiplying the distances [4050b] by the masses M, m, m', &c.; and taking the sum of these products, it will become equal to nothing, by means of the first of the equations [124]; hence we get the first of the equations [4050]. In like manner, by multiplying the distances, measured on the axis of Y, by M, m, m', &c., respectively, and putting the sum of the products equal to nothing, we get the second of the equations [4050]. The third of these equations is formed by a similar sum, corresponding to the axis of Z. From these three equations, we may find the values of X, $\frac{Y}{R}$, $\frac{Z}{R}$, as in [4051, 4052, 4053]; and as the radius R, or axis X, passes through the place of the *common centre of gravity*, it is evident that these quantities X, $\frac{Y}{R}$, $\frac{Z}{R}$ will represent, respectively, the perturbations of the radius vector, of the longitude and of the latitude, conformably to what is said above. [4050e] [4050f] [4050g]

Lastly, the perturbation of the motion of the planet in latitude, as seen from the sun, is very nearly $\frac{Z}{R}$; hence it is nearly equal to

[4053]
$$-\frac{m}{M}\cdot\frac{rs}{R}-\frac{m'}{M}\cdot\frac{r's'}{R}-\&c.=\text{Perturbation in latitude.}$$

These different perturbations are sensible only in the earth, disturbed by the moon. The masses of Jupiter's satellites are very small in comparison with that of the planet, and their elongations, seen from the sun, are so very
[4054] *small, that these perturbations of Jupiter are insensible. There is every reason to believe that this is also the ease for Saturn and Uranus.*

CHAPTER V.

CONSIDERATIONS ON THE ELLIPTICAL PART OF THE RADIUS VECTOR, AND ON THE MOTION OF A PLANET.

20. We have determined, in [1017, &c.], the arbitrary constant quantities, so that the mean motion and the equation of the centre may not be changed by the mutual action of the planets. Now we have, in the elliptical hypothesis,* $\frac{1+m}{a^3}=n^2$, *the mass of the sun being put equal to unity.* Hence we obtain [4055]

$$a=n^{-\frac{2}{3}}.(1+\tfrac{1}{3}m)\,; \qquad [4056]$$

for the semi-transverse axis, which must be used in the elliptical part of the radius vector.

If we suppose, in conformity to the principles assumed in [4078—4079, &c.], that

$$a=n^{-\frac{2}{3}}\,; \qquad a'=n'^{-\frac{2}{3}}, \quad \&c.\,; \qquad [4057]$$

we must increase a, a', &c. in the calculation of the elliptical part of the

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* (2550) This is the same as [3700], putting, as in [3709a], $\mu=M+m$, and $M=1$, as in [4055]. From this we get

$$a=n^{-\frac{2}{3}}.(1+m)^{\frac{1}{3}}=n^{-\frac{2}{3}}.(1+\tfrac{1}{3}m-\tfrac{1}{18}m^2+\&c.)\,; \qquad [4056a]$$

which, by neglecting terms of the order m^2, becomes as in [4056].

[4058] radius vector by the quantities $\frac{1}{3}ma$, $\frac{1}{3}m'a'$, &c. respectively; but this augmentation is only sensible in the orbits of Jupiter and Saturn.*

Increment of the radius vector

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* (2551) The values of a^{iv}, a^{v}, for Jupiter and Saturn [4079], are respectively augmented by the correction [4058], in the expressions [4451, 4510]. The similar augmentation, corresponding to the other great planet Uranus, is $\frac{1}{3}m^{vi}a^{vi}$, which, by using
[4058a] m^{vi} [4061], becomes $\frac{a^{vi}}{58512}$. If this quantity were an arc of the planet's orbit,
[4058b] *perpendicular* to the radius vector, it would subtend only an angle of $3^s,6$, when viewed from the sun; but being in the direction of the radius vector, it produces no change in the longitude, seen from the sun; or from the earth, when the planet is in conjunction or in opposition. The most favorable situation for augmenting the effect of this correction, in the geocentric longitude of the planet, is when the earth is nearly at its greatest angle of elongation from the sun, as seen from the planet. This angle for the planet Uranus
[4058c] is quite small, its sine being represented by $\frac{a''}{a^{vi}}=\frac{1}{19}$ nearly [4079]; and as the above correction $3^s,6$ is to be diminished in the same ratio, it produces only $0^s,2$ for the greatest possible effect of this augmentation of the radius, in changing the place of the planet Uranus,
[4058d] as seen from the earth; consequently this correction is wholly insensible.

We have already observed in the commentary in Vol. I, page 561, that Mr. Plana makes some objections to the introduction of the constant quantity g, in the integral [1012'],
[4058e] and he has also urged similar remarks against the use of the constant quantities $f_{\prime}$, $f'_{\prime}$ [1015'], in finding the integral δu [1015]; but a little consideration will show, that these objections do not apply to the accuracy of the results, or to the astronomical tables founded upon them; but merely to the most convenient way of ascertaining, *as a mere matter of curiosity*, the orbit a body would describe if it were not acted upon by the disturbing force, or of computing the whole effect of the disturbing force in a given time. This subject has been discussed very ably by Mr. Poisson, in the *Connaissance des Tems* for the year 1831,
[4058f] pag. 23—33; and we shall, in the remaining part of this note, avail ourselves of his remarks. The complete integrals of the three differential equations [545], which determine the co-ordinates x, y, z, of the planet referred to the sun's centre as their origin, contain *six* arbitrary constant quantities [571a], which we shall denote by a, b, c, &c.; and the same is true in using the polar co-ordinates r, v, s; as we have already seen, in [602''],
[4058g] in the *first approximation*, where the disturbing forces are neglected, and the simple elliptical motion obtained. In a *second approximation*, in which we notice only the first power of the disturbing forces, we may put δr, δv, δs for the increments of r, v, s; and then
[4058h] the integrations being made, as in [1015, &c., 1021, 1030], will introduce *six* new arbitrary constant quantities, a', b', c', &c.; these accented letters being taken for symmetry, instead of g, $f_{\prime}$, $f'_{\prime}$, &c., used by La Place. A third approximation includes terms of the second
[4058i] order of the disturbing forces, and by similar integrations, produces *six* other constant quantities a'', b'', c'', &c., and so on successively. *If we restrict ourselves to the second*

We must then apply to the radius vector the corrections given by the

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approximation, neglecting terms of the order of the square of the disturbing forces, the polar co-ordinates will be $r+\delta r$, $v+\delta v$, $s+\delta s$, containing the twelve constant quantities a, b, c, &c.; a', b', c', &c., which must, by the nature of the question, be reduced to six only, or to *six distinct functions* A, B, C, D, E, F, of these twelve quantities. The values of A, B, C, &c. may be determined by the position, velocity, and direction of the planet at a given moment; or by the comparison of the values of $r+\delta r$, $v+\delta v$, $s+\delta s$, with those deduced from observation; in each case the result will be fixed and determined. On the contrary, we may assume at pleasure any values of a', b', c', &c.; and the values thus assigned to these terms, will determine absolutely the quantities a, b, c, &c., which differ but little from A, B, C, &c. on account of the smallness of the disturbing forces. [4058k] [4058l] [4058m]

If we wish that δr, δv, δs *should express the effects produced by the disturbing forces on the radius vector, the longitude and the latitude* of the disturbed planet; we must determine a, b, c, &c. so that the elliptical co-ordinates r, v, s, and their differential coefficients $\frac{dr}{dt}$, $\frac{dv}{dt}$, $\frac{ds}{dt}$, may represent the position, the velocity, and the direction of the planet at the commencement of this interval of time; and afterwards determine a', b', c', &c., so that we may have at the same epoch [4058n]

$$\delta r=0, \qquad \delta v=0, \qquad \delta s=0; \qquad \frac{d.\delta r}{dt}=0, \qquad \frac{d.\delta v}{dt}=0, \qquad \frac{d.\delta s}{dt}=0. \qquad [4058o]$$

At the end of the time t, counted from the same epoch, r will be the distance of the planet from the sun, which will obtain, if the disturbing force cease to act from the commencement, and δr will be the augmentation of distance produced by this force. Similar remarks may be made relative to the quantities v, δv; or s, δs. *If we determine* a', b', c' *by other conditions, the perturbations of the troubled orbit will no longer be wholly expressed by the quantities* δr, δv, δs; *because the elliptical parts* r, v, s, *are also affected by means of the constant quantities* a, b, c, &c., *which partake of the disturbing forces*, and are different from what they would be if these forces were suppressed. But this is not attended with any inconvenience; since it does not prevent these complete values of $r+\delta r$, $v+\delta v$, $s+\delta s$, from representing, at every instant, the true position of the planet, which is the object of the tables of its motion, into which these values are finally reduced. [4058p] [4058q] [4058r]

Instead of considering directly the increments δr, δv, δs, of the elliptical orbit, we may use the method depending on the variation of the arbitrary constant quantities; supposing δa, δb, δc, &c. to be the increments of the constant quantities a, b, c, &c., contained in r, v, s. These six variable quantities δa, δb, δc, &c. will be given by direct integration of formulas similar to [1177], or like those collected together in the appendix [5786—5791], *supposing that we neglect the second and higher powers of the disturbing forces.* These values will then be of the forms, [4058s] [4058s']

$$\delta a=a_{\prime}+\alpha; \qquad \delta b=b_{\prime}+\beta; \qquad \delta c=c_{\prime}+\gamma, \quad \&c. \qquad [4058t]$$

formulas of Book II, § 50 [1020, &c.], and by the preceding articles

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$a_{,}$, $b_{,}$, $c_{,}$ being new arbitrary constant quantities, and α, β, γ, &c. functions of t, and of a, b, c, &c. Substituting $a+\delta a$, $b+\delta b$, $c+\delta c$, &c. for a, b, c, &c. in the values
[4058u] of r, v, s, we shall obtain for the co-ordinates of the disturbed planet, expressions which are equivalent to the preceding values of $r+\delta r$, $v+\delta v$, $s+\delta s$. The constant
[4058v] quantities $a_{,}$, $b_{,}$, $c_{,}$, &c., as well as α, β, γ, &c., are of the order of the disturbing forces; therefore, by neglecting terms of the second order, as in [4058s'], we may put, in the values of α, β, γ, &c.; $a+a_{,}$ for α, $b+b_{,}$ for β, $c+c_{,}$ for c, &c.; by which means $a+a_{,}$, $b+b_{,}$, $c+c_{,}$, &c. will be the six arbitrary constant quantities, which occur in the values of $r+\delta r$, $v+\delta v$, $s+\delta s$. This shows how the arbitrary constant quantities, contained in the co-ordinates of the disturbed planet, as found
[4058w] by the two first approximations, are reduced to the number corresponding to the system of differential equations upon which they depend.

If we wish to determine the total effect of the disturbing forces upon each of the elliptical elements, during a given time, we must find, as above, the constant quantities
[4058x] a, b, c, &c.; by means of the position, the velocity, and the direction of the planet at the commencement of this interval of time; and then the constant quantities $a_{,}$, $b_{,}$, $c_{,}$, by means of the equations

[4058y] $$a_{,}+\alpha=0, \qquad b_{,}+\beta=0, \qquad c_{,}+\gamma=0, \qquad \&c.,$$

corresponding to the same instant. The effect of the disturbing force at the end of any proposed time t, will be expressed by means of the quantities δa, δb, δc, &c., which will then contain nothing arbitrary. This is practised in the theory of comets, in which the
[4058z] values of δa, δb, δc, &c. are calculated, by quadratures, for the interval of time between the two successive appearances of a comet.

These general considerations agree with the method used by La Place in the second book of this work. In the abovementioned paper of the *Connaissance des Tems* for the year 1831, page 29, &c., Mr. Poisson has applied these principles to the investigation of the effect of the whole disturbing force of a planet m', upon another planet m, moving
[4059a] in the same plane. The radius vector and the longitude of the planet m being affected by this action, but not its latitude, because the bodies m, m' move in the same plane. In this case, the *six* arbitrary constant quantities mentioned in [4058l], are reduced to *four.*
[4059b] If we neglect terms of the order e^2 in the elliptical motion of the body m, the expressions of the radius vector and longitude [669, 605'], become

[4059c] $$r=a-ae\,.\,\cos.\,(nt+\varepsilon-\varpi)\,;$$

[4059d] $$v=nt+\varepsilon+2e\,.\,\sin.\,(nt+\varepsilon-\varpi)\,;$$

[4059e] $$n^2a^3=M+m=\mu.$$

If we suppose the body m' to begin to disturb the motion of m at the commencement

[3706—4058]. The expression of δr [1020] contains these two terms,

$$\delta r = -m'a.fe.\cos.(nt+\varepsilon-\varpi) - m'a.f'e'.\cos.(nt+\varepsilon-\varpi');$$ [4059]

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of the time t, we may determine the effect of the perturbation of the radius vector by means of the value of δr [1016], in which the arbitrary constant quantities are retained. [4059f] The expression of δv [1021] would give the perturbations in longitude, if particular values had not been assigned to the arbitrary constant quantities g, f, f'. To obviate this objection, we must retain these arbitrary quantities as they are found in the functions [1021b, c, d, e], whose sum is assumed in the first line of the note in page 556, Vol. I [4059g] [1021e—f], for the value of δv. In order to simplify this calculation, it will be convenient to change the form of the terms depending on f, f'; by developing the sines and cosines of the angles $nt+\varepsilon-\varpi$, $nt+\varepsilon-\varpi'$, into terms depending on $\sin.nt$, $\cos.nt$, by the method used in [1023a]; and changing the values of the arbitrary constant quantities f, f', so that the part of the expression of $\frac{\delta r}{a}$ [1016], depending upon them, [4059h] may be put under the form $f.\cos.nt+f'.\sin.nt$. The corresponding terms of the value [4059i] of δv may be found by multiplying this expression by 2, and changing the angle nt into $nt+90^d$; as is evident, by comparing the terms of $\frac{\delta r}{a}$ [1016], depending on f, f', with those of δv [1021b]; hence these terms of δv become $-2f.\sin.nt+2f'.\cos.nt$. [4059k] We may also add an arbitrary constant quantity h, to the part of δv, computed in either of the integrations [1021d, e], and retain the terms

$$m'.ant.\left\{3g+a.\left(\frac{d.A^{(0)}}{da}\right)\right\} \quad [1021d,\, e],$$ [4059l]

which were put equal to nothing in [1021f]. Making these changes in the expressions of $\frac{\delta r}{a}$, δv [1016, 1021]; neglecting the other terms of the order e or e', because this degree of accuracy is sufficient in our present calculation, which is only designed for the purpose of illustration; and supposing also, for brevity, as in [1018a],

$$\nu = n - n'; \qquad T = n't - nt + \varepsilon' - \varepsilon; \qquad G = a^2.\left(\frac{d.A^{(i)}}{da}\right) + \frac{2n}{\nu}.a\,A^{(i)},$$ [4059m]

we get

$$\frac{\delta r}{a} = -2m'.ag - \tfrac{1}{2}m'.a^2.\left(\frac{d.A^{(0)}}{da}\right) + \tfrac{1}{2}m'.n^2.\Sigma.\frac{G}{i^2\nu^2-n^2}.\cos.iT + f.\cos.nt + f'.\sin.nt;$$ [4059n]

$$\delta v = h - 2f.\sin.nt + 2f'.\cos.nt + m'.nt.\left\{3ag + a^2.\left(\frac{d.A^{(0)}}{da}\right)\right\}$$
$$+\tfrac{1}{2}m'.\Sigma.\left\{\frac{n^2}{i\nu^2}.aA^{(i)}. + \frac{2n^3.G}{i\nu.\{i^2\nu^2-n^2\}}\right\}.\sin.iT;$$ [4059o]

which are substantially the same as the equations (5), (6), of Mr. Poisson, in the paper

f and f' being determined by the two following equations, given in [1018],

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abovementioned; observing, that i includes all integral numbers, *positive* and *negative*, except $i=0$ [1012']; whereas he only uses the positive values of i. Now if we use the expression of g [1017], the terms depending on nt will vanish from δv, and then

[4059p] δr [1020] will contain the constant part $\frac{1}{6}m'.a^3.\left(\frac{dA^{(0)}}{da}\right)$; but this is not the whole effect of the disturbing force upon the radius vector; because a part of this perturbation is

[4059q] introduced in the value of n, which is affected by the value of g, assumed in [1017], and n is connected with a by means of the equation [4059e].

We shall, for greater simplicity, take, as the epoch, the instant of the mean conjunction

[4059r] of the planets m, m'; so that we shall then have $t=0$, $T=0$; also $\varepsilon'=\varepsilon$. We shall also suppose that the body m', at that instant, commences its action upon the radius vector, and upon the longitude of the body m. Now we may find, from the tables of the planet's motion, the numerical values of r, v, $\frac{dr}{dt}$, $\frac{dv}{dt}$, when $t=0$; and these are to be put equal to the values deduced from [4059c, d]. These four equations, being combined

[4059s] with [4059e], determine the constant quantities n, a, e, ε, ϖ; and then the formulas [4059c, d] determine the elliptical motion, which obtains, if the disturbing force cease to act at the epoch $t=0$. This being premised, we must put $t=0$, $T=0$ [4059r], in the four equations [4058o],

[4059t] $$\delta r=0; \qquad \delta v=0; \qquad \frac{d.\delta r}{dt}=0; \qquad \frac{d.\delta v}{dt}=0;$$

and by substituting in them the values [4059n, o], we may obtain the values of the four arbitrary constant quantities g, f, f', h, introduced by the second approximation.

If we substitute these values of g, f, f', h, in δr, δv [4059n,o], they will express, at the end of the time t, the effect of the disturbing force during that time. Now the differential of δr [4059n], relative to t, being found, and substituted in the third equation [4059t], gives $f'=0$, when $t=0$, $T=0$ [4059r]. With this value of f', and those of δv [4059o, t], together with $t=0$, $T=0$, we get $h=0$. Substituting these values

[4059u] of t, h, f', in the equations $\delta r=0$, $\frac{d.\delta v}{dt}=0$ [4059t], using also the values [4059n, o], we obtain the following equations,

[4059v] $$0=-2m'.ag-\tfrac{1}{2}m'.a^2.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}m'.n^2.\Sigma.\frac{G}{i^2\nu^2-n^2}+f;$$

[4059w] $$0=-2fn+m'.n.\left\{3ag+a^2.\left(\frac{dA^{(0)}}{da}\right)\right\}-\tfrac{1}{2}m'.\Sigma.\left\{\frac{n^2}{\nu}.aA^{(i)}+\frac{2n^3.G}{i^2\nu^2-n^2}\right\}.$$

Multiplying the equation [4059v] by $2n$, and adding the product to [4059w] we find that the terms depending on f, G, $\left(\frac{dA^{(0)}}{da}\right)$, vanish from the sum, which becomes

[4059x] $$0=-m'.nag-\tfrac{1}{2}m'.\Sigma.\frac{n^2}{\nu}.aA^{(i)};$$

$$f = \tfrac{2}{3} a^2 . \left(\frac{d\,A^{(0)}}{d\,a}\right) + \tfrac{1}{4} a^3 . \left(\frac{d\,d\,A^{(0)}}{d\,a^2}\right);$$

$$f' = \tfrac{1}{4} . \left\{ a\,A^{(1)} - a^2 . \left(\frac{d\,A^{(1)}}{d\,a}\right) - a^3 . \left(\frac{d\,d\,A^{(1)}}{d\,a^2}\right) \right\}. \qquad [4060]$$

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whence $g = -\frac{n}{2\nu} . \Sigma . A^{(i)}$. Substituting this in [4059v], we get

$$f = -\frac{m' . a\,n}{\nu} . \Sigma . A^{(i)} + \tfrac{1}{2} m' . a^2 . \left(\frac{d\,A^{(0)}}{d\,a}\right) - \tfrac{1}{2} m' . n^2 . \Sigma . \frac{G}{i^2 \nu^2 - n^2} . \qquad [4059y]$$

By means of the values of f', h, g, f [4059u, y], the expressions [4059n, o] become

$$\frac{\delta r}{a} = \frac{m' . a\,n}{\nu} . \Sigma . A^{(i)} . (1 - \cos . n\,t) - \tfrac{1}{2} m' . a^2 . \left(\frac{d\,A^{(0)}}{d\,a}\right) . (1 - \cos . n\,t)$$

$$+ \tfrac{1}{2} m' . n^2 . \Sigma . \frac{G}{i^2 \nu^2 - n^2} . (\cos . i\,T - \cos . n\,t); \qquad [4059z]$$

$$\delta v = m' . n\,t . \left\{ -\frac{3\,a\,n}{2\,\nu} . \Sigma . A^{(i)} + a^2 . \left(\frac{d\,A^{(0)}}{d\,a}\right) \right\}$$

$$+ m' . \left\{ \frac{2\,a\,n}{\nu} . \Sigma . A^{(i)} - a^2 . \left(\frac{d\,A^{(0)}}{d\,a}\right) + n^2 . \Sigma . \frac{G}{i^2 \nu^2 - n^2} \right\} . \sin . n\,t \qquad [4059z']$$

$$+ \tfrac{1}{2} m' . \Sigma . \left\{ \frac{n^2}{i\,\nu^2} . a\,A^{(i)} + \frac{2\,n^3\,G}{i\,\nu . (i^2 \nu^2 - n^2)} \right\} . \sin . i\,T.$$

If we retain merely the non-periodical parts of r, v, δr, δv [4059c, d, z, z'], and resubstitute the value of ν [4059m], we shall get

$$r + \delta r = a + \frac{m' . a^2 n}{n - n'} . \Sigma . A^{(i)} - \tfrac{1}{2} m' . a^3 . \left(\frac{d\,A^{(0)}}{d\,a}\right); \qquad [4060a]$$

$$v + \delta v = n\,t + \varepsilon + m' . n\,t . \left\{ -\frac{3\,a\,n}{2 . (n - n')} . \Sigma . A^{(i)} + a^2 . \left(\frac{d\,A^{(0)}}{d\,a}\right) \right\}; \qquad [4060b]$$

for the expressions of the mean distance and mean longitude of the planet m.

The expressions of the same mean distance and mean longitude, according to La Place's calculation [1020, 1021], are

$$r + \delta r = a + \tfrac{1}{6} m' . a^3 . \left(\frac{d\,A^{(0)}}{d\,a}\right); \qquad v + \delta v = n\,t. \qquad [4060c]$$

The differences between these values, and those in [4060a, b], are merely apparent, and arise from using different values of n, a, in [4060c] from those in [4060a, b]. To render this evident, we shall suppose, for a moment, that $n_{,} t$ represents the mean motion of the planet m, derived from observation; then, by putting the coefficient of t, in the equation [4060b], equal to $n_{,} t$, we shall have

$$n_{,} = n + m' . n . \left\{ -\frac{3\,a\,n}{2 . (n - n')} . \Sigma . A^{(i)} . + a^2 . \left(\frac{d\,A^{(0)}}{d\,a}\right) \right\}. \qquad [4060d]$$

[4060′] The preceding part of the radius vector [4059] may be united in the same table with the elliptical part of the radius.*

[4060e] Let $a_{\prime}$ be the value of a, deduced from the equation $a=\mu^{\frac{1}{3}}.n^{-\frac{2}{3}}$ [4059e], when $n_{\prime}$ is substituted for n; so that this equation holds good for a, n, and also for $a_{\prime}$, $n_{\prime}$; we shall have successively, by development, neglecting the square of $n_{\prime}-n$,

[4060f]
$$a_{\prime}=\mu^{\frac{1}{3}}.n_{\prime}^{-\frac{2}{3}}=\mu^{\frac{1}{3}}.\{n+(n_{\prime}-n)\}^{-\frac{2}{3}}=\mu^{\frac{1}{3}}.n^{-\frac{2}{3}}.\left\{1+\left(\frac{n_{\prime}-n}{n}\right)\right\}^{-\frac{2}{3}}$$
$$=a.\left\{1+\left(\frac{n_{\prime}-n}{n}\right)\right\}^{-\frac{2}{3}}=a-\frac{2a}{3n}.(n_{\prime}-n).$$

Substituting in this the value of $n_{\prime}-n$ [4060d], we get, by transposition,

[4060g]
$$a=a_{\prime}+\tfrac{2}{3}m'.a.\left\{-\frac{3an}{2.(n-n')}.\Sigma.A^{(i)}+a^2.\left(\frac{dA^{(0)}}{da}\right)\right\}.$$

This value of a being substituted in [4060a], we find, that the parts depending on $A^{(i)}$ destroy each other, and we have

[4060h]
$$r+\delta r=a_{\prime}+\tfrac{1}{6}m'.a^3.\left(\frac{dA^{(0)}}{da}\right).$$

Now as we neglect terms of the order m'^2, we may change a into $a_{\prime}$, in the part depending on $A^{(0)}$; and then the expression [4060h] becomes of the same form as in [4060c]; being equivalent to that found by La Place. This calculation serves to illustrate and confirm his
[4060i] method of calculation; and shows, at the same time, how we can dispose of the additional arbitrary constant quantities, which are introduced by the integrations of δr, δv; so as to conform to the actual situations and motions of the attracting bodies; and to investigate the part of the effect of the disturbing forces, that we have particularly considered in this note.

* (2552) We have here omitted a clause, in which the author directs, that the sign
[4060k] of the term of f', depending on $ddA^{(1)}$, should be changed; because we have previously corrected the mistake, and given the accurate expression of f' in [1021g], which agrees with that in [4060].

CHAPTER VI.

NUMERICAL VALUES OF THE DIFFERENT QUANTITIES WHICH ENTER INTO THE EXPRESSIONS OF THE PLANETARY INEQUALITIES.

21. To reduce to numbers, the formulas contained in the second book and in the preceding chapters, we shall use the following data;

*Masses of the Sun and Planets.**

Masses of the planets, the mass of the sun being unity.

Sun, ... $M = 1$;

Mercury, ... $m = \frac{1+\mu}{2025810}$; $\log. m = 93{,}6934013$;

Venus, ... $m' = \frac{1+\mu'}{383130}$; $\log. m' = 94{,}4166538$;

The Earth, ... $m'' = \frac{1+\mu''}{329630}$; $\log. m'' = 94{,}4819733$;

Mars, ... $m''' = \frac{1+\mu'''}{1846082}$; $\log. m''' = 93{,}7337490$; [4061]

Jupiter, ... $m^{iv} = \frac{1+\mu^{iv}}{1067{,}09}$; $\log. m^{iv} = 96{,}9717990$;

Saturn, ... $m^{v} = \frac{1+\mu^{v}}{3359{,}40}$; $\log. m^{v} = 96{,}4737383$;

Uranus, ... $m^{vi} = \frac{1+\mu^{vi}}{19504}$; $\log. m^{vi} = 95{,}7098763$.

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* (2553) The factors $1+\mu$, $1+\mu'$, &c. in the values of m, m', &c. [4061], are not inserted in the original work; but as they are introduced in [4230'], and frequently [4061a]

Of all these masses, that of Jupiter is the most accurately determined; it is obtained by means of the formula [709]. If we put T for the time

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used in computing the perturbations of the motions of the planets, it was thought best,
[4061b] for the sake of convenient reference, to insert them in this place. When the author printed this part of the work, he supposed, in conformity with the best observations, which could then be procured, that the masses of the planets were as in the table [4061], putting each
[4061c] of the quantities μ, μ', &c. equal to zero. Since that time, he has been induced, by other observations, to make successive corrections in these masses, as in [4605, 4608, 9161, &c.]. In his last edition of the *Système du Monde*, he adopts the following

Corrected Masses of the Planets.

Masses finally adopted by the author.

[4061d]

Mercury, ...	$m = \frac{1}{2025810}$;	$\mu = 0$;	$\log. m = 93{,}6934013$;
Venus, ...	$m' = \frac{1}{405871}$;	$\mu' = -0{,}056030$;	$\log. m' = 94{,}3916120$;
The Earth, . . .	$m'' = \frac{1}{354936}$;	$\mu'' = -0{,}071297$;	$\log. m'' = 94{,}4498499$;
Mars, ...	$m''' = \frac{1}{2546320}$;	$\mu''' = -0{,}275000$;	$\log. m''' = 93{,}5940870$;
Jupiter, ...	$m^{\text{iv}} = \frac{1}{1070{,}5}$;	$\mu^{\text{iv}} = -0{,}003136$;	$\log. m^{\text{iv}} = 96{,}9704133$;
Saturn, ...	$m^{\text{v}} = \frac{1}{3512}$;	$\mu^{\text{v}} = -0{,}013451$;	$\log. m^{\text{v}} = 96{,}4544455$;
Uranus, ...	$m^{\text{vi}} = \frac{1}{17918}$;	$\mu^{\text{vi}} = 0{,}088514$;	$\log. m^{\text{vi}} = 95{,}7467105$.

The alterations here made in the values of m', m''', are in conformity with the results of the calculations of Burckhardt, in his late solar tables, by comparing the observed perturbations
[4061e] of the earth's orbit with the theory. The change in the value of m'', arises from the supposition, that the sun's horizontal parallax is nearly equal to $8^s{,}6$ [5589], instead of $8^s{,}8$, assumed in [4073]. Lastly, the values of m^{iv}, m^{v}, m^{vi}, are obtained, by Mr. Bouvard, from the observations used in constructing his new tables of Jupiter, Saturn, and Uranus, by comparing the theory with the actual perturbations depending upon their mutual attractions.
[4061f] Putting the values in [4061] equal to those in [4061d], respectively, we get the corresponding values of μ, μ', &c. [4061d]. Lindeneau, in his tables of Mercury, printed
[4061g] in 1813, supposes that the mass of Venus ought to be increased to $\frac{1}{374440}$; making $\mu' = 0{,}08613$ nearly; to satisfy the perturbations of Mercury, by the action of Venus. Encke, in his *Astronomisches Jahrbuch* for 1831, states, that the mass of Jupiter $\frac{1}{1053{,}924}$, deduced by Nicolai, from the perturbations of Juno, agrees better with the observations
[4061h] of Pallas and Vesta, than the mass adopted by La Place [4061, 4065], and that it probably

of the sideral revolution of the planet m'; T for that of one of its satellites; q for the sine of the greatest angle, under which the mean radius of the orbit of this satellite appears, when viewed from the centre of the sun, [4062]
at the mean distance of the planet from that centre; then the mass of the sun being taken for unity, that of the planet will be expressed by*

$$\frac{q^3 \cdot \left(\frac{\mathrm{T}}{T}\right)^2}{1 - q^3 \cdot \left(\frac{\mathrm{T}}{T}\right)^2} = \text{mass of the planet.} \qquad [4063]$$

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agrees also better for Vesta. Comparing this with [4061], we get $\mu^{iv} = 0{,}012492$. When [4061i]
we take into consideration that the *first* value of $\mu^{iv} = 0$ [4061, 4065] is obtained from the observed elongations of the satellites of Jupiter; the *second* value, $\mu^{iv} = -0{,}003186$ [4061d], [4061k]
from the perturbations of Saturn and Uranus; the *third* value, $\mu^{iv} = 0{,}012492$ [4061i], from the perturbations of the newly discovered planets; we shall not be surprised in finding [4061l]
these small differences in the results of methods, which are so wholly independent of each other. Nothing is known relatively to the masses of these new planets or the masses of the [4061m]
comets, except that they are all very small; so that their action on the other bodies of the system is wholly insensible.

* (2554) This is deduced from [709], $\frac{m'+p}{M} = \frac{h^3}{a^3} \cdot \left(\frac{\mathrm{T}}{T}\right)^2$, in which we must write [4062a]
μ for M, as is evident from [706']; and as m' represents the mass of the planet, in the present notation, we have $\mu = M + m'$. Moreover p is the mass of the satellite [707'], and M that of the sun [706']; h the mean distance of the satellite from the planet; a the mean distance of the planet from the sun; so that $\frac{h}{a}$ represents the quantity [4062b]
q [4062]; hence the preceding equation [4062a] becomes $\frac{m'+p}{M+m'} = q^3 \cdot \left(\frac{\mathrm{T}}{T}\right)^2$. If we [4062c]
neglect p in comparison with m', and put $M=1$; also, for brevity, $q^3 \cdot \left(\frac{\mathrm{T}}{T}\right)^2 = \frac{1}{k}$, we get, as in [4063], $m' = \frac{\frac{1}{k}}{1-\frac{1}{k}} = \frac{1}{k-1}$. If we put ρ^{iv}, ρ^{v} for the mean densities of the [4062d]
bodies m^{iv}, m^{v}; also R^{iv}, R^{v} for the radii; we shall have nearly, as in [2106],

$$m^{iv} = \tfrac{4}{3}\pi \cdot \rho^{iv} \cdot (R^{iv})^3; \qquad m^{v} = \tfrac{4}{3}\pi \cdot \rho^{v} \cdot (R^{v})^3. \qquad [4062e]$$

Hence we easily obtain the relative densities of these two bodies, $\frac{\rho^{iv}}{\rho^{v}} = \frac{m^{iv}}{m^{v}} \cdot \left(\frac{R^{v}}{R^{iv}}\right)^3$. [4062f]
This may be used for ascertaining the densities of all the bodies, whose masses are known, and whose apparent diameters have been well observed.

We have, relatively to the fourth satellite,*

$$q = \sin.\ 1530'',38 = \sin.\ 495^{s},84;$$

[4064] $$\mathrm{T} = 4332^{\text{days}},602208 = 4332^{d}\,14^{h}\,27^{m}\,10^{s},8;$$

$$T = 16^{\text{days}},6890 = 16^{d}\,16^{h}\,32^{m}\,09^{s},6.$$

From [4063, 4064], we obtain

[4065] $$m^{\text{iv}} = \frac{1}{1067,09}.$$

The mass of Saturn is found by the same method; supposing the sideral revolution of its sixth satellite to be $15^{\text{days}},9453 = 15^{d}\,22^{h}\,41^{m}\,13^{s},9$, and the greatest angle, under which the mean radius of the orbit of this satellite appears, when viewed from the sun, in the mean distances of Saturn,
[4066] $552'',47 = 179^{s}$. The mass of Uranus has, in like manner, been obtained, by supposing, conformably to the observations of Herschel, that the duration of the sideral revolution of its fourth satellite, is $13^{\text{days}},4559 = 13^{d}\,10^{h}\,56^{m}\,29^{s},8$;
[4067] and the mean radius of the orbit of this satellite, viewed from the sun, at the mean distance of Uranus, $136'',512 = 44^{s},23$. But the greatest elongations of the satellites of Saturn and Uranus have not been so accurately ascertained as that of the fourth satellite of Jupiter. Observations of these elongations deserve the careful attention of astronomers.

The mass of the earth is found in the following manner. If we take the mean distance of the earth from the sun for unity, the arc described by the earth, in a centesimal second of time, will be obtained by dividing the circumference of a circle, whose radius is unity, by the number of
[4068] seconds in a sideral year, $36525638^{\text{sec.}},4$. Dividing the square of this arc
[4068′] by the diameter, we obtain its versed sine $= \frac{1479565}{10^{20}}$,† which is the space the earth falls towards the sun in a centesimal second, by means of its relative motion about the sun. On the parallel of latitude, whose sine is

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[4064a] * (2555) The values of q, T [4061], are nearly the same as those used in the theory of this satellite [6781, 6785]; the value of T corresponds to the mean motion n^{iv} [4077].

† (2556) The radius of the orbit being 1, its circumference is 6,28318 nearly; if we
[4068a] divide this by 36525638,4, and take half the square of the product, we get the expression of the versed sine, corresponding to this arc, as in [4068′].

equal to $\sqrt{\frac{1}{3}}$, the attraction of the earth causes a body to fall through $3^{\text{met.}},66553$* in one centesimal second. To deduce from this the earth's [4069]
attraction at the mean distance of the earth from the sun, we must multiply it by the square of the sine of the sun's parallax; and divide the product by the number of metres contained in that distance. Now the earth's radius on the proposed parallel, is† $6369374^{\text{met.}}$; therefore, by dividing this [4069′]
number by the sine of the sun's parallax, supposing it to be $27'',2 = 8'',8$, [4070]
we obtain the mean radius of the earth's orbit, expressed in metres. Hence it follows, that the effect of the attraction of the earth, at a distance equal to that of the mean distance of the earth from the sun, is equal to the product of the fraction $\frac{3,66553}{6369374}$, by the cube of the sine of $27'',2$; [4071]
consequently it is equal to‡ $\frac{4,4885}{10^{20}}$. Subtracting this fraction from [4071′]
$\frac{1479565}{10^{20}}$, we obtain $\frac{1479560,5}{10^{20}}$ for the effect of the attraction of the sun,

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* (2557) This computation varies a little from that in [388″] or in [388*a*]; probably owing to a small difference in the ellipticity, used in reducing the observations. [4069*a*]

† (2558) Using the polar and equatorial semi-axes of the earth, $6356677^{\text{met.}}$, $6375709^{\text{met.}}$ [2035′], whose difference is $19032^{\text{met.}}$, we find the radius corresponding to [4070*a*]
the latitude, whose sine is $\sqrt{\frac{1}{3}}$, to be $6375709^{\text{met.}} - \frac{1}{3} \times 19032^{\text{met.}} = 6369365^{\text{met.}}$, agreeing nearly with [4069′].

‡ (2559) Gravity decreases, in proceeding from the earth's surface, inversely, as the square of the distance of the attracted point; or as the square of the sine of the horizontal [4071*a*]
parallax of that point nearly. Hence the earth's attraction, at the distance of the sun, will cause a body to fall through a space represented by $3^{\text{met.}},66553 \times (\sin. \odot\text{'s par.})^2$, [4071*b*]
in one centesimal second of time. To reduce this from metres to parts of the mean distance of the earth from the sun, we must divide it by that distance, which is evidently equal to $\frac{\text{earth's radius}}{\sin. \odot\text{'s par.}} = \frac{6369374^{\text{met.}}}{\sin. 27'',2}$, so that the space fallen through in a second, becomes
$\frac{3,66553}{6369374} \cdot (\sin. \odot\text{'s par.})^3 = \frac{4,4885}{10^{20}}$, as in [4071′]. Now in [4068′], we have found, that [4071*c*]
the earth falls towards the sun, in the same time, by the *combined action of the sun and earth* $\frac{1479565}{10^{20}}$; hence the effect of the sun alone is $\frac{1479565 - 4,4885}{10^{20}} = \frac{1479560,5}{10^{20}}$
nearly; and as that of the earth is $\frac{4,4885}{10^{20}}$, the mass of the earth is to that of the sun as $\frac{4,4885}{10^{20}}$ to $\frac{1479560,5}{10^{20}}$, or 1 to 329630 nearly, as in [4072]. [4071*d*]

at the same distance. Hence the masses of the sun and earth are in the ratio of the numbers 1479560,5 to 4,4885; consequently the mass of
[4072] the earth is $\frac{1}{329630}$. If the sun's parallax differ a little from the quantity we have assumed in [4070], the value of the earth's mass will vary as
[4073] the cube of that parallax, compared with the cube of $27'',2 = 8^s,8$ [4071c].

We have computed the mass of Venus from the formulas [4251, 4332, &c.], which express the secular diminution of the obliquity of the ecliptic to the
[4074] equator; supposing it, by observation, to be $154^s,30 = 50^s$. This diminution is obtained from those observations which appear the most to be relied upon.* With respect to the masses of Mercury and Mars, we have supposed, according to observation, that the mean diameters of Mercury, Mars, and Jupiter, viewed at the mean distance of the earth from the sun, are, respectively,
[4075] $21'',60 = 7^s$; $35'',19 = 11^s,4$; $626'',04 = 202^s,84$. Now Jupiter's mass being ascertained, we could, by means of these diameters, obtain the masses of Mercury and Mars, if the relative densities of these three planets were known. If we compare the masses of the Earth, Jupiter, and Saturn, with their magnitudes, respectively, we find, that the densities of these planets are very nearly in the inverse ratio of their mean distances from the

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* (2560) If we change γ, Λ [3102c] into φ'', θ'', respectively, to conform to the
[4074a] notation used in [4082, 4083]; we shall find, that the arc $FG = \gamma \,.\, \cos.\, \Lambda$ [3109c], which represents the difference between the inclinations of the equator to the fixed ecliptic of 1750 and to the variable ecliptic of $1750 + t$, is equal to $\varphi''.\cos.\theta''$, or q'' [4249].
[4074b] The value of q'' is found by integrating the second equation [4251]. In this expression of q'', the coefficients of μ, μ''', μ^{v}, μ^{vi}, are small, and the value of μ^{iv} [4061d] is small and tolerably well ascertained; therefore we need only retain μ', so that the integral
[4074c] becomes $q'' = -(0^s,500955 + 0^s,309951 \,.\, \mu') \,.\, t$. If we suppose $\mu' = 0$, the annual
[4074d] decrement becomes $0^s,500955$, being nearly as in [4074]. The action of the planet Venus has more effect in producing this change of obliquity, than that of all the other planets taken together; as is evident from the inspection of the value of $d\,q''$ [4251]; in which
[4074e] we find, that the coefficient of μ' exceeds the sum of the coefficients of the other quantities, μ, μ''', μ^{iv}, μ^{v}, μ^{vi}. We have already remarked, in [3380n—q], that the author increased the annual variation to $0^s,521154$ [4613]; on the other hand, Mr. Poisson uses $0^s,45692$
[4074f] [3380p], and Mr. Bessel $0^s,48368$ [3380q]; each of them varying the values of μ, μ', &c., so as to conform to their assumed decrements.

sun;* we shall therefore adopt the same hypothesis, relatively to the three planets Mercury, Mars, and Jupiter; whence we obtain the preceding values of the masses of Mercury and Mars [4061]. The irradiation and the other difficulties attending the measures of the diameters of the planets, taken in connexion with the uncertainty of the hypothesis adopted on the law of their densities, render these estimated values somewhat doubtful, and this uncertainty seems to be increased from the circumstance, that the hypothesis is not correct relative to the masses of Venus and Uranus. Fortunately, Mercury and Mars have only a very small influence on the planetary system; and it will be easy to correct the following results, so far as they are affected by this cause, whenever the development of the secular inequalities shall make known exactly the values of these masses. [4076] [4076']

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* (2561) The densities of the Earth, Jupiter, and Saturn, given by the author in the *Système du Monde*, are 3,93; 0,99; 0,55; respectively, being found as in [4062f, &c.]. [4076a] These densities of Jupiter and Saturn are nearly in the inverse ratio of the distances a^{iv}, a^{v} [4079]; but the density of the earth differs considerably from this rule. If we [4076b] suppose this ratio of the densities to hold good for the three planets Mercury, Mars, Jupiter, and represent their apparent diameters [4075], by $D=21'',60$, $D'''=35'',19$, $D^{\text{iv}}=626'',04$; [4076c] the corresponding masses will be $m=b\cdot\frac{D^3}{a}$; $m'''=b\cdot\frac{D'''^3}{a'''}$; $m^{\text{iv}}=b\cdot\frac{D^{\text{iv}\,3}}{a^{\text{iv}}}$; b being [4076d] a constant quantity, to be found by means of the value of m^{iv} [4061]; which gives $b=\frac{a^{\text{iv}}}{D^{\text{iv}\,3}}\cdot\frac{1}{1067,09}$. Hence we get [4076e]

$$m=\frac{1}{1067,09}\cdot\left(\frac{D}{D^{\text{iv}}}\right)^3\cdot\frac{a^{\text{iv}}}{a};\qquad m'''=\frac{1}{1067,09}\cdot\left(\frac{D'''}{D^{\text{iv}}}\right)^3\cdot\frac{a^{\text{iv}}}{a'''};\qquad [4076f]$$

and by substituting the values [4076c, 4079], we get, for m, m''', rather greater values than those in [4061]. These differences probably arise from having used different values of D, D''', D^{iv}, which cannot be obtained, by observation, to a great degree of accuracy. [4076g]

In some of the subsequent calculations, it will be sufficiently accurate to use the values of n, n', &c. to the nearest degree; and for convenience of reference we have here inserted these approximate values;

$$n=1661^\circ;\quad n'=650^\circ,\quad n''=400^\circ,\quad n'''=212^\circ,7,\quad n^{\text{iv}}=33^\circ,7,$$
$$n^{\text{v}}=13^\circ,6,\quad n^{\text{vi}}=4^\circ,8.\qquad [4076h]$$

22. *Mean sideral motions of the Planets in a Julian year of $365\frac{1}{4}$ days, or the values of n, n', &c.*

Mean motions of the planets.

[4077]

			Sexagesimals.	
Mercury, .	n	$=16608076'',50$	$=5381016^s,786$;	$\log. n =6,7308643$;
Venus, . . .	n'	$=6501980'',00$	$=2106641^s,520$;	$\log. n' =6,3235906$;
The Earth,	n''	$=3999930'',09$	$=1295977^s,349$;	$\log. n'' =6,1125974$;
Mars, ...	n'''	$=2126701'',00$	$=689051^s,124$;	$\log. n''' =5,8382514$;
Jupiter, . .	n^{iv}	$=337210'',78$	$=109256^s,293$;	$\log. n^{iv} =5,0384465$;
Saturn, . .	n^{v}	$=135792'',34$	$=43996^s,718$;	$\log. n^{v} =4,6434203$;
Uranus, . .	n^{vi}	$=47606'',62$	$=15424^s,545$;	$\log. n^{vi} =4,1882124$.

[4078] If we use these values of n, n', &c., the *time t will be represented in Julian years;* hence if we put the *mean distance of the earth from the sun equal to unity,* we shall obtain, from Kepler's law [385'''], the following mean distances of the planets from the sun.

The time t is expressed in Julian years.

*Mean distances of the Planets from the Sun, or the semi-major axes of their orbits.**

Mean distances of the planets from the sun.

[4079]

Mercury, ...	a	$= 0,38709812$;	$\log. a = 9,5878211$;
Venus, ...	a'	$= 0,72333230$;	$\log. a' = 9,8593379$;
The Earth, ...	a''	$= 1,00000000$;	$\log. a'' = 0,0000000$;
Mars, ...	a'''	$= 1,52369352$;	$\log. a''' = 0,1828976$;
Jupiter, ...	a^{iv}	$= 5,20116636$;	$\log. a^{iv} = 0,7161007$;
Saturn, ...	a^{v}	$= 9,53787090$;	$\log. a^{v} = 0,9794514$;
Uranus, ...	a^{vi}	$= 19,18330500$;	$\log. a^{vi} = 1,2829234$.

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* (2562) These values of a, a', &c. are deduced from [4077], by putting them,

[4079a] respectively, equal to $\left(\frac{n''}{n}\right)^{\frac{2}{3}}$, $\left(\frac{n''}{n'}\right)^{\frac{2}{3}}$, $\left(\frac{n''}{n''}\right)^{\frac{2}{3}}$, &c.

The elements of the orbits of the newly discovered planets, Ceres, Pallas, Vesta, and Juno, were first computed by Gauss, and have since been repeatedly corrected by him,

The mutual action of the planets alters a little their mean distances; we shall, in [4451, 4510], determine these alterations.

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and by other astronomers; taking notice of the most important perturbations, from the action of the nearest planets; so that we can now compute the places of these bodies with a considerable degree of accuracy. The usual methods of finding the perturbations can be applied to these small planets; but the great excentricities and inclinations of some of their orbits, will make it necessary, when great accuracy is required, to notice the terms depending on the powers and products of these two elements, of a higher order than is generally used with the other planets. The laborious task of ascertaining all the inequalities of these four planets, was not performed by the author of this work; and it will probably be a long while before it can be done completely, on account of the small imperfections in the present estimated values of the elements, which have not yet been determined with perfect accuracy in the short period since the bodies have been observed. It is evident, also, that until these elements have been found very nearly, it will not be of much use to compute several of the very small inequalities, with the *extreme minuteness* which is used relatively to the other planets. [4079b] [4079c] [4079d] [4079e]

In computing the *Jahrbuch*, it has been found most convenient by Encke to apply the corrections directly to the elements of the orbit, rather than to the elliptical places of the bodies; in a manner similar to that which is used in finding the elements of a comet, in two successive returns. He finds, when the elements are thus adjusted to any particular moment of time, that they will give, tolerably well, the places of the planet for a considerable period, on each side of this epoch. The elements of the orbits obtained by him, for these four planets, about the time of the opposition of Pallas, in the year 1831, are as in the following table; which will serve to give an idea of the relative positions of the orbits *at that time*; remarking, that these elements must not be confounded with the *mean* values. [4079f] [4079g] [4079h]

Epoch 1831, *July* 23 *d*, 0*h*, *mean time at Berlin.*

	Vesta.	Juno.	Pallas.	Ceres.	
Mean longitude, ...	$81^d\,17^m\,03^s$	$74^d\,39^m\,41^s$	$290^d\,38^m\,12^s$	$307^d\,03^m\,26^s$	
Mean anomaly, ...	195 35 26	20 22 31	169 33 11	159 22 02	Elements of Vesta, Juno, Pallas, and Ceres.
Longitude of the perihelion, ...	249 11 37	54 17 13	121 05 01	147 41 23	
Longitude of the ascending node, .	103 20 28	170 52 31	172 38 30	80 53 50	
Inclination, ...	7 07 57	13 02 10	34 35 49	10 36 56	
Excentricity, ...	0,0885601	0.2555592	0,2419986	0,0767379	
Mean daily sideral motion, ...	977".75540	813".52533	768".54121	769".26059	[4079i]
Semi-major axis, ...	2.361484	2.669464	2,772631	2.770907	
Periodic revolution corresponding, .	1325,5 days	1593,1 days	1686,3 days	1684,7 days	

Ratios of the excentricities to the mean distances, or the values of e, e', *&c. for the year* 1750.

Excentricities of the orbits of the planets.

	Mercury, ... $e = 0{,}20551320$;	$\log. e = 9{,}3128397$;
	Venus, ... $e' = 0{,}00688405$;	$\log. e' = 7{,}8378440$;
	The Earth, ... $e'' = 0{,}01681395$;	$\log. e'' = 8{,}2256698$;
	Mars, ... $e''' = 0{,}09308767$;	$\log. e''' = 8{,}9688922$;
[4080]	Jupiter, ... $e^{\text{iv}} = 0{,}04807670$;	$\log. e^{\text{iv}} = 8{,}6819346$;
	Saturn, ... $e^{\text{v}} = 0{,}05622460$;	$\log. e^{\text{v}} = 8{,}7499264$;
	Uranus, ... $e^{\text{vi}} = 0{,}04669950$;	$\log. e^{\text{vi}} = 8{,}6693122$.

[4079k] The distances of the planets Pallas and Ceres from the sun, are so nearly equal to each other, that it may sometimes happen, in finding the apparent orbits, in the preceding manner, that the order of the bodies will be inverted, relative their distances from the sun, by means of the perturbations.

[4079l] Besides these planets, there are four comets, whose periodical revolutions have been discovered by Halley, Olbers, Encke, and Biela. They have been usually called by the names of the discoverers respectively. That of Olbers has been observed only once, at the time of its return to the perihelion in 1815: the others have been observed in several successive revolutions.

Elements of the orbits of the four known periodical comets.

		Halley's.	Olbers's.	Encke's.	Biela's.
	Periodic revolution, ...	76 years	74 years	1204 days	6,7 years
	Time of perihelion, ...	Nov. 7, 1835	April 26, 1815	Jan. 10, 1829	Nov. 27, 1832
	Longitude of perihelion on the orbit,	$304^d 31^m 43^s$	$149^d\ 2^m$	$157^d 18^m 35^s$	$109^d 56^m 45^s$
	Longitude of the ascending node,	55 30	83 29	334 24 15	248 12 24
	Inclination, ...	17 44 21	44 30	13 22 34	13 13 13
[4079m]	Excentricity, ...	0.9675212	0,9313	0,8446862	0,751748
	Semi-major axis, ...	17,98705	17,7	2,224346	3,53683

[4079n] Of the seven periodical bodies, which have been made known to astronomers since the commencement of the present century, three were discovered by Dr. Olbers of Bremen; namely, Vesta, Pallas, and the comet of 1815. His great success in the discovery of these remarkable bodies, which had silently performed their revolutions in the heavens for ages, unperceived by astronomers, induced an eminent German writer to style him, *the fortunate Columbus of the planetary world.*

Longitudes of the perihelia in the year 1750, *or the values of* ϖ, ϖ', *&c.*

Longitudes of the perihelia in 1750.

Mercury, ...	$\varpi = 81°,7401 =$	$73^d 33^m 58^s$;	
Venus, ...	$\varpi' = 142°,1241 =$	127 54 42 ;	
The Earth, ...	$\varpi'' = 109°,5790 =$	98 37 16 ;	
Mars, ...	$\varpi''' = 368°,3037 =$	331 28 24 ;	
Jupiter, ...	$\varpi^{iv} = 11°,5012 =$	10 21 04 ;	[4081]
Saturn, ...	$\varpi^{v} = 97°,9466 =$	88 09 07 ;	
Uranus, ...	$\varpi^{vi} = 185°,1262 =$	166 36 49.	

Inclinations of the orbits to the ecliptic in the year 1750, *or the values of* φ, φ', *&c.*

Inclinations of the orbits to the fixed ecliptic of 1750.

Mercury, ...	$\varphi = 7°,7778 =$	$7^d 00^m 00^s$;	
Venus, ...	$\varphi' = 3°,7701 =$	3 23 35 ;	
The Earth, ...	$\varphi'' = 0°$;		
Mars, ...	$\varphi''' = 2°,0556 =$	1 51 00 ;	
Jupiter, ...	$\varphi^{iv} = 1°,4636 =$	1 19 02 ;	[4082]
Saturn, ...	$\varphi^{v} = 2°,7762 =$	2 29 55 ;	
Uranus, ...	$\varphi^{vi} = 0°,8596 =$	0 46 25.	

Longitudes of the ascending nodes on the ecliptic of the year 1750, *or the values of* θ, θ', *&c.*

Longitudes of the ascending nodes of the orbits on the fixed ecliptic of 1750.

Mercury, ...	$\theta = 50°,3336 =$	$45^d 20^m 43^s$;	
Venus, ...	$\theta' = 82°,7093 =$	74 26 18 ;	
The Earth, ...	θ'' as in [4249—4251] ;		
Mars, ...	$\theta''' = 52°,9376 =$	47 38 38 ;	
Jupiter, ...	$\theta^{iv} = 108°,7846 =$	97 54 22 ;	[4083]
Saturn, ...	$\theta^{v} = 123°,8960 =$	111 30 23 ;	
Uranus, ...	$\theta^{vi} = 80°,7015 =$	72 37 53.	

Epoch. [4084] *All these longitudes are counted from the mean vernal equinox, at the epoch of December 31st, 1749, mid-day, mean time at Paris.* We may here observe, that *by the longitude of the perihelion, is to be understood, the distance of the perihelion from the ascending node, counted on the orbit, increased by the longitude of that node.*

Longitude of the perihelion. [4084']

23. We have obtained the following results, by the formulas of § 49, Book II.

MERCURY AND VENUS,

[4085] $$\alpha = \frac{a}{a'} = 0{,}53516076;$$

hence we deduce

[4086]
$$b_{-\frac{1}{2}}^{(0)} = 2{,}145969210;$$
$$b_{-\frac{1}{2}}^{(1)} = -\,0{,}515245873.$$

Then we obtain*

Mercury and Venus. [4087]

$$b_{\frac{1}{2}}^{(0)} = 2{,}1721751; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}6057052; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}2465877;$$
$$b_{\frac{1}{2}}^{(3)} = 0{,}1107665; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}0520855; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}0251378;$$
$$b_{\frac{1}{2}}^{(6)} = 0{,}0123166; \qquad b_{\frac{1}{2}}^{(7)} = 0{,}0060633; \qquad b_{\frac{1}{2}}^{(8)} = 0{,}0029287;$$
$$b_{\frac{1}{2}}^{(9)} = 0{,}0012758.$$

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* (2563) From a, a' [4079], we have $\alpha = \frac{a}{a'}$, as in [4085]. Then from [989],
[4086a] we find, $b_{-\frac{1}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)}$, as in [4086]; from these we get $b_{\frac{1}{2}}^{(0)}$, $b_{\frac{1}{2}}^{(1)}$ [4087], by means of the formulas [990, 991]. Then putting, in [966], $s=\frac{1}{2}$, and successively, $i=2$, $i=3$, $i=4$, &c. we obtain the remaining terms of [4087]. From these last, we get those
[4086b] in [4088], by putting, successively, $i=0$, $i=1$, &c., and $s=\frac{1}{2}$, in [981]. The same values, being substituted in [982], give [4089]; also [983] gives [4090]. Lastly, by taking the partial differential of [983], relative to α, we shall get an expression
[4086c] of $\frac{d^4 b_s^{(i)}}{d\alpha^4}$; in which we must put $s=\frac{1}{2}$; then $i=0$; $i=1$, &c.; and we shall get [4091]. Again, the formulas [992] give $b_{\frac{3}{2}}^{(0)}$, $b_{\frac{3}{2}}^{(1)}$, [4092]; from these two

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,780206\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,457891\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 1,070071\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,691487\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0,423818\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} = 0,252376\,; \qquad [4088]$$

$$\frac{d\,b_{\frac{1}{2}}^{(6)}}{d\,\alpha} = 0,147708\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(7)}}{d\,\alpha} = 0,085953\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(8)}}{d\,\alpha} = 0,050726.$$

$$\frac{d^2\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 2,756285\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 2,426165\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 3,395022\,;$$

$$\frac{d^2\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} = 3,381072\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha^2} = 2,826559\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha^2} = 2,137906\,; \qquad [4089]$$

Mercury and Venus

$$\frac{d^2\,b_{\frac{1}{2}}^{(6)}}{d\,\alpha^2} = 1,511016\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(7)}}{d\,\alpha^2} = 1,014134\,;$$

$$\frac{d^3\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha^3} = 11,308703\,; \qquad \frac{d^3\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^3} = 12,064245\,; \qquad \frac{d^3\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^3} = 11,983424\,;$$

$$\frac{d^3\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha^3} = 14,584366\,; \qquad \frac{d^3\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha^3} = 16,067040\,; \qquad \frac{d^3\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha^3} = 15,617274\,; \qquad [4090]$$

$$\frac{d^3\,b_{\frac{1}{2}}^{(6)}}{d\,\alpha^3} = 13,720218.$$

$$\frac{d^4\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^4} = 69,60594\,; \qquad \frac{d^4\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha^4} = 82,36773\,; \qquad \frac{d^4\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha^4} = 92,72610\,; \qquad [4091]$$

$$\frac{d^4\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha^4} = 105,33962.$$

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terms, we may obtain the others of [4092], by means of the formula [966]; putting $s=\frac{3}{2}$, and, successively, $i=2$, $i=3$, &c. The values [4093] are found from [981], by putting $s=\frac{3}{2}$, and $i=2$, $i=3$, &c. Those in [4094] are deduced from [982], by using similar values of s, i; observing to substitute, in any of these formulas, the values of b, or its differentials, which occur, and have been found in the preceding parts of the calculation. All the other terms of this article, §23, are found in the same manner, except those in [4113, 4119, 4124, &c.], where α is very small; and there is no difficulty in the calculation, except the *ennui*, arising from a long and uninteresting numerical calculation. [4086d] [4086e]

$$b_{\frac{3}{2}}^{(0)} = 4,214154; \qquad b_{\frac{3}{2}}^{(1)} = 3,035376; \qquad b_{\frac{3}{2}}^{(2)} = 1,950536;$$

[4092] $$b_{\frac{3}{2}}^{(3)} = 1,192372; \qquad b_{\frac{3}{2}}^{(4)} = 0,708667; \qquad b_{\frac{3}{2}}^{(5)} = 0,413762;$$

$$b_{\frac{3}{2}}^{(6)} = 0,238807.$$

Mercury and Venus.

[4093] $$\frac{d b_{\frac{3}{2}}^{(2)}}{d\alpha} = 12,50630; \qquad \frac{d b_{\frac{3}{2}}^{(3)}}{d\alpha} = 9,76666; \qquad \frac{d b_{\frac{3}{2}}^{(4)}}{d\alpha} = 7,08399;$$

$$\frac{d b_{\frac{3}{2}}^{(5)}}{d\alpha} = 4,88781.$$

[4094] $$\frac{d^2 b_{\frac{3}{2}}^{(3)}}{d\alpha^2} = 78,09476; \qquad \frac{d^2 b_{\frac{3}{2}}^{(4)}}{d\alpha^2} = 67,14764.$$

MERCURY AND THE EARTH.

[4095] $$\alpha = \frac{a}{a''} = 0,38709812;$$

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2,07565247;$$

[4096]

$$b_{-\frac{1}{2}}^{(1)} = -0,37970591.$$

Then we get

Mercury and the Earth.

$$b_{\frac{1}{2}}^{(0)} = 2,081980; \qquad b_{\frac{1}{2}}^{(1)} = 0,411140; \qquad b_{\frac{1}{2}}^{(2)} = 0,120178;$$

[4097] $$b_{\frac{1}{2}}^{(3)} = 0,038900; \qquad b_{\frac{1}{2}}^{(4)} = 0,013202; \qquad b_{\frac{1}{2}}^{(5)} = 0,004603;$$

$$b_{\frac{1}{2}}^{(6)} = 0,001629; \qquad b_{\frac{1}{2}}^{(7)} = 0,000573; \qquad b_{\frac{1}{2}}^{(8)} = 0,000177.$$

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\alpha} = 0{,}464378\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}199633\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\alpha} = 0{,}665739\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\alpha} = 0{,}316756\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\alpha} = 0{,}141792\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\alpha} = 0{,}061433\,;$$ [4098]

$$\frac{d\,b_{\frac{1}{2}}^{(6)}}{d\alpha} = 0{,}026130\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(7)}}{d\alpha} = 0{,}011153.$$

$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\alpha^2} = 1{,}672199\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\alpha^2} = 1{,}220775\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\alpha^2} = 2{,}235935\,;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\alpha^2} = 1{,}852364\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(4)}}{d\alpha^2} = 1{,}197245\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(5)}}{d\alpha^2} = 0{,}670874.$$ [4099]

Mercury and the Earth.

$$\frac{d^3 b_{\frac{1}{2}}^{(2)}}{d\alpha^3} = 5{,}49232\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(3)}}{d\alpha^3} = 5{,}45663\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(4)}}{d\alpha^3} = 6{,}51373.$$ [4100]

$$b_{\frac{3}{2}}^{(0)} = 2{,}871833\,; \qquad b_{\frac{3}{2}}^{(1)} = 1{,}576062\,; \qquad b_{\frac{3}{2}}^{(2)} = 0{,}747619\,;$$

$$b_{\frac{3}{2}}^{(3)} = 0{,}334212\,; \qquad b_{\frac{3}{2}}^{(4)} = 0{,}153779.$$ [4101]

$$\frac{d\,b_{\frac{3}{2}}^{(3)}}{d\alpha} = 3{,}05535.$$ [4102]

MERCURY AND MARS.

$$\alpha = \frac{a}{a'''} = 0{,}25405312\,;$$ [4103]

hence we deduce

Mercury and Mars.

$$b_{-\frac{1}{2}}^{(0)} = 2{,}03240384\,;$$

$$b_{-\frac{1}{2}}^{(1)} = -\,0{,}25198657.$$ [4104]

Then we have

[4105]
$$b_{\frac{1}{2}}^{(0)} = 2,033500; \qquad b_{\frac{1}{2}}^{(1)} = 0,260462; \qquad b_{\frac{1}{2}}^{(2)} = 0,049765;$$
$$b_{\frac{1}{2}}^{(3)} = 0,010546; \qquad b_{\frac{1}{2}}^{(4)} = 0,002331; \qquad b_{\frac{1}{2}}^{(5)} = 0,000538.$$

Mercury and Mars.

[4106]
$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,273829; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,077839; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0,402980;$$
$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,127139; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0,037781.$$

[4107]
$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 1,244725; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 0,656780; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 1,778641;$$
$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} = 1,050458.$$

[4108]
$$b_{\frac{3}{2}}^{(0)} = 2,322536; \qquad b_{\frac{3}{2}}^{(1)} = 0,863876; \qquad b_{\frac{3}{2}}^{(2)} = 0,272085.$$

MERCURY AND JUPITER.

[4109]
$$\alpha = \frac{a}{a^{\mathrm{iv}}} = 0,07412555;$$

Mercury and Jupiter.

hence we deduce

[4110]
$$b_{-\frac{1}{2}}^{(0)} = 2,00277053;$$
$$b_{-\frac{1}{2}}^{(1)} = -\,0,07437397.$$

In computing the values of $b_{\frac{1}{2}}^{(0)}$, $b_{\frac{1}{2}}^{(1)}$, &c., by means of the formulas [966—983], it is found, that the successive terms of the series become more inaccurate, particularly if α be rather small; because these values

are the differences of numbers, which vary but little from each other; so that we are under the necessity of computing them to an extreme degree of exactness, to enable us to determine correctly their differences,* and this requires the use of tables of logarithms to ten or twelve places of decimals. To obviate this inconvenience, we may have recourse to the value of $b_s^{(i)}$, developed in a series, by means of the formulas [976, 984—985],† [4111]

$$b_s^{(i)}=2.\frac{s.(s+1).(s+2).....(s+i-1)}{1.2.3.....i}.\alpha^i.\left\{\begin{array}{l}1+\frac{s}{1}.\frac{(s+i)}{i+1}.\alpha^2+\frac{s.(s+1)}{1.2}.\frac{(s+i).(s+i+1)}{(i+1).(i+2)}.\alpha^4\\ +\frac{s.(s+1).(s+2)}{1.2.3}.\frac{(s+i).(s+i+1).(s+i+2)}{(i+1).(i+2).(i+3)}.\alpha^6+\&c.\end{array}\right\}. \qquad [4112]$$

This value of $b_s^{(i)}$ is, in the present case, very converging, on account of the smallness of α. We shall hereafter use it, in finding the values of $b_{\frac{1}{2}}^{(0)}$, $b_{\frac{1}{2}}^{(1)}$, &c.; $b_{\frac{3}{2}}^{(0)}$, &c., in all cases where α is rather small. By this method we have computed, for Mercury and Jupiter, the following values;

$$b_{\frac{1}{2}}^{(0)}=2,002778; \qquad b_{\frac{1}{2}}^{(1)}=0,074581; \qquad b_{\frac{1}{2}}^{(2)}=0,004164; \qquad [4113]$$

Mercury and Jupiter.

$$b_{\frac{1}{2}}^{(3)}=0,000258; \qquad b_{\frac{1}{2}}^{(4)}=0,000017.$$

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* (2564) Thus, if we put $s=\frac{1}{2}$ and $i=2$, in [966], it becomes

$$b_{\frac{1}{2}}^{(2)}=\frac{(1+\alpha^2).b_{\frac{1}{2}}^{(1)}-\frac{1}{2}\alpha.b_{\frac{1}{2}}^{(0)}}{\frac{3}{2}\alpha}. \qquad [4111a]$$

Now $b_{\frac{1}{2}}^{(2)}$, is much smaller than $b_{\frac{1}{2}}^{(0)}$ or $b_{\frac{1}{2}}^{(1)}$ [4105], and the preceding value of $b_{\frac{1}{2}}^{(2)}$ is divided by the small quantity $\frac{3}{2}\alpha$. Hence it necessarily follows, that the terms $(1+\alpha^2).b_{\frac{1}{2}}^{(1)}$ and $-\frac{1}{2}\alpha.b_{\frac{1}{2}}^{(0)}$, in the numerator of this expression, must be very nearly equal to each other; and their difference, which is to be divided by a quantity of the order α, must therefore be very accurately computed. The same takes place in $b_{\frac{3}{2}}^{(2)}$, &c. [4111b]

† (2565) The quantity $b_s^{(i)}$ is the coefficient of $\cos.i\theta$, in λ^{-s} [976]; and λ^{-s} is the product of the two factors [985]. If we multiply these factors, and retain only terms of the form $c^{\pm i\theta\sqrt{-1}}$, putting $c^{i\theta\sqrt{-1}}+c^{-i\theta\sqrt{-1}}=2.\cos.i\theta$ [12] Int., it becomes as in [4112]. [4112a]

[4114] $\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0{,}074891$; $\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1{,}006269$; $\frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0{,}111380$;

Mercury and Jupiter. $\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0{,}010428$;

[4115] $\frac{d^2\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 1{,}018876$; $\frac{d^2\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 0{,}171781$; $\frac{d^2\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 1{,}499780$;

[4116] $b_{\frac{3}{2}}^{(0)} = 2{,}025143$; $b_{\frac{3}{2}}^{(1)} = 0{,}225613$; $b_{\frac{3}{2}}^{(2)} = 0{,}020984$.

MERCURY AND SATURN.

[4117] $$\alpha = \frac{a}{a^{v}} = 0{,}04058547;$$

hence we deduce

[4118] $$b_{-\frac{1}{2}}^{(0)} = 2{,}00082368;$$ $$b_{-\frac{1}{2}}^{(1)} = -\,0{,}04057711.$$

Mercury and Saturn. Then we find

[4119] $b_{\frac{1}{2}}^{(0)} = 2{,}000823$; $b_{\frac{1}{2}}^{(1)} = 0{,}040610$; $b_{\frac{1}{2}}^{(2)} = 0{,}001236$;

$b_{\frac{1}{2}}^{(3)} = 0{,}000042$; $b_{\frac{1}{2}}^{(4)} = 0{,}000001$.

[4120] $\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0{,}040662$; $\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1{,}001841$; $\frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0{,}060919$;

$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0{,}003085$.

[4121] $\frac{d^2\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 1{,}003904$; $\frac{d^2\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 0{,}091840$; $\frac{d^2\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 1{,}469188$.

MERCURY AND URANUS.

$$\alpha = \frac{a}{a^{\mathrm{vi}}} = 0{,}02017895\,; \qquad [4122]$$

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2{,}00020360\,;$$

$$b_{-\frac{1}{2}}^{(1)} = -\,0{,}02017792. \qquad [4123]$$

Mercury and Uranus.

Then we find

$$b_{\frac{1}{2}}^{(0)} = 2{,}000182\,; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}020183\,; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}000306\,; \qquad [4124]$$

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0{,}020196\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1{,}000913. \qquad [4125]$$

VENUS AND THE EARTH.

$$\alpha = \frac{a'}{a''} = 0{,}72333230\,; \qquad [4126]$$

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2{,}27159162\,;$$

$$b_{-\frac{1}{2}}^{(1)} = -\,0{,}67226315. \qquad [4127]$$

Then we obtain

Venus and the Earth.

$$b_{\frac{1}{2}}^{(0)} = 2{,}386343\,; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}942413\,; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}527589\,;$$

$$b_{\frac{1}{2}}^{(3)} = 0{,}323359\,; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}206811\,; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}135616\,; \qquad [4128]$$

$$b_{\frac{1}{2}}^{(6)} = 0{,}090412\,; \qquad b_{\frac{1}{2}}^{(7)} = 0{,}061101\,; \qquad b_{\frac{1}{2}}^{(8)} = 0{,}041731.$$

[4129]

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha}=1{,}643709\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha}=2{,}272414\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha}=2{,}069770\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha}=1{,}738781\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha}=1{,}407491\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha}=1{,}113704\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(6)}}{d\,\alpha}=0{,}867147\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(7)}}{d\,\alpha}=0{,}668830.$$

[4130]

$$\frac{d^2\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2}=7{,}719923\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2}=7{,}531096\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2}=8{,}558595\,;$$

$$\frac{d^2\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2}=9{,}112527\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha^2}=9{,}107400\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha^2}=8{,}634030\,;$$

$$\frac{d^2\,b_{\frac{1}{2}}^{(6)}}{d\,\alpha^2}=7{,}842733.$$

Venus and the Earth.

[4131]

$$\frac{d^3\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha^3}=56{,}55335\,; \qquad \frac{d^3\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^3}=57{,}35721\,; \qquad \frac{d^3\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^3}=58{,}19633\,;$$

$$\frac{d^3\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha^3}=62{,}87646\,; \qquad \frac{d^3\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha^3}=66{,}32409\,; \qquad \frac{d^3\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha^3}=70{,}54326.$$

[4132]

$$b_{\frac{3}{2}}^{(0)}=9{,}992539\,; \qquad b_{\frac{3}{2}}^{(1)}=8{,}871894\,; \qquad b_{\frac{3}{2}}^{(2)}=7{,}386580\,;$$

$$b_{\frac{3}{2}}^{(3)}=5{,}953940\,; \qquad b_{\frac{3}{2}}^{(4)}=4{,}704321\,; \qquad b_{\frac{3}{2}}^{(5)}=3{,}652052.$$

[4133]

$$\frac{d\,b_{\frac{3}{2}}^{(3)}}{d\,\alpha}=56{,}65440\,; \qquad \frac{d\,b_{\frac{3}{2}}^{(4)}}{d\,\alpha}=50{,}90290.$$

VENUS AND MARS.

[4134]

$$\alpha=\frac{a'}{a''}=0{,}47472320\,;$$

hence we deduce

Venus and Mars.

[4135]

$$b_{-\frac{1}{2}}^{(0)}=2{,}11436649\,;$$

$$b_{-\frac{1}{2}}^{(1)}=-\,0{,}46094390.$$

Then we find

$$b_{\frac{1}{2}}^{(0)} = 2{,}129668\,; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}521624\,; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}187726\,;$$

$$b_{\frac{1}{2}}^{(3)} = 0{,}074675\,; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}031127\,; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}013337\,;$$ [4136]

$$b_{\frac{1}{2}}^{(6)} = 0{,}005829.$$

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0{,}631752\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1{,}330781\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0{,}884106\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0{,}510976\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0{,}279002\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} = 0{,}147606.$$ [4137]

$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 2{,}192778\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 1{,}815836\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 2{,}795574\,;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} = 2{,}628516\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^2} = 2{,}004429.$$ [4138]

Venus and Mars.

$$\frac{d^3 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^3} = 7{,}65440\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^3} = 8{,}45655\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^3} = 8{,}17676\,;$$

$$\frac{d^3 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^3} = 10{,}66513.$$ [4139]

$$b_{\frac{3}{2}}^{(0)} = 3{,}523572\,; \qquad b_{\frac{3}{2}}^{(1)} = 2{,}304481\,; \qquad b_{\frac{3}{2}}^{(2)} = 1{,}325959\,;$$

$$b_{\frac{3}{2}}^{(3)} = 0{,}722687.$$ [4140]

$$\frac{d\,b_{\frac{3}{2}}^{(2)}}{d\,\alpha} = 8{,}47521.$$ [4141]

VENUS AND JUPITER.

Venus and Jupiter.

$$\alpha = \frac{a'}{a^{\mathrm{iv}}} = 0{,}13907116\,;$$ [4142]

hence we deduce

[4143]
$$b_{-\frac{1}{2}}^{(0)} = 2,00968215\,;$$
$$b_{-\frac{1}{2}}^{(1)} = -\,0,13873412.$$

Then we have

[4144]
$$b_{\frac{1}{2}}^{(0)} = 2,009778\,; \qquad b_{\frac{1}{2}}^{(1)} = 0,140092\,; \qquad b_{\frac{1}{2}}^{(2)} = 0,014623\,;$$
$$b_{\frac{1}{2}}^{(3)} = 0,001695\,; \qquad b_{\frac{1}{2}}^{(4)} = 0,000206\,; \qquad b_{\frac{1}{2}}^{(5)} = 0,000026.$$

Venus and Jupiter.

[4145]
$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,142160\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,022206\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0,212046\,;$$
$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,036783\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0,006111.$$

[4146]
$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 1,067532\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 0,325369\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 1,575190\,;$$
$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} = 0,533951.$$

[4147]
$$b_{\frac{3}{2}}^{(0)} = 2,089736\,; \qquad b_{\frac{3}{2}}^{(1)} = 0,432801\,; \qquad b_{\frac{3}{2}}^{(2)} = 0,075054.$$

VENUS AND SATURN.

[4148]
$$\alpha = \frac{a'}{a^{\text{v}}} = 0,07583790\,;$$

Venus and Saturn. hence we deduce

[4149]
$$b_{-\frac{1}{2}}^{(0)} = 2,00287673\,;$$
$$b_{-\frac{1}{2}}^{(1)} = -\,0,07578334.$$

Then we obtain

$$b_{\frac{1}{2}}^{(0)} = 2,002886 ; \qquad b_{\frac{1}{2}}^{(1)} = 0,076002 ; \qquad b_{\frac{1}{2}}^{(2)} = 0,004323 ;$$ [4150]

$$b_{\frac{1}{2}}^{(3)} = 0,000273 ; \qquad b_{\frac{1}{2}}^{(4)} = 0,000018.$$ [4151]

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,076331 ; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,006490 ; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0,114267 ;$$ [4152]

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,011085.$$

Venus and Saturn.

$$\frac{d^2\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 1,019629 ; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 0,172510 ; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 1,419950.$$ [4153]

$$b_{\frac{3}{2}}^{(0)} = 2,026116 ; \qquad b_{\frac{3}{2}}^{(1)} = 0,229988 ; \qquad b_{\frac{3}{2}}^{(2)} = 0,021791.$$ [4154]

VENUS AND URANUS.

$$\alpha = \frac{a'}{a^{\text{vi}}} = 0,03770634 ;$$ [4155]

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2,00071095 ;$$

$$b_{-\frac{1}{2}}^{(1)} = -\,0,03769964.$$ [4156]

Venus and Uranus.

Then we find

$$b_{\frac{1}{2}}^{(0)} = 2,000712 ; \qquad b_{\frac{1}{2}}^{(1)} = 0,037725 ; \qquad b_{\frac{1}{2}}^{(2)} = 0,001067 ;$$ [4157]

$$b_{\frac{1}{2}}^{(3)} = 0,000034.$$

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,716690 ; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,000829 ; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0,056634.$$ [4158]

THE EARTH AND MARS.

[4159] $$\alpha = \frac{a''}{a'''} = 0{,}65630030\,;$$

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2{,}22192172\,;$$

[4160] $$b_{-\frac{1}{2}}^{(1)} = -\,0{,}61874262.$$

Then

[4161] $$b_{\frac{1}{2}}^{(0)} = 2{,}291132\,; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}804563\,, \qquad b_{\frac{1}{2}}^{(2)} = 0{,}405584\,;$$

$$b_{\frac{1}{2}}^{(3)} = 0{,}224598\,; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}129973\,; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}077170\,;$$

$$b_{\frac{1}{2}}^{(6)} = 0{,}046595\,; \qquad b_{\frac{1}{2}}^{(7)} = 0{,}028480\,; \qquad b_{\frac{1}{2}}^{(8)} = 0{,}0175565.$$

The Earth and Mars.

[4162] $$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\alpha} = 1{,}228078\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}871211\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\alpha} = 1{,}601236\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\alpha} = 1{,}240990\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\alpha} = 0{,}920710\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\alpha} = 0{,}666207\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(6)}}{d\alpha} = 0{,}473942\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(7)}}{d\alpha} = 0{,}333444.$$

[4163] $$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\alpha^2} = 4{,}985108\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\alpha^2} = 4{,}744671\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\alpha^2} = 5{,}731111\,;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\alpha^2} = 6{,}057860\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(4)}}{d\alpha^2} = 5{,}776483\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(5)}}{d\alpha^2} = 5{,}141993\,;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(6)}}{d\alpha^2} = 4{,}388001.$$

[4164] $$\frac{d^3 b_{\frac{1}{2}}^{(0)}}{d\alpha^3} = 29{,}03400\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(1)}}{d\alpha^3} = 29{,}78930\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(2)}}{d\alpha^3} = 30{,}18848\,;$$

$$\frac{d^3 b_{\frac{1}{2}}^{(3)}}{d\alpha^3} = 33{,}29381\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(4)}}{d\alpha^3} = 36{,}32093\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(5)}}{d\alpha^3} = 37{,}23908.$$

$$b_{\frac{3}{2}}^{(0)} = 6,856336\,; \qquad b_{\frac{3}{2}}^{(1)} = 5,727893\,; \qquad b_{\frac{3}{2}}^{(2)} = 4,404530\,;$$

$$b_{\frac{3}{2}}^{(3)} = 3,255964\,; \qquad b_{\frac{3}{2}}^{(4)} = 2,351254\,; \qquad b_{\frac{3}{2}}^{(5)} = 1,671668\,; \qquad [4165]$$

$$b_{\frac{3}{2}}^{(6)} = 1,174650.$$

$$\frac{d\,b_{\frac{3}{2}}^{(2)}}{d\,\alpha} = 31,80897\,; \qquad \frac{d\,b_{\frac{3}{2}}^{(3)}}{d\,\alpha} = 32,26285\,; \quad \ldots\ldots \quad \frac{d\,b_{\frac{3}{2}}^{(5)}}{d\,\alpha} = 18,25867. \qquad [4166]$$

THE EARTH AND JUPITER.

$$\alpha = \frac{a''}{a^{\text{iv}}} = 0,19226461\,; \qquad [4167]$$

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2,01852593\,;$$

$$b_{-\frac{1}{2}}^{(1)} = -\,0,19137205. \qquad [4168]$$

Then

The Earth and Jupiter.

$$b_{\frac{1}{2}}^{(0)} = 2,018885\,; \qquad b_{\frac{1}{2}}^{(1)} = 0,195003\,; \qquad b_{\frac{1}{2}}^{(2)} = 0,028195\,;$$

$$b_{\frac{1}{2}}^{(3)} = 0,004516\,; \qquad b_{\frac{1}{2}}^{(4)} = 0,000779\,; \qquad b_{\frac{1}{2}}^{(5)} = 0,000132\,; \qquad [4169]$$

$$b_{\frac{1}{2}}^{(6)} = 0,000023.$$

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,200586\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,043204\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0,297995\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,070932\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0,016369\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} = 0,003448\,; \qquad [4170]$$

[4171]

$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\alpha^2} = 1,132355; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\alpha^2} = 0,466165; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\alpha^2} = 1,628667;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\alpha^2} = 0,746681.$$

The Earth and Jupiter.

[4172]

$$\frac{d^3 b_{\frac{1}{2}}^{(0)}}{d\alpha^3} = 1,472714; \qquad \frac{d^3 b_{\frac{1}{2}}^{(1)}}{d\alpha^3} = 2,874986; \qquad \frac{d^3 b_{\frac{1}{2}}^{(2)}}{d\alpha^3} = 1,418830.$$

[4173]

$$b_{\frac{3}{2}}^{(0)} = 2,176460; \qquad b_{\frac{3}{2}}^{(1)} = 0,619063; \qquad b_{\frac{3}{2}}^{(2)} = 0,148198;$$

$$b_{\frac{3}{2}}^{(3)} = 0,032493.$$

THE EARTH AND SATURN.

[4174]

$$\alpha = \frac{a''}{a^{\text{v}}} = 0,10484520;$$

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2,00550004;$$

[4175]

$$b_{-\frac{1}{2}}^{(1)} = -0,10470094.$$

Then

The Earth and Saturn

[4176]

$$b_{\frac{1}{2}}^{(0)} = 2,005535; \qquad b_{\frac{1}{2}}^{(1)} = 0,105283; \qquad b_{\frac{1}{2}}^{(2)} = 0,008282;$$

$$b_{\frac{1}{2}}^{(3)} = 0,000724; \qquad b_{\frac{1}{2}}^{(4)} = 0,000066.$$

[4177]

$$\frac{d b_{\frac{1}{2}}^{(0)}}{d\alpha} = 0,106155; \qquad \frac{d b_{\frac{1}{2}}^{(1)}}{d\alpha} = 1,012536; \qquad \frac{d b_{\frac{1}{2}}^{(2)}}{d\alpha} = 0,158723;$$

$$\frac{d b_{\frac{1}{2}}^{(3)}}{d\alpha} = 0,020779.$$

$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\alpha^2} = 1,037816 ; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\alpha^2} = 0,246193 ; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\alpha^2} = 1,526303. \qquad [4178]$$

$$b_{\frac{3}{2}}^{(0)} = 2,050321 ; \qquad b_{\frac{3}{2}}^{(1)} = 0,321144 ; \qquad b_{\frac{3}{2}}^{(2)} = 0,041977. \qquad [4179]$$

THE EARTH AND URANUS.

$$\alpha = \frac{a''}{a^{\mathrm{vi}}} = 0,05212866 ; \qquad [4180]$$

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2,00135893 ;$$

$$b_{-\frac{1}{2}}^{(1)} = -0,05211095. \qquad [4181]$$

Then we find

The Earth and Uranus.

$$b_{\frac{1}{2}}^{(0)} = 2,001355 ; \qquad b_{\frac{1}{2}}^{(1)} = 0,052182 ; \qquad b_{\frac{1}{2}}^{(2)} = 0,002040 ;$$

$$b_{\frac{1}{2}}^{(3)} = 0,000089. \qquad [4182]$$

$$\frac{d b_{\frac{1}{2}}^{(0)}}{d\alpha} = 0,052288 ; \qquad \frac{d b_{\frac{1}{2}}^{(1)}}{d\alpha} = 1,003060 ; \qquad \frac{d b_{\frac{1}{2}}^{(2)}}{d\alpha} = 0,078449. \qquad [4183]$$

MARS AND JUPITER.

$$\alpha = \frac{a'''}{a^{\mathrm{iv}}} = 0,29295212. \qquad [4184]$$

hence we deduce

[4185]
$$b_{-\frac{1}{2}}^{(0)} = 2,04314576;$$
$$b_{-\frac{1}{2}}^{(1)} = -0,28977479.$$

Then

[4186]
$$b_{\frac{1}{2}}^{(0)} = 2,045112; \qquad b_{\frac{1}{2}}^{(1)} = 0,302922; \qquad b_{\frac{1}{2}}^{(2)} = 0,066812;$$
$$b_{\frac{1}{2}}^{(3)} = 0,016357; \qquad b_{\frac{1}{2}}^{(4)} = 0,004192; \qquad b_{\frac{1}{2}}^{(5)} = 0,001109;$$
$$b_{\frac{1}{2}}^{(6)} = 0,000297; \qquad b_{\frac{1}{2}}^{(7)} = 0,000081.$$

[4187]
$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,324004; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,105998; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0,473717;$$
$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,172096; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0,058420; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} = 0,019258;$$

Mars and Jupiter
$$\frac{d\,b_{\frac{1}{2}}^{(6)}}{d\,\alpha} = 0,006173.$$

[4188]
$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 1,338759; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 0,794557; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 1,871538;$$
$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} = 1,258858; \qquad \frac{d^2 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^2} = 0,623184.$$

[4189]
$$\frac{d^3 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^3} = 2,69358; \qquad \frac{d^3 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^3} = 3,77722; \qquad \frac{d^3 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^3} = 2,91068;$$
$$\frac{d^3 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^3} = 5,47068.$$

[4190]
$$b_{\frac{3}{2}}^{(0)} = 2,444762; \qquad b_{\frac{3}{2}}^{(1)} = 1,040206; \qquad b_{\frac{3}{2}}^{(2)} = 0,376693;$$
$$b_{\frac{3}{2}}^{(3)} = 0,127942.$$

$$\frac{d\,b_{\frac{3}{2}}^{(0)}}{d\,\alpha} = 3{,}48815\,; \qquad \frac{d\,b_{\frac{3}{2}}^{(1)}}{d\,\alpha} = 4{,}80540\,; \qquad \frac{d\,b_{\frac{3}{2}}^{(2)}}{d\,\alpha} = 2{,}99684. \qquad [4191]$$

MARS AND SATURN.

$$\alpha = \frac{a'''}{a^{\rm v}} = 0{,}15975187\,; \qquad [4192]$$

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2{,}01278081\,;$$

$$b_{-\frac{1}{2}}^{(1)} = -\,0{,}15924060. \qquad [4193]$$

Then we find

Mars and Saturn.

$$b_{\frac{1}{2}}^{(0)} = 2{,}012945\,; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}161305\,; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}019347\,;$$

$$b_{\frac{1}{2}}^{(3)} = 0{,}002577\,; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}000360\,; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}000052. \qquad [4194]$$

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0{,}164463\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1{,}029493\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0{,}244843\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0{,}048740\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0{,}009065. \qquad [4195]$$

$$\frac{d^2\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 1{,}090095\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 0{,}379322\,; \qquad \frac{d^2\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 1{,}596248\,;$$

$$\frac{d^2\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} = 0{,}620632. \qquad [4196]$$

$$b_{\frac{3}{2}}^{(0)} = 2{,}119585\,; \qquad b_{\frac{3}{2}}^{(1)} = 0{,}503071\,; \qquad b_{\frac{3}{2}}^{(2)} = 0{,}100136\,; \qquad [4197]$$

MARS AND URANUS.

[4198] $$\alpha = \frac{a'''}{a^{\text{vi}}} = 0,07942807;$$

hence we deduce

[4199] $$b_{-\frac{1}{2}}^{(0)} = 2,00315565;$$
$$b_{-\frac{1}{2}}^{(1)} = -0,07936538.$$

Mars and Uranus.

Then we find

[4200] $$b_{\frac{1}{2}}^{(0)} = 2,003167; \qquad b_{\frac{1}{2}}^{(1)} = 0,079617; \qquad b_{\frac{1}{2}}^{(2)} = 0,004746;$$
$$b_{\frac{1}{2}}^{(3)} = 0,000314; \qquad b_{\frac{1}{2}}^{(4)} = 0,000022.$$

[4201] $$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,079995; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,007144; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0,119822;$$
$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,011982.$$

JUPITER AND SATURN.

[4202] $$\alpha = \frac{a^{\text{iv}}}{a^{\text{v}}} = 0,54531725;$$

Jupiter and Saturn.

hence we deduce

[4203] $$b_{-\frac{1}{2}}^{(0)} = 2,15168241;$$
$$b_{-\frac{1}{2}}^{(1)} = -0,52421272.$$

Then we have

$$b_{\frac{1}{2}}^{(0)} = 2,1802348; \qquad b_{\frac{1}{2}}^{(1)} = 0,6206406; \qquad b_{\frac{1}{2}}^{(2)} = 0,2576379;$$

$$b_{\frac{1}{2}}^{(3)} = 0,1179750\,; \qquad b_{\frac{1}{2}}^{(4)} = 0,0565522\,; \qquad b_{\frac{1}{2}}^{(5)} = 0,0278360\,;$$

$$b_{\frac{1}{2}}^{(6)} = 0,0139345\,; \qquad b_{\frac{1}{2}}^{(7)} = 0,0070481\,; \qquad b_{\frac{1}{2}}^{(8)} = 0,0035837\,;$$ [4204]

$$b_{\frac{1}{2}}^{(9)} = 0,0018056\,; \qquad b_{\frac{1}{2}}^{(10)} = 0,0008632\,; \qquad b_{\frac{1}{2}}^{(11)} = 0,0003223.$$

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,808789\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,483154\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 1,105160\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,726550\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0,453285\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} = 0,274717\,;$$

$$\frac{d\,b_{\frac{1}{2}}^{(6)}}{d\,\alpha} = 0,163506\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(7)}}{d\,\alpha} = 0,096019\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(8)}}{d\,\alpha} = 0,056171\,;$$ [4205]

$$\frac{d\,b_{\frac{1}{2}}^{(9)}}{d\,\alpha} = 0,033083\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(10)}}{d\,\alpha} = 0,020265.$$

Jupiter and Saturn.

$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 2,875229\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 2,552788\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 3,521040\,;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} = 3,533622\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^2} = 2,995647\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(5)}}{d\,\alpha^2} = 2,302428\,;$$ [4206]

$$\frac{d^2 b_{\frac{1}{2}}^{(6)}}{d\,\alpha^2} = 1,664586\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(7)}}{d\,\alpha^2} = 1,144377\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(8)}}{d\,\alpha^2} = 0,760603\,;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(9)}}{d\,\alpha^2} = 0,485135.$$

$$\frac{d^3 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^3} = 12,128630\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^3} = 12,878804\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^3} = 12,832050\,;$$

$$\frac{d^3 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^3} = 15,454850\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(4)}}{d\,\alpha^3} = 17,058155\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(5)}}{d\,\alpha^3} = 16,655445\,;$$ [4207]

$$\frac{d^3 b_{\frac{1}{2}}^{(6)}}{d\,\alpha^3} = 14,958762\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(7)}}{d\,\alpha^3} = 12,234874\,; \qquad \frac{d^3 b_{\frac{1}{2}}^{(8)}}{d\,\alpha^3} = 9,566420.$$

$$\frac{d^4 b_{\frac{1}{2}}^{(0)}}{d\alpha^4} = 84{,}40159\,; \qquad \frac{d^4 b_{\frac{1}{2}}^{(1)}}{d\alpha^4} = 83{,}94825\,; \qquad \frac{d^4 b_{\frac{1}{2}}^{(2)}}{d\alpha^4} = 87{,}3027\,;$$

[4208] $$\frac{d^4 b_{\frac{1}{2}}^{(3)}}{d\alpha^4} = 89{,}8615\,; \qquad \frac{d^4 b_{\frac{1}{2}}^{(4)}}{d\alpha^4} = 101{,}3809\,; \qquad \frac{d^4 b_{\frac{1}{2}}^{(5)}}{d\alpha^4} = 113{,}5238\,;$$

$$\frac{d^4 b_{\frac{1}{2}}^{(6)}}{d\alpha^4} = 118{,}6607\,; \qquad \frac{d^4 b_{\frac{1}{2}}^{(7)}}{d\alpha^4} = 115{,}9588.$$

$$\frac{d^5 b_{\frac{1}{2}}^{(0)}}{d\alpha^5} = 747{,}480\,; \qquad \frac{d^5 b_{\frac{1}{2}}^{(1)}}{d\alpha^5} = 753{,}417\,; \qquad \frac{d^5 b_{\frac{1}{2}}^{(2)}}{d\alpha^5} = 761{,}843\,;$$

[4209] $$\frac{d^5 b_{\frac{1}{2}}^{(3)}}{d\alpha^5} = 785{,}884\,; \qquad \frac{d^5 b_{\frac{1}{2}}^{(4)}}{d\alpha^5} = 819{,}180\,; \qquad \frac{d^5 b_{\frac{1}{2}}^{(5)}}{d\alpha^5} = 884{,}505\,;$$

$$\frac{d^5 b_{\frac{1}{2}}^{(6)}}{d\alpha^5} = 912{,}301.$$

Jupiter and Saturn.

$$b_{\frac{3}{2}}^{(0)} = 4{,}358387\,; \qquad b_{\frac{3}{2}}^{(1)} = 3{,}185493\,; \qquad b_{\frac{3}{2}}^{(2)} = 2{,}082131\,;$$

$$b_{\frac{3}{2}}^{(3)} = 1{,}295672\,; \qquad b_{\frac{3}{2}}^{(4)} = 0{,}784084\,; \qquad b_{\frac{3}{2}}^{(5)} = 0{,}466047\,;$$

[4210] $$b_{\frac{3}{2}}^{(6)} = 0{,}273629\,; \qquad b_{\frac{3}{2}}^{(7)} = 0{,}158799\,; \qquad b_{\frac{3}{2}}^{(8)} = 0{,}092290\,;$$

$$b_{\frac{3}{2}}^{(9)} = 0{,}053922.$$

$$\frac{d b_{\frac{3}{2}}^{(0)}}{d\alpha} = 14{,}681324\,; \qquad \frac{d b_{\frac{3}{2}}^{(1)}}{d\alpha} = 15{,}239657\,; \qquad \frac{d b_{\frac{3}{2}}^{(2)}}{d\alpha} = 13{,}416026\,;$$

[4211] $$\frac{d b_{\frac{3}{2}}^{(3)}}{d\alpha} = 10{,}598611\,; \qquad \frac{d b_{\frac{3}{2}}^{(4)}}{d\alpha} = 7{,}802247\,; \qquad \frac{d b_{\frac{3}{2}}^{(5)}}{d\alpha} = 5{,}470398\,;$$

$$\frac{d b_{\frac{3}{2}}^{(6)}}{d\alpha} = 3{,}710043\,; \qquad \frac{d b_{\frac{3}{2}}^{(7)}}{d\alpha} = 2{,}426079\,; \qquad \frac{d b_{\frac{3}{2}}^{(8)}}{d\alpha} = 1{,}563695.$$

$$\frac{d^2 b_{\frac{3}{2}}^{(0)}}{d\alpha^2} = 96{,}68536\,; \qquad \frac{d^2 b_{\frac{3}{2}}^{(1)}}{d\alpha^2} = 94{,}91701\,; \qquad \frac{d^2 b_{\frac{3}{2}}^{(2)}}{d\alpha^2} = 93{,}19282\,;$$

$$\frac{d^2 b_{\frac{3}{2}}^{(3)}}{d\alpha^2} = 86{,}90215 ; \qquad \frac{d^2 b_{\frac{3}{2}}^{(4)}}{d\alpha^2} = 75{,}08115 ; \qquad \frac{d^2 b_{\frac{3}{2}}^{(5)}}{d\alpha^2} = 61{,}10115 ;$$

$$\frac{d^2 b_{\frac{3}{2}}^{(6)}}{d\alpha^2} = 47{,}48185 ; \qquad \frac{d^2 b_{\frac{3}{2}}^{(7)}}{d\alpha^2} = 35{,}74355.$$ [4212]

Jupiter and Saturn.

$$\frac{d^3 b_{\frac{3}{2}}^{(0)}}{d\alpha^3} = 830{,}0586 ; \qquad \frac{d^3 b_{\frac{3}{2}}^{(1)}}{d\alpha^3} = 830{,}1580 ; \qquad \frac{d^3 b_{\frac{3}{2}}^{(2)}}{d\alpha^3} = 810{,}1045 ;$$

$$\frac{d^3 b_{\frac{3}{2}}^{(3)}}{d\alpha^3} = 785{,}5855 ; \qquad \frac{d^3 b_{\frac{3}{2}}^{(4)}}{d\alpha^3} = 740{,}6775 ; \qquad \frac{d^3 b_{\frac{3}{2}}^{(5)}}{d\alpha^3} = 666{,}4080 ;$$ [4213]

$$\frac{d^3 b_{\frac{3}{2}}^{(6)}}{d\alpha^3} = 574{,}9115.$$

JUPITER AND URANUS.

$$\alpha = \frac{a^{\text{iv}}}{a^{\text{v}}} = 0{,}27112980 ;$$ [4214]

hence we deduce

$$b_{-\frac{1}{2}}^{(0)} = 2{,}03692776 ;$$

$$b_{-\frac{1}{2}}^{(1)} = -0{,}26861497.$$ [4215]

Then we get

Jupiter and Uranus.

$$b_{\frac{1}{2}}^{(0)} = 2{,}038359 ; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}278966 ; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}056906 ;$$

$$b_{\frac{1}{2}}^{(3)} = 0{,}012879 ; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}003058 ; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}000745 ;$$ [4216]

$$b_{\frac{1}{2}}^{(6)} = 0{,}000185.$$

$$\frac{d b_{\frac{1}{2}}^{(0)}}{d\alpha} = 0{,}295410 ; \qquad \frac{d b_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}089551 ; \qquad \frac{d b_{\frac{1}{2}}^{(2)}}{d\alpha} = 0{,}433630 ;$$ [4217]

Jupiter and Uranus.

$$\frac{d\,b_{\frac{1}{2}}^{(3)}}{d\,\alpha} = 0,145398\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(4)}}{d\,\alpha} = 0,045930\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(5)}}{d\,\alpha} = 0,015410.$$

[4218]

$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d\,\alpha^2} = 1,283434\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2} = 0,714932\,; \qquad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\,\alpha^2} = 1,815451\,;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d\,\alpha^2} = 1,133359.$$

[4219]

$$b_{\frac{3}{2}}^{(0)} = 2,372983\,; \qquad b_{\frac{3}{2}}^{(1)} = 0,938794\,; \qquad b_{\frac{3}{2}}^{(2)} = 0,315186\,;$$

$$b_{\frac{3}{2}}^{(3)} = 0,099260.$$

SATURN AND URANUS.

[4220]

$$\alpha = \frac{a^{\mathrm{v}}}{a^{\mathrm{vi}}} = 0,49719638\,;$$

hence we deduce

[4221]

$$b_{-\frac{1}{2}}^{(0)} = 2,12564287\,;$$

$$b_{-\frac{1}{2}}^{(1)} = -\,0,48131675.$$

Then we get

Saturn and Uranus.

[4222]

$$b_{\frac{1}{2}}^{(0)} = 2,144440\,; \qquad b_{\frac{1}{2}}^{(1)} = 0,552007\,; \qquad b_{\frac{1}{2}}^{(2)} = 0,208313\,;$$

$$b_{\frac{1}{2}}^{(3)} = 0,086834\,; \qquad b_{\frac{1}{2}}^{(4)} = 0,037909\,; \qquad b_{\frac{1}{2}}^{(5)} = 0,016990\,;$$

$$b_{\frac{1}{2}}^{(6)} = 0,007728\,; \qquad b_{\frac{1}{2}}^{(7)} = 0,003522\,; \qquad b_{\frac{1}{2}}^{(8)} = 0,001547.$$

$$\frac{d\,b_{\frac{1}{2}}^{(0)}}{d\,\alpha} = 0,683055\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha} = 1,373806\,; \qquad \frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha} = 0,949128\,;$$

$$\frac{d b_{\frac{1}{2}}^{(3)}}{d \alpha} = 0{,}572896 ; \quad \frac{d b_{\frac{1}{2}}^{(4)}}{d \alpha} = 0{,}327198 ; \quad \frac{d b_{\frac{1}{2}}^{(5)}}{d \alpha} = 0{,}181370 ;$$

$$\frac{d b_{\frac{1}{2}}^{(6)}}{d \alpha} = 0{,}098799 ; \quad \frac{d b_{\frac{1}{2}}^{(7)}}{d \alpha} = 0{,}053642. \qquad [4223]$$

$$\frac{d^2 b_{\frac{1}{2}}^{(0)}}{d \alpha^2} = 2{,}377102 ; \quad \frac{d^2 b_{\frac{1}{2}}^{(1)}}{d \alpha^2} = 2{,}017767 ; \quad \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d \alpha^2} = 2{,}992245 ;$$

$$\frac{d^2 b_{\frac{1}{2}}^{(3)}}{d \alpha^2} = 2{,}881218 ; \quad \frac{d^2 b_{\frac{1}{2}}^{(4)}}{d \alpha^2} = 2{,}278077 ; \quad \frac{d^2 b_{\frac{1}{2}}^{(5)}}{d \alpha^2} = 1{,}616470 ; \qquad [4224]$$

$$\frac{d^2 b_{\frac{1}{2}}^{(6)}}{d \alpha^2} = 1{,}067430.$$

Saturn and Uranus.

$$\frac{d^3 b_{\frac{1}{2}}^{(0)}}{d \alpha^3} = 8{,}798999 ; \quad \frac{d^3 b_{\frac{1}{2}}^{(1)}}{d \alpha^3} = 9{,}578267 ; \quad \frac{d^3 b_{\frac{1}{2}}^{(2)}}{d \alpha^3} = 9{,}425450 ;$$

$$\frac{d^3 b_{\frac{1}{2}}^{(3)}}{d \alpha^3} = 11{,}904140 ; \quad \frac{d^3 b_{\frac{1}{2}}^{(4)}}{d \alpha^3} = 12{,}988670 ; \quad \frac{d^3 b_{\frac{1}{2}}^{(5)}}{d \alpha^3} = 12{,}135721. \qquad [4225]$$

$$b_{\frac{3}{2}}^{(0)} = 3{,}750905 ; \quad b_{\frac{3}{2}}^{(1)} = 2{,}547992 ; \quad b_{\frac{3}{2}}^{(2)} = 1{,}530452 ;$$

$$b_{\frac{3}{2}}^{(3)} = 0{,}872105 ; \quad b_{\frac{3}{2}}^{(4)} = 0{,}482564 ; \quad b_{\frac{3}{2}}^{(5)} = 0{,}262146. \qquad [4226]$$

$$\frac{d b_{\frac{3}{2}}^{(2)}}{d \alpha} = 9{,}75656 ; \quad \frac{d b_{\frac{3}{2}}^{(3)}}{d \alpha} = 7{,}24097 ; \quad \frac{d b_{\frac{3}{2}}^{(4)}}{d \alpha} = 4{,}95052. \qquad [4227]$$

CHAPTER VII.

NUMERICAL EXPRESSIONS OF THE SECULAR VARIATIONS OF THE ELEMENTS OF THE PLANETARY ORBITS.

24. We shall now give the numerical values of the secular variations of the elements of the planetary orbits. For this purpose we shall resume the differential variations of the excentricities, perihelia and inclinations of the orbits [1122, 1126, 1142, 1143, 1146]. To reduce these formulas to numbers, we must previously determine the numerical values of the quantities $(0,1)$, $\boxed{0,1}$, &c. These are obtained by computing, in the first place, the values of $(0,1)$, $\boxed{0,1}$; by means of the formulas [1076, 1082],*

Formulas for the computation of (0,1), &c.

[4228]
$$(0,1) = -\frac{3m'.n\alpha^2.b_{-\frac{1}{2}}^{(1)}}{4.(1-\alpha^2)^2};$$

[4228']
$$\boxed{0,1} = -\frac{3m'.n\alpha.\{(1+\alpha^2).b_{-\frac{1}{2}}^{(1)}+\frac{1}{2}\alpha.b_{-\frac{1}{2}}^{(0)}\}}{2.(1-\alpha^2)^2}.$$

From these we have deduced the values of $(1,0)$ $\boxed{1,0}$, by means of the equations [1093, 1094].

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* 2566. The values of m', n, α, $b_{-\frac{1}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)}$ to be used in these formulas are given in
[4228a] [4061 — 4222]. By the formula [4228] we must compute the values corresponding to the exterior planets, namely; (0,1), (0,2), (0,3), (0,4), (0,5), (0,6); (1,2), (1,3), (1,4), (1,5), (1,6); (2,3), (2,4), (2,5), (2,6); (3,4), (3,5), (3,6); (4,5), (4,6); (5,6); and the similar ones of [4228'], namely; $\boxed{0,1}$ &c.; $\boxed{1,2}$ &c.; $\boxed{2,3}$ &c.; $\boxed{3,4}$ &c.; $\boxed{4,5}$
[4228b] &c.; $\boxed{5,6}$. The remaining terms corresponding to interior planets are to be deduced from these by the formulas [1229]. Thus, if it be required to compute (4,5), $\boxed{4,5}$ corresponding to the action of Saturn upon Jupiter. The value of m' to be used in [4228],

$$(1,0)=\frac{m\,.\sqrt{a}}{m'.\sqrt{a'}}.(0,1)\,; \qquad [\boxed{1,0}]=\frac{m.\sqrt{a}}{m'.\sqrt{a'}}.[\boxed{0,1}].$$ [4229]

By this means we have obtained the following results, in seconds, supposing the numerical characters 0, 1, 2, 3, 4, 5, 6 to refer respectively to Mercury, Venus, the Earth, Mars, Jupiter, Saturn, and Uranus. [4230] *The preceding masses of the planets* [4061, 4061*d*], *have been multiplied by* $1+\mu$, $1+\mu'$, $1+\mu''$, [4230'] &c. respectively, *in order that these results may be immediately corrected, for any change in the values of the masses, which may hereafter be found necessary.*

$$\begin{array}{llll}
(0,1) = (1+\mu'\,).3^s{,}052453\,; & [\boxed{0,1}] = (1+\mu'\,).1^s{,}961407\,; \\
(0,2) = (1+\mu'').0^s{,}963818\,; & [\boxed{0,2}] = (1+\mu'').0^s{,}457195\,; \\
(0,3) = (1+\mu''').0^s{,}040631\,; & [\boxed{0,3}] = (1+\mu''').0^s{,}012797\,; \\
(0,4) = (1+\mu^{\rm iv}).1^s{,}575473\,; & [\boxed{0,4}] = (1+\mu^{\rm iv}).0^s{,}146329\,; \\
(0,5) = (1+\mu^{\rm v}).0^s{,}080560\,; & [\boxed{0,5}] = (1+\mu^{\rm v}).0^s{,}004086\,; \\
(0,6) = (1+\mu^{\rm vi}).0^s{,}001702\,; & [\boxed{0,6}] = (1+\mu^{\rm vi}).0^s{,}000042.
\end{array}$$ [4231] Mercury.

$$\begin{array}{llll}
(1,0) = (1+\mu\,).0^s{,}422318\,; & [\boxed{1,0}] = (1+\mu\,).0^s{,}271367\,; \\
(1,2) = (1+\mu'').7^s{,}416280\,; & [\boxed{1,2}] = (1+\mu'').6^s{,}174974\,; \\
(1,3) = (1+\mu''').0^s{,}148161\,; & [\boxed{1,3}] = (1+\mu''').0^s{,}085252\,; \\
(1,4) = (1+\mu^{\rm iv}).4^s{,}131166\,; & [\boxed{1,4}] = (1+\mu^{\rm iv}).0^s{,}716427\,; \\
(1,5) = (1+\mu^{\rm v}).0^s{,}207370\,; & [\boxed{1,5}] = (1+\mu^{\rm v}).0^s{,}019641\,; \\
(1,6) = (1+\mu^{\rm vi}).0^s{,}004354\,; & [\boxed{1,6}] = (1+\mu^{\rm vi}).0^s{,}000205.
\end{array}$$ [4232] Venus.

$$\begin{array}{llll}
(2,0) = (1+\mu\,).0^s{,}097574\,; & [\boxed{2,0}] = (1+\mu\,).0^s{,}046285\,; \\
(2,1) = (1+\mu'\,).5^s{,}426695\,; & [\boxed{2,1}] = (1+\mu'\,).4^s{,}518397\,; \\
(2,3) = (1+\mu''').0^s{,}432999\,; & [\boxed{2,3}] = (1+\mu''').0^s{,}332961\,;
\end{array}$$ The Earth. [4233]

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is that of Saturn, $m^{\rm v}=\frac{(1+\mu^{\rm v})}{3359{,}40}$ [4061], the value of n is that of $n^{\rm iv}=109256^s293$ [4077]; the value of α is 0,54531725, [4202]; then we have $b_{-\frac{1}{2}}^{(0)}=2.15168241$, [4228c] $b_{-\frac{1}{2}}^{(1)}=-0{,}52421272$ [4203]. Substituting these in [4228, 4228'] we get the values of (4,5), $[\boxed{4,5}]$ as in [4235]. Lastly the formulas [4229] give $(5,4)=\frac{m^{\rm iv}.\sqrt{a^{\rm iv}}}{m^{\rm v}.\sqrt{a^{\rm v}}}.(4,5)$; $[\boxed{5,4}]=\frac{m^{\rm iv}.\sqrt{a^{\rm iv}}}{m^{\rm v}.\sqrt{a^{\rm v}}}.[\boxed{4,5}]$; hence we obtain (5,4), $[\boxed{5,4}]$ as in [4236], using the factor [4228d] $1+\mu^{\rm iv}$ instead of $1+\mu^{\rm v}$. In like manner the other formulas [4231 — 4237] are to be computed.

	$(2,4) = (1+\mu^{iv}).\ 6^s{,}947861;$	$[\overline{2,4}] = (1+\mu^{iv}).\ 1^s{,}662036;$
The Earth.	$(2,5) = (1+\mu^{v}).\ 0^s{,}340441;$	$[\overline{2,5}] = (1+\mu^{v}).\ 0^s{,}044514;$
	$(2,6) = (1+\mu^{vi}).\ 0^s{,}007095;$	$[\overline{2,6}] = (1+\mu^{vi}).\ 0^s{,}000463.$
	$(3,0) = (1+\mu).\ 0^s{,}018662;$	$[\overline{3,0}] = (1+\mu).\ 0^s{,}005878;$
	$(3,1) = (1+\mu').\ 0^s{,}491880;$	$[\overline{3,1}] = (1+\mu').\ 0^s{,}283029;$
[4234]	$(3,2) = (1+\mu'').\ 1^s{,}964546;$	$[\overline{3,2}] = (1+\mu'').\ 1^s{,}510657;$
Mars.	$(3,4) = (1+\mu^{iv}).\ 14^s{,}411136;$	$[\overline{3,4}] = (1+\mu^{iv}).\ 5^s{,}219092;$
	$(3,5) = (1+\mu^{v}).\ 0^s{,}658341;$	$[\overline{3,5}] = (1+\mu^{v}).\ 0^s{,}131041;$
	$(3,6) = (1+\mu^{vi}).\ 0^s{,}013436;$	$[\overline{3,6}] = (1+\mu^{vi}).\ 0^s{,}001333.$
	$(4,0) = (1+\mu).\ 0^s{,}000226;$	$[\overline{4,0}] = (1+\mu).\ 0^s{,}000021;$
	$(4,1) = (1+\mu').\ 0^s{,}004291;$	$[\overline{4,1}] = (1+\mu').\ 0^s{,}000744;$
[4235]	$(4,2) = (1+\mu'').\ 0^s{,}009862;$	$[\overline{4,2}] = (1+\mu'').\ 0^s{,}002359;$
Jupiter.	$(4,3) = (1+\mu''').\ 0^s{,}004509;$	$[\overline{4,3}] = (1+\mu''').\ 0^s{,}001633;$
	$(4,5) = (1+\mu^{v}).\ 7^s{,}701937;$	$[\overline{4,5}] = (1+\mu^{v}).\ 5^s{,}034195;$
	$(4,6) = (1+\mu^{vi}).\ 0^s{,}096647,$	$[\overline{4,6}] = (1+\mu^{vi}).\ 0^s{,}032446.$
	$(5,0) = (1+\mu).\ 0^s{,}000027;$	$[\overline{5,0}] = (1+\mu).\ 0^s{,}000001;$
	$(5,1) = (1+\mu').\ 0^s{,}000501;$	$[\overline{5,1}] = (1+\mu').\ 0^s{,}000047;$
[4236]	$(5,2) = (1+\mu'').\ 0^s{,}001123;$	$[\overline{5,2}] = (1+\mu'').\ 0^s{,}000147;$
Saturn.	$(5,3) = (1+\mu''').\ 0^s{,}000479;$	$[\overline{5,3}] = (1+\mu''').\ 0^s{,}000095;$
	$(5,4) = (1+\mu^{iv}).\ 17^s{,}905446;$	$[\overline{5,4}] = (1+\mu^{iv}).\ 11^s{,}703495;$
	$(5,6) = (1+\mu^{vi}).\ 0^s{,}355214;$	$[\overline{5,6}] = (1+\mu^{vi}).\ 0^s{,}213356.$
	$(6,0) = (1+\mu).\ 0^s{,}000002;$	$[\overline{6,0}] = (1+\mu).\ 0^s{,}000000;$
	$(6,1) = (1+\mu').\ 0^s{,}000043;$	$[\overline{6,1}] = (1+\mu').\ 0^s{,}000002;$
[4237]	$(6,2) = (1+\mu'').\ 0^s{,}000096;$	$[\overline{6,2}] = (1+\mu'').\ 0^s{,}000006;$
Uranus.	$(6,3) = (1+\mu''').\ 0^s{,}000040;$	$[\overline{6,3}] = (1+\mu''').\ 0^s{,}000004;$
	$(6,4) = (1+\mu^{iv}).\ 0^s{,}919814;$	$[\overline{6,4}] = (1+\mu^{iv}).\ 0^s{,}308803;$
	$(6,5) = (1+\mu^{v}).\ 1^s{,}454176;$	$[\overline{6,5}] = (1+\mu^{v}).\ 0^s{,}873434.$

25. By means of these values and the formulas [1122, 1126, 1142, 1143, 1146] the following results have been obtained;
[4237'] *which exhibit, at the epoch of* 1750, *the annual variations of the elements, during a year of* $365\frac{1}{4}$ *days, namely,*

$\frac{d\varpi}{dt}$ = the annual sideral motion of the perihelion in longitude in 1750;* [4238]

$\frac{2de}{dt}$ = the annual variation of the equation of the centre, or that of double the excentricity in 1750;† [4238′]

$\frac{d\varphi}{dt}$ = the annual variation of the inclination of the orbit to the fixed ecliptic of 1750; [4239]

Symbols.

$\frac{d\varphi_{,}}{dt}$ = the annual variation of the inclination of the orbit to the apparent ecliptic; [4239′]

$\frac{d\theta}{dt}$ = the annual sideral motion of the ascending node of the orbit upon the fixed ecliptic of 1750; [4240]

$\frac{d\theta_{,}}{dt}$ = the annual sideral motion of the same node upon the apparent ecliptic.‡ [4241]

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* (2567) Neglecting terms of the order t^2, we get $u = U + t.\frac{dU}{dt}$, by Taylor's [4238a] theorem [3850a]. The time t is counted in Julian years [4078] and the values of n, n', n'' &c. [4077] are taken to conform to this *unit* of time, so that $n''t$, which represents generally the motion of the earth in the time t, will become simply n'', in one year, or when $t=1$. Now U being the value of u when $t=0$, if we subtract it from the value for [4238b] the case of $t=1$, which by [4238a] is $U+\frac{dU}{dt}$, we shall get the annual variation of u equal to $\frac{dU}{dt}$. Therefore if we write successively ϖ, $2e$, φ, $\varphi_{,}$, θ, $\theta_{,}$, for u, we shall obtain the annual variations of these quantities respectively, namely, $\frac{d\varpi}{dt}$, [4238c] $2.\frac{de}{dt}$, $\frac{d\varphi}{dt}$, $\frac{d\varphi_{,}}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\theta_{,}}{dt}$. Now in [4080—4083] ϖ represents the longitude of the perihelion, e the excentricity of the orbit, φ the inclination of the orbit, and θ the longitude [4238d] of the ascending node of m, upon the *fixed* ecliptic. Moreover, $\varphi_{,}$ is, as in [1143$^{\text{iv}}$], the inclination, and $\theta_{,}$ the longitude of the node counted upon the apparent ecliptic. With *one* [4238e] accent *above* these quantities, they correspond to the body m'; and with *two* accents to the body m'', &c.

† (2568) Neglecting terms of the order e^2, in the equation of the centre [3748], it becomes $2e.\sin.(nt+\varepsilon-\varpi)$; the maximum value being $2e$, whose annual variation is [4239a] $2.\frac{de}{dt}$ [4238c].

‡ (2569) The formulas used for computing the values [4242—4248] are as follows. [4242a]

MERCURY.

$$\frac{d\varpi}{dt} = 5^s,627032 + 3^s,014032\,.\,\mu' + 0^s,929932\,.\,\mu'' + 0^s,041845\,.\,\mu'''$$
$$+ 1^s,560043\,.\,\mu^{\rm iv} + 0^s,079478\,.\,\mu^{\rm v} + 0^s,001702\,.\,\mu^{\rm vi}.$$

$$2\,.\frac{de}{dt} = 0^s,013690 + 0^s,021948\,.\,\mu' + 0^s,006511\,.\,\mu'' - 0^s,002330\,.\,\mu'''$$
$$- 0^s,012560\,.\,\mu^{\rm iv} + 0^s,000116\,.\,\mu^{\rm v} + 0^s,000004\,.\,\mu^{\rm vi}.$$

Mercury.

$$\frac{d\varphi}{dt} = - 0^s,119993 - 0^s,087951\,.\,\mu' - 0^s,000052\,.\,\mu''' - 0^s,028764\,.\,\mu^{\rm iv}$$
$$- 0^s,003215\,.\,\mu^{\rm v} - 0^s,000011\,.\,\mu^{\rm vi}.$$

[4242]

$$\frac{d\varphi_{\prime}}{dt} = + 0^s,177408 + 0^s,068409\,.\,\mu' + 0^s,000508\,.\,\mu''' + 0^s,098085\,.\,\mu^{\rm iv}$$
$$+ 0^s,010373\,.\,\mu^{\rm v} + 0^s,000033\,.\,\mu^{\rm vi}.$$

$$\frac{d\theta}{dt} = - 4^s,224994 - 1^s,764590\,.\,\mu' - 0^s,963817\,.\,\mu'' - 0^s,029951\,.\,\mu'''$$
$$- 1^s,396112\,.\,\mu^{\rm iv} - 0^s,068989\,.\,\mu^{\rm v} - 0^s,001535\,.\,\mu^{\rm vi}.$$

$$\frac{d\theta_{\prime}}{dt} = - 7^s,566802 - 0^s,097574\,.\,\mu - 4^s,054426\,.\,\mu' - 0^s,963817\,.\,\mu''$$
$$- 0^s,143774\,.\,\mu''' - 2^s,187093\,.\,\mu^{\rm iv} - 0^s,117889\,.\,\mu^{\rm v}$$
$$- 0^s,002228\,.\,\mu^{\rm vi}.$$

The values of $\frac{d\varpi}{dt}$, $\frac{d\varpi'}{dt}$, &c. are given in [1126]; $2\,.\frac{de}{dt}$, $2\,.\frac{de'}{dt}$, &c. are derived from

[4242*b*] [1122]; $\frac{d\varpi}{dt}$, $\frac{d\varpi'}{dt}$, &c. and $\frac{d\theta}{dt}$, $\frac{d\theta'}{dt}$, &c. from [1142, 1143]. Lastly $\frac{d\varphi_{\prime}}{dt}$, $\frac{d\varphi_{\prime}'}{dt}$, &c. and $\frac{d\theta_{\prime}}{dt}$, $\frac{d\theta_{\prime}'}{dt}$, &c. are obtained from [1146]. If we put i for the number of accents *over* φ, ϖ, &c. so that $\varphi^{(i)}$, $\varpi^{(i)}$, &c. represent the values of φ, ϖ, &c. corresponding to the planet which is numbered i, according to the notation adopted in [4230]; and suppose the sign Σ *of finite integrals to include all the values of* k, *contained in the series* of numbers, 0, 1, 2, 3,

General expressions of the annual variations of the elements of the orbits.

[4242*c*] 4, 5, 6 [4230], excepting $i = k$; then the four first of the preceding equations, may be put under the following forms, as is evident by mere inspection,

[4242*d*] $$\frac{d\varpi^{(i)}}{dt} = \Sigma\,.\left\{(i,k) - \left[\overline{i,k}\right].\frac{e^{(k)}}{e^{(i)}}.\cos.(\varpi^{(i)} - \varpi^{(k)})\right\}; \qquad [1126]$$

[4242*e*] $$2.\frac{de^{(i)}}{dt} = -2\,\Sigma\,.\left[\overline{i,k}\right].e^{(k)}.\sin.(\varpi^{(i)} - \varpi^{(k)}); \qquad [1122]$$

VENUS.

$$\frac{d\varpi'}{dt} = -2^s{,}343127 - 4^s{,}315177\,.\,\mu - 5^s{,}754638\,.\,\mu'' + 1^s{,}203777\,.\,\mu'''$$
$$+ 6^s{,}435827\,.\,\mu^{\text{iv}} + 0^s{,}083814\,.\,\mu^{\text{v}} + 0^s{,}003269\,.\,\mu^{\text{vi}}.$$

$$2\,.\frac{de'}{dt} = -0^s{,}260567 - 0^s{,}090479\,.\,\mu - 0^s{,}101170\,.\,\mu'' - 0^s{,}006378\,.\,\mu'''$$
$$-0^s{,}061143\,.\,\mu^{\text{iv}} - 0^s{,}001409\,.\,\mu^{\text{v}} + 0^s{,}000012\,.\,\mu^{\text{vi}}.$$

$$\frac{d\varphi'}{dt} = -0^s{,}015950 + 0^s{,}025200\,.\,\mu + 0^s{,}002157\,.\,\mu''' - 0^s{,}037854\,.\,\mu^{\text{iv}}$$ Venus.
$$-0^s{,}005455\,.\,\mu^{\text{v}} + 0^s{,}000002\,.\,\mu^{\text{vi}}.$$

$$\frac{d\varphi_{\prime}'}{dt} = 0^s{,}044538 + 0^s{,}019377\,.\,\mu - 0^s{,}004148\,.\,\mu''' + 0^s{,}025810\,.\,\mu^{\text{iv}}$$ [4243]
$$+ 0^s{,}003500\,.\,\mu^{\text{v}} - 0^s{,}000001\,.\,\mu^{\text{vi}}.$$

$$\frac{d\theta'}{dt} = -9^s{,}900996 + 0^s{,}342053\,.\,\mu - 7^s{,}416280\,.\,\mu'' - 0^s{,}076112\,.\,\mu'''$$
$$-2^s{,}661705\,.\,\mu^{\text{iv}} - 0^s{,}085589\,.\,\mu^{\text{v}} - 0^s{,}003363\,.\,\mu^{\text{vi}}.$$

$$\frac{d\theta_{\prime}'}{dt} = -18^s{,}387762 + 0^s{,}165450\,.\,\mu - 5^s{,}426693\,.\,\mu' - 7^s{,}416280\,.\,\mu''$$
$$-0^s{,}286675\,.\,\mu''' - 5^s{,}133067\,.\,\mu^{\text{iv}} - 0^s{,}285519\,.\,\mu^{\text{v}}$$
$$-0^s{,}004978\,.\,\mu^{\text{vi}}.$$

$$\frac{d\varphi^{(i)}}{dt} = \Sigma\,.\,\boxed{i,k}\,.\,\text{tang.}\,\varphi^{(k)}\,.\,\sin.\,(\theta^{(i)} - \theta^{(k)});$$ [4242f]

[1142, 1143]

$$\frac{d\theta^{(i)}}{dt} = -\Sigma\,.(i,k) + \Sigma\,.(i,k)\,.\frac{\text{tang.}\,\varphi^{(k)}}{\text{tang.}\,\varphi^{(i)}}\,.\cos.\,(\theta^{(i)} - \theta^{(k)}).$$ [4242g]

In like manner the expressions [1146] may be reduced to the forms [4242*i*, *k*], *supposing the orbits of all the other planets to be referred to that which is numbered* l [4230]; $\varphi_l^{(i)}$ being the inclination, and $\theta_l^{(i)}$ the longitude of the node of the orbit denoted by i referred to that which is denoted by l; conformably to the notation [1143iv]; the fixed plane being the orbit of l, at the epoch 1750, [4242h]

$$\frac{d\varphi_l^{(i)}}{dt} = \Sigma\,.\{(i,k) - (l,k)\}\,.\,\text{tang.}\,\varphi^{(k)}\,.\sin.\,(\theta^{(i)} - \theta^{(k)});$$ [4242i]

$$\frac{d\theta_l^{(i)}}{dt} = -(l,i) - \Sigma.(i,k) + \Sigma\,.\{(i,k) - (l,k)\}\,.\frac{\text{tang.}\,\varphi^{(k)}}{\text{tang.}\,\varphi^{(i)}}\,.\cos.\,(\theta^{(i)} - \theta^{(k)}).$$ [4242k]

THE EARTH.

The Earth

$$* \frac{d\varpi''}{dt} = 11'',949588 - 0'',414923\,.\,\mu + 3'',813276\,.\,\mu' + 1'',546163\,.\,\mu''' + 6'',804392\,.\,\mu^{iv} + 0'',194066\,.\,\mu^{v} + 0'',006614\,.\,\mu^{vi}.$$

[4244]

$$2\,.\frac{de''}{dt} = -\,0'',187638 - 0'',008057\,.\,\mu + 0'',030435\,.\,\mu' - 0'',049410\,.\,\mu''' - 0'',159738\,.\,\mu^{iv} - 0'',000909\,.\,\mu^{v} + 0'',000040\,.\,\mu^{vi}.$$

Instead of excepting $k=i$ [4242c], we may suppose the sign Σ to include all the numbers
[4242l] 0, 1, 2, 3, 4, 5, 6 [4230]; putting $(i,i)=0$, $\left[\overline{i,i}\right]=0$, in all the formulas [4242d — k]; observing also that the first term of [4242k], namely $-(l,i)$ is that which arises from the
[4242m] value $k=i$, under the sign Σ; because then $\frac{\text{tang.}\,\varphi^{(k)}}{\text{tang.}\,\varphi^{(i)}}=1$; $\cos.(\theta^{(i)}-\theta^{(k)})=1$. We may moreover remark, that as the orbit of the planet l, in 1750, is taken for the fixed plane [4232h], tang. $\varphi^{(l)}$ must be of the order m, and since this is multiplied, in [4242i], by quanti-
[4242n] ties of the same order, the product will be of the order m^2, which is neglected; likewise the term depending on tang. $\varphi^{(i)}$ vanishes, because it is multiplied by sin. $(\theta^{(i)}-\theta^{(i)})=0$. If we now substitute in [4242d — k] the values [4080 — 4083, 4231 — 4237], we shall
[4242o] obtain the expressions [4242 — 4248]. For the sake of illustration, we shall give a few examples of the numerical calculations in the following notes.

* (2570) As an example of the formula [4242d], we shall compute the action of Mercury on the Earth, in which case $i=2$, $k=0$. and the corresponding terms of this formula
[4244a] are $(2,0)-\left[\overline{2,0}\right].\frac{e}{e''}.\cos.(\varpi''-\varpi)$. Substituting the values of (2,0), $\left[\overline{2,0}\right]$, e, e'', ϖ, ϖ'' [4233, 4080, 4081], it becomes,

[4244b]

$$(1+\mu).\left\{0'',097574 - 0'',046285\,.\frac{0,20551320}{0,01681395}.\cos.(98^d\,37^m\,16^s - 73^d\,33^m\,58^s)\right\}$$
$$= (1+\mu).\{0'',097574 - 0'',512497\} = -\,0'',414923 - 0'',414923\,.\,\mu;$$

in which the part depending on μ is the same as in $\frac{d\varpi''}{dt}$ [4244], the other part $-0'',414923$ is included in the constant term 11'',949588, which is the sum of all the coefficients of μ, $\mu^{(1)}$,
[4244c] &c. noticing their signs. This constant quantity represents the value of $\frac{d\varpi''}{dt}$, supposing μ, μ', &c. to vanish, or the numerical values of the masses [4061] to be correct.

MARS.

$$\frac{d\varpi'''}{dt} = 15^s,677160 + 0^s,015944\,.\,\mu + 0^s,511046\,.\,\mu' + 2^s,129320\,.\,\mu''$$

$$+ 12^s,312891\,.\,\mu^{iv} + 0^s,693878\,.\,\mu^{v} + 0^s,014082\,.\,\mu^{vi}.$$

$$2\,.\frac{de'''}{dt} = 0^s,372537 + 0^s,002363\,.\,\mu + 0^s,001566\,.\,\mu' + 0^s,040492\,.\,\mu''$$

$$+ 0^s,314982\,.\,\mu^{iv} + 0^s,013167\,.\,\mu^{v} - 0^s,000032\,.\,\mu^{vi}.$$

$$\frac{d\varphi'''}{dt} = -0^s,293800 + 0^s,000092\,.\,\mu - 0^s,013146\,.\,\mu' - 0^s,254879\,.\,\mu^{iv}$$

$$-0^s,025790\,.\,\mu^{v} - 0^s,000076\,.\,\mu^{vi}.$$ Mars.

$$\frac{d\varphi_{\prime}'''}{dt} = -0^s,012984 - 0^s,000388\,.\,\mu + 0^s,131893\,.\,\mu' - 0^s,131999\,.\,\mu^{iv}$$

$$-0^s,012454\,.\,\mu^{v} - 0^s,000036\,.\,\mu^{vi}.$$ [4245]

$$\frac{d\theta'''}{dt} = -9^s,728234 + 0^s,052224\,.\,\mu + 0^s,314067\,.\,\mu' - 1^s,964546\,.\,\mu''$$

$$-7^s,855103\,.\,\mu^{iv} - 0^s,266532\,.\,\mu^{v} - 0^s,008345\,.\,\mu^{vi}.$$

$$\frac{d\theta_{\prime}'''}{dt} = -22^s,789674 - 0^s,318395\,.\,\mu - 8^s,577599\,.\,\mu' - 1^s,964546\,.\,\mu''$$

$$-0^s,432999\,.\,\mu''' - 11^s,015955\,.\,\mu^{iv} - 0^s,469146\,.\,\mu^{v}$$

$$-0^s,011033\,.\,\mu^{vi}.$$

In like manner the terms of $2\,.\,\frac{de''}{dt}$ [4242e], depending on Mercury, become by using [4244d]
the same values as above,

$$-(1+\mu)\,.\,[\boxed{2,0}]\,.\,2\,e\,.\,\sin.\,(\varpi''-\varpi)$$

$$=-(1+\mu)\,.\,0,046285\times 2\times 0,20551320\,.\,\sin.\,(98^d\ 37^m\ 16^s - 73^d\,33^m\ 58^s)$$ [4244e]

$$=-(1+\mu)\,.\,0^s,008057 = -0^s,008057 - 0^s,008057\,.\,\mu,$$

in which the coefficient of μ is the same as in [4214], and the quantity $-0^s,008057$ forms part of the constant quantity $-0^s,187638$ [4244], as in the case of $\frac{d\varpi''}{dt}$ [4244c]. In like manner we may compute any other values $\frac{d\varpi^{(i)}}{dt}$, $2\,.\frac{de^{(i)}}{dt}$.

JUPITER.

$$\frac{d\varpi^{iv}}{dt} = 6^s{,}599770 + 0^s{,}000186\,.\,\mu + 0^s{,}004330\,.\,\mu' + 0^s{,}009837\,.\,\mu''$$
$$+ 0^s{,}002047\,.\,\mu''' + 6^s{,}457871\,.\,\mu^{v} + 0^s{,}125498\,.\,\mu^{vi}.$$
$$2\,.\frac{de^{iv}}{dt} = 0^s{,}554418 - 0^s{,}000008\,.\,\mu + 0^s{,}000009\,.\,\mu' + 0^s{,}000079\,.\,\mu''$$
$$- 0^s{,}000191\,.\,\mu''' + 0^s{,}553308\,.\,\mu^{v} + 0^s{,}001220\,.\,\mu^{vi}.$$
$$* \quad \frac{d\varphi^{iv}}{dt} = -0^s{,}078140 + 0^s{,}000022\,.\,\mu + 0^s{,}000101\,.\,\mu' + 0^s{,}000112\,.\,\mu'''$$

Jupiter.

$$- 0^s{,}078933\,.\,\mu^{v} + 0^s{,}000557\,.\,\mu^{vi}.$$
$$\frac{d\varphi_{\prime}^{iv}}{dt} = -0^s{,}223178 - 0^s{,}009491\,.\,\mu - 0^s{,}128114\,.\,\mu' - 0^s{,}010645\,.\,\mu'''$$

[4246]

$$- 0^s{,}075444\,.\,\mu^{v} + 0^s{,}000516\,.\,\mu^{vi}.$$
$$\frac{d\theta^{iv}}{dt} = 6^s{,}456281 + 0^s{,}000509\,.\,\mu + 0^s{,}005857\,.\,\mu' - 0^s{,}009862\,.\,\mu''$$
$$- 0^s{,}000461\,.\,\mu''' + 6^s{,}505571\,.\,\mu^{v} - 0^s{,}045332\,.\,\mu^{vi}.$$
$$\frac{d\theta_{\prime}^{iv}}{dt} = -14^s{,}663377 - 0^s{,}316227\,.\,\mu - 12^s{,}828736\,.\,\mu' - 0^s{,}009862\,.\,\mu''$$
$$- 0^s{,}389153\,.\,\mu''' - 6^s{,}947861\,.\,\mu^{iv} + 5^s{,}877561\,.\,\mu^{v}$$
$$- 0^s{,}049100\,.\,\mu^{vi}.$$

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* (2571) As an example of the use of the formula [4242f], we shall compute the

[4246a] part of $\frac{d\varphi^{iv}}{dt}$ depending on the action of Mars. In this case $i=4$, $k=3$, and the corresponding terms of the formula become, by using the values [4080—4083, 4231—4237];

$$(4{,}3)\,.\,\text{tang.}\,\varphi'''.\,\sin.\,(\theta^{iv} - \theta''')$$

[4246b]

$$= (1+\mu''')\,.\,0^s{,}004509 \times \text{tang.}\,1^d\,51^m \times \sin.\,(97^d\,51^m\,22^s - 47^d\,38^m\,38^s)$$
$$= (1+\mu''')\,.\,0^s{,}000112 = 0^s{,}000112 + 0^s{,}000112\,.\,\mu'''$$

of which the part depending on μ''' is the same as in $\frac{d\varphi^{iv}}{dt}$ [4246], and the other term $0^s{,}000112$ forms part of the constant quantity $-0^s{,}078140$ of this formula.

[4246c] In like manner by putting $i=4$, $k=3$, $l=2$ in [4242i], and using the same data,

SATURN.

$$\frac{d\varpi^{v}}{dt}=16^{s},112726+0^{s},000022\,.\,\mu+0^{s},000496\,.\,\mu'+0^{s},001080\,.\,\mu''$$
$$+0^{s},000550\,.\,\mu'''+15^{s},790810\,.\,\mu^{iv}+0^{s},319768\,.\,\mu^{vi}.$$
$$2\,.\frac{de^{v}}{dt}=-1^{s},080409-0^{s},000000\,.\,\mu+0^{s},000000\,.\,\mu'+0^{s},000001\,.\,\mu''$$
$$-0^{s},000016\,.\,\mu'''-1^{s},099919\,.\,\mu^{iv}+0^{s},019524\,.\,\mu^{vi}.$$
$$\frac{d\varphi^{v}}{dt}=0^{s},099740+0^{s},000003\,.\,\mu+0^{s},000018\,.\,\mu'+0^{s},000014\,.\,\mu'''$$ Saturn.
$$+0^{s},096696\,.\,\mu^{iv}+0^{s},003010\,.\,\mu^{vi}.$$
$$\frac{d\varphi_{,}^{v}}{dt}=-0^{s},155290-0^{s},010955\,.\,\mu-0^{s},193918\,.\,\mu'-0^{s},012542\,.\,\mu'''$$ [4247]
$$+0^{s},059175\,.\,\mu^{iv}+0^{s},002950\,.\,\mu^{vi}.$$
$$*\ \frac{d\theta^{v}}{dt}=-9^{s},005292+0^{s},000004\,.\,\mu+0^{s},000042\,.\,\mu'-0^{s},001123\,.\,\mu''$$
$$-0^{s},000323\,.\,\mu'''-8^{s},734249\,.\,\mu^{iv}-0^{s},269642\,.\,\mu^{vi}.$$
$$\frac{d\theta_{,}^{v}}{dt}=-19^{s},041499-0^{s},110961\,.\,\mu-5^{s},883249\,.\,\mu'-0^{s},001123\,.\,\mu''$$
$$-0^{s},141414\,.\,\mu'''-12^{s},292960\,.\,\mu^{iv}-0^{s},340441\,.\,\mu^{v}$$
$$-0^{s},271351\,.\,\mu^{vi}.$$

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we get the part of $\frac{d\varphi_{3}^{iv}}{dt}$, or as it is called $\frac{d\varphi_{,}^{iv}}{dt}$ [4246], depending on Mars, equal to

$$\{(4,3)-(2,3)\}\,.\,\text{tang.}\,\varphi'''\,.\,\text{sin.}(\theta^{iv}-\theta''')$$
$$=(1+\mu''').\{0^{s},004509-0^{s},432999\}\times\text{tang.}\,1^{d}\,51^{m}\times\text{sin.}\,(97^{d}\,54^{m}\,22^{s}-47^{d}\,38^{m}\,38^{s})$$ [4246d]
$$=-(1+\mu''')\,.\,0^{s},010643=-0^{s},010643-0^{s},010643\,.\,\mu''',$$

which agree very nearly with the corresponding terms of $\frac{d\varphi_{,}^{iv}}{dt}$ [4246].

* (2572) Putting $i=5$ in [4242g], we get the expression of $\frac{d\theta^{v}}{dt}$, and the terms corresponding to the action of any one of the planets, is found by using the value of k corresponding to it; thus for Mars $k=3$, and the terms depending on this planet become, by using the data [4080 — 4083, 4231 — 4237], [4247a]

URANUS.

$$\frac{d\varpi^{\text{vi}}}{dt} = 2^s{,}454851 + 0^s{,}000003\,.\,\mu + 0^s{,}000043\,.\,\mu' + 0^s{,}000095\,.\,\mu''$$
$$+ 0^s{,}000048\,.\,\mu''' + 1^s{,}210830\,.\,\mu^{\text{iv}} + 1^s{,}243833\,.\,\mu^{\text{v}}.$$

$$2\,.\,\frac{de^{\text{vi}}}{dt} = -\,0^s{,}108184 - 0^s{,}000000\,.\,\mu - 0^s{,}000000\,.\,\mu' - 0^s{,}000000\,.\,\mu''$$
$$+ 0^s{,}000000\,.\,\mu''' - 0^s{,}011952\,.\,\mu^{\text{iv}} - 0^s{,}096232\,.\,\mu^{\text{v}}.$$

$$\frac{d\varphi^{\text{vi}}}{dt} = -\,0^s{,}048861 + 0^s{,}000000\,.\,\mu + 0^s{,}000000\,.\,\mu' + 0^s{,}000000\,.\,\mu'''$$

Uranus.

$$-\,0^s{,}009036\,.\,\mu^{\text{iv}} - 0^s{,}039826\,.\,\mu^{\text{v}}.$$

$$\frac{d\varphi_{\prime}^{\text{vi}}}{dt} = -\,0^s{,}027460 - 0^s{,}005492\,.\,\mu + 0^s{,}010145\,.\,\mu' - 0^s{,}005907\,.\,\mu'''$$

[4248]

$$+ 0^s{,}059217\,.\,\mu^{\text{iv}} - 0^s{,}030502\,.\,\mu^{\text{v}}.$$

$$\frac{d\theta^{\text{vi}}}{dt} = 2^s{,}700876 + 0^s{,}000017\,.\,\mu + 0^s{,}000146\,.\,\mu' - 0^s{,}000096\,.\,\mu''$$
$$+ 0^s{,}000047\,.\,\mu''' + 0^s{,}496382\,.\,\mu^{\text{iv}} + 2^s{,}204381\,.\,\mu^{\text{v}}.$$

$$\frac{d\theta_{\prime}^{\text{vi}}}{dt} = -\,34^s{,}403396 - 0^s{,}788517\,.\,\mu - 23^s{,}815885\,.\,\mu' - 0^s{,}000096\,.\,\mu''$$
$$-\,0^s{,}938767\,.\,\mu''' - 10^s{,}200902\,.\,\mu^{\text{iv}} + 1^s{,}347866\,.\,\mu^{\text{v}}$$
$$-\,0^s{,}007096\,.\,\mu^{\text{vi}}.$$

$$-(5,3) + (5,3)\,.\,\frac{\text{tang.}\,\varphi'''}{\text{tang.}\,\varphi^{\text{v}}}\,.\,\cos.\,(\theta^{\text{v}} - \theta''')$$
$$= (1+\mu''')\,.\left\{-0^s{,}000479 + 0^s{,}000179\,.\,\frac{\text{tang.}\,1^d\,51^m\,0^s}{\text{tang.}\,2^d\,29^m\,55^s}\,.\,\cos.\,(111^d\,30^m\,23^s - 47^d\,38^m\,38^s)\right\}$$

[4247b]

$$= (1+\mu''')\,.\,\{-0^s{,}000479 + 0^s{,}000156\}$$
$$= -\,0^s{,}000323 - 0^s{,}000323\,.\,\mu''', \quad \text{as in} \quad \frac{d\varpi^{\text{v}}}{dt} \quad [4247].$$

Putting $i=5$, $l=2$, in [4242k], we obtain the expression of $\frac{d\theta_{\prime}^{\text{v}}}{dt}$, in the notation

[4247c] of [4247]. The term of this expression corresponding to Mars, is found by putting $k=3$, and using the above data, by which means it becomes,

The variations of the earth's orbit are not included in the preceding formulas; they may be determined by the equations *

$$\text{tang.}\,\varphi''.\,\sin.\,\theta''=p''; \qquad \text{tang.}\,\varphi''.\,\cos.\,\theta''=q''. \qquad [4249]$$

With respect to the values of p'', q'', we may determine them by the formulas [1132, &c.], and we have, *by taking the ecliptic of* 1750 *for the fixed plane*,† [4249']

$$p''=t.\frac{d\,\mathrm{p}''}{dt}+\frac{t^2}{2}.\frac{dd\,\mathrm{p}''}{dt^2}+\&c.;$$
$$q''=t.\frac{d\,\mathrm{q}''}{dt}+\frac{t^2}{2}.\frac{dd\,\mathrm{q}''}{dt^2}+\&c.; \qquad [4250]$$

in which t *is the number of Julian years elapsed since* 1750, *and* $\frac{d\,\mathrm{p}''}{dt}$, $\frac{d\,\mathrm{q}''}{dt}$, $\frac{dd\,\mathrm{p}''}{dt^2}$, &c. *are taken to correspond to that epoch.* It is only necessary [4250'] to notice the first power of t in these formulas, if t be less than 300. If t do not exceed 1000 or 1200, we may reject the third and higher powers [4250''] of t; and we may do the same even with the most ancient observations,

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$$-(5,3)+\{(5,3)-(2,3)\}.\frac{\text{tang.}\,\varphi'''}{\text{tang.}\,\varphi^{\mathrm{v}}}.\cos.(\theta^{\mathrm{v}}-\theta''')$$

$$=(1+\mu''').\left\{-0^{s},000479+(0^{s},000479-0^{s},432999).\frac{\text{tang.}\,1^{d}\,51^{m}\,0^{s}}{\text{tang.}\,2^{d}\,29^{m}\,55^{s}}.\cos.63^{d}\,51^{m}\,45^{s}\right\} \qquad [4247d]$$

$$=(1+\mu''').\{-0^{s},000179-0^{s},141035\}=-0^{s},141514-0^{s},141514.\mu''',$$

which differs $0^{s},0001$ from that given by the author. We have thus given an example of the numerical calculations of each of the formulas [4212d—k].

* (2573) The formulas [4249] are similar to [1032], accenting p, q, &c. with *two* accents, in order to conform to the case now under consideration. [4249a]

† (2574) Putting successively $u=p''$, $U=\mathrm{p}''$; or $u=q''$, $U=\mathrm{q}''$, in the formula [3850a], we get the following expressions of p'', q'',

$$p''=\mathrm{p}''+t.\frac{d\,\mathrm{p}''}{dt}+\tfrac{1}{2}t^2.\frac{d^2\,\mathrm{p}''}{dt^2}+\&c.; \qquad q''=\mathrm{q}''+t.\frac{d\,\mathrm{q}''}{dt}+\tfrac{1}{2}t^2.\frac{d^2\,\mathrm{q}''}{dt^2}+\&c.; \qquad [4250a]$$

in which the quantities p'', q'', and their differentials, in the second members, correspond to the epoch of 1750. Now at that epoch we have $\varphi''=0$ [4249']; substituting this in [4250b] [4249], we get $\mathrm{p}''=0$, $\mathrm{q}''=0$; hence the formulas [4250a] become as in [4250].

taking into view their imperfections. We obtain from the formulas [4250], the following results.*

Values corresponding to the earth's variable orbit.

[4251]
$$\frac{dp''}{dt} = 0^s{,}076721 + 0^s{,}008420\,.\,\mu + 0^s{,}086316\,.\,\mu' + 0^s{,}009423\,.\,\mu''' - 0^s{,}022021\,.\,\mu^{iv} - 0^s{,}005446\,.\,\mu^{v} + 0^s{,}000029\,.\,\mu^{vi}.$$
$$\frac{dq''}{dt} = -0^s{,}500955 - 0^s{,}008522\,.\,\mu - 0^s{,}309951\,.\,\mu' - 0^s{,}010335\,.\,\mu''' - 0^s{,}158234\,.\,\mu^{iv} - 0^s{,}013821\,.\,\mu^{v} - 0^s{,}000091\,.\,\mu^{vi}.$$

Motion of the perihelion depending on the ellipticity of the sun.

26. We have seen, in [4037], that the oblateness of the sun produces, in the perihelia of the planetary orbits, a small motion, which is represented by,

[4252]
$$\delta\varpi = (\rho - \tfrac{1}{2}q)\,.\,\frac{D^2}{a^2}\,.\,nt.$$

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* (2575) If we substitute the values p'', q'' [4250], in the terms of $\frac{dp''}{dt}$, $\frac{dq''}{dt}$ [1132], depending upon p'', or q'', they produce terms of the order $\{(2,0)+(2,1)+\&c.\}\,.\,\frac{dp''}{dt}$;

[4251a] or of the order m in comparison with $\frac{dp''}{dt}$, $\frac{dq''}{dt}$, which occur in the first members of these equations; therefore these terms may be neglected, and then the values of $\frac{dp''}{dt}$, $\frac{dq''}{dt}$ [1132], become,

[4251b]
$$\frac{dp''}{dt} = (2,0)\,.\,q + (2,1)\,.\,q' + (2,3)\,.\,q''' + \&c.;$$
$$\frac{dq''}{dt} = -(2,0)\,.\,p - (2,1)\,.\,p' - (2,3)\,.\,p''' - \&c.$$

[4251c] Substituting $p = \text{tang.}\,\varphi\,.\,\sin.\theta$, $p' = \text{tang.}\,\varphi'\,.\,\sin.\theta'$, &c.; $q = \text{tang.}\,\varphi\,.\cos.\theta$, &c. we get

[4251d]
$$\frac{dp''}{dt} = (2,0)\,.\,\text{tang.}\,\varphi\,.\cos.\theta + (2,1)\,.\,\text{tang.}\,\varphi'\,.\cos.\theta' + (2,3)\,.\,\text{tang.}\,\varphi'''\,.\cos.\theta''' + \&c.;$$

[4251e]
$$\frac{dq''}{dt} = -(2,0)\,.\,\text{tang.}\,\varphi\,.\sin.\theta - (2,1)\,.\,\text{tang.}\,\varphi'\,.\sin.\theta' - (2,3)\,.\,\text{tang.}\,\varphi'''\,.\sin.\theta''' - \&c.;$$

and by using the values [4082, 4083, 4233], they become as in [4251] nearly. Thus the term of $\frac{dp''}{dt}$, depending on Mars, is

[4251f]
$$(2,3)\,.\,\text{tang.}\,\varphi'''\,.\cos.\theta''' = (1+\mu''')\,.\,0^s{,}432999 \times \text{tang.}\,1^d\,51^m \times \cos.\,47^d\,38^m\,38^s$$
$$= (1+\mu''')\,.\,0^s{,}009423,$$

We shall consider the motion relatively to Mercury. Now q *is the ratio of the centrifugal force to gravity at the solar equator* [4028]; and if mt be the sun's angular rotary motion, the centrifugal force at the solar equator will be $m^2 D$.* Putting the mass of the sun equal to S, we have† $\frac{S}{a''^3} = n''^2$, or $S = n''^2 . a''^3$, which gives the gravity at the solar equator, [4253] [4253'] [4254]

$$\frac{S}{D^2} = \frac{n''^2 . a''^3}{D^2}; \qquad [4255]$$

therefore we have ‡

$$q = \frac{m^2}{n''^2} \cdot \frac{D^3}{a''^3} = \left(\frac{m}{n''}\right)^2 \cdot \left(\frac{D}{a''}\right)^3 . \qquad [4256]$$

The time of the sun's revolution about its axis, according to observations, is nearly equal to $25^{\text{days}},417$. The duration of the earth's sideral revolution is $365^{\text{days}},256$; hence we obtain, [4257]

$$\frac{m}{n''} = \frac{365,256}{25,417} . \qquad [4258]$$

The apparent semidiameter of the sun, at its mean distance, is $961^s,632$; which gives [4259]

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in which the coefficient of μ''' is the same as in the value of $\frac{dp''}{dt}$ [4251]. In like manner we find the other terms of $\frac{dp''}{dt}$, $\frac{dq''}{dt}$ [4251].

* (2576) The angular rotary velocity being m, and the equatorial radius D; the actual velocity of a point of the surface of the equator will be represented by mD. The square of this, divided by the radius D, gives the centrifugal force [54'], equal to $m^2 D$, as in [4253]. [4253a]

† (2577) We have $n^2 = \frac{\mu}{a^3} = \frac{M+m}{a^3}$ [3700, 3709a]; and in like manner $n''^2 = \frac{M+m''}{a''^3}$. [4254a]
Now changing M into S to conform to the notation [4254], neglecting also m'' in comparison with S, we obtain $\frac{S}{a''^3} = n''^2$ [4254]; multiplying by $\frac{a''^3}{D^2}$ we get [4255]. [4254b]

‡ (2578) The centrifugal force $m^2 D$ [4253'], divided by the gravity $\frac{n''^2 . a''^3}{D^2}$, gives q [4253], as in [4256]; substituting the values [4258, 4260] it becomes [4255a]

$$q = \left(\frac{365,256}{25,417}\right)^2 . (\sin. 961^s,632)^3 = 0,000020926, \text{ as in } [4261].$$

[4260] $$\frac{D}{a''} = \sin. 961^s{,}632\,;$$

therefore we have

[4261] $$q = 0{,}0000209268.$$

[4262] If the sun be homogeneous, we have $\rho = \frac{5}{4} q$ [1590′, 1592′], in which case the motion of Mercury's perihelion [4252], produced by the ellipticity of the sun, is*

[4263] $$\delta\varpi = (\rho - \tfrac{1}{2} q) \cdot \frac{D^2}{a^2} \cdot nt = \tfrac{3}{4} q \cdot \frac{D^2}{a^2} \cdot nt,$$

or the equivalent expression,

[4264] $$\delta\varpi = \tfrac{3}{4} q \cdot (\sin. 961^s{,}632)^2 \cdot \left(\frac{a''}{a}\right)^2 \cdot nt.$$

If we substitute in this formula the values of n, a, a'' [4077, 4079], it

[4265] becomes $\delta\varpi = 0^s{,}012250 \,.\, t$; so that it increases $\frac{d\varpi}{dt}$ [4242] by the quantity $0^s{,}012250$, which is nearly insensible. This must be still farther decreased if the sun be formed of strata whose densities increase from the surface to the centre, as there is reason to believe is the case.† Hence we may neglect this

[4266] expression for Mercury, and much more so for the other planets. The variations of the nodes and inclinations of the orbits, depending on the same cause, may also be rejected on account of their smallness [4045′]

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* (2579) The density of the sun being supposed uniform, we have $\lambda^2 = \frac{5}{2} q$ nearly

[4262a] [1590′]. Moreover by [1592′] the polar semiaxis being 1, the equatorial semiaxis is $\sqrt{(1+\lambda^2)} = 1 + \frac{1}{2}\lambda^2 = 1 + \frac{5}{4} q$ nearly; so that the ellipticity ρ is nearly equal to $\frac{5}{4} q$, as in [4262]; substituting this in [4252] we get [4263]. Now we have

[4262b] $$\frac{D^2}{a^2} = \frac{D^2}{a''^2} \cdot \frac{a''^2}{a^2} = (\sin. 961^s{,}632)^2 \cdot \left(\frac{a''}{a}\right)^2 \quad [4260]\,;$$

hence [4263] becomes as in [4264]; and by using the values of q, a, a'', n [4261, 4079, 4077], it becomes as in [4265], namely,

[4262c] $$\delta\varpi = \tfrac{3}{4} \times (0{,}0000209268) \times (\sin. 961^s{,}632)^2 \times (0{,}38709812)^{-2} \times 5381016^s . t = 0^s{,}01250 . t.$$

† (2580) The effect of increasing the density towards the centre is seen, in the extreme

[4266a] case, when the whole mass is collected in the centre, and $\rho = \frac{1}{2}\alpha\varphi$ [1732‴]; or in the present notation $\rho = \frac{1}{2} q$ [1726′, 4253]. Substituting this in [4252], we get $\delta\varpi = 0$; so that in this case the ellipticity has no effect on the motion of the perihelion; hence it

[4266b] appears that this increase of density, towards the centre, decreases the motion of the perihelion. We have supposed, in this example, that D remains unaltered, the density being considered as infinitely rare, from the surface towards the centre.

CHAPTER VIII.

THEORY OF MERCURY.

27. The inequalities of the planets which are independent of the excentricities, and those which depend on the first power of the excentricities, were computed by means of the formulas [1020, 1021, 1030], having previously ascertained the values of $A^{(0)}$, $A^{(1)}$ &c. and their differences, by the formulas [963^{iv} — 1008]. The results of these calculations are contained in this, and in the following chapters, neglecting the perturbations of the radius vector, whose effect on the geocentric longitude of the planet is less than one centesimal second. To determine * [4267]

Terms which may be neglected on account of their smallness.

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* (2581) Let S be the sun, E the earth, M Mercury, supposing it to move in the plane of the ecliptic; S ♈ the line drawn from the sun towards the first point of Aries in the heavens, being the line from which the longitude v, v'' are counted. Then $SE=r''$ $SM=r$, $ES♈=v''$, $MS♈=v$, $ESM=v-v''$. [4268a]

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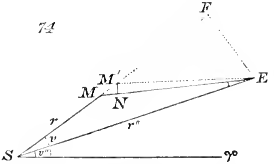

Hence the longitude of the sun, as it appears from the earth, is $180^d+v''$; and if from this we subtract the angle of elongation $SEM=E$, we shall obtain the geocentric longitude of Mercury $V=180^d+v''-E$. Now if $SM=r$ be increased by the quantity $MM'=\delta r$, the angle E will increase by the quantity $MEM'=\delta E$, while v, v'' remain unaltered; therefore the variation of the preceding value of V will be $\delta V=-\delta E$. If we draw $M'N$, EF, perpendicular to EM, SM respectively, we shall have in the similar triangles MNM', MFE; $ME:EF::MM':M'N$; hence $M'N=\delta r.\frac{EF}{ME}$. Dividing this by $M'E$, or ME, we obtain very nearly the angle $MEM'=\delta E=-\delta V=\delta r.\frac{EF}{ME^2}$; substituting $EF=SE.\sin.ESM$ $=r''.\sin.(v-v'')$, and $ME^2=r''^2-2r''r.\cos.(v-v'')+r^2=r''^2.\{1-2\alpha.\cos.(v-v'')+\alpha^2\}$ [62 *Int.* 4268], we get [4269]. [4268b] [4268c] [4268d] [4268e]

the limit, which an inequality in the radius vector must attain, to produce one second in the geocentric longitude of Mercury, we shall observe that if
[4268] we put this longitude equal to V, and $r = r''\alpha$, we shall have for the variation δV corresponding to δr,

[4269]
$$\delta V = -\frac{\delta r}{r''} \cdot \frac{\sin(v-v'')}{1-2\alpha.\cos.(v-v'')+\alpha^2}.$$

The maximum of the function

[4270]
$$\frac{\sin.(v-v'')}{1-2\alpha.\cos.(v-v'')+\alpha^2}$$

corresponds to*

[4271]
$$\cos.(v-v'') = \frac{2\alpha}{1+\alpha^2};$$

[4271′] which gives $\frac{1}{1-\alpha^2}$ [4270e] for this maximum; therefore we shall then have,†

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* (2582) The maximum of [4270] is found, by taking the differential, supposing v to
[4270a] be the variable quantity; putting it equal to zero, and dividing by $d.(v-v'')$. This differential expression being multiplied by $\{1-2\alpha.\cos.(v-v'')+\alpha^2\}^2$ becomes, without
[4270b] reduction, as in the first of the following expressions, and this is easily reduced to the last form [4270d];

[4270c]
$$0 = \cos.(v-v'').\{1-2\alpha.\cos.(v-v'')+\alpha^2\} - 2\alpha.\sin.^2(v-v'')$$
$$= (1+\alpha^2).\cos.(v-v'') - 2\alpha.\{\cos.^2(v-v'')+\sin.^2(v-v'')\}$$

[4270d]
$$= (1+\alpha^2).\cos.(v-v'') - 2\alpha.$$

From this we easily obtain [4271]; thence

[4270e]
$$\sin.(v-v'') = \left(1-\frac{4\alpha^2}{(1+\alpha^2)^2}\right)^{\frac{1}{2}} = \frac{1-\alpha^2}{1+\alpha^2}.$$
$$1-2\alpha.\cos.(v-v'')+\alpha^2 = 1-\frac{4\alpha^2}{1+\alpha^2}+\alpha^2 = \frac{(1-\alpha^2)^2}{1+\alpha^2}.$$

Dividing the first of these expressions by the second, we get the value of the maximum of the function [4270], as in [4271′].

[4271a] † (2583) Substituting in [4269] the value of the function [4270], at its maximum [4271′], we find $\delta V = -\frac{\delta r}{r''} \cdot \frac{1}{1-\alpha^2}$; hence we get δr [4272].

$$\delta r = -r''.(1-\alpha^2).\delta V.$$ [4272]

If we suppose $\delta V = \pm 1'' = \pm 0^s,324$, and take for r, r'', the mean distances of Mercury and the earth from the sun [4079], we shall have by what precedes $r''=1$; $\alpha = 0,38709812$ [4095]; hence we obtain* [4273]

$$\delta r = \mp 0,000001335;$$ [4274]

therefore we may neglect all the inequalities of the radius vector of Mercury, in which the coefficient is less than $\pm 0,000001$. Among the inequalities of the motion in longitude, we shall retain generally only those whose coefficients exceed a quarter of a centesimal second [$0^s,081$]; but as the inequalities depending on the simple angular distances of the planets can be introduced into the same table with those of greater magnitude, they are retained. [4275]

Inequalities of the radius vector which may be neglected

Inequalities of Mercury, independent of the excentricities.

$$\delta v = (1+\mu').\left\{\begin{array}{l} \quad 0^s,662353.\sin.\,(n't-nt+\varepsilon'-\varepsilon) \\ -1^s,457111.\sin.2(n't-nt+\varepsilon'-\varepsilon) \\ -0^s,128075.\sin.3(n't-nt+\varepsilon'-\varepsilon) \\ -0^s,029264.\sin.4(n't-nt+\varepsilon'-\varepsilon) \\ -0^s,008905.\sin.5(n't-nt+\varepsilon'-\varepsilon) \end{array}\right\}$$

[4276]

$$+(1+\mu'').\left\{\begin{array}{l} \quad 0^s,201688.\sin.\,(n''t-nt+\varepsilon''-\varepsilon) \\ -0^s,165645.\sin.2(n''t-nt+\varepsilon''-\varepsilon) \\ -0^s,016901.\sin.3(n''t-nt+\varepsilon''-\varepsilon) \\ -0^s,003127.\sin.4(n''t-nt+\varepsilon''-\varepsilon) \end{array}\right\}$$

Inequalities independent of the excentricities.

$$+(1+\mu^{iv}).\left\{\begin{array}{l} \quad 0^s,569336.\sin.\,(n^{iv}t-nt+\varepsilon^{iv}-\varepsilon) \\ -0^s,118384.\sin.2(n^{iv}t-nt+\varepsilon^{iv}-\varepsilon) \\ -0^s,003118.\sin.3(n^{iv}t-nt+\varepsilon^{iv}-\varepsilon) \end{array}\right\}$$

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* (2584) Using the mean values $r=a$, $r''=a''$ [4079], we get α [4095], substituting these and $\delta V = \pm 1''$, or $\delta V = \pm \sin. 1'' = \pm 0,324.\sin. 1^s$, we obtain [4274] [4274a]

[4277] $$\delta r = -(1+\mu') . \left\{ \begin{array}{l} 0,0000000376^* \\ -0,0000004094 . \cos. \, (n't - nt + \varepsilon' - \varepsilon) \\ +0,0000015545 . \cos.2(n't - nt + \varepsilon' - \varepsilon) \\ +0,0000001702 . \cos.3(n't - nt + \varepsilon' - \varepsilon) \\ +0,0000000437 . \cos.4(n't - nt + \varepsilon' - \varepsilon) \end{array} \right\}$$

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* (2585) The parts of δr, δv [1023, 1024] independent of the excentricities are, by using T [3702a],

[4277a] $$\delta r = \frac{m'}{6} . a^3 . \left(\frac{d A^{(0)}}{da}\right) + \frac{m' n^2}{2} . \Sigma . \left\{ \frac{a^3 . \left(\frac{d A^{(i)}}{da}\right) + \frac{2n}{n-n'} . a^2 A^{(i)}}{i^2 (n-n')^2 - n^2} \right\} . \cos. i\, T;$$

[4277b] $$\delta v = \frac{m'}{2} . \Sigma . \left\{ \frac{n^2}{i(n-n')^2} . a A^{(i)} + \frac{2n^3 . \left\{ a^2 . \left(\frac{d A^{(i)}}{da}\right) + \frac{2n}{n-n'} . a A^{(i)} \right\}}{i(n-n') . \{ i^2 (n-n')^2 - n^2 \}} \right\} . \sin. i\, T;$$

[4277c] in which m, a, n, ε correspond to the *disturbed* planet, and m', a' n', ε', to the *disturbing* planet. These expressions must be accented so as to conform to the notation
[4277d] [4061, 4077 — 4083], taking for i all integral numbers from $i = -\infty$ to $i = \infty$. For example, if we wish to calculate the action of Mars on the earth, we must, in the formulas [4277a, b], change m, a, n, ε into m'', a'', n'', ε'', &c. corresponding to the
[4277e] disturbed planet; and m', a', n', ε', &c. into m''', a''', n''', ε''', &c. respectively, for the disturbing planet.

[4277f] As an example of the use of these formulas we shall apply them to the computation of the perturbations of Mercury by the action of Venus. The constant part of δr deduced from the first term of [4277a] is as in the first expression [4277h]. This is successively reduced, by the substitution of the values

$$\left(\frac{d A^{(0)}}{da}\right) = -\frac{1}{a'^2} . \frac{d b_{\frac{1}{2}}^{(0)}}{d\alpha} \quad [999], \qquad a = 0,38709812 \quad [4079],$$

[4277g] $$\frac{a}{a'} = \alpha = 0,53516076 \quad [4085], \qquad \frac{d b_{\frac{1}{2}}^{(0)}}{d\alpha} = 0,780206 \; [4088], \quad m' = \frac{1+\mu'}{383130} \; [4061];$$

[4277h] $$\begin{aligned} \delta r &= \frac{m'}{6} . a^3 . \left(\frac{d A^{(0)}}{da}\right) = -\frac{m'}{6} . a . \frac{a^2}{a'^2} . \frac{d b_{\frac{1}{2}}^{(0)}}{d\alpha} \\ &= -\frac{m'}{6} . a \alpha^2 . \frac{d b_{\frac{1}{2}}^{(0)}}{d\alpha} = -(1+\mu') . 0,0000000376, \text{ as in } [4277]. \end{aligned}$$

Again by putting successively $i=1$, $i=-1$, $A^{(1)} = A^{(-1)}$ [954''], in [4277a], and connecting the two terms, we obtain the part of δr depending on $\cos. T$, namely,

*Inequalities depending on the first power of the excentricities.**

$$\delta v = (1+\mu')\cdot\left\{\begin{array}{l} 0^s{,}295201\,.\sin.(n't+\varepsilon'-\varpi) \\ -4^s{,}030852\,.\sin.(2n't-\ \ nt+2\varepsilon'-\ \ \varepsilon-\varpi) \\ -1^s{,}686174\,.\sin.(3n't-2nt+3\varepsilon'-2\varepsilon-\varpi) \\ +0^s{,}093989\,.\sin.(3n't-2nt+3\varepsilon'-2\varepsilon-\varpi') \\ +0^s{,}293992\,.\sin.(4n't-3nt+4\varepsilon'-3\varepsilon-\varpi) \\ -0^s{,}176820\,.\sin.(2nt-\ \ n't+2\varepsilon-\ \ \varepsilon'-\varpi) \\ +0^s{,}394486\,.\sin.(3nt-2n't+3\varepsilon-2\varepsilon'-\varpi) \end{array}\right\}$$ [4276]

$$\delta r = m'n^2.a.\left\{\frac{a^2.\left(\frac{d.A^{(1)}}{da}\right)+\frac{2n}{n-n'}.a.A^{(1)}}{(n-n')^2-n^2}\right\}.\cos.T;$$ [4277i]

in which we must substitute $a A^{(1)}=\alpha^2-\alpha\,.\,b_{\frac{1}{2}}^{(1)}$, $a^2.\left(\frac{d.A^{(1)}}{da}\right)=\alpha^2-\alpha^2.\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}$ [4277k]
[997,1000,963^{iv}], and use the values [4277c] corresponding to the disturbing and disturbed planets. Thus in computing the action of Venus upon Mercury, we must use the values a, α, m' [4277g], $n=5381016^s{,}786$, $n'=2106641^s{,}520$ [4077], $b_{\frac{1}{2}}^{(1)}$, [4087], $\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}$ [4088], and we shall get $\delta r=0{,}0000004094\,.\cos.T$, as in the second [4277l]
line of [4277]. The terms depending on $\cos.2T$, $\cos.3T$, $\cos.4T$, &c. are found from [4277a], by using successively, $i=\mp 2$, $i=\mp 3$, $i=\mp 4$, &c. [4277m]

In like manner, the part of δv [4277b], depending on $\sin.T$, is found by using $i=\pm 1$; hence we have

$$\delta v = m'.\left\{\frac{n^2}{(n-n')^2}.aA^{(1)}+\frac{2n^3\left\{a^2.\left(\frac{d.A^{(1)}}{da}\right)+\frac{2n}{n-n'}.a.A^{(1)}\right\}}{(n-n').\{(n-n')^2-n^2\}}\right\}.\sin.T.$$ [4277n]

Substituting the values of the elements given in [4277g, l], it becomes $0^s{,}6623.\sin.T$, as in the first line of [4276]; the other terms depending on $\sin.2T$, $\sin.3T$, &c. are found [4277o]
in like manner, from [4277b], by using successively $i=\pm 2$, $i=\pm 3$, &c. The similar terms, corresponding to the other planets, are computed by means of the same formulas [4277a, b], altering the accents as in [4277c]. The results of these calculations are given in [4289, 4290; 4305, 4306; 4373, 4374; 4388, 4389; 4463, 4464; 4523, 4524]. [4277p]

* (2586) The terms depending on the first power of the excentricities are those parts of δr, δv, [1020, 1021], containing e and e'. The calculation of these terms is made as [4278a]
in the preceding note; using for e the excentricity [4080], corresponding to the disturbed

$$+(1+\mu'').\left\{\begin{array}{l} 0'',095418.\sin.(n''t+\varepsilon''-\varpi) \\ -0'',461708.\sin.(2n''t-\quad nt+2\varepsilon''-\quad\varepsilon-\varpi) \\ +0\ ,244148.\sin.(3n''t-2nt+3\varepsilon''-2\varepsilon-\varpi) \end{array}\right\}$$

Inequalities depending on the first power of the excentricities.

$$+(1+\mu\ ).\left\{\begin{array}{l} 0'',236346.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi) \\ -0'',572172.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv}) \\ -3\ ,278687.\sin.(2n^{iv}t-nt+2\varepsilon^{iv}-\varepsilon-\varpi) \end{array}\right\}$$

$$-(1+\mu^{v}).\left\{\begin{array}{l} 0'',084167\ .\sin.(n^{v}t+\varepsilon^{v}-\varpi^{v}) \\ +0'',395493\ .\sin.(2n^{v}t-nt+2\varepsilon^{v}-\varepsilon-\varpi) \end{array}\right\}.$$

[4279]
$$\begin{array}{rl}\delta r= & -(1+\mu').0,0000013482.\cos.(3n't-2nt+3\varepsilon'-2\varepsilon-\varpi) \\ & -(1+\mu^{iv}).0,0000029625.\cos.(2n^{iv}t-\quad nt+2\varepsilon^{iv}-\quad\varepsilon-\varpi).\end{array}$$

Inequalities depending on the squares and products of the excentricities and inclinations of the orbits.

These inequalities have been calculated by the formulas of [3711—3755]. Now twice the motion of Mercury differs but very little from five times that of Venus;* so that $5(n'-n)+2n$ is very nearly equal to $-n$; we must
[4280] therefore, as in [3732], notice the inequality depending on $3nt-5n't$. The angle $3n't-nt$ varies quite slowly, therefore it is necessary to notice the inequality depending on it [3733]. Moreover the motion of Mercury is
[4281] very nearly equal to four times that of the earth, so that $4.(n''-n)+2n$ differs but little from $-n$; therefore, we must, as in [3732], notice the inequality depending on $2nt-4n''t$. Hence we obtain,

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planet; and for e' the value [4080] corresponding to the disturbing planet; these symbols being accented so as to conform to these two bodies.

[4282a] * (2587) Using the values [4076h] we have very nearly $2n-5n'=72^{\circ}=\frac{n}{23}$, $3n'-n=289^{\circ}=\frac{n}{6}$, and $n-4n''=61^{\circ}=\frac{n}{27}$; so that these three quantities are small in comparison with n, as is observed above. Hence $5(n'-n)+2n$ is very nearly
[4282b] equal to $-n$, and must be noticed as in [3732]; also $3(n'-n)+2n$ is very small, and must be noticed as in [3733]; lastly $4(n''-n)+2n$ is very nearly equal to $-n$, and must be noticed as in [3732]. The terms of R [3745—3745''] depending on these angles

$$\delta v = -(1+\mu').\left\{\begin{array}{l} 1'',690443 \,.\, \sin.(3\,n\,t - 5\,n'\,t + 3\,\varepsilon - 5\,\varepsilon' - 43^{d}18^{m}32^{s}) \\ +\, 0'',597664 \,.\, \sin.(3\,n'\,t - \quad n\,t + 3\,\varepsilon' - \quad \varepsilon + 40^{d}36^{m}35^{s}) \end{array}\right\}$$ Inequalities of the second order.

$$-(1+\mu'') \,.\, 0'',263474 \,.\, \sin.(2\,n\,t - 4\,n''\,t + 2\,\varepsilon - 4\,\varepsilon'' - 41^{d}\,11^{m}\,46^{s})$$ [4282]

$$\delta r = (1+\mu') \,.\, 0{,}0000016056 \,.\, \cos.(3\,n\,t - 5\,n'\,t + 3\,\varepsilon - 5\,\varepsilon' - 42^{d}\,58^{m}\,04^{s}).$$

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are found by putting in the first case $i=5$; in the second $i=3$, and in the third $i=4$. The values of $M^{(0)}$, $M^{(1)}$, $M^{(2)}$, $M^{(3)}$, corresponding to these values of i, are successively [4282c] obtained from [3750, 3755, 3755', 3750''']; and they may be reduced to terms of $b^{(i)}$, $\frac{d\,b^{(i)}}{d\alpha}$, &c. by means of the formulas [996—1001]. These values are to be substituted separately for M in the expressions of $\frac{r\,\delta r}{a^2}$, δv, [3711, 3715], and we shall [4282d] obtain very nearly the terms of $\frac{\delta r}{a}$, δv, having the small divisors $5\,n' - 2\,n$, $3\,n' - n$, $4\,n'' - n$, which are the only ones necessary to be noticed in this place. Now [4282e] if we use, for a moment, the abridged symbol, $T_{,} = i.(n't - n\,t + \varepsilon' - \varepsilon) + 2\,n\,t + 2\,\varepsilon$ [4282f] [3711g], the resulting terms of $\frac{\delta r}{a}$ or δr [3711, &c.] will be of the form [4282h]. Developing this by [24], Int. it becomes as in [4282i]; substituting $A_1 \sin. B_1$ for the coefficient of $\sin. T_{,}$, also $A_1 \cos. B_1$, for the coefficient of $\cos. T_{,}$, it changes into [4282g] [4282k], and is finally reduced to the form [4282l], by means of [24], Int.

$$\delta r = M_{,}^{(0)}.\cos.(T_{,} - 2\varpi) + M_{,}^{(1)}.\cos\,(T_{,} - \varpi - \varpi') + M_{,}^{(2)}.\cos.(T_{,} - 2\varpi') + M_{,}^{(3)}.\cos.(T_{,} - 2\Pi)$$ [4282h]

$$= \{M_{,}^{(0)}.\cos.2\,\varpi + M_{,}^{(1)}.\cos.(\varpi + \varpi') + M_{,}^{(2)}.\cos.2\,\varpi' + M_{,}^{(3)}.\cos.2\,\Pi\}.\cos.T_{,}$$
$$+ \{M_{,}^{(0)}.\sin.2\,\varpi + M_{,}^{(1)}.\sin.(\varpi + \varpi') + M_{,}^{(2)}.\sin.2\,\varpi' + M_{,}^{(3)}.\sin.2\,\Pi\}.\sin.T_{,}$$ [4282i]

$$= A_1.\{\cos.B_1.\cos.T_{,} + \sin.B_1.\sin.T_{,}\}$$ [4282k]

$$= A_1.\cos.(T_{,} - B_1), \text{ as in } [4282].$$ [4282l]

In like manner the several terms of δv may be reduced to the form $A_2.\sin.(T_{,} - B_2)$; there is no other difficulty than the tediousness of the numerical calculation, arising from its [4282m] length.

We may observe that the quantities γ^2, $2\,\Pi$, which occur in [3745'''], are not explicitly included among the data [4077—4083], but must be computed from the formulas [4282n] [1032, 1033].

$$\gamma.\sin.\Pi = \text{tang}.\varphi'.\sin.\theta' - \text{tang}.\varphi.\sin.\theta; \quad \gamma.\cos.\Pi = \text{tang}.\varphi'.\cos.\theta' - \text{tang}.\varphi.\cos.\theta;$$ [4282o]

supposing φ, θ to correspond to the *disturbed* planet, and φ', θ' to the *disturbing* planet; these symbols being accented so as to conform to the notation [4230]; then using the [4282p] values [4082, 4083] we get the required values of γ, Π.

Inequalities depending on the cubes and products of three dimensions of the excentricities and inclinations of the orbits.

The first of these inequalities, depending on the angle $2nt-5n't$, is
[4282'] computed by means of the formula [3844];* the second, depending on the angle $nt-4n''t$, is found by means of [3882];† hence we obtain,

[4283]
$$\delta v = -(1+\mu')\,.\,8^{s},483765\,.\,\sin.(2nt-5n't+2\varepsilon-5\varepsilon'+30^{d}\,13^{m}\,36^{s})$$
$$-(1+\mu'')\,.\,0^{s},690612\,.\,\sin.(nt-4n''t+\varepsilon-4\varepsilon''+19^{d}\,02^{m}\,13^{s}).$$

Inequalities of the third order.

The inequalities of Mercury's motion in latitude, may be calculated by means of the formula [1030]; but as they are insensible, being less than
[4283'] a quarter of a centesimal second, it was thought unnecessary to insert them.

[4283a] * (2588) The first line of [4283] is obtained from the formula [3844], connecting all the terms into one, as in [4282h—l].

† (2589) The second line of [4283] is obtained from [3882], reducing all the terms
[4283b] into one, as in [4282h—l]. We have already seen in [3883h], that the correction, as it is given by the author, in [4283], is rather too great; his method of computation [3882] being
[4283c] merely an approximation. The direct method of computation has already been explained in the previous notes [3876a—3883w]; and it is unnecessary to say more upon the subject
[4283d] in this place. There is a similar equation in the earth's motion [4311, 3883y].

CHAPTER IX.

THEORY OF VENUS.

28. If we put $\frac{r'}{r''} = \alpha$, and V' equal to the geocentric longitude [4284]
of Venus, we shall find that the equation [4272],

$$\delta r = -r''.(1-\alpha^2).\delta V, \qquad [4285]$$

will become, relatively to Venus,

$$\delta r' = -r''.(1-\alpha^2).\delta V'. \qquad [4286]$$

Taking for r', r'', the mean distances of Venus and the earth from the sun [4079], we shall have, as in [4126], $\alpha = 0,72333230$; therefore by [4287]
putting $\delta V' = \pm 1'' = \pm 0^s,324$, we shall obtain,

$$\delta r' = \mp 0,0000007489. \qquad [4288]$$

Terms which may be neglected on account of their smallness.

Therefore we shall neglect those inequalities of the radius vector whose coefficients are less than 0,0000007. We shall also neglect the inequalities of the motion in longitude, which are less than a quarter of a centesimal second, or $0^s,081$.

Inequalities of Venus, independent of the excentricities.

$$\delta v' = (1+\mu'') . \left\{ \begin{array}{l} +\ 5^s,015931 . \sin. (n''t - n't + \varepsilon'' - \varepsilon') \\ +11^s,424392 . \sin.2(n''t - n't + \varepsilon'' - \varepsilon') \\ -\ 7^s,253867 . \sin.3(n''t - n't + \varepsilon'' - \varepsilon') \\ -\ 1^s,056720 . \sin.4(n''t - n't + \varepsilon'' - \varepsilon') \\ -\ 0^s,345898 . \sin.5(n''t - n't + \varepsilon'' - \varepsilon') \\ -\ 0^s,145382 . \sin.6(n''t - n't + \varepsilon'' - \varepsilon') \\ -\ 0^s,069726 . \sin.7(n''t - n't + \varepsilon'' - \varepsilon') \\ -\ 0^s,036207 . \sin.8(n''t - n't + \varepsilon'' - \varepsilon') \end{array} \right\} \qquad [4289]$$

$$+(1+\mu''') \cdot \left\{\begin{array}{l} 0^s{,}079908 \,.\, \sin.\,(n'''t - n't + \varepsilon''' - \varepsilon') \\ -0^s{,}105987 \,.\, \sin.2(n'''t - n't + \varepsilon''' - \varepsilon') \\ -0^s{,}010853 \,.\, \sin.3(n'''t - n't + \varepsilon''' - \varepsilon') \\ -0^s{,}002332 \,.\, \sin.4(n'''t - n't + \varepsilon''' - \varepsilon') \end{array}\right\}^{*}$$

[4289]

$$+(1+\mu^{iv}) \cdot \left\{\begin{array}{l} 2^s{,}891136 \,.\, \sin.\,(n^{iv}t - n't + \varepsilon^{iv} - \varepsilon') \\ -0^s{,}877624 \,.\, \sin.2(n^{iv}t - n't + \varepsilon^{iv} - \varepsilon') \\ -0^s{,}040034 \,.\, \sin.3(n^{iv}t - n't + \varepsilon^{iv} - \varepsilon') \\ -0^s{,}002754 \,.\, \sin.4(n^{iv}t - n't + \varepsilon^{iv} - \varepsilon') \end{array}\right\}$$

$$+(1+\mu^{v}) \cdot \left\{\begin{array}{l} 0^s{,}190473 \,.\, \sin.\,(n^{v}t - n't + \varepsilon^{v} - \varepsilon') \\ -0^s{,}039859 \,.\, \sin.2(n^{v}t - n't + \varepsilon^{v} - \varepsilon') \\ -0^s{,}001306 \,.\, \sin.3(n^{v}t - n't + \varepsilon^{v} - \varepsilon') \end{array}\right\}.$$

Inequalities independent of the excentricities.

[4290]

$$\delta r' = (1+\mu'') \cdot \left\{\begin{array}{l} -0{,}0000003145 \\ +0{,}0000038362 \,.\, \cos.\,(n''t - n't + \varepsilon'' - \varepsilon') \\ +0{,}0000165050 \,.\, \cos.2(n''t - n't + \varepsilon'' - \varepsilon') \\ -0{,}0000140155 \,.\, \cos.3(n''t - n't + \varepsilon'' - \varepsilon') \\ -0{,}0000024255 \,.\, \cos.4(n''t - n't + \varepsilon'' - \varepsilon') \\ -0{,}0000008873 \,.\, \cos.5(n''t - n't + \varepsilon'' - \varepsilon') \\ -0{,}0000004021 \,.\, \cos.6(n''t - n't + \varepsilon'' - \varepsilon') \\ -0{,}0000002033 \,.\, \cos.7(n''t - n't + \varepsilon'' - \varepsilon') \\ -0{,}0000001094 \,.\, \cos.8(n''t - n't + \varepsilon'' - \varepsilon') \end{array}\right\}$$

$$+(1+\mu^{iv}) \cdot \left\{\begin{array}{l} -0{,}0000003106 \\ +0{,}0000048903 \,.\, \cos.\,(n^{iv}t - n't + \varepsilon^{iv} - \varepsilon') \\ -0{,}0000021911 \,.\, \cos.2(n^{iv}t - n't + \varepsilon^{iv} - \varepsilon') \\ -0{,}0000001155 \,.\, \cos.3(n^{iv}t - n't + \varepsilon^{iv} - \varepsilon') \\ -0{,}0000000098 \,.\, \cos.4(n^{iv}t - n't + \varepsilon^{iv} - \varepsilon') \end{array}\right\}.$$

[4289a] * (2590) The values $\delta v'$, $\delta r'$ [4289, 4290], were computed from the formulas [4277a, b], accenting the symbols as in [4277c], so as to conform to the present case.

*Inequalities depending on the first power of the excentricities.**

Inequalities depending on the first power of the excentricities.

$$\delta v' = (1+\mu)\,.\,0^s{,}800933\,.\,\sin.(2n't - nt + 2\varepsilon' - \varepsilon - \varpi)$$

$$+(1+\mu'')\,.\left\{\begin{array}{l}
\;0^s{,}073206\,.\,\sin.\,(n''t + \varepsilon'' - \varpi') \\
-\,0^s{,}127720\,.\,\sin.\,(n''t + \varepsilon'' - \varpi'') \\
+\,0^s{,}163115\,.\,\sin.\,(2n''t - n't + 2\varepsilon'' - \varepsilon' - \varpi') \\
-\,0^s{,}113443\,.\,\sin.\,(2n''t - n't + 2\varepsilon'' - \varepsilon' - \varpi'') \\
-\,1^s{,}549550\,.\,\sin.\,(3n''t - 2n't + 3\varepsilon'' - 2\varepsilon' - \varpi') \\
+\,4^s{,}766332\,.\,\sin.\,(3n''t - 2n't + 3\varepsilon'' - 2\varepsilon' - \varpi'') \\
-\,0^s{,}299478\,.\,\sin.\,(4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \varpi') \\
+\,0^s{,}947648\,.\,\sin.\,(4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \varpi'') \\
-\,0^s{,}691744\,.\,\sin.\,(5n''t - 4n't + 5\varepsilon'' - 4\varepsilon' - \varpi') \\
+\,2^s{,}196527\,.\,\sin.\,(5n''t - 4n't + 5\varepsilon'' - 4\varepsilon' - \varpi'') \\
+\,0^s{,}106435\,.\,\sin.\,(3n't - 2n''t + 3\varepsilon' - 2\varepsilon'' - \varpi')
\end{array}\right\}$$ [4291]

$$-(1+\mu''')\,.\,1^s{,}092755\,.\,\sin.\,(3n'''t - 2n't + 3\varepsilon''' - 2\varepsilon' - \varpi''')$$

$$+(1+\mu^{\text{iv}})\,.\left\{\begin{array}{l}
-\,1^s{,}503893\,.\,\sin.\,(n^{\text{iv}}t + \varepsilon^{\text{iv}} - \varpi^{\text{iv}}) \\
-\,0^s{,}321108\,.\,\sin.\,(2n^{\text{iv}}t - n't + 2\varepsilon^{\text{iv}} - \varepsilon' - \varpi') \\
+\,0^s{,}232430\,.\,\sin.\,(2n^{\text{iv}}t - n't + 2\varepsilon^{\text{iv}} - \varepsilon' - \varpi^{\text{iv}}) \\
-\,0^s{,}163470\,.\,\sin.\,(3n^{\text{iv}}t - 2n't + 3\varepsilon^{\text{iv}} - 2\varepsilon' - \varpi^{\text{iv}})
\end{array}\right\}$$

$$-(1+\mu^{\text{v}})\,.\,0^s{,}218743\,.\,\sin.\,(n^{\text{v}}t + \varepsilon^{\text{v}} - \varpi^{\text{v}})\,;$$

$$\delta r' = (1+\mu)\,.\,0{,}0000008831\,.\,\cos.\,(2n't - nt + 2\varepsilon' - \varepsilon - \varpi)$$

$$+(1+\mu'')\,.\left\{\begin{array}{l}
\;0{,}0000016482\,.\,\cos.\,(3n''t - 2n't + 3\varepsilon'' - 2\varepsilon' - \varpi'') \\
-\,0{,}0000011406\,.\,\cos.\,(5n''t - 4n't + 5\varepsilon'' - 4\varepsilon' - \varpi') \\
+\,0{,}0000036421\,.\,\cos.\,(5n''t - 4n't + 5\varepsilon'' - 4\varepsilon' - \varpi'')
\end{array}\right\}$$ [4292]

$$-(1+\mu''')\,.\,0{,}0000019404\,.\,\cos.\,(3n'''t - 2n't + 3\varepsilon''' - 2\varepsilon' - \varpi''').$$

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* (2591) The terms of $\delta v'$, $\delta r'$ [4291, 4292] are computed from the parts of δv, δr [1021, 1020] depending upon the excentricities e, e'; in the same manner as the calculation is made for Mercury in [4278a]. [4291a]

Inequalities depending on the squares and products of two dimensions of the excentricities and inclinations of the orbits.

$$\delta v' = -(1+\mu) \,.\, 0^s{,}333596 \,.\, \sin.\,(4n't - 2nt + 4\varepsilon' - 2\varepsilon - 39^d\,30^m\,30^s)$$

[4293] $$-(1+\mu'') \,.\, \left\{ \begin{array}{l} 1^s{,}505036 \,.\, \sin.\,(5n''t - 3n't + 5\varepsilon'' - 3\varepsilon' + 20^d\,54^m\,26^s) \\ +0^s{,}039351 \,.\, \sin.\,(4n''t - 2n't + 4\varepsilon'' - 2\varepsilon' + 26^d\,56^m\,32^s) \end{array} \right\}$$

$$+ (1+\mu''') \,.\, 2^s{,}009677 \,.\, \sin.\,(3n'''t - n't + 3\varepsilon''' - \varepsilon' + 65^d\,53^m\,09^s).$$

Inequalities of the second order.

The mean motions of Mercury, Venus, the earth and Mars, bear such proportions to each other that the quantities $2n - 5n'$, $5n'' - 3n'$ and

[4293′] $n' - 3n'''$ are very small in comparison with n';* hence it follows from the remarks made in [3732, &c.], that the preceding inequalities [4293] are the only ones of the order of the square of the excentricities which can become sensible.

Inequalities depending on terms of the third order, relative to the powers and products of the excentricities and inclinations of the orbits.

[4294] $$\delta v' = (1+\mu) \,.\, 1^s{,}184842 \,.\, \sin.\,(2nt - 5n't + 2\varepsilon - 5\varepsilon' + 30^d\,13^m\,36^s).\dagger$$

Inequalities of the third order.

Inequalities of the motion of Venus in latitude.

The formulas of § 51. Book I. give ‡

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* (2592) The values [4076h] give, very nearly, $2n - 5n' = 72^\circ = \frac{n'}{9}$;

[4293a] $5n'' - 3n' = 50^\circ = \frac{n'}{13}$; $n' - 3n''' = 12^\circ = \frac{n'}{54}$; all of which are small. The first of these gives $4n' - 2n$ nearly equal to $-n'$, and corresponds to the first form mentioned in [3732]. The second quantity $5n'' - 3n'$, and the third $n' - 3n'''$, being nearly equal to zero, correspond to the second form [3733]. The

[4293b] terms of $\delta v'$ [4293] corresponding to these quantities are to be computed from [3715], and reduced as in [4282h—l]. The term depending on $4n''t - 2n' = 300^\circ = \frac{1}{2}n'$ nearly, is computed for the same reasons as that in [4310′].

[4294a] † (2593) This is obtained from [3817], reducing the several terms to one, as in [4282h—l].

[4295a] ‡ (2594) If we change, in [1030], n, a, ε, n', a', ε', into n', a', ε', n'', a'', ε''.

Inequalities in the latitude.

$$\delta s' = -(1+\mu'').\left\{\begin{array}{l} 0^s{,}124804.\sin.(n''t+\varepsilon''-\theta') \\ +0^s{,}090932.\sin.(2n''t-\;\;n't+2\varepsilon''-\;\;\varepsilon'-\theta') \\ +0^s{,}073443.\sin.(3n''t-2n't+3\varepsilon''-2\varepsilon'-\theta') \\ +0^s{,}081481.\sin.(4n''t-3n't+4\varepsilon''-3\varepsilon'-\theta') \\ +0^s{,}312535.\sin.(5n''t-4n't+5\varepsilon''-4\varepsilon'-\theta') \\ -0^s{,}078119.\sin.(2n't-\;\;n''t+2\varepsilon'-\;\;\varepsilon''-\theta') \end{array}\right\} \quad [4295]$$

$$-(1+\mu''').0^s{,}148701.\sin.(3n'''t-2n't+3\varepsilon'''-2\varepsilon-\Pi''')$$

$$+(1+\mu^{iv}).0^s{,}161414.\sin.(2n^{iv}t-n't+2\varepsilon^{iv}-\varepsilon'-\Pi^{iv}).$$

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respectively, we shall obtain the value of $\delta s'$ corresponding to Venus disturbed by the earth; and by neglecting the term containing the arc of a circle nt without the signs of sine and cosine, as is done in [1051]; also excluding $i=0$ [1028, &c.] from the sign Σ, we get,

$$\begin{array}{l} \delta s' = -\dfrac{m''n'^2}{n'^2-n''^2}\cdot\dfrac{a'^2}{a''^2}\cdot\gamma\,.\sin.(n''t+\varepsilon''-\Pi) \\ \quad +\dfrac{m''.n'^2.a'^2a''}{2}\,.\,\Sigma\,.\,\dfrac{B^{(i-1)}}{n'^2-\{n'-i(n'-n'')\}^2}.\gamma\,.\sin.\{i(n''t-n't+\varepsilon''-\varepsilon')+n't+\varepsilon'-\Pi\}. \end{array} \quad [4295b]$$

In this formula, γ [1026'] represents the inclination, and Π the longitude of the ascending node of the orbit of the *disturbing* planet, above that of the *disturbed* planet. These quantities for the earth's action upon Venus are, nearly $\gamma=\text{tang}.\varphi'$, and $\Pi=180^d+\theta'$; φ' being the inclination of the orbit of Venus to the fixed orbit of the earth; and θ' the longitude of the ascending node of the orbit of Venus upon that of the earth [4082, 4083]. For Mars they become γ''', Π'''; for Jupiter γ^{iv}, Π^{iv}, &c. In the expression [4295b] we must include all positive and negative integral values of i, except $i=0$ [1028, &c.]. The values of γ, γ', &c. Π, Π', &c. are deduced from those of φ, φ', &c. θ, θ', &c. [4082, 4083]; by means of formulas similar to those in [4282o]. Thus if we wish to find the part of $\delta s'$ depending on the angle $2n''t-n't$, we must put $i=2$, in [4295b], and the term in question becomes,

[4295c]
[4295d]
[4295e]
[4295f]

$$\frac{m''.n'^2.a'^2.a''}{2}\cdot\frac{B^{(1)}.\gamma}{n'^2-(2n''-n')^2}.\sin.(2n''t-n't+2\varepsilon''-\varepsilon'-\Pi). \quad [4295g]$$

Now the factor $n'^2-(2n''-n')^2=4n''.(n'-n'')$; also $B^{(1)}=\frac{1}{a''^3}\cdot b^{(1)}_{\frac{3}{2}}$ [1006]; substituting these and γ, Π [4295c], in [4295g], it becomes,

$$\begin{array}{l} \dfrac{-m''.n'^2.a'^2a''}{2}\cdot\dfrac{b^{(1)}_{\frac{3}{2}}.\text{tang}.\varphi'}{4n''.(n'-n'').a''^3}.\sin.(2n''t-n't+2\varepsilon''-\varepsilon'-\theta') \\ =-\dfrac{m''.n'^2.b^{(1)}_{\frac{3}{2}}}{8n''.(n'-n'')}\cdot\left(\dfrac{a'}{a''}\right)^2.\text{tang}.\varphi'.\sin.(2n''t-n't+2\varepsilon''-\varepsilon'-\theta'). \end{array} \quad [4295h]$$

[4295'] Π''' being here the longitude of the ascending node of the orbit of Mars upon that of Venus,* and Π^{iv} the longitude of the ascending node of the

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If in this we substitute $m''=\frac{1+\mu''}{329630}$ [4061], $n''=1295977^s$, $n'=2106641^s$ [4077],

[4295i] $b_{\frac{3}{2}}^{(1)}=8,871894$ [4132], $\frac{a'}{a''}=0,72333230$ [4126], $\varphi'=3^d\,23^m\,35^s$ [4082]; it is

[4295k] reduced to $-0^s.090932.(1+\mu'').\sin.(2n''t-n't+2\varepsilon''-\varepsilon'-\theta')$, as in [4295]. In the same way we may compute other terms. If we suppose $i=1$, there will be found two corresponding terms in [4295b]; namely,

[4295l] $$\frac{m''.n'^2}{n'^2-n''^2}\cdot\frac{a'^2}{a''^2}\cdot\text{tang.}\varphi'.\{1-\tfrac{1}{2}a''^3.B^{(0)}\}.\sin.(n''t+\varepsilon''-\theta').$$

But by changing a' into a'', in [1006], to conform to this case, we have $a''^3B^{(0)}=b_{\frac{3}{2}}^{(0)}$;

[4295m] hence the preceding expression becomes $\frac{m''.n'^2}{n'^2-n''^2}\cdot\left(\frac{a'}{a''}\right)^2.\text{tang.}\varphi'.\left(1-\frac{1}{2}.b_{\frac{3}{2}}^{(0)}\right)$. If we use the values of m'', n', n'', $\frac{a'}{a''}$ [4295i]; also $b_{\frac{3}{2}}^{(0)}=9,992539$ [4132]; we get $0^s,031231$, for the part independent of $b_{\frac{3}{2}}^{(0)}$; and $-0^s,156035$, for the part

[4295n] depending on $b_{\frac{3}{2}}^{(0)}$; the sum is $-0^s,124804.\sin.(n''t+\varepsilon''-\theta')$; as in the first line of [4295].

* (2595) A small inequality in the mean motion of Venus, depending on terms of the fifth order of the powers and products of the excentricities, has lately been discovered by

[4296a] Mr. Airy, arising from the action of the earth upon that planet. This inequality affects the mean motion, the radius vector, the perihelion, the excentricity, and the latitude; its period

[4296b] is nearly 239 years; being the time required for the argument $8n't-13n''t$ to increase from 0^d to 360^d. This appears from the values of n', n'' [4077]; from which we

[4296c] get $8n'-13n''=5427^s=\frac{n''}{239}$ nearly; and as this quantity is very small, it follows that the mean motions of Venus and the earth must be affected by inequalities, depending

[4296d] upon the argument $8n't-13n''t$; in like manner as the mutual attraction of Jupiter and Saturn produces the great inequalities of these planets in [1196, 1204]; supposing the accents on the letters a, n, &c. to be increased to conform to the present notation, and putting $i'=8$, $i''=13$. The variations in the excentricities and in the motions of the perihelia, similar to those of Jupiter and Saturn [1298 — 1302], are in the present case nearly

[4296e] insensible. The inequalities of the mean motions of Venus and the earth, ζ', ζ'' depending on the argument $8n't-13n''t$, are of the order $13-8=5$ [957^{viii}, &c.], or of the fifth order relative to the powers and products of the excentricities. Now e', e'' are

[4296f] both quite small, so that the largest of them e'' gives $e''^5=\frac{1}{744000000}$ nearly; but this

orbit of Jupiter upon that of Venus.

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very minute fraction is multiplied, in [1197], by $\frac{3\,i''\,n''^2}{(i''n''-i'n')^2}=3\times 13\times(239)^2=2200000$ [4296g]
nearly, in finding the value of ζ''; and by this means the correction is very much increased. The theory and numerical computation of this inequality are given by Mr. Airy, in an elaborate paper on this subject, in the Philosophical Transactions of the Royal Society of London for [4296h]
1832; using the data [4061—4083]; and putting $\mu'=-0{,}045$, $\mu''=0$, so that [4296i]
$m'=\frac{1}{401211}$. He finds the correction ζ' of the mean motion of Venus, to be represented by [4296k]

$$\zeta'=\{2^s{,}946-t\,.\,0^s{,}0002970\}\,.\sin.\{8n't-13n''t+8\varepsilon'-13\varepsilon''+220^d\,44^m\,34^s-t\,.\,10^s{,}76\}. \quad [4296l]$$

He also obtains the following equations, depending on the same cause, and similar to those given in [1298—1302];

$$\delta\varpi'=-5^s{,}70\,.\cos.(8n't-13n''t+8\varepsilon'-13\varepsilon''); \quad [4296m]$$

$$\delta e'=-0{,}000000190\,.\sin.(8n't-13n''t+8\varepsilon'-13\varepsilon''); \quad [4296n]$$

$$\delta s=0^s{,}0151\,.\sin.(9n't-13n''t+9\varepsilon'-13\varepsilon''+140^d\,31^m). \quad [4296o]$$

These corrections of $\delta\varpi'$, δe, δs, may be generally neglected, as insensible; as also that in the radius vector, similar to [1197]. We shall give, in [4310c—f], the corresponding corrections of the earth's motion. The expressions of ζ', ζ'' [4296l, 4310c], are subject [4296p]
to the noted equation [1208], which in the present case becomes

$$m'.\sqrt{a'}.\zeta'+m''\sqrt{a''}.\zeta''=0. \quad [4296q]$$

CHAPTER X.

THEORY OF THE EARTH'S MOTION.

29. If we suppose the geocentric longitude of Venus to be represented
[4296] by V', and $\frac{r'}{r''}=\alpha$; V'* will be a function of α and $v'-v''$.
Then we shall have, by [4269],

[4297]
$$\delta V' = -\frac{\delta\alpha\,.\,\sin.(v'-v'')}{1-2\alpha\,.\,\cos.(v'-v'')+\alpha^2};$$

which gives, as in [4272], where $\delta V'$ is at its maximum,

[4298]
$$\delta V' = -\frac{\delta\alpha}{1-\alpha^2}.$$

[4297a] * (2596) In strictness it is not the angle V' which is to be considered as a function of α and $v'-v''$ exclusively, but the angle of elongation E of Venus, as seen from the earth. This will appear by referring to fig. 74, page 229; supposing M to represent the place of Venus; $SM=r'$, $♈SM=v'$. For it is evident that the angle of elongation $E=SEM$ will remain the same, if the angle $ESM=v'-v''$ and the
[4297b] ratio $\alpha=\frac{SM}{SE}=\frac{r'}{r''}$ do not vary, whatever changes may be made in the absolute lengths of the lines SM, SE. This inadvertence of the author, in using V' for E does not
[4297c] however affect the result of his calculation [4297, &c.]; because the differentials only of these quantities are used; and we have, as in [4268c] $\delta V'=-\delta E$. Now in [4268, 4269] we have supposed r'' to be invariable, so that the variation of $\frac{r}{r''}=\alpha$ is $\frac{\delta r}{r''}=\delta\alpha$; substituting this in [4269], and accenting the letters r', v', so as to correspond to the
[4297d] planet Venus, we get the expression [4297]. This is reduced to the form [4298], by the substitution of the maximum value of the coefficient of $-\delta\alpha$ [4271'], in the second member of [4297].

Supposing r'' only to vary in $\delta\alpha$, we have $\delta\alpha = -\frac{\alpha\,\delta r''}{r''}$;* therefore, [4299]

$$\delta r'' = r''.\frac{(1-\alpha^2)}{\alpha}.\delta V'. \qquad [4300]$$

If we put $\delta V' = \pm 1'' = \pm 0^s{,}324$, and take for r' and r'', the mean distances of Venus and the earth from the sun [4079], we shall get, [4300']

$$\delta r'' = \pm\, 0{,}000001035. \qquad [4301]$$

If we put V''' for the geocentric longitude of Mars, and $\frac{r''}{r'''} = \alpha$, we shall have, by [4272],† [4301']

$$\delta r'' = -r'''.(1-\alpha^2).\delta V'''. \qquad [4302]$$

If we take for r'', r''', the mean distances of the earth and Mars from the sun, we shall have,

$$\alpha = 0{,}65630030 \qquad [4159];$$
$$r''' = 1{,}52369352 \qquad [4079]; \qquad [4303]$$

and if we put $\delta V''' = \pm 1'' = \pm 0^s{,}324$, we shall obtain, Terms which may be

$$\delta r'' = \mp\, 0{,}000001363; \qquad [4304]$$

therefore, we *may neglect the inequalities of* $\delta r''$, *whose coefficients are* neglected on account of their smallness.

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* (2597) If we suppose r' to be invariable in the value of α [4296], we shall get $\delta\alpha = -\frac{r'\delta r''}{r''^2} = -\frac{\alpha\delta r''}{r''}$ [4299]; substituting this in [4298], we obtain [4300]; which is reduced to the form [4301], by the substitution of $\delta V' = \pm 1''$ [4300'], $r'' = 1$ [4079] and $\alpha = 0{,}7233323$ [4126]. [4299a]

† (2598) Venus, being an inferior planet to the earth, is situated in the same relative position as the earth is to Mars; therefore the equation [4286], which obtains relatively to Venus and the earth, may be applied to the earth and Mars, by substituting in [4286] the value of α [4284], and then adding one more accent to each of the symbols r', r'', V'; [4301a] by which means we shall obtain $\delta r'' = -r'''.\left(1-\frac{r''^2}{r'''^2}\right).\delta V''$ [4286]. In this case $\delta V''$ is the change of the longitude of the earth, as seen from Mars, arising from the increment $\delta r''$; and is evidently equal to the increment of the geocentric longitude of Mars, depending upon the same cause, which is represented by $\delta V'''$; hence we get $\delta r'' = -r'''.\left(1-\frac{r''^2}{r'''^2}\right).\delta V'''$, as in [4302]. [4301b] [4301c] [4301d]

less than $\pm 0{,}000001$.* *We shall also neglect those inequalities of the*
[4304′] *earth's motion in longitude, which are less than a quarter of a centesimal second, or* $0^s{,}081$.

Inequalities of the Earth, independent of the excentricities.†

Inequalities independent of the excentricities.

[4305]

$$\delta v'' = (1+\mu').\left\{\begin{array}{l} 5^s{,}290878\,.\,\sin.\,\ (n't-n''t+\varepsilon'-\varepsilon'') \\ -6^s{,}015891\,.\,\sin.\,2(n't-n''t+\varepsilon'-\varepsilon'') \\ -0^s{,}743445\,.\,\sin.\,3(n't-n''t+\varepsilon'-\varepsilon'') \\ -0^s{,}225439\,.\,\sin.\,4(n't-n''t+\varepsilon'-\varepsilon'') \\ -0^s{,}091210\,.\,\sin.\,5(n't-n''t+\varepsilon'-\varepsilon'') \\ -0^s{,}042805\,.\,\sin.\,6(n't-n''t+\varepsilon'-\varepsilon'') \\ -0^s{,}022027\,.\,\sin.\,7(n't-n''t+\varepsilon'-\varepsilon'') \\ -0^s{,}012053\,.\,\sin.\,8(n't-n''t+\varepsilon'-\varepsilon'') \end{array}\right\}$$

$$+(1+\mu''').\left\{\begin{array}{l} 0^s{,}427214\,.\,\sin.\,\ (n'''t-n''t+\varepsilon'''-\varepsilon'') \\ 3^s{,}483037\,.\,\sin.\,2(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -0^s{,}215249\,.\,\sin.\,3(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -0^s{,}047022\,.\,\sin.\,4(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -0^s{,}015871\,.\,\sin.\,5(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -0^s{,}006458\,.\,\sin.\,6(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -0^s{,}002923\,.\,\sin.\,7(n'''t-n''t+\varepsilon'''-\varepsilon'') \end{array}\right\}$$

$$+(1+\mu^{iv}).\left\{\begin{array}{l} 7^s{,}059053\,.\,\sin.\,\ (n^{iv}t-n''t+\varepsilon^{iv}-\varepsilon'') \\ -2^s{,}674257\,.\,\sin.\,2(n^{iv}t-n''t+\varepsilon^{iv}-\varepsilon'') \\ -0^s{,}167770\,.\,\sin.\,3(n^{iv}t-n''t+\varepsilon^{iv}-\varepsilon'') \\ -0^s{,}016549\,.\,\sin.\,4(n^{iv}t-n''t+\varepsilon^{iv}-\varepsilon'') \end{array}\right\}$$

$$+(1+\mu^{v}).\left\{\begin{array}{l} 0^s{,}439410\,.\,\sin.\,\ (n^{v}t-n''t+\varepsilon^{v}-\varepsilon'') \\ -0^s{,}111010\,.\,\sin.\,2(n^{v}t-n''t+\varepsilon^{v}-\varepsilon'') \\ -0^s{,}004145\,.\,\sin.\,3(n^{v}t-n''t+\varepsilon^{v}-\varepsilon'') \end{array}\right\}.$$

* (2599) This quantity, independent of its sign, is less than either of the values [4301, 4301], corresponding to the *nearest* inferior and superior planets; and for the more
[4304a] distant planets this degree of accuracy is more than is absolutely requisite, in the present state of astronomy.

† (2600) The quantities [4305, 4306] are deduced from [4277a, b]; accenting the
[4305a] symbols so as to correspond to the present case, and using the data [4061, &c.].

$$\delta r'' = (1+\mu') \cdot \left\{ \begin{array}{l} \quad 0,0000015553 \\ -\,0,0000060012 \,.\, \cos. \;\; (n't - n''t + \varepsilon' - \varepsilon'') \\ +\,0,0000171431 \,.\, \cos. 2(n't - n''t + \varepsilon' - \varepsilon'') \\ +\,0,0000027072 \,.\, \cos. 3(n't - n''t + \varepsilon' - \varepsilon'') \\ +\,0,0000009358 \,.\, \cos. 4(n't - n''t + \varepsilon' - \varepsilon'') \\ +\,0,0000004086 \,.\, \cos. 5(n't - n''t + \varepsilon' - \varepsilon'') \\ +\,0,0000002008 \,.\, \cos. 6(n't - n''t + \varepsilon' - \varepsilon'') \end{array} \right\}$$

Inequalities independent of the excentricities.

$$+ (1+\mu''') \cdot \left\{ \begin{array}{l} -\,0,0000000478 \\ +\,0,0000005487 \,.\, \cos. \;\; (n'''t - n''t + \varepsilon''' - \varepsilon'') \\ +\,0,0000080620 \,.\, \cos. 2(n'''t - n''t + \varepsilon''' - \varepsilon'') \\ -\,0,0000006475 \,.\, \cos. 3(n'''t - n''t + \varepsilon''' - \varepsilon'') \\ -\,0,0000001643 \,.\, \cos. 4(n'''t - n''t + \varepsilon''' - \varepsilon'') \end{array} \right\}$$

$$+ (1+\mu^{\text{iv}}) \cdot \left\{ \begin{array}{l} -\,0,0000011581 \\ +\,0,0000159384 \,.\, \cos. \;\; (n^{\text{iv}}t - n''t + \varepsilon^{\text{iv}} - \varepsilon'') \\ -\,0,0000090986 \,.\, \cos. 2(n^{\text{iv}}t - n''t + \varepsilon^{\text{iv}} - \varepsilon'') \\ -\,0,0000006550 \,.\, \cos. 3(n^{\text{iv}}t - n''t + \varepsilon^{\text{iv}} - \varepsilon'') \\ -\,0,0000000704 \,.\, \cos. 4(n^{\text{iv}}t - n''t + \varepsilon^{\text{iv}} - \varepsilon'') \end{array} \right\} \qquad [4306]$$

$$+ (1+\mu^{\text{v}}) \cdot \left\{ \begin{array}{l} -\,0,0000000580 \\ +\,0,0000010337 \,.\, \cos. \;\; (n^{\text{v}}t - n''t + \varepsilon^{\text{v}} - \varepsilon'') \\ -\,0,0000003859 \,.\, \cos. 2(n^{\text{v}}t - n''t + \varepsilon^{\text{v}} - \varepsilon'') \end{array} \right\}.$$

In the solar tables of La Caille, Mayer, La Lande, Delambre and Zach, published before the year 1803, the chief correction of the radius vector of the earth's orbit, arising from the action of Jupiter, is given with a wrong sign; in consequence of taking, for $n''t + \varepsilon''$, the sun's longitude, instead of that of the earth, in finding the argument corresponding to the terms which were used, namely, [4305b]

$$+0,0000159384 \,.\, \cos.(n^{\text{iv}}t - n''t + \varepsilon^{\text{iv}} - \varepsilon'') - 0,0000090986 \,.\, \cos.2(n^{\text{iv}}t - n''t + \varepsilon^{\text{iv}} - \varepsilon''). \quad [4305c]$$

This mistake was first made known in a letter communicated by me to La Lande, and noticed in vol. 8, p. 449, of the *Monatliche Correspondenz* for 1803. [4305d]

*Inequalities depending on the first power of the excentricities.**

Inequalities depending on the first power of the excentricities.

[4307]

$$\delta v'' = (1+\mu') . \left\{ \begin{array}{l} 0^s,075910 . \sin. (n't + \varepsilon' - \varpi'') \\ - 0^s,129675 . \sin. (2n't - n''t + 2\varepsilon' - \varepsilon'' - \varpi'') \\ + 0^s,145179 . \sin. (2n''t - n't + 2\varepsilon'' - \varepsilon' - \varpi'') \\ - 0^s,168981 . \sin. (2n''t - n't + 2\varepsilon'' - \varepsilon' - \varpi') \\ - 3^s,667112 . \sin. (3n''t - 2n't + 3\varepsilon'' - 2\varepsilon' - \varpi'') \\ + 1^s,186390 . \sin. (3n''t - 2n't + 3\varepsilon'' - 2\varepsilon' - \varpi') \\ - 2^s,342956 . \sin. (4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \varpi'') \\ + 0^s,722424 . \sin. (4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \varpi') \\ + 0^s,216368 . \sin. (5n''t - 4n't + 5\varepsilon'' - 4\varepsilon' - \varpi'') \end{array} \right\}$$

$$+ (1+\mu''') . \left\{ \begin{array}{l} - 1^s,095603 . \sin. (2n'''t - n''t + 2\varepsilon''' - \varepsilon'' - \varpi'') \\ + 2^s,137658 . \sin. (2n'''t - n''t + 2\varepsilon''' - \varepsilon'' - \varpi''') \\ - 0^s,087400 . \sin. (3n'''t - 2n''t + 3\varepsilon''' - 2\varepsilon'' - \varpi'') \\ + 0^s,661950 . \sin. (3n'''t - 2n''t + 3\varepsilon''' - 2\varepsilon'' - \varpi''') \\ - 0^s,103758 . \sin. (4n'''t - 3n''t + 4\varepsilon''' - 3\varepsilon'' - \varpi'') \\ + 0^s,807111 . \sin. (4n'''t - 3n''t + 4\varepsilon''' - 3\varepsilon'' - \varpi''') \\ - 0^s,134915 . \sin. (5n'''t - 4n''t + 5\varepsilon''' - 4\varepsilon'' - \varpi''') \end{array} \right\}$$

$$+ (1+\mu^{iv}) . \left\{ \begin{array}{l} 0^s,302092 . \sin. (n^{iv}t + \varepsilon^{iv} - \varpi'') \\ - 2^s,539884 . \sin. (n^{iv}t + \varepsilon^{iv} - \varpi^{iv}) \\ - 1^s,492044 . \sin. (2n^{iv}t - n''t + 2\varepsilon^{iv} - \varepsilon'' - \varpi'') \\ + 0^s,606399 . \sin. (2n^{iv}t - n''t + 2\varepsilon^{iv} - \varepsilon'' - \varpi^{iv}) \\ - 0^s,543364 . \sin. (3n^{iv}t - 2n''t + 3\varepsilon^{iv} - 2\varepsilon'' - \varpi^{iv}) \\ - 0^s,148925 . \sin. (2n''t - n^{iv}t + 2\varepsilon'' - \varepsilon^{iv} - \varpi'') \\ - 0^s,093643 . \sin. (2n''t - n^{iv}t + 2\varepsilon'' - \varepsilon^{iv} - \varpi^{iv}) \end{array} \right\}$$

$$+ (1+\mu^{v}) . \left\{ \begin{array}{l} - 0^s,359921 . \sin. (n^{v}t + \varepsilon^{v} - \varpi^{v}) \\ - 0^s,151752 . \sin. (2n^{v}t - n''t + 2\varepsilon^{v} - \varepsilon'' - \varpi'') \end{array} \right\} .$$

[4307a] * (2601) The terms of $\delta v''$, $\delta r''$ [4307, 4308] are computed as in the theory of Mercury [4278a].

$$\delta r'' = (1+\mu').\left\{\begin{array}{l} -0,0000030439 \,.\, \cos.\,(3n''t - 2n't + 3\varepsilon'' - 2\varepsilon' - \varpi'') \\ -0,0000049815 \,.\, \cos.\,(4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \varpi'') \\ +0,0000015895 \,.\, \cos.\,(4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \varpi') \end{array}\right\}$$

$$+ (1+\mu''') \,.\, 0,0000017707 \,.\, \cos.\,(4n'''t - 3n''t + 4\varepsilon''' - 3\varepsilon'' - \varpi''') \qquad [4308]$$

$$+(1+\mu^{iv}).\left\{\begin{array}{l} -0,0000030410 \,.\, \cos.\,(2n^{iv}t - n''t + 2\varepsilon^{iv} - \varepsilon'' - \varpi'') \\ +0,0000012652 \,.\, \cos.\,(2n^{iv}t - n''t + 2\varepsilon^{iv} - \varepsilon'' - \varpi^{iv}) \\ -0,0000018101 \,.\, \cos.\,(3n^{iv}t - 2n''t + 3\varepsilon^{iv} - 2\varepsilon'' - \varpi^{iv}) \end{array}\right\}.$$

*Inequalities depending on the squares and products of the excentricities and inclinations of the orbits.**

Inequalities of the second order.

$$\delta v'' = (1+\mu') \,.\, 1^s,125575 \,.\, \sin.\,(5n''t - 3n't + 5\varepsilon'' - 3\varepsilon' + 21^d 02^m 18^s)$$

$$+ (1+\mu''') \,.\left\{\begin{array}{l} +0^s,993935 \,.\, \sin.\,(4n'''t - 2n''t + 4\varepsilon''' - 2\varepsilon'' + 67^d 48^m 56^s) \\ +0^s,351796 \,.\, \sin.\,(5n'''t - 3n''t + 5\varepsilon''' - 3\varepsilon'' + 68^d 25^m 09^s) \end{array}\right\}. \qquad [4309]$$

The mean motions of Venus, the earth and Mars bear such proportions to each other, that the quantities $5n'' - 3n'$, $4n''' - 2n''$ are small in comparison with n''; hence it follows, from [3733], that the two first of these inequalities are the only ones of this order which are deserving of notice. However we have calculated the third; because $3n'' - 5n'''$, being very nearly equal to $\frac{1}{3}n''$, it is satisfactory to show, by a direct calculation, that this inequality acquires by integration only a very insensible value.† [4310] [4310′]

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* (2602) From [4076h] we get, very nearly, $5n'' - 3n' = 50^\circ = \frac{n''}{8}$; $4n''' - 2n'' = 50^\circ = \frac{n''}{8}$; $3n'' - 5n''' = 137^\circ = \frac{n''}{3}$. These angles ought therefore to be noticed, as in [3733]; and by making the computation, as for Mercury [4282a–p], we may reduce the terms, depending on each angle, to one single term, as in [4282h–l]. [4309a]

† (2603) We have already mentioned, in [4296p], that Mr. Airy had discovered an inequality in the earth's motion, depending on terms of the fifth order of the excentricities and inclinations, connected with the angle $8n't - 13n''t$. He has given in the paper mentioned in [4296h] the numerical values of the inequalities of the mean motion ζ'', of the perihelion $\delta\varpi''$, of the excentricity $\delta e''$, and of the latitude $\delta s''$, namely, [4310a] [4310b]

Inequalities depending on the powers and products of three dimensions of the excentricities and inclinations of the orbits.

Inequalities of the third order.

[4311] $$\delta v'' = (1+\mu) \,.\, 0^s{,}069915 \,.\, \sin.(nt - 4n''t + \varepsilon - 4\varepsilon'' + 19^d\,02^m\,13^s).*$$

Periodical inequalities of the Earth's motion in latitude.

Inequalities in the latitude.

We find, by formula [1030],†

[4312] $$\delta s'' = (1+\mu') \,.\, \left\{ \begin{array}{l} 0^s{,}099180 \,.\, \sin.\,(2n''t - n't + 2\varepsilon'' - \varepsilon' - \theta') \\ 0^s{,}234256 \,.\, \sin.\,(4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \theta') \end{array} \right\}$$
$$+ (1+\mu^{iv}) \,.\, 0^s{,}164703 \,.\, \sin.\,(2n^{iv}t - n''t + 2\varepsilon^{iv} - \varepsilon'' - \theta^{iv}).$$

Inequalities of the Earth depending upon the Moon.

30. If we put

Symbols.

U = the longitude of the moon, as viewed from the centre of the earth;

v'' = the longitude of the earth, as viewed from the centre of the sun;

[4313] R = the radius vector of the moon; its origin being the earth's centre;

r'' = the radius vector of the earth; its origin being the sun's centre;

m = the mass of the moon;

M = the mass of the earth;

s = the latitude of the moon, as viewed from the earth's centre,

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[4310c] $\zeta'' = (2^s{,}059 - t.0^s{,}0002076).\sin.(8n't - 13n''t + 8\varepsilon' - 13\varepsilon'' + 40^d 44^m 34^s - t.10^s{,}76)$;

[4310d] $\delta\varpi'' = 2^s{,}268 \,.\, \sin.(8n't - 13n''t + 8\varepsilon' - 13\varepsilon'' + 60^d\,16^m)$;

[4310e] $\delta e'' = -\,0{,}0000001849 \,.\, \cos.(8n't - 13n''t + 8\varepsilon' - 13\varepsilon'' + 60^d\,16^m)$;

[4310f] $\delta s'' = 0^s{,}0105 \,.\, \sin.(8n't - 12n''t + 8\varepsilon' - 12\varepsilon'' - 39^d\,29^m)$.

* (2604) The direct calculation of this inequality can be made, by a process like that [4311a] which is used for Mercury, in [3881e, &c.]; but it is probable that the author deduced it from the similar inequality of Mercury [4283], by the method given in [3883y].

† (2605) The terms of [4312] are computed by means of the formula [4295b]; changing, in the first place, n', a', ε', into n'', a'', ε'', respectively. Then changing [4312a] m'', n'', a'', ε'' into m', n', a', ε', in computing the action of Venus on the earth; or into m^{iv}, n^{iv}, a^{iv}, ε^{iv}, respectively, in computing the action of Jupiter on the earth.

we shall have, for the inequality of the earth's motion in longitude [4052], produced by the action of the moon,*

The moon's action produces a perturbation in the longitude;

$$\delta v'' = -\frac{m}{M} \cdot \frac{R}{r''} \cdot \sin.(U - v''). \qquad [4314]$$

The inequality of the radius vector of the earth [4051] is

in the radius;

$$\delta r'' = -\frac{m}{M} \cdot R \cdot \cos.(U - v''); \qquad [4315]$$

and the inequality of the earth's motion in latitude [4053] is

and in the latitude.

$$\delta s'' = -\frac{m}{M} \cdot \frac{R}{r''} \cdot s. \qquad [4316]$$

For greater accuracy, we must substitute† $\frac{m}{M+m}$ for $\frac{m}{M}$, in the expressions of these three inequalities.

We shall suppose conformably to the phenomena of the tides [2706, 2768],

$$\frac{m}{R^3} = \frac{3S}{r''^3}; \qquad [4317]$$

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* (2606) The moon's action upon the earth produces, in the radius vector, the longitude and the latitude of the earth, the inequalities given in [4051, 4052, 4053]; namely,

$$-\frac{m}{M} \cdot r \cdot \cos.(v - U); \qquad -\frac{m}{M} \cdot \frac{r}{R} \cdot \sin.(v - U); \qquad -\frac{m}{M} \cdot \frac{rs}{R}; \qquad [4314a]$$

and by comparing the notation used in [4047, 4048], with that in [4313], it appears that we must change R, r, v, U, into r'', R, U, v'', respectively, to conform nearly to the notation of this article. By this means the preceding expressions become, [4314b]

$$-\frac{m}{M} \cdot R \cdot \cos.(U - v''); \qquad -\frac{m}{M} \cdot \frac{R}{r''} \cdot \sin.(U - v''); \qquad -\frac{m}{M} \cdot \frac{Rs}{r''}; \qquad [4314c]$$

corresponding respectively to the formulas [4315, 4314, 4316]. In the original work the divisor r'', by mistake, is omitted in [4314], and inserted in [4315]; we have rectified this mistake. [4314d]

† (2607) The radius r [4048] has for its origin the *common centre of gravity of the earth and moon.* This is changed into R, in [4314b], to conform to the present notation; but as the origin of R [4313] is in the *centre of the earth*, the value of the radius is too great, and must be decreased in the ratio of M to $M+m$; which is equivalent [4316a] to the multiplication of the perturbations [4314 — 4316] by $\frac{M}{M+m}$; or in other words [4316b] to change the divisor M into $M+m$, in all three of these formulas.

S being the sun's mass. Now, by the theory of central forces [3700],* we have,

[4318] $$\frac{M+m}{R^3} = n_{,}^2; \qquad \frac{S}{r''^3} = n''^2;$$

$n_{,}t$ being the moon's mean motion; hence we obtain,

[4319] $$\frac{m}{M+m} = \frac{3\,n''^2}{n_{,}^2}.$$

[4319′] We have by observation $\frac{n''}{n_{,}} = 0{,}0748013$ [5117, 4835]; hence we get,

[4320] $$\frac{m}{M+m} = \frac{1}{59{,}6};$$

Mass of the moon. consequently,

[4321] $$\frac{m}{M} = \frac{1}{58{,}6} \qquad [4318d].$$

[4322] If we suppose the sun's horizontal parallax to be $27'',2 = 8^s,8$, and the moon's mean horizontal parallax $10661'' = 3454^s = 57^m\,34^s$,† we shall have,

[4323] $$\frac{R}{r''} = \frac{\text{sun's hor. par.}}{\text{moon's hor. par.}} = \frac{8{,}8}{3454{,}0};$$

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* (2608) Substituting $\mu = M+m$ [3709a] in [3700], then changing a, n, into

[4318a] R, $n_{,}$, respectively, we get the first of the equations [4318], corresponding to the moon's motion about the earth. Changing in this, M, m, R, n, into S, M, r'', n'',

[4318b] and neglecting M in comparison with S, we get the second of the equations [4318]; corresponding to the earth's motion about the sun. Multiplying the first of the equations

[4318c] [4318], by $\frac{m}{M+m}$, and the second by 3; then substituting the products in [4317] we

[4318d] get $\frac{m}{M+m}\cdot n_{,}^2 = 3\,n''^2$; dividing this by $n_{,}^2$, we obtain [4319]; substituting in this the value [4319′], we finally get the expression of the mass of the moon [4321]. This was afterwards found to be too great [4631, 1190b, &c.], as we have already observed in [3380b, &c.].

Instead of supposing, as in [2706], that the ratio of the mean force of the moon on the

[4318e] tides, is to that of the sun as 3 to 1, we may express it more generally by $3(1-\beta)$ to 1; by which means the second members of the equations [4317, 4319, 4320], will be

[4318f] multiplied by $1-\beta$; and the last of these expressions will become $\frac{m}{M+m} = \frac{1-\beta}{59{,}6}$;

[4318g] whence we get the following expression, which will be used hereafter, $\frac{m}{M} = \frac{1-\beta}{58{,}6+\beta}$.

[4322a] † (2609) This parallax, taken for the mean between the *greatest* and *least* values,

consequently,*

Perturbations in the longitude,

$$\delta v'' = -\,27'',2524\,.\,\sin.\,(U - v'') = -\,8^s,8298\,.\,\sin.\,(U - v'')\,;$$ [4324]

$$\delta r'' = -\,0,000042808\,.\,\cos.\,(U - v'').$$

and in the radius.

Then taking for s the greatest inequality of the moon in latitude, which we shall suppose to be $18543^s.\sin.(U - \theta)$ [5308]; $U - \theta$ being the moon's distance from her ascending node; we shall obtain† [4325]

Perturbation of the earth in latitude.

$$\delta s'' = -\,0^s,7938\,.\,\sin.\,(U - \theta),$$ [4326]

for the inequality of the earth's motion in latitude. We must add it to the terms of $\delta s''$ [4312], to obtain the complete value of $\delta s''$; and *by taking this sum, with a contrary sign, we have the inequalities of the sun's apparent motion in latitude.* These inequalities in the latitude have an influence on the obliquity of the ecliptic, deduced from the observations of the meridian altitudes of the sun near the solstices. They have also an influence upon the time of the equinox, deduced from observations of the sun, when near the equinoxes, as well as upon the right-ascensions and declinations of the stars, determined by comparing directly their places in

Sun's latitude.

[4327]

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exceeds, by 33^s, the constant quantity in Burg's tables [5603], and is nearly conformable to the result given by La Lande in § 1698 of the third edition of his astronomy. For the purpose of illustration, we may neglect all the inequalities of the moon's parallax, except those depending on the moon's mean anomaly; then taking the coefficients to the nearest second, we have, from Burg's tables [5603], [4322b]

$$☽\text{'s hor. par.} = 3421^s + 187^s\,.\,\cos.\,(\text{mean anom.}) + 10^s\,.\,\cos.\,(2\,\text{mean anom.}).$$ [4322c]

The *greatest* value of this expression, corresponding to the perigee, or the mean anom. $=0$, is $3421^s + 187^s + 10^s$; and the *least* value, in the apogee is $3421^s - 187^s + 10^s$. The *mean* of these two values $3421^s + 10^s$, *exceeds* by 10^s, the constant term 3421^s; and it is from causes similar to this, that the difference above-mentioned depends. [4322d]

* (2610) The inequalities [4324] are deduced from [4314, 4315], by using the values [4321, 4323], and multiplying the value of $\delta v''$ by the expression of the radius in seconds $206264^s,8$. [4324a]

† (2611) Substituting the values [4321, 4323], and s [4325], in [4316], we get $\delta s''$ [4326]; changing M into $M+m$, in all these calculations, as in [4316b]. [4326a]

the heavens with that of the sun. On account of the great accuracy of modern observations, it is necessary to notice these inequalities. It is evident that this correction increases the apparent declination of the sun, by the quantity,*

Perturbation of the sun in declination,

[4328] $$\text{Increment of } \odot\text{'s declination} = -\frac{\delta s'' . \cos . \text{(obliquity of the ecliptic)}}{\cos . \text{(sun's declination)}};$$

and its apparent right-ascension is also increased, by the following expression,

and in right-ascension.

[4329] $$\text{Inc. of } \odot\text{'s right-ascen.} = \frac{\delta s'' . \sin . \text{(obliquity of the ecliptic)} . \cos . \text{(sun's right-ascension)}}{\cos . \text{(sun's declination)}}.$$

[4329'] *We must therefore decrease, by these quantities, the observed declinations and right-ascensions of the sun, to obtain those which would be observed, if the earth did not quit the plane of the ecliptic.*

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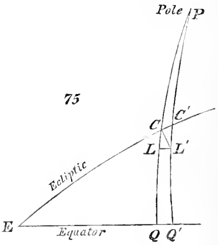
* (2612) Let $E\,C\,C'$ be the ecliptic, $E\,Q\,Q'$ the equator, P the north pole of the equator; then if the earth's latitude, north of the ecliptic, be $\delta s''$, that of the sun will be south, and may be represented by $C\,L' = \delta s''$ perpendicular to the ecliptic. $P\,C\,L\,Q$,
[4328a] $P\,C'\,L'\,Q'$, are circles of declination, perpendicular to the equator, and $L\,L'$ is parallel to the equator. The small differential triangle $C\,L\,L'$, may be supposed rectangular in L, and angle $L\,C\,L' = 90^{\circ} -$ angle $E\,C\,Q$. Then in the spherical triangle $E\,C\,Q$, we have, by [1315^{32}], $\cos . E\,C\,Q = \sin . L\,C\,L' = \sin . C\,E\,Q . \cos . E\,Q$;
[4328b] $\sin . E\,C\,Q = \cos . L\,C\,L' = \frac{\cos . C\,E\,Q}{\cos . C\,Q}$. Now the declination is decreased by the quantity $C\,L$; the right-ascension is
[4328c] increased by the quantity $Q\,Q' = \frac{L\,L'}{\sin . P\,L} = \frac{L\,L'}{\cos . \text{dec.}}$; and we have



[4328d] $$L\,L' = C\,L' . \sin . L\,C\,L' = \delta s'' . \sin . C\,E\,Q . \cos . E\,Q;$$

hence we get,

[4328e] $$\text{Increm. dec.} = -C\,L = -C\,L' . \cos . L\,C\,L' = -\delta s'' . \frac{\cos . C\,E\,Q}{\cos . C\,Q}, \quad \text{as in [4328]; and}$$

[4328f] $$\text{Increm. right-ascen. } Q\,Q' = \frac{L\,L'}{\cos . \text{dec.}} = \delta s'' . \frac{\sin . C\,E\,Q . \cos . E\,Q}{\cos . \text{dec.}}, \quad \text{as in [4329].}$$

On the secular variations in the Earth's orbit, in its equator, and in the length of the year.

31. We have given, in [4244, 4249, &c.], the secular variations of the elements of the earth's orbit; but the influence of these variations on the most [4329″] important phenomena of astronomy has been an inducement to compute them with greater accuracy, noticing the square of the *time* t;* *supposing* t *to denote the number of Julian years elapsed since* 1750. We have found by the methods given in [1096–1126], and using the values of the masses of [4329‴] the planets [4061], that the coefficient of the *equation of the centre* of the earth's orbit is represented by,†

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* (2613) The values of e^2, tang. ϖ [1109, 1110], give those of e''^2, tang. ϖ''; [4329a] by changing the quantities corresponding to m, into those relative to m'', and the contrary. The formulas, thus found, may be developed in series, ascending according to the powers of t, by Taylor's theorem [3850a]; hence we easily deduce the values of e'', ϖ'', in similar [4329b] forms. The calculation may also be made by the method pointed out in the following note.

† (2614) We have, by Taylor's theorem, as in [1126‴],

$$2e'' = 2E + \frac{2de''}{dt}\cdot t + \frac{dde''}{dt^2}\cdot t^2, \qquad [4330a]$$

neglecting the higher powers of t; the values of $\frac{de''}{dt}$, $\frac{dde''}{dt^2}$, being taken to correspond to the epoch 1750. The differential of $\frac{de''}{dt}$ [1122], taken according to the directions [4330b] in [1126^{iv}], or as in note 768, vol. I. p. 612, and divided by dt, gives $\frac{dde''}{dt^2}$, in terms [4330c] of e, e', &c. ϖ, ϖ', &c. and of their first differentials. Substituting in this expression, the values of these first differentials, given in [1122, 1126], it changes into a function of the finite quantities e, e', &c. ϖ, ϖ', &c.; and by substituting the values of these quantities, [4330d] for the year 1750, given in [4080, 4081], we obtain the expression of $\frac{dde''}{dt^2}$. Moreover, by similar substitutions, we get the value of the expression of $\frac{de''}{dt}$ [1122]. These values, [4330e] being substituted in [4330a], give the expression of $2e''$ [4330]. The formulas [4330–4360] are so frequently referred to in the work, that we have given the numerical [4330f] values in centesimal, as well as in sexagesimal seconds. The values given in [4330, 4331, 4332], are altered, in [4610–4612], by reason of the changes in the masses of Venus and Mars.

We have seen in vol. I. p. 612, note 468, that terms of the order $m'e'$ are retained, and those of the order $m'e'^3$, which are of the *first* order relative to the mass m', are

[4330] $$\text{Coeff. equa. centre} = 2E - t\,.\,0'',579130 - t^2\,.\,0'',0000207446$$
$$= 2E - t\,.\,0^s,187638 - t^2\,.\,0^s,0000067213,$$

Secular equations of the earth's orbit.

$2E$ *being this coefficient at the beginning of the year* 1750, when t is nothing. We have also found the sideral longitude of the perihelion of the earth's orbit, namely,*

[4331] $$\text{Long. perih. of the earth} = \varpi'' + t\,.\,36'',881443 + t^2\,.\,0'',0002454382$$
$$= \varpi'' + t\,.\,11^s,949588 + t^2\,.\,0^s,0000795220.$$

Lastly, the values of p'', q'', at any time t, have been found respectively equal to,†

[4332] $$p'' = t\,.\,0'',236793 + t^2\,.\,0'',0000665275$$
$$= t\,.\,0^s,076721 + t^2\,.\,0^s,0000215549;$$
$$q'' = -t\,.\,1'',546156 + t^2\,.\,0,0000208253$$
$$= -t\,.\,0^s,500955 + t^2\,.\,0^s,0000067474.$$

[4330g] neglected, in the expression of $\frac{de''}{dt}$ [1122]. If we suppose, for a rough estimate, that $e' = \frac{1}{20}$, the neglected terms will be of the order of $\frac{1}{400}$ part of those retained; so that the neglected part in the coefficient of t [4330], may be considered as of the order
[4330h] $\frac{1}{400} \times 0^s,187638 = 0^s,0001$, which is much greater than the coefficient of t^2 in [4330]; and at the first view it might be thought strange that we should neglect this, and yet notice the much smaller coefficient of t^2, which is of the order of the *square* of the disturbing masses. But the reason will appear very evident from the consideration, that when t is large, the term depending on t^2 becomes very great in comparison with these neglected
[4330i] terms. Thus, if $t = 2500$, the neglected term $0,0001t$ is only one second, while the term depending on t^2, exceeds 42^s. Similar remarks may be made relative to the quantities ϖ'', p'', q'' [4331, 4332].

* (2615) Proceeding as in the last note, we may deduce from [3850a], by changing
[4331a] u into ϖ'', $\varpi'' = \varpi'' + t\,.\,\frac{d\varpi''}{dt} + \frac{1}{2}t^2.\frac{dd\varpi''}{dt^2}$; the quantities in the second member referring to the epoch of 1750. The differential of $\frac{d\varpi''}{dt}$ [1126], divided by dt, gives $\frac{dd\varpi''}{dt^2}$,
[4331b] in terms of e, e', &c. ϖ, ϖ', and their first differentials. Substituting in this expression the values of the differentials [1122, 1126], it changes into a function of the finite quantities e, e', &c. ϖ, ϖ', &c.; and by using the numerical values [4080, 4081], we get the
[4331c] values of $\frac{d\varpi''}{dt}$, $\frac{dd\varpi''}{dt^2}$, to be substituted in [4331a], to obtain [4331].

† (2616) The expressions of $\frac{dp''}{dt}$, $\frac{dq''}{dt}$, are in [4251b]; their differentials taken

We have given, in [3100—3110], the expressions of the precession of the equinoxes,* and of the inclination of the equator, referred to the fixed ecliptic, and to the apparent ecliptic. In these formulas, we have supposed the values of p'', q'', to be given under the forms [4333]

$$p''=\Sigma\,.\,c\,.\,\sin.\,(g\,t+\beta)\,;\qquad q''=\Sigma\,.\,c\,.\cos.\,(g\,t+\beta)\ [3068b].$$ [4334]

Moreover, we have seen, in [1133], that the finite expressions of p'', q'', appear under these forms, and we may determine, by the method explained in [1098, &c.], the values of c, g, β. To obtain these quantities accurately, by this method, we must know the correct values of the masses of the planets; and there is considerable uncertainty relative to some of them, as we have observed in [4076, &c.]. Therefore, instead of making the tedious calculation, required by this method, it is preferable to simplify it, so as to embrace a period of ten or twelve hundred years, before and after the epoch of 1750; which is sufficient for all the purposes of astronomy. We may easily rectify these calculations as often as the development of the secular variations shall make known, with greater accuracy, the masses of the planets. We shall give to the values of p'' and q'' the following forms, which are comprised in those mentioned in [4334].† [4335] [4336]

Assumed forms of p'', q''.

$$p''=\Sigma\,.\,c.\sin.(g t+\beta)=c\,.\,\sin.\beta-c\,.\cos.\beta\,.\,\sin.g t-c.\sin.\beta\,.\,\sin.(g't+\tfrac{1}{2}\pi)\,;$$
$$q''=\Sigma\,.\,c.\cos.(g t+\beta)=c\,.\cos.\beta-c.\cos.\beta.\cos.g t-c.\sin.\beta\,.\cos.(g't+\tfrac{1}{2}\pi)\,;$$ [4337]

π being the semi-circumference of a circle whose radius is unity. If we

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relatively to t, and divided by $d\,t$, give $\frac{d\,d\,p''}{d\,t^2}$, $\frac{d\,d\,q''}{d\,t^2}$, in terms of $\frac{d\,p}{d\,t}$, $\frac{d\,p'}{d\,t}$, &c. $\frac{d\,q}{d\,t}$, $\frac{d\,q'}{d\,t}$, &c.; substituting the values of these last quantities [1132], we get $\frac{d\,d\,p''}{d\,t^2}$, $\frac{d\,d\,q''}{d\,t^2}$, expressed in finite terms of p, p', &c. q, q', &c. The values of p, p', &c. q, q', &c. are given in [4251c], in terms of φ, φ', &c. θ, θ', &c.; and the numerical values of these last quantities, in the year 1750, are in [4082, 4083]; hence we obtain the numerical values of p, p', &c. q, q', &c. at that epoch. Substituting these in [4251d, e], and in the preceding values of $\frac{d\,d\,p''}{d\,t^2}$, $\frac{d\,d\,q''}{d\,t^2}$, we get the numerical values of $\frac{d\,p''}{d\,t}$, $\frac{d\,q''}{d\,t}$, $\frac{d\,d\,p''}{d\,t^2}$, $\frac{d\,d\,q''}{d\,t^2}$, at the same epoch, 1750; these are to be substituted in the general values of p'', q'' [4250], to obtain [4332]. [4332a] [4332b] [4332c] [4332d]

* (2617) The formulas, here referred to, are [3100, 3101, 3107, 3110]. [4333a]

† (2618) The three terms of the second member of the value of p'' or q'' [4337],

develop these two functions relatively to the powers of the time t, we shall find, by comparing them with the preceding series [4332],*

Values of c, g, g', β.

[4338]
$$c\,g\,.\cos.\beta = -0^s,076721\,;$$
$$c\,g'.\sin.\beta = -0^s,500955\,;$$
$$c\,g^2.\cos.\beta = 0^s,0000134948\,;$$
$$c\,g'^2.\sin.\beta = 0^s,0000431098.$$

Hence we easily obtain,†

[4339]
$$g = -36^s,2808\,;$$
$$g' = -17^s,7502\,;$$
$$c\,.\sin.\beta = 5821^s,308\,;$$
$$c\,.\cos.\beta = 436^s,17.$$

[4337a] are deduced from those of p'' or q'' [4331], by changing c, g, β, respectively, into c, 0, β, in the first term; $-c\,.\cos.\beta$, g, 0, in the second term; and $-c\,.\sin.\beta$, g', $\frac{1}{2}\pi$, in the third term. These expressions of p'', q'', being developed according to the powers of
[4337b] t, and compared with those in [4332], give, as in [4339], values of c, β, g, g', which satisfy the numerical expressions of p'', q'', [4332], neglecting t^3, and the higher powers of t; and as the values [4332] will answer for ten or twelve centuries from the
[4337c] epoch, it will follow, that the forms assumed in [4337] will answer for the same period, by using these values of c, β, g, g'.

* (2619) We have by development, using the formulas [43, 44] Int. and neglecting terms of
[4338a] the order t^3. $\sin.g\,t = g\,t$; $\cos.g\,t = 1 - \frac{1}{2}g^2t^2$; $\sin.(g'\,t + \frac{1}{2}\pi) = \cos.g'\,t = 1 - \frac{1}{2}g'^2t^2$; $\cos.(g'\,t + \frac{1}{2}\pi) = -\sin.g'\,t = -g'\,t$; substituting these in [4337], we get,

[4338b]
$$p'' = \Sigma\,.\,c\,.\sin.(g\,t+\beta) = c\,.\sin.\beta - c\,.\,g\,t\,.\cos.\beta - c\,.(1-\tfrac{1}{2}g'^2t^2).\sin.\beta$$
$$= -t\,.(c\,g\,.\cos.\beta) + t^2.(\tfrac{1}{2}c\,g'^2.\sin.\beta)\,;$$
$$q'' = \Sigma\,.\,c\,.\cos.(g\,t+\beta) = c\,.\cos.\beta - c\,.(1-\tfrac{1}{2}g^2t^2).\cos.\beta + c\,g'\,t\,.\sin.\beta$$
$$= t\,.(c\,g'.\sin.\beta) + t^2.(\tfrac{1}{2}c\,g^2.\cos.\beta).$$

Comparing the coefficients of t, in these expressions, with the corresponding ones in [4332],
[4338c] we get, without any reduction, the two first equations [4338]. In like manner, by comparing the coefficients of $\frac{1}{2}t^2$, in [4332, 4338b], we get the other two equations [4338].

† (2620) Dividing the square of the first equation [4338], by the third, we get $c\,.\cos.\beta$ [4339]; and the square of the second, divided by the fourth, gives $c\,.\sin.\beta$ [4339].
[4339a] Now, dividing the values of $c\,g^2.\cos.\beta$, $c\,g'^2.\sin.\beta$ [4338], by those of $c\,g\,.\cos.\beta$, $c\,g'.\sin.\beta$ [4338], respectively, and multiplying the products by the radius in seconds, 206265^s, we get g, g' [4339].

Now we have seen, in [3100], that *the precession of the equinoxes* ψ, *relative to the fixed ecliptic of* 1750, noticing only the secular variations, is,

Precession relative to the fixed ecliptic of 1750.

$$\psi = l\,t + \zeta + \Sigma \, . \left\{ \left(\frac{l}{f} - 1 \right) . \,\text{tang.}\, h + \text{cot.}\, h \right\} . \frac{l\,c}{f} . \,\text{sin.}\, (f\,t + \beta). \qquad [4340]$$

First form.

To obtain $\Sigma \, . \, c \, . \,\text{sin.}\, (f\,t + \beta)$, we must increase the angle $g\,t + \beta$, in $\Sigma \, . \, c \, . \,\text{sin.}\, (g\,t + \beta)$, by the quantity $l\,t$ [3073′, &c.];* making $f = g + l$ [3113*a*]; then we shall have, [4341]

$$\Sigma \, . \, c \, . \,\text{sin.}\, (f\,t + \beta) = c \, . \,\text{sin.}\, (l\,t + \beta) - c \, . \,\text{cos.}\, \beta \, . \,\text{sin.}\, (g\,t + l\,t) \\ - c \, . \,\text{sin.}\, \beta \, . \,\text{sin.}\, (g'\,t + l\,t + \tfrac{1}{2}\pi); \qquad [4342]$$

consequently,†

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* (2621) If we increase the angle $g\,t$, by the quantity $l\,t = (f - g)\,t$ [3113*a*], the function $\Sigma \, . \, c \, . \,\text{sin.}\, (g\,t + \beta)$ will become $\Sigma \, . \, c \, . \,\text{sin.}\, (f\,t + \beta)$, as in [4341]; and the first equation [4337], will change into [4342]; observing that we have $g = 0$ [4337*a*], in the first term, or $c \, . \,\text{sin.}\, \beta = c \, . \,\text{sin.}\, (0 \, . \, t + \beta)$, which becomes $c \, . \,\text{sin.}\, (l\,t + \beta)$, as in the first term of [4342]. [4341*a*]

† (2622) The expression $\Sigma \, . \, c \, . \,\text{sin.}\, (f\,t + \beta)$, in the form assumed [4342], consists of *three* terms. In the *first* of these terms, the general symbols c, f, β, of the first member, become c, l, β; or in other words, f is changed into l, while c, β, are unaltered; and the corresponding term of [4340] becomes, [4342*a*]

$$\left\{ \left(\frac{l}{l} - 1 \right) . \,\text{tang.}\, h + \text{cot.}\, h \right\} . \frac{l\,c}{l} . \,\text{sin.}\, (l\,t + \beta); \quad \text{or simply,} \quad c . \text{cot.}\, h \, . \,\text{sin.}\, (l\,t + \beta); \qquad [4342b]$$

which is the first term of ψ [4343], depending on c. The *second* term of [4342], $-c \, . \,\text{cos.}\, \beta \, . \,\text{sin.}\, (g\,t + l\,t)$, being compared with the general expression $c \, . \,\text{sin.}\, (f\,t + \beta)$, in the first member of [4342], shows that c, f, β, must be changed into $-c \, . \,\text{cos.}\, \beta$, $g + l$, 0, respectively; and the corresponding term of [4340] becomes, [4342*c*]

$$-\left\{ \left(\frac{l}{g + l} - 1 \right) . \,\text{tang.}\, h + \text{cot.}\, h \right\} . \frac{l\,c \, . \,\text{cos.}\, \beta}{l + g} . \,\text{sin.}\, (g\,t + l\,t); \qquad [4342d]$$

which is easily reduced to the same form as the term of [4343], depending on the angle $g\,l + l\,t$. Lastly, the *third* term of [4342], $-c \, . \,\text{sin.}\, \beta \, . \,\text{sin.}\, (g'\,t + l\,t + \frac{1}{2}\pi)$, being compared with the general term, in the first member of [4342], gives for c, f, β, the corresponding expressions, $-c \, . \,\text{sin.}\, \beta$, $g' + l$, $\frac{1}{2}\pi$, respectively; and the resulting term of [4340] is, [4342*e*] [4342*f*]

$$-\left\{ \left(\frac{l}{g' + l} - 1 \right) . \,\text{tang.}\, h + \text{cot.}\, h \right\} . \frac{l\,c \, . \,\text{sin.}\, \beta}{l + g'} . \,\text{sin.}\, (g'\,t + l\,t + \tfrac{1}{2}\pi); \qquad [4342g]$$

which is easily reduced to the form of the last term of [4343]. The two first terms of [4340, 4343], represented by $l\,t + \zeta$, are the same in both formulas.

Precession relative to the fixed ecliptic of 1750.

[4343] Second form.

$$\psi = l\,t + \zeta + c\,.\,\mathrm{cot.}\,h\,.\,\mathrm{sin.}\,(l\,t+\beta)$$
$$-\frac{l}{l+g}\,.\,c\,.\,\mathrm{cos.}\,\beta\,.\left\{\mathrm{cot.}\,h - \frac{g}{l+g}\,.\,\mathrm{tang.}\,h\right\}.\,\mathrm{sin.}\,(g\,t+l\,t)$$
$$-\frac{l}{l+g'}\,.\,c\,.\,\mathrm{sin.}\,\beta\,.\left\{\mathrm{cot.}\,h - \frac{g'}{l+g'}\,.\,\mathrm{tang.}\,h\right\}.\,\mathrm{sin.}\,(g'\,t+l\,t+\tfrac{1}{2}\,\pi).$$

Inclination relative to the fixed ecliptic of 1750.

Then by putting V *for the inclination of the equator to the fixed ecliptic of* 1750, we shall have, as in [3101],*

[4344] First form;

$$\mathrm{V} = h - \Sigma\,.\,\frac{l\,c}{f}\,.\,\mathrm{cos.}\,(f\,t+\beta).$$

To obtain $\Sigma\,.\,c\,.\,\mathrm{cos.}\,(f\,t+\beta)$, we must increase the angle $g\,t+\beta$ in $\Sigma\,.\,c\,.\,\mathrm{cos.}\,(g\,t+\beta)$ by $l\,t$† [3073, &c.]; hence we shall have,

[4345]

$$\Sigma\,.\,c\,.\,\mathrm{cos.}\,(f\,t+\beta) = c\,.\,\mathrm{cos.}\,(l\,t+\beta) - c\,.\,\mathrm{cos.}\,\beta\,.\,\mathrm{cos.}\,(g\,t+l\,t)$$
$$- c\,.\,\mathrm{sin.}\,\beta\,.\,\mathrm{cos.}\,(g'\,t+l\,t+\tfrac{1}{2}\,\pi)\,;$$

therefore,‡

Second form.

[4346]

$$\mathrm{V} = h - c\,.\,\mathrm{cos.}\,(l\,t+\beta) + \frac{l}{l+g}\,.\,c\,.\,\mathrm{cos.}\,\beta\,.\,\mathrm{cos.}\,(g\,t+l\,t)$$
$$+\frac{l}{l+g'}\,.\,c\,.\,\mathrm{sin.}\,\beta\,.\,\mathrm{cos.}\,(g'\,t+l\,t+\tfrac{1}{2}\,\pi).$$

[4347] ψ' *denoting the precession of the equinoxes relative to the apparent ecliptic,*

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* (2623) This is the same as [3101], putting V for the part of θ, depending on
[4344a] h and Σ; or in other words, neglecting the periodical terms depending on the angles $f'\,t+\beta'$, $2\,v$, $2\,v'$.

† (2624) This is done upon the principles used in [4341, &c.]; and in the same
[4345a] manner as [4342] was deduced from the first of the equations [4337], we may derive [4345] from the second of [4337].

‡ (2625) Proceeding as in [4342a—f]; and comparing the general form of the first
[4346a] member of [4345], with the three terms of the second member, we find, that c, f, β, become, respectively, c, l, β, in the *first* term; $-c\,.\,\mathrm{cos.}\,\beta$, $g+l$, 0, in the
[4346b] *second* term; and $-c\,.\,\mathrm{sin.}\,\beta$, $g'+l$, $\frac{1}{2}\,\pi$, in the *third* term. Substituting these values in the terms under the sign Σ [4344], we get the three terms containing c, in [4346]; the first term h, is the same in both expressions [4344, 4346].

and V′ *the inclination of the equator to this ecliptic;* we shall have, as in [3107, 3110],* [4347‴]

Precession and obliquity relative to the apparent ecliptic.

$$\psi' = l\,t + \zeta + \frac{g}{l+g} \cdot c \cdot \cos.\beta \cdot \left\{\cot.h + \frac{l}{l+g} \cdot \operatorname{tang}.h\right\} \cdot \sin.(g\,t + l\,t)$$
$$+ \frac{g'}{l+g'} \cdot c \cdot \sin.\beta \cdot \left\{\cot.h + \frac{l}{l+g'} \cdot \operatorname{tang}.h\right\} \cdot \sin.(g'\,t + l\,t + \tfrac{1}{2}\pi); \quad [4348]$$

$$V' = h - \frac{g}{l+g} \cdot c \cdot \cos.\beta \cdot \cos.(g\,t + l\,t) - \frac{g'}{l+g'} \cdot c \cdot \sin.\beta \cdot \cos.(g't + lt + \tfrac{1}{2}\pi). \quad [4349]$$

The expression of ψ' gives,†

$$\frac{d\psi'}{dt} = l + c\,g \cdot \cos.\beta \cdot \left\{\cot.h + \frac{l}{l+g} \cdot \operatorname{tang}.h\right\} \cdot \cos.(g\,t + l\,t)$$
$$+ c\,g' \cdot \sin.\beta \cdot \left\{\cot.h + \frac{l}{l+g'} \cdot \operatorname{tang}.h\right\} \cdot \cos.(g't + l\,t + \tfrac{1}{2}\pi). \quad [4350]$$

If we subtract from this value of $\frac{d\psi'}{dt}$, when t is nothing, its value at any [4350′] other epoch, and reduce the difference of these two expressions to time; considering the whole circumference as equal to one tropical year; we shall get the increment of the length of the tropical year since 1750. We see, [4350″] by this formula, and by the differential of the general expression of

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* (2626) Retaining only the secular inequalities in ψ', θ' [3107, 3110], changing also θ' into V′ [3108, 4317′], we get, by a slight reduction in the term of ψ', under [4347a] the sign Σ,

$$\psi' = l\,t + \zeta + \Sigma \cdot \left\{\cot.h + \frac{l}{f} \cdot \operatorname{tang}.h\right\} \cdot \left(\frac{l-f}{f}\right) \cdot c \cdot \sin.(f\,t + \beta); \quad [4347b]$$

$$V' = h + \Sigma \cdot \left(\frac{f-l}{f}\right) \cdot c \cdot \cos.(f\,t + \beta). \quad [4347c]$$

In the terms under the sign Σ [4347b], we must substitute, successively, the values of the triplets of terms c, f, β, given in [4342a, c, f], and we shall obtain [4348]; observing [4347d] that the first term vanishes, because the factor $\frac{l-f}{f} = 0$. In like manner the substitution of the same triplets of values [4316a–b], in [4347c], gives h [4349]; the first term vanishing, [4347e] on account of the factor $\frac{f-l}{f} = 0$.

† (2627) The differential of ψ' [4348], taken relatively to t, and divided by dt, [4349a] gives [4350].

[4350‴] ψ' [3107],* that the action of the sun and moon changes considerably the law of the variation of the length of the year. In the most probable hypothesis on the masses of the planets, the whole variations, in the length of the year, and in the obliquity of the ecliptic, are reduced to nearly a
[4351] quarter part† of what they would be without that action [3115, 3113w].

[4351′] According to observation, we have in 1750, $\frac{d\psi'}{dt} = 154'',63 = 50^s,1$; but, by what has been said, we get at this epoch,‡

[4352] $$\frac{d\psi'}{dt} = l + cg\,.\cos.\beta\,.\left\{\cot.h + \frac{l}{l+g}\,.\,\mathrm{tang}.\,h\right\};$$

hence we obtain,

[4353] $$l + cg\,.\cos.\beta\,.\left\{\cot.\,h + \frac{l}{l+g}\,.\,\mathrm{tang}.\,h\right\} = 154'',63 = 50^s,1.$$

[4353′] If we neglect the square of c, in this equation, we may substitute for h, the obliquity of the ecliptic to the equator in 1750.§ This obliquity
[4353″] was then, by observation, $26^{\circ},0796 = 23^d\,28^m\,17^s,9$; hence we deduce,**

[4354] $$l = 155'',542 = 50^s,396;$$

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* (2628) This differential is found in [3118], and by reducing it into time, as in [3118′],
[4350a] we get the *decrement* of the year, using $f = g + l$ [3113a]; or the *increment* of the year, by changing its sign, as in [4350″].

† (2629) This subject has already been discussed in [3113a—z]; and we have merely
[4351a] to remark in this place, that the values *arbitrarily assumed* in [4337—4339] do not produce such essential alterations in these variations of ψ', V', as are mentioned in [3113w, 4351].
[4351b] This difference is what might be expected, taking into consideration, that the results, obtained in [4338, 4339], are restricted to values of t, which are less than 1200 [4335]; and that for much greater values of t, the results cannot be relied upon.

‡ (2630) At the epoch 1750, we have $t = 0$ [4329″], and then $\cos.(gt + lt) = 1$,
[4352a] $\cos.(g't + lt + \frac{1}{2}\pi) = \cos.\frac{1}{2}\pi = 0$; substituting these in [4350], it becomes as in [4352]; putting this equal to $50^s,1$ [4351′], derived from observation, we get [4353].

§ (2631) The expression of V [4346] differs from h, by terms of the order c;
[4353a] hence it is evident that if we neglect terms of the order c^2, we may substitute indifferently, the value of V or h, for h, in [4353].

** (2632) Substituting in [4353] the values $h = 23^d\,28^m\,17^s,9$ [4353″], also the values
[4354a] of $cg\,.\cos.\beta$, g [4338, 4339], it becomes, as in the following equation, from which we easily obtain the value of l [4354],

then we have in 1750,*

$$V' = h - \frac{g}{l+g} \cdot c \cdot \cos.\beta\,;$$ [4355]

which gives,

$$h = 26^{\circ},0796 - 3460'',3 = 23^{d}\,28^{m}\,17^{s},9 - 1121^{s},1.$$ [4356]

By means of these values we obtain the following expressions,† [which are altered in 4614 — 4617],

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$$l - 0^{s},076721 \cdot \cot.23^{d}\,28^{m}\,17^{s},9 - \frac{0^{s},076721 \cdot l}{l - 36^{s},2808} \cdot \text{tang.}\,23^{d}\,28^{m}\,17^{s},9 = 154^{s},63.$$ [4354b]

* (2633) Putting $t=0$ in [4349], it becomes as in [4355]. Substituting in this, $V' = 23^{d}\,28^{m}\,17^{s},9$ [4353″], also the values of l, g, $c \cdot \cos.\beta$ [4354, 4339], it becomes, $23^{d}\,28^{m}\,17^{s},9 = h + 1121^{s},1$; hence we get h [4356]. [4356a]

† (2634) Dividing the value of $c \cdot \sin.\beta$ [4339] by that of $c \cdot \cos.\beta$ [4339], we get $\text{tang.}\,\beta = 13,34636 = \text{tang.}\,85^{d}\,42^{m}\,54^{s}$; hence $\beta = 85^{d}\,42^{m}\,54^{s}$; substituting this in the expression of $c \cdot \sin.\beta$ [4339], we obtain $c = 5821^{s},308 \cdot \text{cosec.}\,\beta = 5837^{s},6$. [4357a]
Using these values of β, c, and these of h, l, g, g' [4356, 4354, 4339], we get, [4357b]

$$c \cdot \cot.h = 13646^{s},3\,;$$

$$-\frac{l}{l+g} \cdot c \cdot \cos.\beta \cdot \left\{\cot.h - \frac{g}{l+g} \cdot \text{tang.}\,h\right\} = -5352^{s},8\,;$$ [4357c]

$$-\frac{l}{l+g'} \cdot c \cdot \sin.\beta \cdot \left\{\cot.h - \frac{g'}{l+g'} \cdot \text{tang.}\,h\right\} = -23097^{s},7\,;$$

$l+g = 14^{s},115$; $l+g' = 32^{s},645$. Substituting these in the third, fourth and fifth terms of [4343], we get the third, fifth and fourth terms of [4357], respectively. The term $l\,t$ [4343, 4354], gives the first term of [4357]. The term ζ [4343], is to be taken so as to render $\psi = 0$ [4357] when $t = 0$; whence [4357d]

$$\zeta = -13646^{s},3 \cdot \sin.85^{d}\,42^{m}\,54^{s} + 23097^{s},7 = 2^{d}\,38^{m}\,9^{s},4.$$ [4357e]

In like manner, we have,

$$\frac{l}{l+g} \cdot c \cdot \cos.\beta = 1557^{s},3\,; \qquad \frac{l}{l+g'} \cdot c \cdot \sin.\beta = 8986^{s},6\,;$$ [4357f]

substituting these and h [4356], also the preceding values [4357c], in [4346], we get [4358].

From the same data, we have,

$$\frac{g}{l+g} \cdot c \cdot \cos.\beta \cdot \left\{\cot.h + \frac{l}{l+g} \cdot \text{tang.}\,h\right\} = -4333^{s},2\,;$$

$$\frac{g'}{l+g'} \cdot c \cdot \sin.\beta \cdot \left\{\cot.h + \frac{l}{l+g'} \cdot \text{tang.}\,h\right\} = -9489^{s},4\,;$$ [4357g]

$$\psi = t\,.\,155'',542 + 2^{\circ},92883 + 42118'',3\,.\,\text{sin.}\,(t\,.\,155'',542 + 95^{\circ},2389)$$
$$- 71289'',2\,.\,\text{cos.}\,t\,.\,(100'',757) - 16521'',1\,.\,\text{sin.}\,(t\,.\,43'',564)$$

[4357] $$= t\,.\,50^{s},396 + 2^{d}38^{m}09^{s},4 + 13646^{s},3\,.\,\text{sin.}\,(t\,.\,50^{s},396 + 85^{d}42^{m}54^{s})$$

Precession and obliquity of the ecliptic for the year 1750.

$$- 23097^{s},7\,.\,\text{cos.}\,(t\,.\,32^{s},645) - 5352^{s},8\,.\,\text{sin.}\,(t\,.\,14^{s},115)\,;$$

$$V = 26^{\circ},0796 - 3460'',3 - 18017'',4\,.\,\text{cos.}\,(t\,.\,155'',542 + 95^{\circ},2389)$$ [Fixed orbit.]
$$+ 4806'',5\,.\,\text{cos.}\,(t\,.\,43'',564) - 27736'',3\,.\,\text{sin.}\,(t\,.\,100'',757)$$

[4358] $$= 23^{d}28^{m}17^{s},9 - 1121^{s},1 - 5837^{s},6\,.\,\text{cos.}\,(t\,.\,50^{s},396 + 85^{d}42^{m}54^{s})$$
$$+ 1557^{s},3\,.\,\text{cos.}\,(t\,.\,14^{s},115) - 8986^{s},6\,.\,\text{sin.}\,(t\,.\,32^{s},645)\,;$$

$$\psi' = t\,.\,155'',542 + 2^{\circ},92883 - 29288'',3\,.\,\text{cos.}\,t\,.\,(100'',757)$$
$$- 13374'',2\,.\,\text{sin.}\,(t\,.\,43'',564)$$

[4359] $$= t\,.\,50^{s},396 + 2^{d}38^{m}09^{s},4 - 9489^{s},4\,.\,\text{cos.}\,(t\,.\,32^{s},645)$$
$$- 4333^{s},2\,.\,\text{sin.}\,(t\,.\,14^{s},115)\,;$$

[Apparent orbit.]

$$V' = 26^{\circ},0796 - 3460'',3\,.\,\{1 - \text{cos.}\,(t\,.\,43'',564)\}$$
$$- 9769'',2\,.\,\text{sin.}\,(t\,.\,100'',757)$$

[4360] $$= 23^{d}28^{m}17^{s},9 - 1121^{s},1\,.\,\{1 - \text{cos.}\,(t\,.\,14^{s},115)\}$$
$$- 3165^{s},2\,.\,\text{sin.}\,(t\,.\,32^{s},645).$$

We may determine, by means of these formulas, the precession of the equinoxes and the obliquity of the ecliptic, in the interval of ten or twelve hundred years

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[4357h] $$\text{sin.}\,(g't + lt + \tfrac{1}{2}\varpi) = \text{cos.}\,(g't + lt) = \text{cos.}\,(t\,.\,32^{s},645)\,;$$

$lt = t\,.\,50^{s},396$. Substituting these in [4348], it becomes as in [4359], the constant quantity ζ, being taken so as to make $\psi' = 0$, when $t = 0$ [4359]; consequently,

[4357i] $$\zeta = 9489^{s},4 = 2^{d}38^{m}9^{s},4.$$

Lastly, by a similar calculation, we have,

[4357k] $$\frac{g}{l+g}\,.\,c\,.\,\text{cos.}\,\beta = -1121^{s},1\,; \qquad \frac{g'}{l+g'}\,.\,c\,.\,\text{sin.}\,\beta = -3165^{s},2\,;$$
$$\text{cos.}\,(g't + lt + \tfrac{1}{2}\varpi) = -\text{sin.}\,(g't + lt) = -\text{sin.}\,(t\,.\,32^{s},645)\,;$$

substituting these and [4356] in [4349], we get [4360]. The numerical values, given in [4357—4360], are varied by the author in [4614—4617], on account of the changes made in the values of the masses of Venus and Mars. We have already given the formulas of Poisson and Bessel, in [3380p,q].

[4357l]

before, or after the epoch of 1750; *observing to make* t *negative, for any time previous to this epoch.* We may indeed apply the formula to the observations made in the time of Hipparchus; taking into consideration the imperfections of these observations. [4361]

The preceding value of ψ', gives, for the increment of the tropical year, counting from 1750, the following expression,*

$$\text{Increment of the year} = -\,0^{\text{day}},000083568\,.\,\{1-\cos.\,(t\,.\,14^{s},115)\} \\ -\,0^{\text{day}},00042327\,.\,\sin.\,(t\,.\,32^{s},645). \qquad [4362]$$

Hence it follows, that in the time of Hipparchus, or one hundred and twenty-eight years before the Christian era, the tropical year was $12^{\text{sec.}},326$ [$= 10^{s},65$ sexages.] longer than in 1750;† the obliquity of the ecliptic was also greater by $2832'',27 = 917^{s},66$. [4363] [4363′]

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* (2635) Using the same data as the preceding note, we get the numerical values of the two functions [4362c, d], expressed in sexagesimal seconds. These are turned into time by supposing the whole circumference, $360^{d} = 1296000^{s}$, to be described in one year, or $365^{\text{days}},242$; hence we have, [4362a] [4362b]

$$c\,g\,.\cos.\,\beta\,.\left\{\cot.\,h+\frac{l}{l+g}\,.\,\mathrm{tang.}\,h\right\} = -\,0^{s},296527 = -\,0^{\text{day}},000083568; \qquad [4362c]$$

$$c\,g'.\,\sin.\,\beta\,.\left\{\cot.\,h+\frac{l}{l+g'}\,.\,\mathrm{tang.}\,h\right\} = -\,1^{s},501877 = -\,0^{\text{day}},00042327. \qquad [4362d]$$

Substituting these and [4357d], in [4350], we get the general expression of $\frac{d\psi'}{dt}$ [4362f]; which becomes as in [4362g], when $t=0$. Subtracting the first of these expressions from the second, we get the increment of the year [4350′], as in [4362], corresponding to any number t, of years after 1750. [4362e]

$$\frac{d\psi'}{dt} = l - 0^{\text{day}},000083568\,.\cos.\,(t\,.\,14^{s},115) + 0^{\text{day}},00042327\,.\sin.\,(t\,.\,32^{s},645); \qquad [4362f]$$

$$\frac{d\psi'}{dt} = l - 0^{\text{day}},000083568. \qquad [4362g]$$

These numerical values are altered in [4618], in consequence of a change in the values of the masses of Venus and Mars. [4362h]

† (2636) In the year 128 before the Christian era, $t = -128 - 1750 = -1878$; substituting this in the two terms of the expression [4362], we find that the first term becomes, $-\,0^{\text{day}},00000069$, and the second, $+\,0^{\text{day}},00012396$; their sum is $0^{\text{day}},00012327$, as in [4363] nearly. The variation of the obliquity of the ecliptic, in the same time, deduced from [4360], is nearly the same as in [4363′], being expressed by, [4363a] [4363b]

$$-\,1121^{s},1\,.\,\{1-\cos.\,(t\,.\,14^{s},115)\} - 3165^{s},2\,.\,\sin.\,(t\,.\,32^{s},645) \\ = -\,9^{s},2 + 926^{s},9 = 917^{s},7 \text{ nearly}. \qquad [4363c]$$

[4363''] *A remarkable astronomical epoch, is that when the greater axis of the earth's orbit was situated in the line of the equinoxes; because the apparent and mean equinoxes then coincided. We find, by the preceding formulas, that this phenomena took place about 4004 years before the Christian era, and at this epoch most of our chronologists place the creation of the world;* so that, in this point of view, we may consider it as an astronomical epoch. For we have,

Remarkable astronomical epoch, when the equinox and sun's apogee coincide.

[4364]

[4364'] at that time, $t = -5754$; and the preceding expression of ψ' gives,*

[4365] $$\psi' = -79^d\,04^m\,04^s;$$

which is the longitude of the fixed equinox of 1750, referred to the equinox

[4365'] of that time t. The preceding expression of ϖ'', gives, for the longitude of the perigee of the earth's orbit, or of the sun's apogee, referred to the fixed equinox of 1750,†

[4366] $$\varpi'' = 80^d\,15^m\,11^s.$$

This longitude, referred to the equinox of the year 4004 before the Christian

[4367] era, is $1^d\,11^m\,07^s$;‡ hence it follows, that *the time when the longitude of the sun's apogee, counted from the moveable equinox, was nothing, precedes, about*

[4367'] *sixty-nine years, the epoch usually assumed for the creation of the world.* This difference will appear very small, if we take into consideration the imperfections of the preceding expressions of ψ', and ϖ'', when applied to so

[4367''] distant a period, and the uncertainty which still remains relatively to the motion of the equinoxes, and to the assumed values of the masses of the planets.

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* (2637) Putting $t = -5754$, we have $t\,.\,32^s{,}645 = 52^d\,10^m\,39^s$;

[4365a] $$t\,.\,14^s{,}115 = 22^d\,33^m\,38^s; \qquad t\,.\,50^s{,}396 = 80^d\,32^m\,59^s;$$

substituting these in [4359], we get the value of ψ' [4365].

† (2638) Substituting $\varpi'' = 98^d\,37^m\,16^s$ [4081], in [4331], it becomes,

[4366a] $$\varpi'' = 98^d 37^m 16^s + t\,.\,11^s{,}949588 + t^2.\,0^s{,}000079522;$$

and by putting $t = -5754$, it is reduced to $98^d37^m16^s - 19^d5^m58^s + 43^m\,53^s = 80^d50^m11^s$, as in [4366].

‡ (2639) Taking, for the fixed point, the equinox of 1750; the longitude of the moveable equinox, and of the solar apogee, corresponding to the year 4004 before Christ,

[4367a] will be respectively $79^d\,4^m\,4^s$, $80^d\,15^m\,11^s$ [4365, 4366]; the difference of these quantities $1^d\,11^m\,7^s$ represents the distance of the perigee from the equinox at that time. The

[4367b] distance of these points, in the year 1750, was $98^d\,37^m\,16^s$ [4081]; so that in the period of 5754 years, they have approached towards each other, by the quantity,

Another remarkable astronomical epoch, is that when the greater axis of the earth's orbit, was perpendicular to the line of equinoxes; for then the apparent ana mean solstices were united. This second epoch is much nearer to our times; it goes back nearly to the year 1250. For if we suppose $t = -500$, the preceding formulas give $90^d\,1^m$,* for the longitude of the sun's apogee, counted from the moveable equinox. Hence the time when this longitude was 90^d, corresponds very nearly to the beginning of the year 1249. The imperfections of the elements used in this calculation, leaves an uncertainty of at least one year in this result.

Another remarkable epoch, when the equinox and sun's apogee are distant 90^d. [4367'''] [4368] [4368'] [4369]

$$98^d\,37^m\,16^s - 1^d\,11^m\,7^s = 97^d\,26^m\,9^s; \qquad [4367c]$$

being at the rate of about 61^s in a year; and at this rate, the arc $1^d 11^m 7^s$ will be described in about 69 years; so that the equinox and solar apogee must have coincided about the year $4004 + 69 = 4073$ [4367'] before the Christian era, according to the data we have used. [4367d]

* (2640) In the year 1250, we have $t = 1250 - 1750 = -500$; and for this value of t, we get, from [4359, 4366a], $\psi' = -6^d\,57^m$; $\varpi'' = 96^d 58^m$; therefore the solar apogee, in 1250, was distant from the equinox of that time, by the quantity [4368a]

$$96^d\,58^m - 6^d\,57^m = 90^d\,1^m; \qquad [4368b]$$

and as the distance of these points, in 1750, was $98^d 37^m 16^s$ [4367b], the variation of distance, in five hundred years, is $98^d\,37^m\,16^s - 90^d\,1^m = 8^d\,36^m\,16^s$, being about 61^s in a year, as in [4367d]; consequently, the distance of these points must have been 90^d, about one year before the year 1250, or in the year 1249. [4368c]

CHAPTER XI.

THEORY OF MARS.

32. We have, in the case of the maximum * of $\delta V'''$,

[4370] $$\delta\alpha = -(1-\alpha^2)\,.\,\delta V''' ;$$

[4370′] supposing $\frac{r''}{r'''} = \alpha$. If we consider r''' as the only variable quantity in α, we shall have,

[4371] $$\delta r''' = \frac{r'''^2}{r''}\,.\,(1-\alpha^2)\,.\,\delta V'''.$$

[4371′] If we take for r'', r''', the mean distances of the earth and Mars from the sun [4079], and suppose $\delta V''' = \pm 1'' = \pm 0^s{,}324$, we shall get,

Terms which may be neglected.

[4372] $$\delta r''' = \pm 0{,}000002076 ;$$

therefore we may neglect the inequalities of the radius vector r''', whose coefficients are less than $\pm 0{,}000002$. We shall also neglect the inequalities of the motion in Mars in longitude, which are less than a quarter of a centesimal second, or $0^s{,}081$.†

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* (2641) The earth is situated, relatively to Venus, in the same manner as Mars is, relatively to the earth; therefore we may obtain $\delta V'''$, corresponding to Mars [4370], from the calculation made for Venus in [4297, 4298], by merely changing the accents on V, in [4298], which makes it become as in [4370], and using α [4370′]. Now the variation of α [4370′], considering α, r''', as the variable quantities in $\delta\alpha = -\frac{\delta r'''.r''}{r'''^2}$; substituting this in [4370], we get [4371]; and by putting $r'' = a''$. $r''' = a'''$ [4079], using also α [4159], $\delta V'''$ [4371′]; it becomes as in [4372]. [4370a] [4370b]

† (2642) The values [4373, 4374] are computed from the functions [4277a, b], accenting the symbols so as to conform to the present example. [4373a]

Inequalities of Mars, independent of the excentricities.

Inequalities independent of the excentricities.

$$\delta v''' = (1+\mu') . \left\{ \begin{array}{l} 0^s{,}208754 . \sin. (n't - n'''t + \varepsilon' - \varepsilon''') \\ - 0^s{,}024915 . \sin. 2(n't - n'''t + \varepsilon' - \varepsilon''') \\ - 0^s{,}005000 . \sin. 3(n't - n'''t + \varepsilon' - \varepsilon''') \\ - 0^s{,}001368 . \sin. 4(n't - n'''t + \varepsilon' - \varepsilon''') \end{array} \right\}$$

$$+ (1+\mu'') . \left\{ \begin{array}{l} 6^s{,}988832 . \sin. (n''t - n'''t + \varepsilon'' - \varepsilon''') \\ - 0^s{,}968689 . \sin. 2(n''t - n'''t + \varepsilon'' - \varepsilon''') \\ - 0^s{,}183012 . \sin. 3(n''t - n'''t + \varepsilon'' - \varepsilon''') \\ - 0^s{,}058242 . \sin. 4(n''t - n'''t + \varepsilon'' - \varepsilon''') \\ - 0^s{,}023099 . \sin. 5(n''t - n'''t + \varepsilon'' - \varepsilon''') \\ - 0^s{,}010339 . \sin. 6(n''t - n'''t + \varepsilon'' - \varepsilon''') \\ - 0^s{,}004992 . \sin. 7(n''t - n'''t + \varepsilon'' - \varepsilon''') \end{array} \right\} \qquad [4373]$$

$$+ (1+\mu^{iv}) . \left\{ \begin{array}{l} 24^s{,}440843 . \sin. (n^{iv}t - n'''t + \varepsilon^{iv} - \varepsilon''') \\ - 13^s{,}598063 . \sin. 2(n^{iv}t - n'''t + \varepsilon^{iv} - \varepsilon''') \\ - 1^s{,}180288 . \sin. 3(n^{iv}t - n'''t + \varepsilon^{iv} - \varepsilon''') \\ - 0^s{,}172768 . \sin. 4(n^{iv}t - n'''t + \varepsilon^{iv} - \varepsilon''') \\ - 0^s{,}033166 . \sin. 5(n^{iv}t - n'''t + \varepsilon^{iv} - \varepsilon''') \\ - 0^s{,}013422 . \sin. 6(n^{iv}t - n'''t + \varepsilon^{iv} - \varepsilon''') \end{array} \right\}$$

$$+ (1+\mu^{v}) . \left\{ \begin{array}{l} 1^s{,}343754 . \sin. (n^{v}t - n'''t + \varepsilon^{v} - \varepsilon''') \\ - 0^s{,}443668 . \sin. 2(n^{v}t - n'''t + \varepsilon^{v} - \varepsilon''') \\ - 0^s{,}023088 . \sin. 3(n^{v}t - n'''t + \varepsilon^{v} - \varepsilon''') \\ - 0^s{,}001879 . \sin. 4(n^{v}t - n'''t + \varepsilon^{v} - \varepsilon''') \end{array} \right\}.$$

$$\delta r''' = (1+\mu') . \left\{ \begin{array}{l} 0{,}0000016104 \\ + 0{,}0000021947 . \cos. (n't - n'''t + \varepsilon' - \varepsilon''') \\ + 0{,}0000001972 . \cos. 2(n't - n'''t + \varepsilon' - \varepsilon''') \\ + 0{,}0000000418 . \cos. 3(n't - n'''t + \varepsilon' - \varepsilon''') \end{array} \right\} \qquad [4374]$$

$$+(1+\mu'').\left\{\begin{array}{l} \quad 0{,}0000023860 \\ -0{,}0000187564.\cos.\ \ (n''t-n'''t+\varepsilon''-\varepsilon''') \\ +0{,}0000052387.\cos.2(n''t-n'''t+\varepsilon''-\varepsilon''') \\ +0{,}0000011969.\cos.3(n''t-n'''t+\varepsilon''-\varepsilon''') \\ +0{,}0000004169.\cos.4(n''t-n'''t+\varepsilon''-\varepsilon''') \\ +0{,}0000001733.\cos.5(n''t-n'''t+\varepsilon''-\varepsilon''') \\ +0{,}0000000796.\cos.6(n''t-n'''t+\varepsilon''-\varepsilon''') \end{array}\right\}$$

Inequalities independent of the excentricities.

[4374]

$$+(1+\mu^{\mathrm{iv}}).\left\{\begin{array}{l} -0{,}0000066174 \\ +0{,}0000784371.\cos.\ \ (n^{\mathrm{iv}}t-n'''t+\varepsilon^{\mathrm{iv}}-\varepsilon''') \\ -0{,}0000679436.\cos.2(n^{\mathrm{iv}}t-n'''t+\varepsilon^{\mathrm{iv}}-\varepsilon''') \\ -0{,}0000069390.\cos.3(n^{\mathrm{iv}}t-n'''t+\varepsilon^{\mathrm{iv}}-\varepsilon''') \\ -0{,}0000010930.\cos.4(n^{\mathrm{iv}}t-n'''t+\varepsilon^{\mathrm{iv}}-\varepsilon''') \\ -0{,}0000002004.\cos.5(n^{\mathrm{iv}}t-n'''t+\varepsilon^{\mathrm{iv}}-\varepsilon''') \\ -0{,}0000000520.\cos.6(n^{\mathrm{iv}}t-n'''t+\varepsilon^{\mathrm{iv}}-\varepsilon''') \end{array}\right\}$$

$$+(1+\mu^{\mathrm{v}}).\left\{\begin{array}{l} -0{,}0000003173 \\ +0{,}0000047062.\cos.\ \ (n^{\mathrm{v}}t-n'''t+\varepsilon^{\mathrm{v}}-\varepsilon''') \\ -0{,}0000023275.\cos.2(n^{\mathrm{v}}t-n'''t+\varepsilon^{\mathrm{v}}-\varepsilon''') \\ -0{,}0000001399.\cos.3(n^{\mathrm{v}}t-n'''t+\varepsilon^{\mathrm{v}}-\varepsilon''') \\ -0{,}0000000125.\cos.4(n^{\mathrm{v}}t-n'''t+\varepsilon^{\mathrm{v}}-\varepsilon''') \end{array}\right\}.$$

Inequalities depending on the first power of the excentricities.

$$\delta v'''=(1+\mu').\left\{\begin{array}{l} \quad 1^{\mathrm{s}}{,}082545\ .\sin.(2n'''t-n't+2\varepsilon'''-\varepsilon'-\varpi''') \\ -0^{\mathrm{s}}{,}252586\ .\sin.(2n'''t-n't+2\varepsilon'''-\varepsilon'-\varpi') \end{array}\right\}$$

Inequalities depending on the first power of the excentricities.

[4375]

$$+(1+\mu'').\left\{\begin{array}{l} \quad 0^{\mathrm{s}}{,}698649\ .\sin.(n''t+\varepsilon''-\varpi''') \\ -\ 0^{\mathrm{s}}{,}134530\ .\sin.(2n''t-n'''t+2\varepsilon''-\varepsilon'''-\varpi''') \\ -10^{\mathrm{s}}{,}114699\ .\sin.(2n'''t-n''t+2\varepsilon'''-\varepsilon''-\varpi''') \\ +\ 5^{\mathrm{s}}{,}123062\ .\sin.(2n'''t-n''t+2\varepsilon'''-\varepsilon''-\varpi'') \\ -\ 6^{\mathrm{s}}{,}516275\ .\sin.(3n'''t-2n''t+3\varepsilon'''-2\varepsilon''-\varpi''') \\ +\ 0^{\mathrm{s}}{,}846004\ .\sin.(3n'''t-2n''t+3\varepsilon'''-2\varepsilon''-\varpi'') \\ +\ 0^{\mathrm{s}}{,}677748\ .\sin.(4n'''t-3n''t+4\varepsilon'''-3\varepsilon''-\varpi''') \\ -\ 0^{\mathrm{s}}{,}079155\ .\sin.(4n'''t-3n''t+4\varepsilon'''-3\varepsilon''-\varpi'') \\ +\ 0^{\mathrm{s}}{,}119926\ .\sin.(5n'''t-4n''t+5\varepsilon'''-4\varepsilon''-\varpi''') \end{array}\right\}$$

$$+(1+\mu^{\text{iv}})\cdot\left\{\begin{array}{l}
+\ 5^s{,}490297\ .\ \sin.\ (n^{\text{iv}}t+\varepsilon^{\text{iv}}-\varpi''')\ *\\
-\ 5^s{,}367005\ .\ \sin.\ (n^{\text{iv}}t+\varepsilon^{\text{iv}}-\varpi^{\text{iv}})\\
-23^s{,}552332\ .\ \sin.\ (2\,n^{\text{iv}}t-n'''t+2\,\varepsilon^{\text{iv}}-\varepsilon'''-\varpi''')\\
+\ 2^s{,}593100\ .\ \sin.\ (2\,n^{\text{iv}}t-n'''t+2\,\varepsilon^{\text{iv}}-\varepsilon'''-\varpi^{\text{iv}})\\
+\ 2^s{,}296703\ .\ \sin.\ (3\,n^{\text{iv}}t-2\,n'''t+3\,\varepsilon^{\text{iv}}-2\,\varepsilon'''-\varpi''')\\
-\ 3^s{,}568875\ .\ \sin.\ (3\,n^{\text{iv}}t-2\,n'''t+3\,\varepsilon^{\text{iv}}-2\,\varepsilon'''-\varpi^{\text{iv}})\\
+\ 0^s{,}220149\ .\ \sin.\ (4\,n^{\text{iv}}t-3\,n'''t+4\,\varepsilon^{\text{iv}}-3\,\varepsilon'''-\varpi''')\\
-\ 0^s{,}352640\ .\ \sin.\ (4\,n^{\text{iv}}t-3\,n'''t+4\,\varepsilon^{\text{iv}}-3\,\varepsilon'''-\varpi^{\text{iv}})\\
-\ 2^s{,}868651\ .\ \sin.\ (2\,n'''t-\ \ n^{\text{iv}}t+2\,\varepsilon'''-\ \ \varepsilon^{\text{iv}}-\varpi''')\\
-\ 0^s{,}204519\ .\ \sin.\ (2\,n'''t-\ \ n^{\text{iv}}t+2\,\varepsilon'''-\ \ \varepsilon^{\text{iv}}-\varpi^{\text{iv}})\\
+\ 1^s{,}853159\ .\ \sin.\ (3\,n'''t-2\,n^{\text{iv}}t+3\,\varepsilon'''-2\,\varepsilon^{\text{iv}}-\varpi''')\\
+\ 0^s{,}198136\ .\ \sin.\ (4\,n'''t-3\,n^{\text{iv}}t+4\,\varepsilon'''-3\,\varepsilon^{\text{iv}}-\varpi''')
\end{array}\right\}$$ [4375]

$$+(1+\mu^{\text{v}})\cdot\left\{\begin{array}{l}
\ \ \ 0^s{,}143758\ .\ \sin.\ (n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi''')\\
-0^s{,}696926\ .\ \sin.\ (n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi^{\text{v}})\\
-\ 1^s{,}798071\ .\ \sin.\ (2\,n^{\text{v}}t-n'''t+2\,\varepsilon^{\text{v}}-\varepsilon'''-\varpi''')\\
+\ 0^s{,}132176\ .\ \sin.\ (2\,n^{\text{v}}t-n'''t+2\,\varepsilon^{\text{v}}-\varepsilon'''-\varpi^{\text{v}})\\
-0^s{,}100246\ .\ \sin.\ (3\,n^{\text{v}}t-2\,n'''t+3\,\varepsilon^{\text{v}}-2\,\varepsilon'''-\varpi^{\text{v}})\\
-0^s{,}156784\ .\ \sin.\ (2\,n'''t-n^{\text{v}}t+2\,\varepsilon'''-\varepsilon^{\text{v}}-\varpi''')
\end{array}\right\}.$$

Inequalities depending on the first power of the excentricities.

$$\delta r'''=(1+\mu')\cdot\left\{\begin{array}{l}
\ \ \ 0{,}0000044700\ .\ \cos.\ (2\,n'''t-n't+2\,\varepsilon'''-\varepsilon'-\varpi''')\\
-0{,}0000009713\ .\ \cos.\ (2\,n'''t-n't+2\,\varepsilon'''-\varepsilon'-\varpi')
\end{array}\right\}$$

$$+(1+\mu'')\cdot\left\{\begin{array}{l}
-\ 0{,}0000022865\ .\ \cos.\ (n''t+\varepsilon''-\varpi''')\\
+\ 0{,}0000086337\ .\ \cos.\ (2\,n'''t-n''t+2\,\varepsilon'''-\varepsilon''-\varpi''')\\
-\ 0{,}0000031269\ .\ \cos.\ (2\,n'''t-n''t+2\,\varepsilon'''-\varepsilon''-\varpi'')\\
-\ 0{,}0000200331\ .\ \cos.\ (3\,n'''t-2\,n''t+3\,\varepsilon'''-2\,\varepsilon''-\varpi''')\\
+\ 0{,}0000025454\ .\ \cos.\ (3\,n'''t-2\,n''t+3\,\varepsilon'''-2\,\varepsilon''-\varpi'')\\
+\ 0{,}0000030863\ .\ \cos.\ (4\,n'''t-3\,n''t+4\,\varepsilon'''-3\,\varepsilon''-\varpi''')\\
+\ 0{,}0000040239\ .\ \cos.\ (4\,n'''t-3\,n''t+4\,\varepsilon'''-3\,\varepsilon''-\varpi'')
\end{array}\right\}$$ [4376]

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* (2643) The computation of the terms [4375, 4376], is made in the same manner as for Mercury, in [4278a]; accenting the symbols so as to conform to the case under consideration. [4375a]

[4376]
$$+(1+\mu^{\mathrm{iv}})\cdot\left\{\begin{array}{l} 0,0000035825\,.\cos.(n'''t+\varepsilon'''-\varpi''') \\ -0,0000107986\,.\cos.(n^{\mathrm{iv}}t+\varepsilon^{\mathrm{iv}}-\varpi''') \\ +0,0000031431\,.\cos.(n^{\mathrm{iv}}t+\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}) \\ -0,0000599470\,.\cos.(2n^{\mathrm{iv}}t-n'''t+2\varepsilon^{\mathrm{iv}}-\varepsilon'''-\varpi''') \\ +0,0000069892\,.\cos.(2n^{\mathrm{iv}}t-n'''t+2\varepsilon^{\mathrm{iv}}-\varepsilon'''-\varpi^{\mathrm{iv}}) \\ +0,0000114352\,.\cos.(3n^{\mathrm{iv}}t-2n'''t+3\varepsilon^{\mathrm{iv}}-2\varepsilon'''-\varpi''') \\ -0,0000169741\,.\cos.(3n^{\mathrm{iv}}t-2n'''t+3\varepsilon^{\mathrm{iv}}-2\varepsilon'''-\varpi^{\mathrm{iv}}) \\ -0,0000020307\,.\cos.(4n^{\mathrm{iv}}t-3n'''t+4\varepsilon^{\mathrm{iv}}-3\varepsilon'''-\varpi^{\mathrm{iv}}) \\ +0,0000087307\,.\cos.(2n'''t-n^{\mathrm{iv}}t+2\varepsilon'''-\varepsilon^{\mathrm{iv}}-\varpi''') \\ -0,0000063983\,.\cos.(3n'''t-2n^{\mathrm{iv}}t+3\varepsilon'''-2\varepsilon^{\mathrm{iv}}-\varpi''') \end{array}\right\}$$

$$-(1+\mu^{\mathrm{v}})\,.\,0,0000061906\,.\cos.(2n^{\mathrm{v}}t-n'''t+2\varepsilon^{\mathrm{v}}-\varepsilon'''-\varpi''').$$

*Inequalities depending on the squares and products of the excentricities and inclinations of the orbits.**

Inequalities of the second order.

$$\delta v'''=-(1+\mu')\,.\,6^{s},899619\,.\sin.(3n'''t-n't+3\varepsilon'''-\varepsilon'+65^{d}26^{m}15^{s})$$

$$-(1+\mu'')\,.\left\{\begin{array}{l} 1^{s},414532\,.\sin.(3n'''t-n''t+3\varepsilon'''-\varepsilon''+73^{d}11^{m}55^{s}) \\ +4^{s},370903\,.\sin.(4n'''t-2n''t+4\varepsilon'''-2\varepsilon''+67^{d}49^{m}0^{s}) \\ +2^{s},665900\,.\sin.(5n'''t-3n''t+5\varepsilon'''-3\varepsilon''+68^{d}23^{m}0^{s}) \end{array}\right\}$$

[4377]
$$+(1+\mu^{\mathrm{iv}})\,.\left\{\begin{array}{l} -0^{s},462779\,.\sin.(n^{\mathrm{iv}}t+n'''t+\varepsilon^{\mathrm{iv}}+\varepsilon'''-53^{d}07^{m}48^{s}) \\ -1^{s},444122\,.\sin.(2n^{\mathrm{iv}}t+2\varepsilon^{\mathrm{iv}}+60^{d}07^{m}02^{s}) \\ +1^{s},295408\,.\sin.(n^{\mathrm{iv}}t-n'''t+\varepsilon^{\mathrm{iv}}-\varepsilon'''+54^{d}41^{m}32^{s}) \end{array}\right\}.$$

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* (2644) Using the values [4076h], we get very nearly, $3n'''-n'=-12^{\circ}=-\frac{n'''}{18}$;

[4377a] also $3n'''-n''=238^{\circ}$, which is nearly equal to n'''; $4n'''-2n''=51^{\circ}=\frac{n'''}{4}$;

[4377b] $5n'''-3n''=-137^{\circ}=-\frac{n'''}{2}$ nearly. Hence it is evident, that if we proceed in the same manner as in the computation of the similar inequalities of Mercury [4282a, &c.], we must notice the angles depending on these coefficients, in computing the terms of [4377—4380]. For the second of these angles comes under the form [3732],

[4377c] $in''+(2-i)\,.\,n'''=n'''$, supposing $i=-1$; and the others under the form [3733], supposing successively, $i=-1$, $i=-2$, $i=-3$. Lastly, as n^{iv} is small in

The last of these expressions may be connected with the following inequality, computed in [4373], and which is independent of the excentricities,

$$(1+\mu^{\rm iv}) \,.\, 24^{\rm s},440843 \,.\, \sin.(n^{\rm iv}t - n'''t + \varepsilon^{\rm iv} - \varepsilon''');$$ [4378]

their sum, by reduction,* gives the following term of $\delta v'''$,

$$\delta v''' = (1+\mu^{\rm iv}) \,.\, 25^{\rm s},211710 \,.\, \sin.(n^{\rm iv}t - n'''t + \varepsilon^{\rm iv} - \varepsilon''' + 2^{d}\,24^{m}\,11^{s}).$$ [4379]

We have also,

Inequalities of the second order.

$$\delta r''' = -(1+\mu') \,.\, 0,0000023461 \,.\, \cos.(3n'''t - n't + 3\varepsilon''' - \varepsilon' + 64^{d}\,47^{m}\,29^{s})$$
$$+(1+\mu'') \,.\, \left\{\begin{array}{l} 0,0000050403 \,.\, \cos.(3n'''t - n''t + 3\varepsilon''' - \varepsilon'' + 72^{d}\,47^{m}\,00^{s}) \\ +0,0000070248 \,.\, \cos.(4n'''t - 2n''t + 4\varepsilon''' - 2\varepsilon'' - 58^{d}\,51^{m}\,50^{s}) \\ -0,0000075032 \,.\, \cos.(5n'''t - 3n''t + 5\varepsilon''' - 3\varepsilon'' - 68^{d}\,27^{m}\,28^{s}) \end{array}\right\}$$ [4380]
$$+(1+\mu^{\rm iv}) \,.\, \left\{\begin{array}{l} +0,0000080002 \,.\, \cos.(2n^{\rm iv}t + 2\varepsilon^{\rm iv} + 60^{d}\,17^{m}\,52^{s}) \\ +0,0000041488 \,.\, \cos.(n^{\rm iv}t - n'''t + \varepsilon^{\rm iv} - \varepsilon''' + 59^{d}\,8^{m}\,57^{s}) \end{array}\right\}.$$

The last of these quantities may be connected with the following inequality, which is independent of the excentricities [4374],

$$(1+\mu^{\rm iv}) \,.\, 0,0000784371 \,.\, \cos.(n^{\rm iv}t - n'''t + \varepsilon^{\rm iv} - \varepsilon''');$$ [4381]

their sum gives the following term of $\delta r'''$,

$$\delta r''' = (1+\mu^{\rm iv}) \,.\, 0,0000806432 \,.\, \cos.(n^{\rm iv}t - n'''t + \varepsilon^{\rm iv} - \varepsilon''' + 2^{d}\,31^{m}\,55^{s}).$$ [4382]

The inequalities of the motion of Mars, in latitude, are hardly sensible.

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comparison with n''', their sum $n^{\rm iv}+n'''$, is very nearly equal to n''', so that this angle comes under the form [3732] $i\,n^{\rm iv}+(2-i)\,.\,n'''$, supposing $i=1$; and [4377d] produces the term of [4377], depending on the angle $n^{\rm iv}t+n'''t$. If we suppose $i=2$, in the same expression [4377d], it becomes $2n^{\rm iv}$; now, as this is small in comparison with n''', it comes under the form [3733], and produces the terms of [4377, 4380], depending on the angle $2n^{\rm iv}t$. The quantity $n^{\rm iv}-n'''$ differs but little from $-n'''$, and comes under the first form [3732], depending on the angle $n^{\rm iv}t-n'''t$ [4377, 4380].

* (2615) The term $(1+\mu^{\rm iv})\,.\,24^{\rm s},440843\,.\,\sin.(n^{\rm iv}t-n'''t+\varepsilon^{\rm iv}-\varepsilon''')$ [4373] may be added to the term $(1+\mu^{\rm iv})\,.\,1^{\rm s},295103\,.\,\sin.(n^{\rm iv}t-n'''t+\varepsilon^{\rm iv}-\varepsilon'''+54^{d}\,41^{m}\,32^{s})$; and the sum reduced to one single term [4379], by a calculation similar to that in [4282h—l]. In like manner the terms of [4374, 4380], depending on the angle $n^{\rm iv}t-n'''t$, may be reduced to one single term of the form [4382]. [4380a]

[4383] Putting Π^{iv} equal to the longitude of the ascending node of Jupiter's orbit upon that of Mars, we find,*

[4384] $$\delta s''' = (1+\mu^{\text{iv}}) \cdot \left\{ \begin{array}{l} 0^{\text{s}},094394 \,.\, \sin.(n^{\text{iv}}t + \varepsilon^{\text{iv}} - \Pi^{\text{iv}}) \\ 0^{\text{s}},403269 \,.\, \sin.(2\,n^{\text{iv}}t - n'''t + 2\,\varepsilon^{\text{iv}} - \varepsilon''' - \Pi^{\text{iv}}) \end{array} \right\}.$$

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* (2646) The term of $\delta s'''$, depending on the attraction of Jupiter, may be derived from
[4384a] the formula [4295b], by adding two accents to the quantities s', a', n', ε', a'', n'', ε'', m''; also supposing γ to represent the inclination, and Π the longitude of the node of Jupiter's orbit upon that of Mars [4295c]. The term independent of Σ produces the first term of
[4384b] [4384], and the term under the sign Σ, corresponding to $i=2$, gives the second term; using $B^{(1)} = \frac{1}{a^{\text{iv}\,3}} \cdot b_{\frac{3}{2}}^{(1)}$ [1006, 4190].

CHAPTER XII.

THEORY OF JUPITER.

33. The reciprocal action of the planets, upon each other, and upon the sun, is most sensible in the theory of Jupiter and Saturn; and we shall now proceed to show that the greatest inequalities of the planetary system depend on this cause. The equation [4371], [4384']

$$\delta r''' = \frac{r'''^2}{r''} \cdot (1 - \alpha^2) \cdot \delta V''', \qquad [4385]$$

corresponding to Mars, becomes for Jupiter,

$$\delta r^{iv} = \frac{r^{iv\,2}}{r''} \cdot (1 - \alpha^2) \cdot \delta V^{iv}. \qquad [4386]$$

If we take for r'', r^{iv}, the mean distances of the earth and Jupiter from the sun [4079], and suppose $\delta V^{iv} = \pm 1'' = \pm 0^s,324$, we shall obtain,

$$\delta r^{iv} = \mp 0,0000409225. \qquad [4387]$$

Therefore we may neglect the inequalities of δr^{iv}, which are below $\mp 0,000041$. We shall also omit the inequalities of Jupiter's motion in longitude, or latitude, which are less than a quarter of a centesimal second, or $0^s,081$. [4387']

*Inequalities of Jupiter, independent of the excentricities.**

Inequalities independent of the excentricities.

$$\delta v^{iv} = (1 + \mu'') \cdot \left\{ \begin{array}{l} 0^s,120833 \,.\, \sin. \ (n''t - n^{iv}t + \varepsilon'' - \varepsilon^{iv}) \\ -0^s,000086 \,.\, \sin. \,2(n''t - n^{iv}t + \varepsilon'' - \varepsilon^{iv}) \end{array} \right\}$$

* (2647) The inequalities [4388, 4389], are deduced from [4277a, b], increasing by four the accents on the symbols, to conform to the present case, and using the data [4388a]

$$+(1+\mu^{v}) \cdot \left\{\begin{array}{l} 82^{s},811711 \,.\, \sin. \;(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,204^{s},406374 \,.\, \sin.2(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,17^{s},071564 \,.\, \sin.3(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,3^{s},926329 \,.\, \sin.4(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,1^{s},210573 \,.\, \sin.5(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0^{s},428420 \,.\, \sin.6(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0^{s},170923 \,.\, \sin.7(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0^{s},076086 \,.\, \sin.8(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0^{s},041273 \,.\, \sin.9(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \end{array}\right\}$$

[4388]

$$+(1+\mu^{vi}) \cdot \left\{\begin{array}{l} 1^{s},051737 \,.\, \sin. \;(n^{vi}t - n^{iv}t + \varepsilon^{vi} - \varepsilon^{iv}) \\ -\,0^{s},427296 \,.\, \sin.2(n^{vi}t - n^{iv}t + \varepsilon^{vi} - \varepsilon^{iv}) \\ -\,0^{s},044085 \,.\, \sin.3(n^{vi}t - n^{iv}t + \varepsilon^{vi} - \varepsilon^{iv}) \\ -\,0^{s},005977 \,.\, \sin.4(n^{vi}t - n^{iv}t + \varepsilon^{vi} - \varepsilon^{iv}) \end{array}\right\}.$$

Inequalities independent of the excentricities.

[4389]

$$\delta r^{iv} = (1+\mu^{v}) \cdot \left\{\begin{array}{l} -\,0,0000620586 \\ +\,0,0006768760 \,.\, \cos. \;(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0,0028966200 \,.\, \cos.2(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0,0003021367 \,.\, \cos.3(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0,0000782514 \,.\, \cos.4(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0,0000258952 \,.\, \cos.5(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0,0000094779 \,.\, \cos.6(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0,0000037560 \,.\, \cos.7(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0,0000014781 \,.\, \cos.8(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -\,0,0000004799 \,.\, \cos.9(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \end{array}\right\}.$$

Inequalities depending on the first power of the excentricities.

Several of these inequalities are of considerable magnitude, so that it becomes necessary to notice the variations of their coefficients; which we

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[4061, &c.]. The term depending on $\sin.(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv})$, being computed, by means of the formula [4277n], is found to be nearly the same as in the first line of this page, and has the same sign; therefore the remark made in the Philosophical Transactions for 1831, page 65, that the sign of this coefficient is negative, is incorrect.

[4388b]

shall do, in those terms of the expression of δv^{iv} which exceed $100''$, or $32^{s},4$. The coefficients of the inequalities depending on ϖ^{iv}, have for a [4389′]
factor the excentricity e^{iv};* therefore, by putting one of these coefficients equal to Ae^{iv}, its variation will be $Ae^{\mathrm{iv}}.\frac{\delta e^{\mathrm{iv}}}{e^{\mathrm{iv}}}$. We shall find, in [4407], [4389″]
that if we include even the quantities depending on the square of the disturbing force [4404, &c.], of which we have given the analytical expression in [3910], we shall have,

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* (2648) The terms of δv^{iv}, δr^{iv} [4392, 4393], were computed from those of δv, δr [1021, 1020], depending on e, e'; changing m, a, e, ϖ, ε, n, into [4390a]
m^{iv}, a^{iv}, e^{iv}, ϖ^{iv}, $\varepsilon^{\mathrm{iv}}$, n^{iv}, respectively. In computing the disturbing force of Saturn, we must also change the symbols m', a', &c. into m^{v}, a^{v}, &c.; and in computing that of Uranus, we must change them into m^{vi}, a^{vi}, &c. We shall neglect the terms containing the arc of circle nt, without the signs of sine and cosine, as is done in [1023, 1024]. In [4390b]
this notation, the angle ϖ^{iv}, is evidently connected with a coefficient having the factor e^{iv}; and the angle ϖ^{v}, with the factor e^{v}; as in [4389′, 4390′]. The variations of e^{iv}, e^{v}, [4390c]
are given in [4407]; and if we retain only the first power of the time t, they will be as in [4390, 4391]. For an example of the method of computing these variations, we shall take the largest term of δv^{iv} [4392], which arises from the substitution of the value of $i=2$, [4390d]
in the term multiplied by e, or e^{iv} [1021]; so that this term becomes,

$$n^{\mathrm{iv}}m^{\mathrm{v}}.\frac{F^{(2)}}{2n^{\mathrm{v}}-n^{\mathrm{iv}}}.e^{\mathrm{iv}}.\sin.(2n^{\mathrm{v}}t-n^{\mathrm{iv}}t+2\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}). \qquad [4390e]$$

Substituting the values of the elements [4061, 4077, 4081], and that of $F^{(2)}$ deduced from $F^{(i)}$ [1019], we find that the coefficient becomes, as in [4392],

$$-138^{s},373337=Ae^{\mathrm{iv}} \quad [4389']. \qquad [4390f]$$

This is to be multiplied by $\frac{\delta e}{e^{\mathrm{iv}}}$, to obtain the expression $A\delta e^{\mathrm{iv}}$. Now, $\delta e^{\mathrm{iv}}=t.0^{s},329487$ [4390], being divided by the radius in seconds 206265^{s}, becomes,

$$\delta e^{\mathrm{iv}}=t.0,0000015974; \qquad [4390g]$$

dividing this by e^{iv} [4080], we get,

$$\frac{\delta e^{\mathrm{iv}}}{e^{\mathrm{iv}}}=t.0,000033226; \qquad [4390h]$$

multiplying this by Ae^{iv} [4390f], we finally obtain,

$$A\delta e^{\mathrm{iv}}=-t.0^{s},004598. \qquad [4390i]$$

Connecting this with Ae^{iv} [4390f], we get the coefficient of the term depending on the angle $2n^{\mathrm{v}}t-n^{\mathrm{iv}}t+2\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}$ [4392]. In the same way the variations of three of

[4390] $$\delta e^{iv} = t \,.\, 0^s{,}329487.$$

In like manner, the coefficients of the inequalities depending on ϖ^{v}, have

[4390'] the factor e^{v}; and by putting $B\,e^{v}$ for one of the coefficients, its variation will be $B\,e^{v}.\frac{\delta e^{v}}{e^{v}}$, and we shall find, as in [4407], that

[4391] $$\delta e^{v} = -t \,.\, 0^s{,}642968.$$

This being premised, we obtain,

Inequalities depending on the first power of the excentricities.

[4392]
$$\delta v^{iv} = (1+\mu^{v}) \,.\, \left\{ \begin{array}{l}
8^s{,}608489 \,.\, \sin.\,(n^{v}t + \varepsilon^{v} - \varpi^{iv}) \\
-\; 9^s{,}692385 \,.\, \sin.\,(n^{v}t + \varepsilon^{v} - \varpi^{v}) \\
-(138^s{,}373337 + t.0^s{,}0045985).\sin.(2n^{v}t - n^{iv}t + 2\varepsilon^{v} - \varepsilon^{iv} - \varpi^{iv}) \\
+\;(56^s{,}634099 - t.0^s{,}0031398).\sin.(2n^{v}t - n^{iv}t + 2\varepsilon^{v} - \varepsilon^{iv} - \varpi^{v}) \\
-\;(44^s{,}460822 + t.0^s{,}0014775).\sin.(3n^{v}t - 2n^{iv}t + 3\varepsilon^{v} - 2\varepsilon^{iv} - \varpi^{iv}) \\
+\;(84^s{,}942569 - t.0^s{,}0047094).\sin.(3n^{v}t - 2n^{iv}t + 3\varepsilon^{v} - 2\varepsilon^{iv} - \varpi^{v}) \\
+\; 7^s{,}925312 \,.\, \sin.\,(4n^{v}t - 3n^{iv}t + 4\varepsilon^{v} - 3\varepsilon^{iv} - \varpi^{iv}) \\
-15^s{,}629621 \,.\, \sin.\,(4n^{v}t - 3n^{iv}t + 4\varepsilon^{v} - 3\varepsilon^{iv} - \varpi^{v}) \\
+\; 1^s{,}047717 \,.\, \sin.\,(5n^{v}t - 4n^{iv}t + 5\varepsilon^{v} - 4\varepsilon^{iv} - \varpi^{iv}) \\
-\; 2^s{,}781664 \,.\, \sin.\,(5n^{v}t - 4n^{iv}t + 5\varepsilon^{v} - 4\varepsilon^{iv} - \varpi^{v}) \\
+\; 0^s{,}497251 \,.\, \sin.\,(6n^{v}t - 5n^{iv}t + 6\varepsilon^{v} - 5\varepsilon^{iv} - \varpi^{iv}) \\
-\; 0^s{,}913302 \,.\, \sin.\,(6n^{v}t - 5n^{iv}t + 6\varepsilon^{v} - 5\varepsilon^{iv} - \varpi^{v}) \\
+\; 0^s{,}149277 \,.\, \sin.\,(7n^{v}t - 6n^{iv}t + 7\varepsilon^{v} - 6\varepsilon^{iv} - \varpi^{iv}) \\
-\; 0^s{,}325592 \,.\, \sin.\,(7n^{v}t - 6n^{iv}t + 7\varepsilon^{v} - 6\varepsilon^{iv} - \varpi^{v}) \\
-\; 5^s{,}208122 \,.\, \sin.\,(2n^{iv}t - n^{v}t + 2\varepsilon^{iv} - \varepsilon^{v} - \varpi^{iv}) \\
-\; 0^s{,}569738 \,.\, \sin.\,(2n^{iv}t - n^{v}t + 2\varepsilon^{iv} - \varepsilon^{v} - \varpi^{v}) \\
+12^s{,}876650 \,.\, \sin.\,(3n^{iv}t - 2n^{v}t + 3\varepsilon^{iv} - 2\varepsilon^{v} - \varpi^{iv}) \\
-\; 0^s{,}352399 \,.\, \sin.\,(3n^{iv}t - 2n^{v}t + 3\varepsilon^{iv} - 2\varepsilon^{v} - \varpi^{v}) \\
+\; 1^s{,}287482 \,.\, \sin.\,(4n^{iv}t - 3n^{v}t + 4\varepsilon^{iv} - 3\varepsilon^{v} - \varpi^{iv}) \\
-\; 0^s{,}172892 \,.\, \sin.\,(4n^{iv}t - 3n^{v}t + 4\varepsilon^{iv} - 3\varepsilon^{v} - \varpi^{v}) \\
+\; 0^s{,}356627 \,.\, \sin.\,(5n^{iv}t - 4n^{v}t + 5\varepsilon^{iv} - 4\varepsilon^{v} - \varpi^{iv}) \\
-\; 0^s{,}083189 \,.\, \sin.\,(5n^{iv}t - 4n^{v}t + 5\varepsilon^{iv} - 4\varepsilon^{v} - \varpi^{v})
\end{array} \right\}$$

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the other large terms of [4392] are computed. The variations of the remaining ones are too small to be noticed.

$$(1+\mu^{vi}) \cdot \left\{ \begin{array}{l} 0^s{,}123506 \,.\, \sin.\, (n^{vi} t + \varepsilon^{vi} - \varpi^{iv}) \\ -0^s{,}235240 \,.\, \sin.\, (n^{vi} t + \varepsilon^{vi} - \varpi^{vi}) \\ -0^s{,}533079 \,.\, \sin.\, (2\, n^{vi} t - n^{iv} t + 2\, \varepsilon^{vi} - \varepsilon^{iv} - \varpi^{iv}) \\ +\, 0^s{,}102673 \,.\, \sin.\, (2\, n^{vi} t - n^{iv} t + 2\, \varepsilon^{vi} - \varepsilon^{iv} - \varpi^{vi}) \\ -0^s{,}127963 \,.\, \sin.\, (3\, n^{vi} t - 2\, n^{iv} t + 3\, \varepsilon^{vi} - 2\varepsilon^{iv} - \varpi^{vi}) \end{array} \right\}.$$

Inequalities depending on the first power of the excentricities.

$$\delta r^{iv} = (1+\mu^{v}) \cdot \left\{ \begin{array}{l} 0{,}0000206111 \,.\, \cos.\, (n^{iv} t + \varepsilon^{iv} - \varpi^{iv}) \\ -0{,}0000795246 \,.\, \cos.\, (n^{v} t + \varepsilon^{v} - \varpi^{iv}) \\ +\, 0{,}0000492096 \,.\, \cos.\, (n^{v} t + \varepsilon^{v} - \varpi^{v}) \\ -0{,}0002922130 \,.\, \cos.\, (2\, n^{v} t - n^{iv} t + 2\, \varepsilon^{v} - \varepsilon^{iv} - \varpi^{iv}) \\ +\, 0{,}0001688085 \,.\, \cos.\, (2\, n^{v} t - n^{iv} t + 2\, \varepsilon^{v} - 2\, \varepsilon^{iv} - \varpi^{v}) \\ -0{,}0004584483 \,.\, \cos.\, (3\, n^{v} t - 2\, n^{iv} t + 3\, \varepsilon^{v} - 2\, \varepsilon^{iv} - \varpi^{iv}) \\ +\, 0{,}0009047822 \,.\, \cos.\, (3\, n^{v} t - 2\, n^{iv} t + 3\, \varepsilon^{v} - 2\, \varepsilon^{iv} - \varpi^{v}) \\ +\, 0{,}0001259429 \,.\, \cos.\, (4\, n^{v} t - 3\, n^{iv} t + 4\, \varepsilon^{v} - 3\, \varepsilon^{iv} - \varpi^{iv}) \\ -0{,}0002424413 \,.\, \cos.\, (4\, n^{v} t - 3\, n^{iv} t + 4\, \varepsilon^{v} - 3\, \varepsilon^{iv} - \varpi^{v}) \\ +\, 0{,}0000268383 \,.\, \cos.\, (5\, n^{v} t - 4\, n^{iv} t + 5\, \varepsilon^{v} - 4\, \varepsilon^{iv} - \varpi^{iv}) \\ -0{,}0000516048 \,.\, \cos.\, (5\, n^{v} t - 4\, n^{iv} t + 5\, \varepsilon^{v} - 4\, \varepsilon^{iv} - \varpi^{v}) \\ +\, 0{,}0000579151 \,.\, \cos.\, (2\, n^{iv} t - n^{v} t + 2\, \varepsilon^{iv} - \varepsilon^{v} - \varpi^{iv}) \\ -0{,}0001346530 \,.\, \cos.\, (3\, n^{iv} t - 2\, n^{v} t + 3\, \varepsilon^{iv} - 2\, \varepsilon^{v} - \varpi^{iv}) \end{array} \right\}. \qquad [4393]$$

*Inequalities depending on the squares and products of the excentricities and inclinations.**

Inequalities of the second order.

$$\delta v^{iv} = (1+\mu^{v}) \cdot \left\{ \begin{array}{l} 1^s{,}003681 \,.\, \sin.\, (n^{v} t + n^{iv} t + \varepsilon^{v} + \varepsilon^{iv} + 45^d\, 29^m\, 22^s) \\ -\; 5^s{,}578707 \,.\, \sin.\, (2\, n^{v} t + 2\, \varepsilon^{v} + 15^d\, 56^m\, 24^s) \\ +\, 11^s{,}724245 \,.\, \sin.\, (3\, n^{v} t - n^{iv} t + 3\, \varepsilon^{v} - \varepsilon^{iv} + 79^d\, 39^m\, 48^s) \\ -18^s{,}075283 \,.\, \sin.\, (4\, n^{v} t - 2\, n^{iv} t + 4\, \varepsilon^{v} - 2\, \varepsilon^{iv} - 57^d\, 12^m\, 26^s) \\ +(169^s{,}265895 - t.0^s{,}004277) \,.\, \sin. \left(\begin{array}{l} 3\, n^{iv} t - 5\, n^{v} t + 3\, \varepsilon^{iv} - 5\, \varepsilon^{v} \\ +55^d 40^m 49^s + t.50^s{,}508360 \end{array} \right) \\ +\; 1^s{,}647140 \,.\, \sin.\, (6\, n^{v} t - 4\, n^{iv} t + 6\, \varepsilon^{v} - 4\, \varepsilon^{iv} - 54^d 25^m 48^s) \\ +\; 2^s{,}476404 \,.\, \sin.\, (n^{v} t - n^{iv} t + \varepsilon^{v} - \varepsilon^{iv} + 43^d 17^m 01^s) \\ -\; 6^s{,}287997 \,.\, \sin.\, (2\, n^{v} t - 2\, n^{iv} t + 2\, \varepsilon^{v} - 2\, \varepsilon^{iv} + 42^d 40^m 44^s) \end{array} \right\}. \qquad [4394]$$

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* (2649) The calculation of the six first terms in [4394] is made in exactly the same way as for Mercury, in [4282a—b]. The coefficient of the angle $3\, n^{iv} t - 5\, n^{v} t$, being [4394a]

These two last inequalities being connected with the two following,

[4395] $$(1+\mu^{v}) \cdot \left\{ \begin{array}{l} 82^{s}{,}811711 \,.\, \sin.\ (n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \\ -204^{s}{,}406374 \,.\, \sin.\, 2\,(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv}) \end{array} \right\};$$

which are found in [4388], among the terms independent of the excentricities, produce the terms,*

[4396] $$\delta v^{iv} = (1+\mu^{v}) \cdot \left\{ \begin{array}{l} -\ 84^{s}{,}628936 \,.\, \sin.\, (n^{iv}t - n^{v}t + \varepsilon^{iv} - \varepsilon^{v} - 1^{d}08^{m}53^{s}) \\ +\ 209^{s}{,}098224 \,.\, \sin.\, (2n^{iv}t - 2n^{v}t + 2\varepsilon^{iv} - 2\varepsilon^{v} - 1^{d}09^{m}58^{s}) \end{array} \right\}.$$

Then we have [4394d],

Inequalities of the second order.

[4397] $$\delta r^{iv} = (1+\mu^{v}) . \left\{ \begin{array}{l} 0{,}0000822415 \,.\, \cos.\, (2n^{v}t + 2\varepsilon^{v} + 11^{d}00^{m}55^{s}) \\ +\ 0{,}0000226252 \,.\, \cos.\, (3n^{v}t - n^{iv}t + 3\varepsilon^{v} - \varepsilon^{iv} - 21^{d}47^{m}18^{s}) \\ -\ 0{,}0001010533 \,.\, \cos.\, (4n^{v}t - 2n^{iv}t + 4\varepsilon^{v} - 2\varepsilon^{iv} - 51^{d}04^{m}04^{s}) \\ -(0{,}0021114502 - t . 0{,}0000000005323) \cdot \cos. \left(\begin{array}{l} 3n^{iv}t - 5n^{v}t + 3\varepsilon^{iv} - 5\varepsilon^{v} \\ +55^{d}35^{m}51^{s} + t . 50^{s}{,}4144 \end{array} \right) \\ -\ 0{,}0000652204 \,.\, \cos.\, (2n^{v}t - 2n^{iv}t + 2\varepsilon^{v} - 2\varepsilon^{iv} + 54^{d}08^{m}52^{s}) \end{array} \right\}.$$

If we connect the last of these inequalities with the following,

[4398] $$-\ (1+\mu^{v}) \,.\, 0{,}0028966200 \,.\, \cos.\, 2\,(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv});$$

which is found in [4389], among the terms that are independent of the excentricities, we obtain the equivalent expression,

[4399] $$\delta r^{iv} = -\,(1+\mu^{v}) \,.\, 0{,}0029251892 \,.\, \cos.\, (2n^{iv}t - 2n^{v}t + 2\varepsilon^{iv} - 2\varepsilon^{v} - 1^{d}02^{m}08^{s}).$$

The preceding inequalities of δv^{iv}, are calculated by the formulas [3711, 3715, 3728, 3729]; excepting, however, that which depends on the angle
[4400] $3n^{iv}t - 5n^{v}t$; observing that $5n^{v} - 2n^{iv}$, is a very small coefficient, as appears from the ratio which obtains between the mean motions of Jupiter

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large, its variations must be noticed and computed by the method pointed out in
[4394b] [4017—4021]. The other coefficients are less than $32^{s}{,}4$, and their variations are neglected, as in [4389', &c.]. The two last terms of [4394] correspond to [3729, 3728];
[4394c] using $i = \pm 1$, or $i = \pm 2$; the values of N being found, by means of the formulas [3753 — 3755'''], and the corresponding terms are to be connected together, like those depending on M, in [4282h — l]. In like manner, the four first terms of [4397] are
[4394d] deduced from [3711]; the last term from [3728]; noticing always the variations of the elements in the greatest coefficients, as is done with the terms of δv.

[4396a] * (2650) This computation is made in the usual manner, as in [4380a].

and Saturn [4076h]; so that the angle $3\,n^{\text{iv}}t - 5\,n^{\text{v}}t$ differs but very little from $n^{\text{iv}}t$, as in [3712, &c.]; in consequence of which, we have used the formulas [3714, 3715], in computing this inequality, by the method given in [4017 — 4021]. [4400′]

Inequalities depending on the powers and products of three and five dimensions of the excentricities and inclinations of the orbits, and on the square of the disturbing force.

The great inequality of Jupiter, is calculated by the formulas [3809—3868; 3910—4027]. We find, from [3836—3841],

$$\begin{aligned} a^{\text{v}}.M^{(0)} &= -\,5{,}2439100\,.\,m^{\text{v}}\,; \\ a^{\text{v}}.M^{(1)} &= \quad 9{,}6074688\,.\,m^{\text{v}}\,; \\ a^{\text{v}}.M^{(2)} &= -\,5{,}8070750\,.\,m^{\text{v}}\,; \\ a^{\text{v}}.M^{(3)} &= \quad 1{,}1620283\,.\,m^{\text{v}}\,; \\ a^{\text{v}}.M^{(4)} &= -\,0{,}6385781\,.\,m^{\text{v}}\,; \\ a^{\text{v}}.M^{(5)} &= \quad 0{,}3320740\,.\,m^{\text{v}}. \end{aligned} \qquad [4401]$$

Inequalities of the third order.

Hence we find, at the epoch 1750,*

$$\begin{aligned} a^{\text{v}}.P &= \quad 0{,}0001093026\,; \\ a^{\text{v}}.P' &= -\,0{,}0010230972. \end{aligned} \qquad [4402]$$

We must find the values of the same quantities in 2250 and 2750. For this purpose it is necessary to determine the values of e^{iv}, e^{v}, ϖ^{iv}, ϖ^{v}, γ, Π, in series, ascending according to the powers of the time; continuing the series so far as to include the second power of t. We must, in the first place, calculate, by the formulas [3910—3924], the secular variations of δe^{iv}, δe^{v}, $\delta\varpi^{\text{iv}}$, $\delta\varpi^{\text{v}}$, depending on the square of the disturbing force; and we shall obtain, for these variations,† [4402′]

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* (2651) The values of $a^{\text{v}}P$, $a^{\text{v}}P'$ [4402], are deduced from [3842, 3843]; adding *four* accents to the letters m, a, e, ϖ, m', a', e', &c. to conform to the present notation, and then using the numerical values [4061, 4077, 4079, 4080, &c.]. [4402a]

† (2652) The value of δe^{iv} [4403], is computed from the part of δe [3910], depending on the time t, without the signs of *sine* and *cosine*; adding *four* accents to the letters m, a, e, m', a', e', &c. to conform to the case now under consideration. $\delta\varpi^{\text{iv}}$ [4403], is [4403a]

[4403]
$$\begin{aligned}
\delta e^{\text{iv}} &= \quad t\,.\,0^{s},052278\,; \\
\delta \varpi^{\text{iv}} &= \quad t\,.\,0^{s},352941\,; \\
\delta e^{\text{v}} &= -t\,.\,0^{s},102763\,; \\
\delta \varpi^{\text{v}} &= \quad t\,.\,3^{s},242722.
\end{aligned}$$

[4404] The coefficients of t, in these expressions, represent the parts of $\frac{de^{\text{iv}}}{dt}$, $\frac{d\varpi^{\text{iv}}}{dt}$, $\frac{de^{\text{v}}}{dt}$, $\frac{d\varpi^{\text{v}}}{dt}$ [4404a, b, c], depending on the square of the disturbing force.* Adding them respectively to the parts of the same quantities, determined in [4246, 4247], we obtain the entire values in 1750,

[4405]
$$\begin{aligned}
\frac{de^{\text{iv}}}{dt} &= \quad 0^{s},329487\,; \\
\frac{d\varpi^{\text{iv}}}{dt} &= \quad 6^{s},952808\,; \\
\frac{de^{\text{v}}}{dt} &= -\,0^{s},642968\,; \\
\frac{d\varpi^{\text{v}}}{dt} &= \quad 19^{s},355448.
\end{aligned}$$

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obtained from the like parts of $\delta\varpi$ [3911]. The expressions δe^{v}, $\delta\varpi^{\text{v}}$ [4403], are deduced from [3922, 3923], by making the same additional number of accents to the letters, and then substituting the values of these elements [4061, 4077, 4079, &c.].

* (2653) We have, as in [4330a], $e^{\text{iv}} = \mathrm{e}^{\text{iv}} + t\,.\,\frac{d\,\mathrm{e}^{\text{iv}}}{dt} + \frac{1}{2}\,t^2.\,\frac{d^2\,\mathrm{e}^{\text{iv}}}{dt^2}$; e^{iv} in the second

[4404a] member, being the value of e^{iv}, at the epoch; and by putting for $e^{\text{iv}} - \mathrm{e}^{\text{iv}}$, its value δe^{iv}, we get,

[4404b]
$$\delta e^{\text{iv}} = t\,.\,\frac{d\,\mathrm{e}^{\text{iv}}}{dt} + \tfrac{1}{2}\,t^2.\,\frac{d^2\,\mathrm{e}^{\text{iv}}}{dt^2}.$$

In like manner we have,

[4404c]
$$\delta e^{\text{v}} = t\,.\,\frac{d\,\mathrm{e}^{\text{v}}}{dt} + \tfrac{1}{2}\,t^2.\,\frac{d^2\,\mathrm{e}^{\text{v}}}{dt^2} + \&c.; \qquad \delta\varpi^{\text{v}} = t\,.\,\frac{d\varpi^{\text{v}}}{dt} + \tfrac{1}{2}\,t^2.\,\frac{d^2\,\varpi}{dt^2} + \&c.$$

The coefficients of t, $\frac{1}{2}t^2$, in the second members of these expressions, correspond to the epoch. The coefficients of the first power of t, in these expressions, are composed of two parts, namely, those computed in [4246, 4247], and those depending on the square of the

[4404d] disturbing masses, computed in [4403]; the sums of the corresponding parts give the coefficients, respectively, as in [4405]. Thus,

$$\frac{de^{\text{iv}}}{dt} = 0^{s},052278 + \tfrac{1}{2}\times 0^{s},554418 = 0^{s},329487, \text{ \&c. as in [4405].}$$

We obtain, by the same method, their values in 1950, and find, at this epoch,*

$$\frac{d\,e^{\text{iv}}}{dt} = 0^s{,}326172\,;$$
$$\frac{d\,\varpi^{\text{iv}}}{dt} = 7^s{,}053178\,;$$
$$\frac{d\,e^{\text{v}}}{dt} = -\,0^s{,}648499\,;$$
$$\frac{d\,\varpi^{\text{v}}}{dt} = 19^s{,}424739.$$
[4406]

From these we get, as in [3850, &c. 3850c], the following expressions of e^{iv}, ϖ^{iv}, e^{v}, ϖ^{v}; for any time whatever;

General values of e^{iv}, e^{v}, ϖ^{iv}, ϖ^{v}.

$$e^{\text{iv}} = e^{\text{iv}} + t\,.\;0^s{,}329487 - t^2.\,0^s{,}0000082871\,;$$
$$\varpi^{\text{iv}} = \varpi^{\text{iv}} + t\,.\;6^s{,}952808 + t^2.\,0^s{,}0002509259\,;$$
$$e^{\text{v}} = e^{\text{v}} - t\,.\;0^s{,}642968 - t^2.\,0^s{,}0000138275\,;$$
$$\varpi^{\text{v}} = \varpi^{\text{v}} + t\,.\,19^s{,}355448 + t^2.\,0^s{,}0001732274\,;$$
[4407]

the values of e^{iv}, ϖ^{iv}, e^{v}, ϖ^{v}, *in the second members of these equations,* [4407′]
correspond to the year 1750.

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* (2654) The calculation of the annual variations of the elements [4406], for the year 1950. is made in the same manner as in [4405], using the expressions of e^{iv}, e^{v}, ϖ^{iv}, ϖ^{v}, [4406a]
corresponding to 1950. These elements are obtained, very nearly, by means of the annual decrements [4405], which give, with sufficient accuracy, the required values, when t does not exceed 200. Thus the increment of e^{iv}, corresponding to $t = 200$, is

$$200 \times 0^s{,}329487 = 65^s{,}8 \text{ nearly } [4405]\,;$$ [4406b]

being the same as the term depending on the first power of t, in the expression of e^{iv} [4407]. The term depending on t^2, in this last expression, is very small, being represented by

$$-\,200^2 \times 0^s{,}0000082871 = -\,0^s{,}3 \text{ nearly}\,;$$ [4406c]

which is about $\frac{1}{200}$ part of the term corresponding to the first power of t. Similar remarks may be made relative to the values of e^{v}, ϖ^{iv}, ϖ^{v}. If these calculations were to be repeated, in consequence of any changes in the assumed values of the masses of the planets, we could take into consideration the parts depending on t^2, as they are given in [4407]; and by this [4406d]
means we might obtain, by successive operations, corrected values of the elements. This process is the same as that so frequently used by astronomers, in re-touching and correcting the elements of the orbits of the heavenly bodies.

Now, from [3850c], we have, $\frac{dd\,e^{\text{iv}}}{2\,dt^2} = \frac{1}{400}\left\{\frac{d\,e^{\text{iv}\prime}}{dt} - \frac{d\,e^{\text{iv}}}{dt}\right\}$; in which we must substitute [4406e]
for $\frac{d\,e^{\text{iv}\prime}}{dt}$, its value $0^s{,}326172$ [4406]; also for $\frac{d\,e^{\text{iv}}}{dt}$, its value $0^s{,}329487$ [4405]; hence

We may find the values of γ, Π, by means of the equations,*

[4408]
$$\gamma \cdot \sin.\Pi = \varphi^{\text{v}} \cdot \sin.\theta^{\text{v}} - \varphi^{\text{iv}} \cdot \sin.\theta^{\text{iv}};$$
$$\gamma \cdot \cos.\Pi = \varphi^{\text{v}} \cdot \cos.\theta^{\text{v}} - \varphi^{\text{iv}} \cdot \cos.\theta^{\text{iv}}.$$

Then we compute the values of $\frac{d\gamma}{dt}$, $\frac{d\Pi}{dt}$, by taking the differentials of

[4408′] these equations, and substituting for $\frac{d\varphi^{\text{iv}}}{dt}$, $\frac{d\varphi^{\text{v}}}{dt}$, $\frac{d\theta^{\text{iv}}}{dt}$, $\frac{d\theta^{\text{v}}}{dt}$, their values [4246, 4247]. We find, in this manner, in 1750,

[4409]
$$\gamma = 1^{d}\,15^{m}\,30^{s};$$
$$\Pi = 125^{d}\,44^{m}\,34^{s};$$
$$\frac{d\gamma}{dt} = -\;0^{s},000106;$$
$$\frac{d\Pi}{dt} = -\,26^{s},094133.$$

The formulas [3935, 3936] give, for the secular variations of γ and Π, depending on the square of the disturbing force,

[4410]
$$\delta\gamma = t \cdot 0^{s},000184;$$
$$\delta\Pi = -\,t \cdot 0^{s},00763.$$

If we add the coefficients of t, in these equations, to those in the preceding values of $\frac{d\gamma}{dt}$, $\frac{d\Pi}{dt}$ [4409], we obtain, for the complete values of these quantities in 1750,

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[4406f] we get $\frac{dde^{\text{iv}}}{2dt^2} = -\frac{0^{s},003315}{400} = -\,0^{s},000008287$. Substituting this value of $\frac{dde^{\text{iv}}}{2dt^2}$, and that of $\frac{de^{\text{iv}}}{dt}$ [4405], in e^{iv} [4404a], we get the first of the equations [4407]. The values

[4406g] of ϖ^{iv}, e^{v}, ϖ^{v}, are found in the same manner, changing e^{iv} [4404a, 4406e], successively, into ϖ^{iv}, e^{v}, ϖ^{v}, and using the values [4405, 4406].

[4409a] * (2655) The equations [4408] are similar to those in [4282o], adding *four* accents to φ, θ, φ', θ', to conform to the present case; and changing tang. φ^{iv}, tang. φ^{v}, into φ^{iv}, φ^{v}, respectively, on account of their smallness. In this case γ [3739] represents the *tangent*

[4409b] of the inclination, or very nearly the inclination itself, of the orbit of Saturn to that of Jupiter; and Π [3746], the longitude of the ascending node of the orbit of Saturn upon that of Jupiter. Substituting in [4408] the values of φ^{iv}, θ^{iv}, φ^{v}, θ^{v}, [4082, 4083], we get γ, Π [4409]. Then taking the differentials of [4408], and substituting the preceding

[4409c] values of φ^{iv}, θ^{v}, &c.; also those of $d\varphi^{\text{iv}}$, $d\theta^{\text{iv}}$, $d\varphi^{\text{v}}$, $d\theta^{\text{v}}$ [4246, 4247], we get the two last equations [4409], by making a few reductions.

$$\frac{d\gamma}{dt} = \quad 0^s{,}000078\,;$$
$$\frac{d\,\Pi}{dt} = -26^s{,}101764. \qquad [4411]$$

We find, by the same process, in 1950,

$$\frac{d\gamma}{dt} = -\ 0^s{,}001487\,;$$
$$\frac{d\,\Pi}{dt} = -26^s{,}402056. \qquad [4412]$$

Hence we obtain, by the method in [3850—3853], for any time whatever t,* Inclination γ, and

$$\gamma = \gamma + t\,.\ \ 0^s{,}000078 - t^2.\ 0^s{,}000003913\,; \qquad [4413]$$
$$\Pi = \Pi - t\,.\,26^s{,}101764 - t^2.\ 0^s{,}000750731. \qquad [4413']$$

longitude Π, of the [4413''] ascending node of the orbit of Saturn on that of Jupiter.

The values of γ, Π, *in the second members of these equations, correspond to* 1750. *This being premised*, we find in 2250,†

$$a^{\mathrm{v}}.\,P = -\ 0{,}000080189\,;$$
$$a^{\mathrm{v}}.\,P' = -\ 0{,}001006510\,; \qquad [4414]$$

and in 2750,

$$a^{\mathrm{v}}.\,P = -\ 0{,}000260997\,;$$
$$a^{\mathrm{v}}.\,P' = -\ 0{,}000954603. \qquad [4415]$$

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* (2656) If we change the symbols γ, Π [4412], for the year 1950, into $\gamma_{,}$, $\Pi_{,}$, respectively, and leave those in [4411], corresponding to the year 1750, without accents, we shall have, as in [4406e],

$$\frac{dd\gamma}{2\,dt^2} = \tfrac{1}{400}\,.\left\{\frac{d\gamma_{,}}{dt} - \frac{d\gamma}{dt}\right\} = \tfrac{1}{400}\,.\{-0^s{,}001487 - 0^s{,}000078\} = -\ 0^s{,}000003913\,; \qquad [4413a]$$

also,

$$\frac{dd\,\Pi}{2\,dt^2} = \tfrac{1}{400}\,.\left\{\frac{d\,\Pi_{,}}{dt} - \frac{d\,\Pi}{dt}\right\} = \tfrac{1}{400}\,.\{-26^s{,}402056 + 26^s{,}101764\} = -0^s{,}00075073. \qquad [4413b]$$

Substituting these and the values of [4411], in the general expressions of γ, Π [4404a], namely,

$$\gamma = \gamma + t\,.\,\frac{d\gamma}{dt} + \tfrac{1}{2}\,t^2.\,\frac{d^2\gamma}{dt^2}, \qquad \Pi = \Pi + t\,.\,\frac{d\,\Pi}{dt} + \tfrac{1}{2}\,t^2.\,\frac{d^2\,\Pi}{dt^2}\,, \qquad [4413c]$$

we get [4413, 4413']; observing that the values, in the second members, correspond to the year 1750.

† (2657) The values of $a^{\mathrm{v}}.\,P$, $a^{\mathrm{v}}.\,P'$, are given in [3842, 3843], in functions of e^{iv}, e^{v}, ϖ^{iv}, ϖ^{v}, γ, Π, &c.; and their values in 1750, have already been given in [4402]. [4414a]

Hence we deduce, by the method of [3850—3856],*

$$a^{\text{v}}.\frac{dP}{dt} = -\,0{,}000000387666\,;$$

$$a^{\text{v}}.\frac{dP'}{dt} = -\,0{,}000000002145\,;$$

[4416]

$$a^{\text{v}}.\frac{ddP}{dt^2} = 0{,}000000000034734\,;$$

$$a^{\text{v}}.\frac{ddP'}{dt^2} = 0{,}0000000001411280.$$

The part of δv^{iv}, given in [4023], is,†

[4417]

$$\delta v^{\text{iv}} = -\frac{6m^{\text{v}}.n^{\text{iv}2}}{(5n^{\text{v}}-2n^{\text{iv}})^2}.\left\{\begin{array}{l}\left\{\begin{array}{l} a^{\text{iv}}.P' + \frac{2a^{\text{iv}}.dP}{(5n^{\text{v}}-2n^{\text{iv}}).dt} - \frac{3a^{\text{iv}}.ddP'}{(5n^{\text{v}}-2n^{\text{iv}})^2.dt^2} \\ + t.\left\{a^{\text{iv}}.\frac{dP'}{dt} + \frac{2a^{\text{iv}}.ddP}{(5n^{\text{v}}-2n^{\text{iv}}).dt^2}\right\} + \frac{1}{2}t^2.a^{\text{iv}}.\frac{ddP'}{dt^2}\end{array}\right\}.\sin.(5n^{\text{v}}t-2n^{\text{iv}}t+5\varepsilon^{\text{v}}-2\varepsilon^{\text{iv}}) \\ \left\{\begin{array}{l} a^{\text{iv}}.P - \frac{2a^{\text{iv}}.dP'}{(5n^{\text{v}}-2n^{\text{iv}}).dt} - \frac{3a^{\text{iv}}.ddP}{(5n^{\text{v}}-2n^{\text{iv}})^2.dt^2} \\ + t.\left\{a^{\text{iv}}.\frac{dP}{dt} - \frac{2a^{\text{iv}}.ddP'}{(5n^{\text{v}}-2n^{\text{iv}}).dt^2}\right\} + \frac{1}{2}t^2.a^{\text{iv}}.\frac{ddP}{dt^2}\end{array}\right\}.\cos.(5n^{\text{v}}t-2n^{\text{iv}}t+5\varepsilon^{\text{v}}-2\varepsilon^{\text{iv}})\end{array}\right\}.$$

Great inequality.

This becomes, by reduction to numbers,

[4418]

$$\delta v^{\text{iv}} = (1263^{s}{,}799671 - t.0^{s}{,}008418 - t^2.0^{s}{,}000019247).\sin.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}})$$
$$+ (119^{s}{,}526951 - t.0^{s}{,}473686 + t^2.0^{s}{,}000078562).\cos.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}).$$

The great inequality of Jupiter includes several other terms; thus, it contains, in [3844], the expression,‡

[4414b] To obtain $a^{\text{v}}.P$, $a^{\text{v}}.P'$ in 2250 [4414], we must put $t=500$, in [4407, 4413, 4413'], and substitute the corresponding values of e^{iv}, ϖ^{iv}, &c. in [3842, 3843]. In like manner, by putting $t=1000$, we get their values in 2750 [4415].

[4416a] * (2658) The values of $a^{\text{v}}.P$ [4402, 4414, 4415], being substituted, respectively, for P, $P_{,}$, $P_{,,}$, in [3856], give the values of $a^{\text{v}}.\frac{dP}{dt}$, $a^{\text{v}}.\frac{d^2P}{dt^2}$ [4416]. In like manner, from $a^{\text{v}}.P'$ [4402, 4414, 4415], we get the terms depending on the differentials of P' [4416].

[4418a] † (2659) The formula [4417], is the same as in [4023], increasing the accents on the elements m, a, e, &c. m', a', e', &c. by *four*, to conform to the case under consideration. Substituting in [4417], the values [4402, 4416], it becomes as in [4418].

[4419a] ‡ (2660) The expression [4419] includes the third and fourth lines of δv^{iv} [3844], the accents being increased as in the last note.

$$\delta v^{\text{iv}} = -\frac{2\,m^{\text{v}}.\,n^{\text{iv}}}{5\,n^{\text{v}} - 2\,n^{\text{iv}}} \cdot \left\{ \begin{array}{l} a^{\text{iv}\,2}.\left(\frac{d\,P}{d\,a^{\text{iv}}}\right).\cos.(5\,n^{\text{v}}t - 2\,n^{\text{iv}}t + 5\,\varepsilon^{\text{v}} - 2\,\varepsilon^{\text{iv}}) \\ -a^{\text{iv}\,2}.\left(\frac{d\,P'}{d\,a^{\text{iv}}}\right).\sin.(5\,n^{\text{v}}t - 2\,n^{\text{iv}}t + 5\,\varepsilon^{\text{v}} - 2\,\varepsilon^{\text{iv}}) \end{array} \right\}. \qquad [4419]$$

To reduce it to numbers, we must calculate the values of $a^{\text{v}\,2}.\left(\frac{d\,M^{(0)}}{d\,a^{\text{iv}}}\right)$; $a^{\text{v}\,2}.\left(\frac{d\,M^{(1)}}{d\,a^{\text{iv}}}\right)$, &c.; and we find,*

$$\begin{aligned} a^{\text{v}\,2}.\left(\frac{d\,M^{(0)}}{d\,a^{\text{iv}}}\right) &= -\,26{,}46390\,.\,m^{\text{v}}; \\ a^{\text{v}\,2}.\left(\frac{d\,M^{(1)}}{d\,a^{\text{iv}}}\right) &= \,65{,}75870\,.\,m^{\text{v}}; \\ a^{\text{v}\,2}.\left(\frac{d\,M^{(2)}}{d\,a^{\text{iv}}}\right) &= -\,50{,}22714\,.\,m^{\text{v}}; \\ a^{\text{v}\,2}.\left(\frac{d\,M^{(3)}}{d\,a^{\text{iv}}}\right) &= \,12{,}14696\,.\,m^{\text{v}}; \\ a^{\text{v}\,2}.\left(\frac{d\,M^{(4)}}{d\,a^{\text{iv}}}\right) &= -\,6{,}75963\,.\,m^{\text{v}}; \\ a^{\text{v}\,2}.\left(\frac{d\,M^{(5)}}{d\,a^{\text{iv}}}\right) &= \,4{,}13173\,.\,m^{\text{v}}. \end{aligned} \qquad [4420]$$

From these we deduce the values of $a^{\text{v}2}.\left(\frac{d\,M^{(0)}}{d\,a^{\text{v}}}\right)$, $a^{\text{v}2}.\left(\frac{d\,M^{(1)}}{d\,a^{\text{v}}}\right)$, &c.; which are necessary in the theory of Saturn, by means of the general equation of homogeneous functions [1001a],†

$$a^{\text{iv}}.\left(\frac{d\,M^{(i)}}{d\,a^{\text{iv}}}\right) + a^{\text{v}}.\left(\frac{d\,M^{(i)}}{d\,a^{\text{v}}}\right) = -\,M^{(i)}. \qquad [4421]$$

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* (2661) The accents being increased as in [4118a], the formulas [3836—3841] give the values of $a^{\text{v}}M^{(0)}$, $a^{\text{v}}M^{(1)}$, &c. in terms of $\alpha = \frac{a^{\text{iv}}}{a^{\text{v}}}$, $b_{\frac{1}{2}}^{(2)}$, $b_{\frac{1}{2}}^{(3)}$, &c. and their differentials. Taking the partial differentials of these expressions relative to a^{iv}, and substituting the values [4202—4211], we get [4420]. Observing that $b_{\frac{1}{2}}^{(2)}$, $b_{\frac{1}{2}}^{(3)}$, &c. are functions of α [964], and if we represent any one of them by b, its partial differential, relative to a^{iv}, will be, [4420a]

$$\left(\frac{d\,b}{d\,a^{\text{iv}}}\right) = \left(\frac{d\,b}{d\,\alpha}\right).\left(\frac{d\,\alpha}{d\,a^{\text{iv}}}\right) = \left(\frac{d\,b}{d\,\alpha}\right).\frac{1}{a^{\text{v}}}. \qquad [4420b]$$

† (2662) The general values of $M^{(0)}$, $M^{(1)}$, $M^{(2)}$, $M^{(3)}$, $M^{(4)}$, $M^{(5)}$ [3836d, 3837c, 3838h, 3840h, &c.], are composed of functions of a^{iv}, a^{v}, of the forms,

Hence we find, in 1750,*

[4422] $$-\frac{2\,m^{\rm v}.\,n^{\rm iv}}{5\,n^{\rm v}-2\,n^{\rm iv}}\cdot\left\{\begin{array}{l} a^{\rm iv\,2}.\left(\frac{d\,P}{d\,a^{\rm iv}}\right).\cos.(5\,n^{\rm v}t-2\,n^{\rm iv}t+5\,\varepsilon^{\rm v}-2\,\varepsilon^{\rm iv}) \\ -a^{\rm iv\,2}.\left(\frac{d\,P'}{d\,a^{\rm iv}}\right).\sin.(5\,n^{\rm v}t-2\,n^{\rm iv}t+5\,\varepsilon^{\rm v}-2\,\varepsilon^{\rm iv}) \end{array}\right\}$$

[4423] $$\begin{array}{l} =-17^{\rm s},228862\,.\sin.(5\,n^{\rm v}t-2\,n^{\rm iv}t+5\,\varepsilon^{\rm v}-2\,\varepsilon^{\rm iv}) \\ +\;\;5^{\rm s},360016\,.\cos.(5\,n^{\rm v}t-2\,n^{\rm iv}t+5\,\varepsilon^{\rm v}-2\,\varepsilon^{\rm iv})\,; \end{array}$$

and in 1950, it becomes,

[4424] $$\begin{array}{l} -16^{\rm s},836801\,.\sin.(5\,n^{\rm v}t-2\,n^{\rm iv}t+5\,\varepsilon^{\rm v}-2\,\varepsilon^{\rm iv}) \\ +\;\;6^{\rm s},449839\,.\cos.(5\,n^{\rm v}t-2\,n^{\rm iv}t+5\,\varepsilon^{\rm v}-2\,\varepsilon^{\rm iv}). \end{array}$$

Hence we obtain the following value of this function, for any time whatever t,

[4425] $$\begin{array}{l} \delta\,v^{\rm iv}=-(17^{\rm s},228862-t\,.\,0^{\rm s},001960)\,.\sin.(5\,n^{\rm v}t-2\,n^{\rm iv}t+5\,\varepsilon^{\rm v}-2\,\varepsilon^{\rm iv}) \\ \qquad +(5^{\rm s},360016+t\,.\,0^{\rm s},005449)\,.\cos.(5\,n^{\rm v}t-2\,n^{\rm iv}t+5\,\varepsilon^{\rm v}-2\,\varepsilon^{\rm iv}). \end{array}$$

[4421a] $$A^{(i)},\quad a^{\rm iv}.\left(\frac{d\,A^{(i)}}{d\,a^{\rm iv}}\right),\quad a^{\rm v}.\left(\frac{d\,A^{(i)}}{d\,a^{\rm v}}\right),\quad a^{\rm iv\,2}.\left(\frac{d^2\,A^{(i)}}{d\,a^{\rm iv\,2}}\right),\ \&c.\,;\quad a^{\rm iv}\,a^{\rm v}\,B^{(i)},\ \&c.\,;$$

all of which are homogeneous, and of the order -1, in $a^{\rm iv}$, $a^{\rm v}$ [1001′, 1007′]; i
[4421b] being any integral number. Hence the general value of $M^{(i)}$ is also homogeneous, and of the degree -1, in $a^{\rm iv}$, $a^{\rm v}$; and the formula [1001a], by changing A, a, a', m, into $M^{(i)}$, $a^{\rm iv}$, $a^{\rm v}$, -1 becomes as in [4421].

* (2663) The values of $m^{\rm v}\,a^{\rm v}\,P$, $m^{\rm v}\,a^{\rm v}\,P'$, are found as in [4402a], by increasing
[4422a] the accents of the elements in [3842, 3843] by *four*. Taking the partial differentials of these expressions, relative to $a^{\rm iv}$, we obtain the values of,

[4422b] $$m^{\rm v}\,a^{\rm v}.\left(\frac{d\,P}{d\,a^{\rm iv}}\right),\qquad m^{\rm v}\,a^{\rm v}.\left(\frac{d\,P'}{d\,a^{\rm iv}}\right),$$

expressed in functions of $a^{\rm iv}$, $e^{\rm iv}$, &c. $a^{\rm v}$, $e^{\rm v}$, &c. and of the terms [4420]. Substituting these in [4419, or 4422], we get [4423], corresponding to the year 1750. Repeating this
[4422c] calculation, with elements computed for the epoch 1950, it becomes as in [4424]; observing that the functions [4420], must also be computed and taken for the year 1950. Comparing the numerical coefficients of the terms [4423, 4424], we find the increments, in 200 years, to be respectively represented by,

$$-16^{\rm s},836801+17^{\rm s},228862=0^{\rm s},392061,$$

[4422d] and

$$6^{\rm s},449839-5,360016=1^{\rm s},089823.$$

Dividing these by 200, we get the annual increments, or the coefficients of t, as in the general expression of $\delta\,v^{\rm iv}$ [4425].

The great inequality of Jupiter [3844] contains also the term,*

$$\delta v^{\text{iv}} = -\tfrac{1}{2} H e^{\text{iv}} . \sin.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}} - \varpi^{\text{iv}} + A); \qquad [4426]$$

which, in 1750, is equal to,

$$\begin{aligned} &0^{s},820290 . \sin.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}) \\ &- 1^{s},837963 . \cos.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}); \end{aligned} \qquad [4427]$$

and in 1950, is,

$$\begin{aligned} &0^{s},701624 . \sin.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}) \\ &- 1^{s},840958 . \cos.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}). \end{aligned} \qquad [4428]$$

Hence we find, that for any time whatever t, this term is represented by,

$$\begin{aligned} \delta v^{\text{iv}} = \; &(0^{s},820290 - t . 0^{s},000593) . \sin.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}) \\ &- (1^{s},837963 + t . 0^{s},000015) . \cos.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}). \end{aligned} \qquad [4429]$$

To determine the part of the great inequality of Jupiter, depending on the products of five dimensions of the excentricities and inclinations of the orbits, we have computed, by the formulas [3860—3860ix], the values of $N^{(0)}$, $N^{(1)}$, &c. for the two epochs 1750 and 1950, and have found,

Terms of the fifth order on e, e', γ.

In 1750.	*In* 1950.
$a^{\text{v}}.N^{(0)} = 0,00000135044;$	$a^{\text{v}}.N^{(0)} = 0,00000129983;$
$a^{\text{v}}.N^{(1)} = 0,00000789719;$	$a^{\text{v}}.N^{(1)} = 0,00000754771;$
$a^{\text{v}}.N^{(2)} = -0,0000198552;$	$a^{\text{v}}.N^{(2)} = -0,0000196012;$
$a^{\text{v}}.N^{(3)} = 0,0000175127;$	$a^{\text{v}}.N^{(3)} = 0,0000172415;$
$a^{\text{v}}.N^{(4)} = -0,0000066540;$	$a^{\text{v}}.N^{(4)} = -0,0000066551;$
$a^{\text{v}}.N^{(5)} = 0,0000009277;$	$a^{\text{v}}.N^{(5)} = 0,0000009408;$
$a^{\text{v}}.N^{(6)} = 0,0000003618;$	$a^{\text{v}}.N^{(6)} = 0,0000003562;$
$a^{\text{v}}.N^{(7)} = 0,0000003643;$	$a^{\text{v}}.N^{(7)} = 0,0000003460;$
$a^{\text{v}}.N^{(8)} = -0,0000001720;$	$a^{\text{v}}.N^{(8)} = -0,0000001712;$
$a^{\text{v}}.N^{(9)} = 0,0000000730.$	$a^{\text{v}}.N^{(9)} = 0,0000000732.$

[4430]

* (2664) The term [4426] is the same as that depending on $-\frac{1}{2} H e$ [3844], accenting the symbols as in [4402a]. In this case H denotes the coefficient of,

By means of these values* we have computed the corresponding inequality in

[4430'] Saturn, in [4487]. Multiplying it by the factor $-\frac{m^{v}\sqrt{a^{v}}}{m^{iv}\sqrt{a^{iv}}}$, we obtain the following inequality of Jupiter,†

[4431]
$$\delta v^{iv} = -(12'',536393 - t \,.\, 0'',001755) \,.\, \sin.(5\,n^{v}t - 2\,n^{iv}t + 5\,\varepsilon^{v} - 2\,\varepsilon^{iv}) + (8'',120963 + t \,.\, 0'',004885) \,.\, \cos.(5\,n^{v}t - 2\,n^{iv}t + 5\,\varepsilon^{v} - 2\,\varepsilon^{iv}).$$

Lastly, we have computed, by the method in [4003], that part of the great inequality of Saturn, which depends on the square of the disturbing force,

[4431'] and is of a sensible magnitude. Then we have deduced from it the corresponding inequality of Jupiter, by multiplying it by $-\frac{m^{v}\sqrt{a^{v}}}{m^{iv}\sqrt{a^{iv}}}$; which gives, for this last inequality, the following expression,‡

[4426a]
$$\cos.(5\,n't - 3\,n\,t + 5\,\varepsilon' - 3\,\varepsilon + A),$$

in the expression [3814], corresponding to Jupiter. Computing the value of $-\frac{1}{2}\,H\,e^{iv}$, for

[4426b] the years 1750, 1950, as in [4427, 4428], we obtain its annual increment, and the general value [4429].

* (2665) The signs of all the terms in [4430, 4431], are different in the original work;

[4430a] we have changed them, in order to correct the mistake in the signs mentioned in [3860a].

† (2666) Changing, in [1208], ζ, ζ', into δv^{iv}, δv^{v}, which represent, respectively, the corresponding parts of the great inequalities of Jupiter and Saturn, we get, by using the notation of [4402a],

[4430b]
$$\delta v^{iv} = -\frac{m^{v}\sqrt{a^{v}}}{m^{iv}\sqrt{a^{iv}}} \,.\, \delta v_{v}.$$

[4430c] Substituting in this, the values m^{iv}, m^{v}, a^{iv}, a^{v}, δv^{v} [4077, 4079, 4487], we get [4431].

‡ (2667) We have already mentioned in [4006t—4007a] the difficulties which occurred in computing this part of the great inequality of Jupiter, and have also observed, that the

[4431a] numbers given by the author, in [4432], are inaccurate; the chief coefficient having a wrong sign, as Mr. Pontécoulant found by computing the most important terms, depending on the

[4431b] arguments contained in the table [4006u], numbered from 1 to 10, and from 1′ to 10′. The parts of δv^{iv}, corresponding to these terms, are given in [4431f], from the abstract, printed by Mr. Pontécoulant, in the *Connaissance des Tems*, for 1833; using, for brevity, the

[4431c] symbol $T_5 = 5\,n^{v}t - 2\,n^{iv}t + 5\,\varepsilon^{v} - 2\,\varepsilon^{iv}$ [3890b]. The first line of the function [4431f]

[4431d] is produced by the term $3\,a^2 ff.\,(n\,d\,t \,.\, \mathrm{d}R \,.\, f\mathrm{d}R)$ [5841]; the other lines arise from the products of the quantities in the table [4006u], marked with the numbers on the same lines

$$\begin{aligned}\delta v^{iv} = &\quad (\;1^s{,}641663 - t\,.\,0^s{,}004688)\,.\,\sin.(5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv}) \\ &- (18^s{,}461954 + t\,.\,0^s{,}001515)\,.\,\cos.(5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv}).\end{aligned}$$ [4432]

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respectively. The sum of all these terms is given in [4431g]; and it differs essentially from that of La Place, in [4432]; particularly in the term depending on $\cos.\,T_5$, which has a different sign, though it is nearly of the same numerical value; an error in the sign having been discovered in the original minutes of the numerical calculation of La Place. [4431e]

Terms of the order of the square of the disturbing forces.

	$\delta v^{iv} = +\,0^s{,}02489\,.\,\sin.\,T_5 + 0^s{,}00266\,.\,\cos.\,T_5$
1	$+\,0^s{,}08628\,.\,\sin.\,T_5 - 0^s{,}01857\,.\,\cos.\,T_5$
1′	$-\,2^s{,}00454\,.\,\sin.\,T_5 + 0^s{,}43757\,.\,\cos.\,T_5$
2	$+\,0^s{,}07587\,.\,\sin.\,T_5 + 0^s{,}08197\,.\,\cos.\,T_5$
2′	$+\,0^s{,}39242\,.\,\sin.\,T_5 + 0^s{,}22555\,.\,\cos.\,T_5$
3	$+\,0^s{,}28829\,.\,\sin.\,T_5 + 0^s{,}19273\,.\,\cos.\,T_5$
3′	$-\,0^s{,}71831\,.\,\sin.\,T_5 - 1^s{,}58658\,.\,\cos.\,T_5$
4	$-\,0^s{,}14619\,.\,\sin.\,T_5 - 0^s{,}09422\,.\,\cos.\,T_5$
5	$-\,0^s{,}76290\,.\,\sin.\,T_5 + 0^s{,}77529\,.\,\cos.\,T_5$ [4431f]
6, 6′	$+\,2^s{,}16304\,.\,\sin.\,T_5 + 16^s{,}97139\,.\,\cos.\,T_5$
7, $i=2$,	$+\,6^s{,}62968\,.\,\sin.\,T_5 - 0^s{,}80829\,.\,\cos.\,T_5$
7, $i=1$,	$-\,2^s{,}49438\,.\,\sin.\,T_5 - 0^s{,}92192\,.\,\cos.\,T_5$
8, $i=2$,	$+\,0^s{,}22613\,.\,\sin.\,T_5 - 0^s{,}53472\,.\,\cos.\,T_5$

$$= \quad 3^s{,}76028\,.\,\sin.\,T_5 + 14^s{,}72286\,.\,\cos.\,T_5.$$ [4431g]

In computing these numbers, the mass of Saturn is supposed to be, as in [4061d], equal to $\frac{1}{3512}$; instead of $\frac{1}{3359{,}4}$, used by La Place [4061]. To compare them with La Place's calculation [4432], given below, in [4431k], we must increase the coefficients [4431g], in the ratio of 3512 to 3359,4; by which means they will become as in [4431i]; the terms depending on t, t^2, being neglected; [4431h]

$$\delta v^{iv} = 3^s{,}93109\,.\,\sin.\,T_5 + 15^s{,}39164\,.\,\cos.\,T_5\,;$$ [4431i]

$$\delta v^{iv} = 1^s{,}64166\,.\,\sin.\,T_5 - 18^s{,}46195\,.\,\cos.\,T_5\,.$$ [4431k]

The difference of the two expressions [4431i,k], which we shall denote by C^{iv}, is a correction, to be applied to the formula [4433 or 4434]; and we shall have,

Correction of the great inequality.

$$C^{iv} = 2^s{,}28943\,.\,\sin.\,T_5 + 33^s{,}85359\,.\,\cos.\,T_5.$$ [4431l]

We may remark, that the number of terms of the forms 7 to 10, and 7′ to 10′, [4006u], is infinite; but it is only necessary to notice a few of them, in which δr, δv, $\delta r'$, or $\delta v'$, have sensible values. Moreover, the terms depending on $\delta\varepsilon$, were not computed by Mr. Pontécoulant, when he published the above results. The effects of the correction C^{iv} [4431l], of the terms depending on $\delta\varepsilon$, and of other quantities of a similar nature, are taken into consideration in book x. chap. viii. [9037, &c.]; where the final results of all these calculations, relative to the inequalities of the motions of Jupiter and Saturn, are given. [4431m] [4431n] [4431o] [4431p] [4431q]

Now, if we connect the several parts of the great inequality of Jupiter, we shall obtain, for its complete value,*

[4433] $$(1+\mu^{v}).\left\{\begin{array}{l}(1261^{s},569155-t.0^{s},013195-t^{2}.0^{s},000019247).\sin.(5n^{v}t-2n^{iv}t+5\varepsilon^{v}-2\varepsilon^{iv})\\ +\ (96^{s},466088-t.0^{s},474651+t^{2}.0^{s},000078564).\cos.(5n^{v}t-2n^{iv}t+5\varepsilon^{v}-2\varepsilon^{iv})\\ +\ \text{function } C^{iv}\ [4431l]+2\,\delta v^{iv}\ [4431]\end{array}\right\}.$$

If we reduce these to one single term, by the method in [4024—4027″], we shall obtain, for δv^{iv}, the following expression,

Great inequality.

[4434] $$(1+\mu^{v}).\left\{\begin{array}{l}(1265^{s},251781-t.0^{s},037090+t^{2}.0^{s},000036669).\sin.\left(\begin{array}{l}5n^{v}t-2n^{iv}t+5\varepsilon^{v}-2\varepsilon^{iv}+4^{d}22^{m}21^{s}\\ -t.77^{s},653+t^{2}.0^{s},012581\end{array}\right)\\ +\ \text{function } C^{iv}\ [4431l]+2\,\delta v^{iv}\ [4431]\end{array}\right\}.$$

This inequality may require some correction, on account of the coefficient μ^{v}, depending on the value of the mass of Saturn; and also on account of the
[4434′] slight imperfection in the assumed value of the divisor $(5n'-2n)^{2}$; a long series of observations will remove this small source of error. *We must apply this great inequality to Jupiter's mean motion, as we have seen in* [4006″].

The square of the disturbing force produces also, in δv^{iv}, the inequality [3890],

[4435] $$\delta v^{iv}=-\frac{\overline{H}^{2}}{8}\cdot\frac{(2m^{v}\sqrt{a^{v}}+5m^{iv}\sqrt{a^{iv}})}{m^{v}\sqrt{a^{v}}}.\sin.(\text{double argument of the great inequality});$$

which, in numbers, is,

[4436] $$\delta v^{iv}=-13^{s},238897.\sin.(\text{double argument of the great inequality});$$

we must also apply the inequality of a long period to the mean motion of Jupiter.

The inequality [3921],

[4437] $$\delta v^{iv}=\tfrac{1}{4}.\frac{(5m^{iv}\sqrt{a^{iv}}+4m^{v}\sqrt{a^{v}})}{m^{v}\sqrt{a^{v}}}.\overline{H}K.\sin.(5n^{iv}t-10n^{v}t+5\varepsilon^{iv}-10\varepsilon^{v}-B-\overline{A}),$$

reduced to numbers, becomes,

[4438] $$\delta v^{iv}=-4^{s},024751.\sin.(5n^{iv}t-10n^{v}t+5\varepsilon^{iv}-10\varepsilon^{v}+51^{d}21^{m}55^{s}).$$

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* (2668) The expression [4433], is the sum of the terms contained in the functions
[4433a] [4418, 4425, 4429, 4431, 4432] multiplied by $(1+\mu^{v})$. Then, by computing this expression for the times, $t=500$, and $t=1000$, we may reduce the whole to one term, as in [4434], by the method explained in [4024—4027″].

We have also, in [3844], the inequality,*

$$\delta v^{\mathrm{iv}} = \tfrac{5}{4} \,.\, K e^{\mathrm{iv}} \,.\, \sin.\, (5\, n^{\mathrm{v}} t - 4\, n^{\mathrm{iv}} t + 5\, \varepsilon^{\mathrm{v}} - 4\, \varepsilon^{\mathrm{iv}} + \varpi^{\mathrm{iv}} + B)\,; \qquad [4439]$$

and by reducing it to numbers, it becomes,

$$\delta v^{\mathrm{iv}} = 10^{s},084660 \,.\, \sin.\, (4\, n^{\mathrm{iv}} t - 5\, n^{\mathrm{v}} t + 4\, \varepsilon^{\mathrm{iv}} - 5\, \varepsilon^{\mathrm{v}} + 45^{d}\, 21^{m}\, 44^{s})\,; \qquad [4440]$$

if we connect this with the two inequalities [4392],†

$$\begin{array}{l} 1^{s},097613 \,.\, \sin.\, (5\, n^{\mathrm{v}} t - 4\, n^{\mathrm{iv}} t + 5\, \varepsilon^{\mathrm{v}} - 4\, \varepsilon^{\mathrm{iv}} - \varpi^{\mathrm{iv}}) \\ -\, 2^{s},781664 \,.\, \sin.\, (5\, n^{\mathrm{v}} t - 4\, n^{\mathrm{iv}} t + 5\, \varepsilon^{\mathrm{v}} - 4\, \varepsilon^{\mathrm{iv}} - \varpi^{\mathrm{v}})\,; \end{array} \qquad [4441]$$

we obtain the single equivalent expression,

$$\delta v^{\mathrm{iv}} = (1 + \mu^{\mathrm{v}}) \,.\, 11^{s},506190 \,.\, \sin.\, (4\, n^{\mathrm{iv}} t - 5\, n^{\mathrm{v}} t + 4\, \varepsilon^{\mathrm{iv}} - 5\, \varepsilon^{\mathrm{v}} + 58^{d}\, 00^{m}\, 36^{s}). \qquad [4442]$$

We have seen, in [3773], that the expression of $d \,.\, \delta v^{\mathrm{iv}}$ contains a secular inequality, depending on the equation,

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* (2669) The inequality [4439], is the same as the last of [3844], augmenting the accents of e, n, n', &c. to conform to the present example. The term K, which occurs in this expression is, by [3824—3826], equal to the constant term of the coefficient of the part of [4394], depending on the angle $3\, n^{\mathrm{iv}} t - 5\, n^{\mathrm{v}} t$; or rather on the angle $5 n^{\mathrm{v}} t - 3 n^{\mathrm{iv}} t$. This part being nearly equal to [4439a]

$$-\, 169^{s},265895 \,.\, \sin.\, (5\, n^{\mathrm{v}} t - 3\, n^{\mathrm{iv}} t + 5\, \varepsilon^{\mathrm{v}} - 3\, \varepsilon^{\mathrm{iv}} - 55^{d}\, 40^{m}\, 49^{s}). \qquad [4439b]$$

If we compare this with [3826], putting $i = 5$, we get,

$$K = -\, 169^{s},265895\,; \qquad B = -\, 55^{d}\, 40^{m}\, 49^{s}\,; \qquad [4439c]$$

and by [4081], $\varpi^{\mathrm{iv}} = 10^{d}\, 21^{m}\, 4^{s}$; hence,

$$\varpi^{\mathrm{iv}} + B = -\, 45^{d}\, 19^{m}\, 45^{s}\,; \qquad [4439d]$$

and [4439] becomes,

$$\begin{array}{l} \tfrac{5}{4} \,.\, K e^{\mathrm{iv}} \,.\, \sin.\, (5\, n^{\mathrm{v}} t - 4\, n^{\mathrm{iv}} t + 5\, \varepsilon^{\mathrm{v}} - 4\, \varepsilon^{\mathrm{iv}} - 45^{d}\, 19^{m}\, 45^{s}) \\ = -\, \tfrac{5}{4} \,.\, K e^{\mathrm{iv}} \,.\, \sin.\, (4\, n^{\mathrm{iv}} t - 5\, n^{\mathrm{v}} t + 4\, \varepsilon^{\mathrm{iv}} - 5\, \varepsilon^{\mathrm{v}} + 45^{d}\, 19^{m}\, 45^{s}). \end{array} \qquad [4439e]$$

Substituting in this, the value of K [4439c], and that of e^{iv} [4080], it becomes nearly as in [4440].

† (2670) These inequalities are found in the ninth and tenth lines of [4392], with a slight and unimportant variation in the first coefficient. These terms [4441] may be connected with [4440], and reduced to one term, of the form [4442], by the method given in [4282h—l]. [4440a]

[4443]
$$\begin{aligned}\frac{d.\delta v^{iv}}{dt} = {} & \frac{m^{v}.n^{iv}}{8}.(h^{iv^2}+l^{iv^2}).\left\{2a^{iv^2}.\left(\frac{d.A^{(0)}}{da^{iv}}\right)+7a^{iv^3}.\left(\frac{dd.A^{(0)}}{da^{iv2}}\right)+2a^{iv^4}.\left(\frac{d^3.A^{(0)}}{da^{iv3}}\right)\right\} \\ & +\frac{m^{v}.n^{iv}}{4}.(h^{v^2}+l^{v^2}).\left\{2a^{iv^2}.\left(\frac{d.A^{(0)}}{da^{iv}}\right)+4a^{iv^3}.\left(\frac{dd.A^{(0)}}{da^{iv2}}\right)+a^{iv^4}.\left(\frac{d^3.A^{(0)}}{da^{iv3}}\right)\right\} \\ & -\frac{m^{v}.n^{iv}}{8}.(h^{iv}h^{v}+l^{iv}l^{v}).\left\{2a^{iv^2}.A^{(1)}-2a^{iv^2}.\left(\frac{d.A^{(1)}}{da^{iv}}\right)+15a^{iv^3}.\left(\frac{dd.A^{(1)}}{da^{iv2}}\right)+4a^{iv^3}.\left(\frac{d^3.A^{(1)}}{da^{iv3}}\right)\right\}.\end{aligned}$$

Hence we deduce,*

[4444] $$\frac{d.\delta v^{iv}}{dt} = -23^s{,}9441.e^{iv2}-27^s{,}7951.e^{v2}+42^s{,}9296.e^{iv}.e^{v}.\cos.(\varpi^{v}-\varpi^{iv}).$$

[4444'] *We may neglect the constant part of the second member of this equation, which is confounded with the mean motion of Jupiter*, and then we shall have,†

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* (2671) We have, as in [3756a, b],

[4443a] $$h^{iv^2}+l^{iv^2}=e^{iv^2}, \quad h^{v^2}+l^{v^2}=e^{v^2}, \quad h^{iv}h^{v}+l^{iv}l^{v}=e^{iv}e^{v}.\cos.(\varpi^{v}-\varpi^{iv}).$$

Substituting these in [4443]; also the values of $A^{(0)}$, $A^{(1)}$, and their differentials, in terms of $b^{(i)}_{\frac{1}{2}}$, and its differentials [996—1001]; then the values of these quantities [4202, &c.]; we finally get the expression [4444].

† (2672) We shall put E for the general expression of the second member of [4444], corresponding to any value whatever of t, and E for its value when $t=0$; then substituting the values e^{iv}, e^{v}, ϖ^{iv}, ϖ^{v} [4407], we shall obtain,

[4445a] $$\frac{d.\delta v^{iv}}{dt}=E=\mathrm{E}+t.\frac{d\mathrm{E}}{dt}+\&c. \quad [4444, 4445a].$$

Multiplying this by dt, and integrating, supposing $\delta v^{iv}=0$ when $t=0$, we get,

[4445b] $$\delta v^{iv}=\mathrm{E}t+\tfrac{1}{2}.\frac{d\mathrm{E}}{dt}.t^2+\&c.$$

of which the first term $\mathrm{E}t$, may be neglected, being confounded with the mean motion of Jupiter; then we have, by neglecting t^3, t^4, &c.

[4445c] $$\delta v^{iv}=\tfrac{1}{2}.\frac{d\mathrm{E}}{dt}.t^2, \quad \text{or} \quad \frac{d.\delta v^{iv}}{dt}=\frac{d\mathrm{E}}{dt}.t, \quad \text{as in [4445].}$$

The coefficient of t, in the second member of this last expression, represents the differential of the second member of [4444], divided by dt, corresponding to the time of the epoch 1750. Substituting in it the values [4405], and dividing by the radius in seconds 206265^s, we get,

[4445d] $$\frac{d.\delta v^{iv}}{dt}=-0^s{,}0000013.t, \quad \text{nearly.}$$

This equation being multiplied by dt, and integrated, gives [4446]; no constant quantity being added, because it is supposed to vanish when $t=0$.

$$\frac{d.\delta v^{iv}}{dt} = -23^s,9441\,.\,t\,.\,2\,e^{iv}.\frac{de^{iv}}{dt} - 27^s,7951\,.\,t\,.\,2\,e^{v}.\frac{de^{v}}{dt}$$

$$+42^s,9296.t.\left\{\left(e^{iv}.\frac{de^{v}}{dt}+e^{v}.\frac{de^{iv}}{dt}\right).\cos.(\varpi^{v}-\varpi^{iv})-e^{iv}.e^{v}.\frac{(d\varpi^{v}-d\varpi^{iv})}{dt}.\sin.(\varpi^{v}-\varpi^{iv})\right\}. \quad [4445]$$

Substituting for $\frac{de^{iv}}{dt}$, $\frac{de^{v}}{dt}$, $\frac{d\varpi^{iv}}{dt}$, $\frac{d\varpi^{v}}{dt}$, their values, given in [4405], and integrating, we obtain,

$$\delta v^{iv} = -t^2.\,0^s,00000065. \quad [4446]$$

This inequality is insensible in the interval of ten or twelve hundred years, and even as it respects the most ancient observations that have been handed down to us; therefore we may neglect it. [4446']

It now remains to consider the radius vector of Jupiter. We have found, in [3845], that the terms depending on the powers and products of the third degree of the excentricities, add, to the expression of this radius, the quantity,*

$$\delta r^{iv} = -H\,a^{iv}.\,e^{iv}.\cos.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}-\varpi^{iv}+A)$$

$$+H\,a^{iv}.\,e^{iv}.\cos.(4\,n^{iv}t-5\,n^{v}t+4\,\varepsilon^{iv}-5\,\varepsilon^{v}-\varpi^{iv}-A) \quad [4447]$$

$$+\frac{4\,m^{v}.\,n^{iv}.\,a^{iv2}}{5\,n^{v}-2\,n^{iv}}.\left\{\begin{array}{l} P.\sin.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \\ +P'.\cos.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \end{array}\right\}.$$

Correction of the radius vector.

Reducing this function to numbers, we obtain,

$$\delta r^{iv}=(1+\mu^{v}).\left\{\begin{array}{l} -0,0003042733.\cos.(5n^{v}t-2n^{iv}t+5\varepsilon^{v}-2\varepsilon^{iv}-12^d08^m49^s) \\ +0,0001001860.\cos.(4n^{iv}t-5n^{v}t+4\varepsilon^{iv}-5\varepsilon^{v}+45^d16^m47^s) \end{array}\right\}. \quad [4448]$$

If we connect this expression with the terms computed in [4393],

$$\delta r^{iv}=(1+\mu^{v}).\left\{\begin{array}{l} 0,0000268383\,.\cos.(5\,n^{v}t-4n^{iv}t+5\varepsilon^{v}-4\,\varepsilon^{iv}-\varpi^{iv}) \\ -0,0000516048.\cos.(5\,n^{v}t-4n^{iv}t+5\varepsilon^{v}-4\,\varepsilon^{iv}-\varpi^{v}) \end{array}\right\}, \quad [4449]$$

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* (2673) The expression [4447] is composed of the three last terms of [3845], increasing the accents as in [4388a]. The value of H is as in [4426a]; those of P, P', as in [4402]; the other elements are given in [4061, 4077, 4079, 4080]; hence the expression [4447] becomes as in [4448]. Connecting this with the two terms of δr^{iv}, given in [4393 or 4449], and reducing by the method [4282h—l], we obtain [4450]. [4447a]

we obtain the following result,

[4450] $\delta r^{\text{iv}} = (1+\mu^{\text{v}}) \,.\, 0{,}0000983161 \,.\, \cos.(4n^{\text{iv}}t - 5n^{\text{v}}t + 4\varepsilon^{\text{iv}} - 5\varepsilon^{\text{v}} - 14^{d}23^{m}19^{s})$.

[4450′] The semi-major axis a^{iv}, which we have used in calculating the elliptical part of the radius vector, must be augmented by the quantity $\frac{1}{3}a^{\text{iv}}.m^{\text{iv}}$ [4058]. Adding this to the expression of a^{iv} [4079], we obtain,

[4451] $$a^{\text{iv}} = 5{,}20279108.$$

Inequalities of Jupiter's motion in latitude.

34. It follows, from [3931, 3931′], that the terms depending on the square of the disturbing force, add to the values of $\frac{d\varphi^{\text{iv}}}{dt}$, $\frac{d\theta^{\text{iv}}}{dt}$, the following quantities,*

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* (2674) In deducing the differentials of $\delta\varphi$, $\delta\theta$, &c. from [3931—3932′], in order

[4452a] to find the increments to be applied to the values of $\frac{d\varphi^{\text{iv}}}{dt}$, $\frac{d\theta^{\text{iv}}}{dt}$, &c. [4246, &c.], we may consider $\delta\gamma$, $\delta\Pi$, $\delta\varphi$, $\delta\theta$, to be the only variable quantities; or, in other words, we may neglect the variations of Π, θ, φ, γ, on account of their smallness. For the expressions of $\delta\gamma$, $\delta\Pi$ [3935, 3936], which are independent of the periodical angles, are of the order

[4452b] m'^{2}; consequently their differential coefficients $\frac{d.\delta\gamma}{dt}$, $\frac{d.\delta\Pi}{dt}$, are of the *same* order, and are therefore much greater than the terms arising from the variations of the angles $\Pi-\theta$, in the differentials of the expressions [3931—3932′]; because these last terms depend on the

[4452c] products $\delta\gamma.\frac{d\Pi}{dt}$, $\delta\gamma.\frac{d\theta}{dt}$, &c. which are evidently of the order m'^{3}; since $\frac{d\gamma}{dt}$, $\frac{d\Pi}{dt}$,

[4452d] [4411] are of the order m'. Hence the differentials of [3931, 3931′] become, by dividing by dt, and increasing the accents, as in [4388a];

[4452e] $$\frac{d.\delta\varphi^{\text{iv}}}{dt} = -\frac{m^{\text{v}}\sqrt{a^{\text{v}}}}{m^{\text{iv}}\sqrt{a^{\text{iv}}}+m^{\text{v}}\sqrt{a^{\text{v}}}} \,.\left\{\frac{d.\delta\gamma}{dt}\,.\cos.(\Pi-\theta^{\text{iv}}) - \gamma\,.\frac{d.\delta\Pi}{dt}.\sin.(\Pi-\theta^{\text{iv}})\right\};$$

[4452f] $$\varphi.\frac{d.\delta\theta^{\text{iv}}}{dt} = -\frac{m^{\text{v}}\sqrt{a^{\text{v}}}}{m^{\text{iv}}\sqrt{a^{\text{iv}}}+m^{\text{v}}\sqrt{a^{\text{v}}}} \,.\left\{\frac{d.\delta\gamma}{dt}\,.\sin.(\Pi-\theta^{\text{iv}}) + \gamma\,.\frac{d.\delta\Pi}{dt}.\cos.(\Pi-\theta^{\text{iv}})\right\}.$$

Now, from [4410], we have,

[4452g] $$\frac{d.\delta\gamma}{dt} = 0^{s}{,}000184 = \frac{\delta\gamma}{t}; \qquad \frac{d.\delta\Pi}{dt} = -0^{s}{,}007631 = \frac{d\Pi}{t};$$

substituting these, in [4452e, f], we get, $\frac{d.\delta\varphi^{\text{iv}}}{dt}$, $\frac{d.\delta\theta^{\text{iv}}}{dt}$, which are changed into $\frac{d\varphi^{\text{iv}}}{dt}$,

[4452h] $\frac{d\theta^{\text{iv}}}{dt}$, in [4452, 4453]; and by using [4452g], also the values of γ, Π [4409], m^{iv}, m^{v},

$$\frac{d\varphi^{iv}}{dt}=\frac{-m^{v}.\sqrt{a^{v}}}{m^{iv}.\sqrt{a^{iv}}+m^{v}.\sqrt{a^{v}}}.\left\{\frac{\delta\gamma}{t}.\cos.(\Pi-\theta^{iv})-\gamma.\frac{\delta\,\Pi}{t}.\sin.(\Pi-\theta^{iv})\right\};$$ [4452]

$$\frac{d\theta^{iv}}{dt}=\frac{-m^{v}.\sqrt{a^{v}}}{(m^{iv}.\sqrt{a^{iv}}+m^{v}.\sqrt{a^{v}}).\varphi}.\left\{\frac{\delta\gamma}{t}.\sin.(\Pi-\theta^{iv})+\gamma.\frac{\delta\,\Pi}{t}.\cos.(\Pi-\theta^{iv})\right\};$$ [4453]

$\delta\gamma$, $\delta\,\Pi$, being computed by the formulas [3931, 3931']. Reducing these functions to numbers, we obtain,

$$\frac{d\varphi^{iv}}{dt}=-0^{s},000073;$$ [4454]

$$\frac{d\theta^{iv}}{dt}=\quad 0^{s},000811.$$ [4455]

The first of these expressions must be added to the values of $\frac{d\varphi^{iv}}{dt}$, $\frac{d\varphi_{,}^{iv}}{dt}$ [4246], and the second to the values of $\frac{d\theta^{iv}}{dt}$, $\frac{d\theta_{,}^{iv}}{dt}$ [4246]; hence we obtain,

$$\frac{d\varphi^{iv}}{dt}=-0^{s},078213;$$

$$\frac{d\varphi_{,}^{iv}}{dt}=-0^{s},223251;$$

$$\frac{d\theta^{iv}}{dt}=\quad 6^{s},457092;$$ [4456]

$$\frac{d\theta_{,}^{iv}}{dt}=-14^{s},662566.$$

Then we find, by means of the formula [4295b],*

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a^{iv}, a^{v}, θ^{iv} [4061, 4079, 4083], they become as in [4454, 4455]. Adding the expression [4454] to the first terms of $\frac{d\varphi^{iv}}{dt}$ and $\frac{d\varphi_{,}^{iv}}{dt}$ [4246], we get their values [4456]; also adding [4455] to the first terms of $\frac{d\theta^{iv}}{dt}$ and $\frac{d\theta_{,}^{iv}}{dt}$ [4246], we obtain the corresponding values [4456]. [4452i]

* (2675) The terms of δs^{iv} [4457], are deduced from those in [4295b], by adding three accents to the symbols m'', n', n'', ε', ε'', a', a'', in order to conform to the case now under consideration. γ, Π, are as in [4409]. The values of $B^{(i-1)}=\frac{1}{a^{v\,3}}\,.\,b_{\frac{3}{2}}^{(i-1)}$ [1006], are given in [4210, 4079]. [4457a]

[4457]
$$\delta s^{iv} = (1+\mu^{v}) \cdot \left\{ \begin{array}{l} 0'',564458 \,.\, \sin.(n^{v}t + \varepsilon^{v} - \Pi^{v}) \\ + 0'',663927 \,.\, \sin.(2n^{v}t - n^{iv}t + 2\varepsilon^{v} - \varepsilon^{iv} - \Pi^{iv}) \\ + 1'',119782 \,.\, \sin.(3n^{v}t - 2n^{iv}t + 3\varepsilon^{v} - 2\varepsilon^{iv} - \Pi^{iv}) \\ - 0'',279382 \,.\, \sin.(4n^{v}t - 3n^{iv}t + 4\varepsilon^{v} - 3\varepsilon^{iv} - \Pi^{iv}) \\ - 0'',269130 \,.\, \sin.(2n^{iv}t - n^{v}t + 2\varepsilon^{iv} - \varepsilon^{v} - \Pi^{iv}) \end{array} \right\};$$

Inequalities in the latitude.

[4457′] Π^{iv}, *in this formula, being the longitude of the ascending node of Saturn's orbit upon that of Jupiter* [4295*b*—*c*]. Lastly, we have, in [3885], the inequality,*

[4458]
$$\delta s^{iv} = 3'',941680 \,.\, \sin.(3n^{iv}t - 5n^{v}t + 3\varepsilon^{iv} - 5\varepsilon^{v} + 59^{d}\,30^{m}\,35^{s}).$$

[4458*a*] * (2676) The quantity [4458], is deduced from [3885], reducing both terms to one, as in [4282*h*—*l*].

[4458*b*] Before concluding the notes on this chapter, we may remark, that the inequalities of the motions of Jupiter and Saturn, computed in this book, are corrected by the author in [5974,&c.], and afterwards more thoroughly, in book x. chap. viii. [9037,&c.]; where he has decreased the assumed value of the mass of Saturn [4061]. He has also computed several small inequalities, which had not been previously noticed, and has given new forms to some of the arguments. Finally, the subject of these inequalities has been treated in a wholly different manner, with a frequent use of definite integrals, by Professor Hansen, Director of the Observatory

[4458*c*] at Seeberg, in a memoir, entitled, "*Untersuchung ueber die gegenseitigen Störungen des Jupiters und Saturns;*" which gained, in 1830, the prize of the Royal Academy of Sciences, of Berlin, relative to the inequalities of these two planets. In this method, the true longitude is computed by means of the elements corresponding to the ***invariable ellipsis at the time of the***

[4458*d*] ***epoch;*** taking instead of t, a function of t, which corrects for the perturbations. As the inequalities of Jupiter's motion had not been completed by Professor Hansen, when he

[4458*e*] published this memoir, we may have occasion to refer to it more particularly, after the completion of his work.

CHAPTER XIII.

THEORY OF SATURN.

35. The equation [4336],

$$\delta r^{\text{iv}} = \frac{r^{\text{iv}\,2}}{r''} \cdot (1 - \alpha^2) \cdot \delta V^{\text{iv}},$$ [4459]

corresponding to Jupiter, becomes for Saturn,

$$\delta r^{\text{v}} = \frac{r^{\text{v}\,2}}{r''} \cdot (1 - \alpha^2) \cdot \delta V^{\text{v}}.$$ [4460]

If we take for r'', and r^{v}, the mean distances of the earth and Saturn from the sun [4079], and suppose $\delta V^{\text{v}} = \pm 1'' = \pm 0^{s},324$, we shall find,

$$\delta r^{\text{v}} = \pm 0{,}000141326.$$ [4461]

Terms which may be neglected.

Therefore we may neglect the inequalities of δr^{v}, below $\mp 0{,}000141$. We shall also neglect the inequalities of Saturn, in longitude and latitude, [4462] which are less than a quarter of a centesimal second, or $0^{s},081$.

*Inequalities of Saturn, independent of the excentricities.**

Inequalities independent of the excentricities.

$$\delta v^{\text{v}} = (1 + \mu^{\text{iv}}) \cdot \left\{ \begin{array}{l} +\ 3^{s},156532 \,.\, \sin. \ \ (n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \\ -31^{s},493729 \,.\, \sin. 2(n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \\ -\ 6^{s},565931 \,.\, \sin. 3(n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \\ -\ 1^{s},965748 \,.\, \sin. 4(n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \\ -\ 0^{s},697047 \,.\, \sin. 5(n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \\ -\ 0^{s},270789 \,.\, \sin. 6(n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \\ -\ 0^{s},116291 \,.\, \sin. 7(n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \\ -\ 0^{s},056126 \,.\, \sin. 8(n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \\ -\ 0^{s},034097 \,.\, \sin. 9(n^{\text{iv}}t - n^{\text{v}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{v}}) \end{array} \right\}$$ [4463]

* (2677) These are computed as in [4277a—o], increasing the accents on a, n, n', &c. so as to conform to the present case. [4463a]

[4463]

$$+(1+\mu^{vi})\cdot\left\{\begin{array}{l}
+\;9^s,248269\,.\,\sin.\;\;(n^{vi}t-n^{v}t+\varepsilon^{vi}-\varepsilon^{v})\\
-14^s,451913\,.\,\sin.\,2(n^{vi}t-n^{v}t+\varepsilon^{vi}-\varepsilon^{v})\\
-\;1^s,427160\,.\,\sin.\,3(n^{vi}t-n^{v}t+\varepsilon^{vi}-\varepsilon^{v})\\
-\;0^s,314960\,.\,\sin.\,4(n^{vi}t-n^{v}t+\varepsilon^{vi}-\varepsilon^{v})\\
-\;0^s,090690\,.\,\sin.\,5(n^{vi}t-n^{v}t+\varepsilon^{vi}-\varepsilon^{v})\\
-\;0^s,047444\,.\,\sin.\,6(n^{vi}t-n^{v}t+\varepsilon^{vi}-\varepsilon^{v})\\
-\;0^s,010686\,.\,\sin.\,7(n^{vi}t-n^{v}t+\varepsilon^{vi}-\varepsilon^{v})\\
-\;0^s,003942\,.\,\sin.\,8(n^{vi}t-n^{v}t+\varepsilon^{vi}-\varepsilon^{v})
\end{array}\right\}.$$

Inequalities independent of the excentricities.

$$\delta r^{v}=(1+\mu^{iv})\cdot\left\{\begin{array}{l}
+0,0039077763\\
+0,0081538400\,.\cos.\;\;(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})\\
+0,0013838330\,.\cos.\,2\,(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})\\
+0,0003200673\,.\cos.\,3\,(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})\\
+0,0000992632\,.\cos.\,4\,(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})\\
+0,0000355919\,.\cos.\,5\,(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})\\
+0,0000135999\,.\cos.\,6\,(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})\\
+0,0000055135\,.\cos.\,7\,(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})\\
+0,0000021631\,.\cos.\,8\,(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})\\
+0,0000006436\,.\cos.\,9\,(n^{iv}t-n^{v}t+\varepsilon^{iv}-\varepsilon^{v})
\end{array}\right\}$$

[4464]

$$+(1+\mu^{vi})\cdot\left\{\begin{array}{l}
-0,0000137622\\
+0,0001491217\,.\cos.\;\;(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi})\\
-0,0003949916\,.\cos.\,2(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi})\\
-0,0000480303\,.\cos.\,3(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi})\\
-0,0000118201\,.\cos.\,4(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi})\\
-0,0000036280\,.\cos.\,5(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi})\\
-0,0000012501\,.\cos.\,6(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi})
\end{array}\right\}.$$

*Inequalities depending on the first power of the excentricities.**

We shall here notice the secular variations in the coefficients of those
[4465] inequalities of Saturn, which exceed 100″, or $32^s,4$; in the same manner as we have done for Jupiter, in [4389′]. Hence we have,

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[4466a] * (2678) The inequalities depending on the first power of the excentricities, are computed in the same manner as for Jupiter [4390a, &c.].

Inequalities depending on the first power of the excentricities.

$$
\begin{aligned}
\delta v^{\mathrm{v}} = (1+\mu^{\mathrm{iv}})\,.\left\{\begin{array}{l}
-11^s,509517\,.\,\sin.\,(n^{\mathrm{iv}}t+\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{v}}) \\
+\ \ 1^s,258041\,.\,\sin.\,(n^{\mathrm{iv}}t+\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}) \\
-\ \ 2^s,064438\,.\,\sin.\,(2n^{\mathrm{iv}}t-n^{\mathrm{v}}t+2\varepsilon^{\mathrm{iv}}-\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{v}}) \\
+\ \ 2^s,672881\,.\,\sin.\,(2n^{\mathrm{iv}}t-n^{\mathrm{v}}t+2\varepsilon^{\mathrm{iv}}-\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{iv}}) \\
-\ \ 0^s,292291\,.\,\sin.\,(3n^{\mathrm{iv}}t-2n^{\mathrm{v}}t+3\varepsilon^{\mathrm{iv}}-2\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{v}}) \\
-\ \ 0^s,223191\,.\,\sin.\,(3n^{\mathrm{iv}}t-2n^{\mathrm{v}}t+3\varepsilon^{\mathrm{iv}}-2\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{iv}}) \\
-\ \ 0^s,090633\,.\,\sin.\,(4n^{\mathrm{iv}}t-3n^{\mathrm{v}}t+4\varepsilon^{\mathrm{iv}}-3\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{iv}}) \\
-(182^s,068686-t\,.\,0^s,0101095)\,.\sin.\left(\begin{array}{c}2n^{\mathrm{v}}t-n^{\mathrm{iv}}t\\+2\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{v}}\end{array}\right) \\
+(417^s,057741+t\,.\,0^s,0138572)\,.\sin.\left(\begin{array}{c}2n^{\mathrm{v}}t-n^{\mathrm{iv}}t\\+2\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}\end{array}\right) \\
+\ (34^s,341627-t\,.\,0^s,0019068)\,.\sin.\left(\begin{array}{c}3n^{\mathrm{v}}t-2n^{\mathrm{iv}}t\\+3\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{v}}\end{array}\right) \\
-17^s,654164\,.\,\sin.\,(3n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+3\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}) \\
+\ \ 4^s,795080\,.\,\sin.\,(4n^{\mathrm{v}}t-3n^{\mathrm{iv}}t+4\varepsilon^{\mathrm{v}}-3\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{v}}) \\
-\ \ 2^s,435410\,.\,\sin.\,(4n^{\mathrm{v}}t-3n^{\mathrm{iv}}t+4\varepsilon^{\mathrm{v}}-3\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}) \\
+\ \ 1^s,393612\,.\,\sin.\,(5n^{\mathrm{v}}t-4n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-4\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{v}}) \\
-\ \ 0^s,703450\,.\,\sin.\,(5n^{\mathrm{v}}t-4n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-4\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}) \\
+\ \ 0^s,537161\,.\,\sin.\,(6n^{\mathrm{v}}t-5n^{\mathrm{iv}}t+6\varepsilon^{\mathrm{v}}-5\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{v}}) \\
-\ \ 0^s,256510\,.\,\sin.\,(6n^{\mathrm{v}}t-5n^{\mathrm{iv}}t+6\varepsilon^{\mathrm{v}}-5\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}}) \\
+\ \ 0^s,216195\,.\,\sin.\,(7n^{\mathrm{v}}t-6n^{\mathrm{iv}}t+7\varepsilon^{\mathrm{v}}-6\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{v}}) \\
-\ \ 0^s,107342\,.\,\sin.\,(7n^{\mathrm{v}}t-6n^{\mathrm{iv}}t+7\varepsilon^{\mathrm{v}}-6\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}})
\end{array}\right\} \\
+(1+\mu^{\mathrm{vi}})\,.\left\{\begin{array}{l}
+\ \ 1^s,142398\,.\,\sin.\,(n^{\mathrm{vi}}t+\varepsilon^{\mathrm{vi}}-\varpi^{\mathrm{v}}) \\
-\ \ 1^s,011647\,.\,\sin.\,(n^{\mathrm{vi}}t+\varepsilon^{\mathrm{vi}}-\varpi^{\mathrm{vi}}) \\
-10^s,033866\,.\,\sin.\,(2n^{\mathrm{vi}}t-n^{\mathrm{v}}t+2\varepsilon^{\mathrm{vi}}-\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{v}}) \\
+\ \ 2^s,766173\,.\,\sin.\,(2n^{\mathrm{vi}}t-n^{\mathrm{v}}t+2\varepsilon^{\mathrm{vi}}-\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{vi}}) \\
-16^s,936280\,.\,\sin.\,(3n^{\mathrm{vi}}t-2n^{\mathrm{v}}t+3\varepsilon^{\mathrm{vi}}-2\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{v}}) \\
+25^s,153348\,.\,\sin.\,(3n^{\mathrm{vi}}t-2n^{\mathrm{v}}t+3\varepsilon^{\mathrm{vi}}-2\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{vi}}) \\
+\ \ 0^s,559336\,.\,\sin.\,(4n^{\mathrm{vi}}t-3n^{\mathrm{v}}t+4\varepsilon^{\mathrm{vi}}-3\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{v}}) \\
-\ \ 0^s,758225\,.\,\sin.\,(4n^{\mathrm{vi}}t-3n^{\mathrm{v}}t+4\varepsilon^{\mathrm{vi}}-3\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{vi}}) \\
-\ \ 0^s,187729\,.\,\sin.\,(5n^{\mathrm{vi}}t-4n^{\mathrm{v}}t+5\varepsilon^{\mathrm{vi}}-4\varepsilon^{\mathrm{v}}-\varpi^{\mathrm{vi}}) \\
-\ \ 0^s,673817\,.\,\sin.\,(2n^{\mathrm{v}}t-n^{\mathrm{vi}}t+2\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}}-\varpi^{\mathrm{v}}) \\
+\ \ 1^s,521577\,.\,\sin.\,(3n^{\mathrm{v}}t-2n^{\mathrm{vi}}t+3\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{vi}}-\varpi^{\mathrm{v}}) \\
+\ \ 0^s,151701\,.\,\sin.\,(4n^{\mathrm{v}}t-3n^{\mathrm{vi}}t+4\varepsilon^{\mathrm{v}}-3\varepsilon^{\mathrm{vi}}-\varpi^{\mathrm{v}})
\end{array}\right\}.
\end{aligned}
\qquad [4466]
$$

[4467]
$$\delta r^{\text{v}}=(1+\mu^{\text{iv}}).\left\{\begin{array}{l}-\,0{,}0003422170\,.\cos.\,(n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi^{\text{iv}})\\ -\,0{,}0020775935\,.\cos.\,(2\,n^{\text{v}}t-n^{\text{iv}}t+2\,\varepsilon^{\text{v}}-\varepsilon^{\text{iv}}-\varpi^{\text{v}})\\ +\,0{,}0053861750\,.\cos.\,(2\,n^{\text{v}}t-n^{\text{iv}}t+2\,\varepsilon^{\text{v}}-\varepsilon^{\text{iv}}-\varpi^{\text{iv}})\\ +\,0{,}0011594872\,.\cos.\,(3\,n^{\text{v}}t-2\,n^{\text{iv}}t+3\,\varepsilon^{\text{v}}-2\,\varepsilon^{\text{iv}}-\varpi^{\text{v}})\\ -\,0{,}0006217670\,.\cos.\,(3\,n^{\text{v}}t-2\,n^{\text{iv}}t+3\,\varepsilon^{\text{v}}-2\,\varepsilon^{\text{iv}}-\varpi^{\text{iv}})\\ +\,0{,}0002117893\,.\cos.\,(4\,n^{\text{v}}t-3\,n^{\text{iv}}t+4\,\varepsilon^{\text{v}}-3\,\varepsilon^{\text{iv}}-\varpi^{\text{v}})\end{array}\right\}$$
$$+\,(1+\mu^{\text{vi}}).\left\{\begin{array}{l}-0{,}0003750767\,.\cos.\,(3\,n^{\text{vi}}t-2\,n^{\text{v}}t+3\,\varepsilon^{\text{vi}}-2\,\varepsilon^{\text{v}}-\varpi^{\text{v}})\\ +\,0{,}0005605490\,.\cos.\,(3\,n^{\text{vi}}t-2\,n^{\text{v}}t+3\,\varepsilon^{\text{vi}}-2\,\varepsilon^{\text{v}}-\varpi^{\text{vi}})\end{array}\right\}.$$

*Inequalities depending on the squares and products of the excentricities and inclinations of the orbits.**

Inequalities of the second order.

$$\delta v^{\text{v}}=(1+\mu^{\text{iv}}).\left\{\begin{array}{l}-(54^{s}{,}847829-t.0^{s}{,}00362).\sin.\left(\begin{array}{l}3\,n^{\text{v}}t-n^{\text{iv}}t+3\,\varepsilon^{\text{v}}-\varepsilon^{\text{iv}}\\ +\,84^{d}36^{m}45^{s}-t.34^{s}{,}55\end{array}\right)\\ +\;\;28^{s}{,}526709\,.\sin.\,(n^{\text{iv}}t-n^{\text{v}}t+\varepsilon^{\text{iv}}-\varepsilon^{\text{v}}+84^{d}16^{m}43^{s})\\ -(669^{s}{,}682372-t.0^{s}{,}015469).\sin.\left(\begin{array}{l}2\,n^{\text{iv}}t-4\,n^{\text{v}}t+2\,\varepsilon^{\text{iv}}-4\,\varepsilon^{\text{v}}\\ +\,56^{d}10^{m}57^{s}+t.49^{s}{,}5\end{array}\right)\\ -\;\;2^{s}{,}935793\,.\sin.\,(5\,n^{\text{v}}t-3\,n^{\text{iv}}t+5\,\varepsilon^{\text{v}}-3\,\varepsilon^{\text{iv}}-57^{d}09^{m}08^{s})\end{array}\right\}$$
[4468]
$$+\,(1+\mu^{\text{vi}})\,.\left\{\begin{array}{l}+\;\;1^{s}{,}923552\,.\sin.\,(3\,n^{\text{vi}}t-3\,n^{\text{v}}t+3\,\varepsilon^{\text{vi}}-3\,\varepsilon^{\text{v}}-67^{d}54^{m}43^{s})\\ +31^{s}{,}025379\,.\sin.\,(3\,n^{\text{vi}}t-n^{\text{v}}t+3\,\varepsilon^{\text{vi}}-\varepsilon^{\text{v}}-85^{d}34^{m}12^{s})\end{array}\right\}.$$

If we connect the inequalities depending on $n^{\text{iv}}t-n^{\text{v}}t$; also those on
[4468′] $3\,n^{\text{vi}}t-3\,n^{\text{v}}t$, with the corresponding terms which are independent of the excentricities [4463], we shall obtain for their sum, the following expression,

[4469]
$$\begin{array}{l}\delta r^{\text{v}}=+\,(1+\mu^{\text{iv}})\,.\,28^{s}{,}967123\,.\sin.\,(n^{\text{iv}}t-n^{\text{v}}t+\varepsilon^{\text{iv}}-\varepsilon^{\text{v}}+78^{d}03^{m}13^{s})\\ \qquad\;\;-\,(1+\mu^{\text{vi}})\,.\;\;1^{s}{,}916292\,.\sin.(3\,n^{\text{vi}}t-3\,n^{\text{v}}t+3\,\varepsilon^{\text{vi}}-3\,\varepsilon^{\text{v}}+68^{d}27^{m}07^{s}).\end{array}$$

Then we have,†

[4470]
$$\delta r^{\text{v}}=(1+\mu^{\text{iv}}).\left\{\begin{array}{l}-\,0{,}0011710805\,.\cos.\,(3\,n^{\text{v}}t-n^{\text{iv}}t+3\,\varepsilon^{\text{v}}-\varepsilon^{\text{iv}}-90^{d}12^{m}35^{s})\\ -\,0{,}0005621901\,.\cos.\,(\;n^{\text{iv}}t-n^{\text{v}}t+\;\varepsilon^{\text{iv}}-\varepsilon^{\text{v}}-83^{d}26^{m}33^{s})\\ +(0{,}0151990624-t\,.\,0{,}0000003370).\cos.\left(\begin{array}{l}2\,n^{\text{iv}}t-4\,n^{\text{v}}t+2\,\varepsilon^{\text{iv}}-4\,\varepsilon^{\text{v}}\\ +\,56^{d}00^{m}33^{s}+t\,.\,49^{s}{,}04\end{array}\right)\end{array}\right\}.$$

[4468a] * (2679) Computed as in [4394a, &c.], for Jupiter.

[4470a] † (2680) This computation is made as in [4394d].

The inequality of the radius vector, depending on the angle $n^{iv}t - n^{v}t$, being connected with the similar term in [4464], which is independent of the excentricities, becomes, [4470′]

$$\delta r^{v} = (1+\mu^{iv}) \,.\, 0{,}0081090035 \,.\, \cos.(n^{v}t - n^{iv}t + \varepsilon^{v} - \varepsilon^{iv} - 3^{d}\,57^{m}\,35^{s}). \qquad [4471]$$

Since $5n^{v} - 2n^{iv}$ is very small, we have computed the inequality depending on $2n^{iv}t - 4n^{v}t$, by the formulas [3714, 3715]. Moreover, as $3n^{vi} - n^{v}$ is very small, we have computed the inequality depending on the angle $3n^{vi}t - n^{v}t$, by the formulas [3711, 3718]. *For greater accuracy, we must apply this last inequality to the mean motion of Saturn, on account of the length of its period.* [4472]

Inequalities depending on the powers and products of three and five dimensions of the excentricities and inclinations of the orbits, and on the square of the disturbing force.

The most considerable part of the great inequality of Saturn, is that which has $(5n^{v} - 2n^{iv})^2$, for a divisor, and depends on P, and P'. It is derived [4472′] from the great inequality of Jupiter, by multiplying it by $-\dfrac{15\,m^{iv}.\,n^{v\,2}.\,a^{v}}{6\,m^{v}.\,n^{iv\,2}.\,a^{iv}}$, in [4473] conformity with the formulas [3844, 3846].* Hence we get, for this part of the inequality of Saturn, the following expression,

$$\begin{aligned}\delta v^{v} = &-\{2957^{s}{,}857566 - t\,.\,0^{s}{,}019701 - t^2.\,0^{s}{,}00004505\}.\sin.(5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv})\\ &-\{279^{s}{,}746590 - t\,.\,1^{s}{,}108638 + t^2.\,0^{s}{,}00018387\}.\cos.(5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv}).\end{aligned} \qquad [4474]$$

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* (2681) If we represent, for brevity, the terms between the braces in the two first lines of [3844], by $a\,P_2$, we shall find, by inspection, that the two first lines of [3846], between the braces, are equal to $a'\,P_2$; and by noticing only those terms of δr, $\delta v'$, which [4472a] have the small divisor $(5n' - 2n)^2$, we shall get, by increasing the accents so as to conform to the case now under consideration,

$$\delta v^{iv} = -\frac{6\,m^{v}.\,n^{iv2}}{(5n^{v} - 2n^{iv})^2}\,.\,a^{iv}.\,P_2; \qquad \delta v^{v} = \frac{15\,m^{iv}.\,n^{v2}}{(5n^{v} - 2n^{iv})^2}\,.\,a^{v}.\,P_2. \qquad [4472b]$$

Hence it is evident that δv^{v} is easily deduced from δv^{iv}, by multiplying this last quantity by the factor [4473]; so that we shall have,

$$\delta v^{v} = -\frac{15\,m^{iv}.\,n^{v2}.\,a^{v}}{6m^{v}.\,n^{iv2}.\,a^{iv}}\,.\,\delta v^{iv}, \qquad [4472c]$$

as in the terms of the fifth dimension of the excentricities [3868a—c].

Inequalities of the third order. The great inequality of Saturn is composed of several other parts: it contains, in [3846], the function,*

[4475] $$\delta v^{v} = -\frac{2\,m^{iv}.n^{v}}{5\,n^{v}-2\,n^{iv}}\cdot\left\{\begin{array}{l} +a^{v\,2}.\left(\frac{dP}{da^{v}}\right).\cos.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \\ -a^{v\,2}.\left(\frac{dP'}{da^{v}}\right).\sin.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \end{array}\right\}.$$

Reducing this quantity to numbers, we find in 1750,

[4476] $$\begin{array}{l} \delta v^{v} = +\,52^{s},138991.\cos.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \\ \qquad -\,11^{s},275407.\sin.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}); \end{array}$$

and in 1950,

[4477] $$\begin{array}{l} \delta v^{v} = +\,51^{s},192839.\sin.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \\ \qquad -\,14^{s},982033.\cos.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}). \end{array}$$

Hence we deduce the value of this function for any time whatever t,

[4478] $$\begin{array}{l} \delta v^{v} = +\,(52^{s},138991-t.0^{s},0047308).\sin.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \\ \qquad -\,(11^{s},275407+t.0^{s},0185331).\cos.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}). \end{array}$$

The great inequality of Saturn contains also, in [3846], the term,

[4479] $$\delta v^{v} = -\tfrac{1}{2}\,H'\,e^{v}.\sin.(5n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}-\varpi^{v}+A').$$

This term, in 1750, is,

[4480] $$\begin{array}{l} \delta v^{v} = +\,7^{s},554290.\sin.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \\ \qquad +\,5^{s},321290.\cos.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}); \end{array}$$

and, in 1950, it is,

[4481] $$\begin{array}{l} \delta v^{v} = +\,7^{s},711294.\sin.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}) \\ \qquad +\,4^{s},825821.\cos.(5\,n^{v}t-2\,n^{iv}t+5\,\varepsilon^{v}-2\,\varepsilon^{iv}). \end{array}$$

---

* (2682) The expression [4475] is similar to [4419], in Jupiter's theory, and is
[4475a] computed in the same manner; namely, by finding the values of $\left(\frac{dM^{(0)}}{da^{v}}\right)$, $\left(\frac{dM^{(1)}}{da^{v}}\right)$, &c. similar to [4420]; which may be easily done, by means of formula [4421], and the values
[4475b] [4420]. Then from [3842, 3843], we get $\left(\frac{dP}{da^{v}}\right)$, $\left(\frac{dP'}{da^{v}}\right)$, &c. It is useless, however, to explain the details of this computation, as it is done in almost exactly the same way as

Hence, for any time t, it becomes,

$$\begin{aligned}\delta v^{\mathrm{v}} = &+ \{7^{\mathrm{s}},554290 + t\,.\,0^{\mathrm{s}},000785\}\,.\,\sin.(5\,n^{\mathrm{v}}t - 2\,n^{\mathrm{iv}}t + 5\,\varepsilon^{\mathrm{v}} - 2\,\varepsilon^{\mathrm{iv}}) \\ &+ \{5^{\mathrm{s}},321290 - t\,.\,0^{\mathrm{s}},002477\}\,.\,\cos.(5\,n^{\mathrm{v}}t - 2\,n^{\mathrm{iv}}t + 5\,\varepsilon^{\mathrm{v}} - 2\,\varepsilon^{\mathrm{iv}}).\end{aligned}$$ [4482]

The part of Saturn's great inequality, depending on *the powers and products of five dimensions of the excentricities and inclinations of the orbits*, is, by [3846,4023],*

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for Jupiter; we shall therefore only observe, that the expressions [4476, 4477, 4478, 4479, 4480, 4481, 4482,] correspond respectively to [4423, 4424, 4425, 4426, 4427, 4428, 4429]. [4475c]

* (2683) From the terms of R, of the *third* dimension, depending on P, P' [3810], we have deduced in the two first lines of [3811], the corresponding terms of δv; which are afterwards developed in [4022,4023], according to the powers of t; and the same process may be applied to the two first lines of $\delta v'$ [3846]. We may also derive these terms of [4483a] $\delta v'$, from the corresponding ones of δv, by multiplying by the factor $-\frac{15\,m\,.\,n'^2\,.\,a'}{6\,m'\,.\,n^2\,.\,a}$, or [4483b] rather by $-\frac{15m^{\mathrm{iv}}.n^{\mathrm{v}\,2}.a^{\mathrm{v}}}{6m^{\mathrm{v}}.n^{\mathrm{iv}\,2}.a^{\mathrm{iv}}}$, as is evident by the inspection of the formulas [3844,3846]. We may [4483c] proceed in exactly the same manner with the terms of R, of the fifth dimension, depending on $P_{\prime}$, $P_{\prime}'$ [3863], or with those of R', depending on $P_{\prime\prime}$, $P_{\prime\prime}'$ [3865]; the only change requisite is to place the accents below the letters P, P'. Now, if we neglect the parts of [4023], depending on t^2, ddP, ddP', and make the above-mentioned changes in the factor and in the accents of the remaining terms; also putting $P_{\prime}$, for $P_{\prime\prime}$, and $P_{\prime}'$, for $P_{\prime\prime}'$ [3864b], [4483d] we shall get, for δv^{v} the expression [4483], depending on quantities of the fifth order in e^{iv}, e^{v}, γ. In finding the values of $P_{\prime}$, $P_{\prime}'$, we may observe that the function R [3859] is easily reduced to the form [3863], by the method explained in [3842b, &c.]; using the values of $\mathcal{N}^{(0)}$, $\mathcal{N}^{(1)}$, &c. [4430], by means of which we obtain the expressions of $a^{\mathrm{v}}.P_{\prime}$, $a^{\mathrm{v}}.P_{\prime}'$, [4484, 4485], for the two epochs of 1750, 1950. The difference of these two [4483e] expressions being found, and divided respectively by 200, give the values [4486]; as is evident from the formula [3723]. Substituting [4484, 4486], in [4483], it becomes as in [4487]. The signs of all the terms [4481—4487], are different in the original work, being changed, as in [4430a], to correct the mistake mentioned in [3860a]. Moreover, to rectify this mistake in the signs, it is necessary to add the expression $2\,\delta v^{\mathrm{v}}$ [4487] to the second member of the great inequality of Saturn [4492, &c.], in the same manner as the similar value of $2\,\delta v^{\mathrm{iv}}$ [4431], is added to the expression of the great inequality of Jupiter [4431, &c.]. The numerical coefficients, in [4431, 4491], are equal to those given by the author; but the corrections C, C^{v}, $2\,\delta v^{\mathrm{iv}}$, $2\,\delta v^{\mathrm{v}}$, in the second members, are not mentioned in the original work. [4483f]

[4483] $$\delta v^{\mathrm{v}}=\frac{15m^{\mathrm{iv}}.n^{\mathrm{v}2}}{(5n^{\mathrm{v}}-2n^{\mathrm{iv}})^2}.\left\{\begin{array}{l}+\left\{a^{\mathrm{v}}.P'_{,}+\dfrac{2a^{\mathrm{v}}.dP_{,}}{(5n^{\mathrm{v}}-2n^{\mathrm{iv}}).dt}+t.a^{\mathrm{v}}.\dfrac{dP'_{,}}{dt}\right\}.\sin.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}})\\ -\left\{a^{\mathrm{v}}.P_{,}-\dfrac{2a^{\mathrm{v}}.dP'_{,}}{(5n^{\mathrm{v}}-2n^{\mathrm{iv}}).dt}+t.a^{\mathrm{v}}.\dfrac{dP_{,}}{dt}\right\}.\cos.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}})\end{array}\right\};$$

Inequalities of the fifth order.

in which $m^{\mathrm{iv}}P_{,}$, $m^{\mathrm{iv}}P'_{,}$ [3863, 4483b], express the coefficients of

$$\sin.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}}), \qquad \cos.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}}),$$

in the development of R, depending on the products of five dimensions of the excentricities and inclinations. We find, in the year 1750,

[4484] $$a^{\mathrm{v}}.P_{,}=0,0000068376;$$
$$a^{\mathrm{v}}.P'_{,}=0,0000100087;$$

and in the year 1950,

[4485] $$a^{\mathrm{v}}.P_{,}=0,0000077132;$$
$$a^{\mathrm{v}}.P'_{,}=0,0000096940;$$

consequently,

[4486] $$a^{\mathrm{v}}.\frac{dP_{,}}{dt}=\quad 0,0000000043780;$$
$$a^{\mathrm{v}}.\frac{dP'_{,}}{dt}=-0,0000000015735.$$

Hence the preceding function [4483], reduced to numbers, is,

[4487] $$\begin{array}{l}\delta v^{\mathrm{v}}=+\{29^{\mathrm{s}},144591-t.0^{\mathrm{s}},004081\}.\sin.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}})\\ \qquad -\{18^{\mathrm{s}},879594+t.0^{\mathrm{s}},011356\}.\cos.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}}).\end{array}$$

Lastly, we have, in [4003], the sensible part of the great inequality of Saturn, *depending on the square of the disturbing force*. This, in 1750, is,*

[4488] $$\begin{array}{l}\delta v^{\mathrm{v}}=-\ 3^{\mathrm{s}},816537.\sin.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}})\\ \qquad +42^{\mathrm{s}},920319.\cos.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}})\\ \qquad +\text{function } C^{\mathrm{v}}\ [4489k];\end{array}$$

and, in 1950,

[4489] $$\begin{array}{l}\delta v^{\mathrm{v}}=-\ 1^{\mathrm{s}},636772.\sin.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}})\\ \qquad +43^{\mathrm{s}},624686.\cos.(5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}})\\ \qquad +\text{function } C^{\mathrm{v}}\ [4489k], \text{ nearly.}\end{array}$$

[4489a] * (2684) The expression of δv^{v} [4003], being developed as in [3842a,b], and then computed as in the last note, becomes, according to the author, in 1750 and 1950, as in [4488, 4489],

Therefore, in the time $1750+t$, this part is expressed by,

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respectively. From these values, the general form [4490] is deduced, by the method used in [4483e, &c.]; but these numerical values, of the function [4003], have the same defects as the similar expression in Jupiter's motion [4432], of which we have treated in [4005a—4007b, 4431a—k]. The corrected value of δv^{v}, given by Mr. Pontécoulant in the paper referred to in [4431c], is as in the following table, which is similar to that of Jupiter [4431f, &c.]. [4489b]

Terms of the order of the square of the disturbing forces.

$$
\begin{array}{lrl}
 & \delta v^{\mathrm{v}} = & 2^s{,}17020\,.\sin.T_5+0^s{,}23185\,.\cos.T_5 \\
1 & & +8^s{,}14230\,.\sin.T_5+1^s{,}88438\,.\cos.T_5 \\
1' & & +4^s{,}89114\,.\sin.T_5-1^s{,}06769\,.\cos.T_5 \\
2' & & -0^s{,}95112\,.\sin.T_5-0^s{,}51669\,.\cos.T_5 \\
2 & & +0^s{,}05488\,.\sin.T_5-0^s{,}83060\,.\cos.T_5 \\
3 & & -0^s{,}25768\,.\sin.T_5-0^s{,}80208\,.\cos.T_5 \\
3' & & +1^s{,}74101\,.\sin.T_5+3^s{,}84548\,.\cos.T_5 \\
4 & & +0^s{,}22091\,.\sin.T_5+0^s{,}23748\,.\cos.T_5 \\
5 & & +1^s{,}85702\,.\sin.T_5-1^s{,}18481\,.\cos.T_5 \\
6,\ 6' & & +3^s{,}46607\,.\sin.T_5-40^s{,}36260\,.\cos.T_5 \\
7,\ i=2, & & -16^s{,}06895\,.\sin.T_5+1^s{,}95914\,.\cos.T_5 \\
7,\ i=1, & & +6^s{,}04586\,.\sin.T_5+2^s{,}23454\,.\cos.T_5 \\
8,\ i=2, & & -0^s{,}54808\,.\sin.T_5+1^s{,}29603\,.\cos.T_5
\end{array}
\qquad [4489c]
$$

$$= 10^s{,}76356\,.\sin.T_5-33^s{,}10557\,.\cos.T_5. \qquad [4489d]$$

This differs very much from the expression given by La Place, in [4488]; which is connected with the other terms of the great inequality [4491], after multiplying it by $1+\mu^{\mathrm{iv}}$. This multiplication, by $1+\mu^{\mathrm{iv}}$, is not strictly correct; because some of the terms depend on $(1+\mu^{\mathrm{iv}}).(1+\mu^{\mathrm{v}})$, and others upon $(1+\mu^{\mathrm{iv}})^2$; but as μ^{iv}, μ^{v}, are small, this difference is not of much importance in this small inequality. We shall therefore adopt this method of the author, as we have already done in the similar inequality of Jupiter [4431h, &c.]; where the factor $1+\mu^{\mathrm{v}}$, is used for all the terms. Proceeding, therefore, as in [4431h, &c.], we shall observe that the mass of Jupiter $\frac{1}{1070{,}5}$ [4061d], is used in computing [4489d]; and the mass $\frac{1}{1067{,}09}$ [4061], is used in computing [4488]; and if we increase the expression [4489d]. in the ratio of 1070,5 to 1067,09, it becomes as in [4489i]. Subtracting the expression [4488] from [4489i], we get very nearly the correction C^{v} [4489k], to be applied to the formula [4491 or 4492]. We must also apply a correction, depending on $\delta\varepsilon'$, similar to that of $\delta\varepsilon$ [4431p], in the great inequality of Jupiter; [4489e] [4489f] [4489g] [4489h]

Correction of the great inequality.

$$\delta v^{\mathrm{v}} = 10^s{,}79796\,.\sin.T_5-33^s{,}21137\,.\cos.T_5; \qquad [4489i]$$

$$C^{\mathrm{v}} = 14^s{,}61450\,.\sin.T_5-76^s{,}13169\,.\cos.T_5. \qquad [4489k]$$

[4490] $$\delta v^{\text{v}} = -\{3^{s},816537 - t\,.\,0^{s},0108988\}.\sin.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}) + \{42^{s},920319 + t\,.\,0^{s},0035218\}.\cos.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}) + \text{function } C^{\text{v}}\ [4489k].$$

Now, if we connect together the different parts of the great inequality of Saturn, we shall obtain its complete value, *which is to be applied to the planet's mean motion*;*

[4491] $$\delta v^{\text{v}} = -(1+\mu^{\text{iv}}).\left\{\begin{array}{l} +\{2931^{s},125145 - t.0^{s},0307355 - t^2.0^{s},0000450\}.\sin.\begin{pmatrix}5n^{\text{v}}t - 2n^{\text{iv}}t\\ +5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}\end{pmatrix} \\ +\ \{223^{s},252793 - t.1^{s},1025051 + t^2.0^{s},0001838\}.\cos.\begin{pmatrix}5n^{\text{v}}t - 2n^{\text{iv}}t\\ +5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}\end{pmatrix} \\ +\ \text{function } C^{\text{v}}\ [4489k] + 2\,\delta v^{\text{v}}\ [4487] \end{array}\right\}.$$

Great inequality. Reducing these two terms to one, by the method in [4025—4027′], we shall obtain,

[4492] $$\delta v^{\text{v}} = -(1+\mu^{\text{iv}}).\left\{\begin{array}{l} (2939^{s},615848 - t.0^{s},085024 + t^2.0^{s},00008421).\sin.\left\{\begin{array}{l}5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}} + 4^{d}21^{m}20^{s}\\ -\,t\,.\,77^{s},629 + t^2\,.\,0^{s},012676\end{array}\right\} \\ +\ \text{function } C^{\text{v}}\ [4489k] + 2\,\delta v^{\text{v}}\ [4487] \end{array}\right\}.$$

The square of the disturbing force produces also, in [3891′], the inequality,†

[4493] $$\delta v^{\text{v}} = \frac{\overline{H}^2}{8}\,.\,\frac{2\,m^{\text{v}}.\sqrt{a^{\text{v}}} + 5\,m^{\text{iv}}.\sqrt{a^{\text{iv}}}}{m^{\text{iv}}.\sqrt{a^{\text{iv}}}}\,.\sin.(\text{double of the argument of the great inequality});$$

which, in numbers, is,

[4494] $$\delta v^{\text{v}} = (30^{s},688957 - t.0^{s},001724).\sin.(\text{double argument of the great inequality});$$

and this must also be applied to the mean motion of Saturn.

Professor Hansen, in the work mentioned in [1158c], makes this part of the great inequality

[4489l] of Saturn, in the year 1800, as in [4489n], using the masses m^{iv}, m^{v} [4061]. The corresponding value of La Place's formula, is found by putting $t=50$, in [4490], by which

[4489m] means it becomes as in [4489o]. The difference of these two expressions represents the value of C^{v} [4489p], corresponding to the calculations of Professor Hansen, noticing all the terms of any importance;

[4489n] $$\delta v^{\text{v}} = \quad 15^{s},476\,.\sin.\,T_5 - 47^{s},531\,.\cos.\,T_5;$$

[4489o] $$\delta v^{\text{v}} = -3^{s},271\,.\sin.\,T_5 + 43^{s},096\,.\cos.\,T_5;$$

[4489p] $$C^{\text{v}} = \quad 18^{s},747\,.\sin.\,T_5 - 90^{s},627\,.\cos.\,T_5.$$

* (2685) The function [4491] is the sum of the expressions [4474, 4478, 4482, 4487, 4490];

[4491a] and this sum is easily reduced to the form [4492], containing but one term, by the method explained in [4025—4027′]. There is a small mistake in the calculation of the term $223^{s},252793$ [4491], which in the preceding sum is $223^{s},900791$; the difference being $0^{s},648 = 2''$.

[4493a] † (2686) The term [4493] is the same as [3891′], $-\overline{H}'$ [3891] being the great

The inequality [3927],*

$$\delta v^{v} = \tfrac{1}{4} \cdot \frac{3\, m^{iv} \sqrt{a^{iv}} + 2\, m^{v} \sqrt{a^{v}}}{m^{iv} \sqrt{a^{iv}}} \cdot \bar{H}' K' . \sin.(4\, n^{iv} t - 9\, n^{v} t + 4\, \varepsilon^{iv} - 9\, \varepsilon^{v} - B' - \bar{A}'), \qquad [4495]$$

reduced to numbers, is,

$$\delta v^{v} = +\, 8^{s},264517 . \sin.(4\, n^{iv} t - 9\, n^{v} t + 4\, \varepsilon^{iv} - 9\, \varepsilon^{v} + 51^{d}\, 49^{m}\, 37^{s}). \qquad [4496]$$

We have also, in [3846], the inequality,†

$$\delta v^{v} = \tfrac{5}{4}\, K' e^{v} . \sin.(3 n^{v} t - 2\, n^{iv} t + 3\, \varepsilon^{v} - 2\, \varepsilon^{iv} + \varpi^{v} + B') ; \qquad [4497]$$

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inequality of Saturn, or

$$\bar{H}' = 2939^{s},615848 - t . 0^{s},085024, \quad \text{and} \quad \bar{A}' = 4^{d}\, 21^{m}\, 20^{s}, \text{ nearly } [4193] : \qquad [4493b]$$

substituting this and the values of m^{iv}, m^{v}, a^{iv}, a^{v} [4061, 4079], and dividing by the radius in seconds 206265^{s}, for the sake of homogenity, we get δv^{v} [4494]. The correction in the value of $\bar{H}'$ [4483f], has a slight effect on this result; and the same may be observed relative to the correction of $\bar{H}$ [4483f], in the term [4436]; and in other terms depending on $\bar{H}$, $\bar{H}'$. [4493c]

* (2687) The inequality [4495] is the same as [3927], increasing the accents as in [4388a]. Now we have nearly as in [4493b],

$$\bar{H} = 2939^{s},615848, \qquad \bar{A}' = 4^{d}\, 21^{m}\, 20^{s} \ [4493b] ; \qquad [4495a]$$

and by comparing the expression [3925] with the third line of [4468], we get, by neglecting the terms depending on t,

$$K' = 669^{s},682372, \qquad B' = -\, 56^{d}\, 10^{m}\, 57^{s}. \qquad [4495b]$$

Substituting these in [4495], it becomes,

$$+\, 9^{s},2107 . \sin.(4\, n^{iv} t - 9\, n^{v} t + 4\, \varepsilon^{iv} - 9\, \varepsilon^{v} + 51^{d}\, 49^{m}\, 37^{s}). \qquad [4495c]$$

In the original work the coefficient has a different sign, being

$$-\, 25'',507770 = -\, 8^{s},264517,$$

also the angle $-B' - \bar{A}'$, as given at first, is,

$$-\, 67^{\circ},3508 = -\, 60^{d}\, 36^{m}\, 57^{s}. \qquad [4495d]$$

These mistakes are corrected by the author in [9105], where the coefficient is made equal to $+\, 8^{s},264517$, and the angle $-B' - \bar{A}' = 51^{d}\, 49^{m}\, 37^{s}$ nearly.

† (2688) This is the same as the last line of [3846], increasing the accents as in [4388a]. [4497a]

and by reduction to numbers, it becomes in 1750,*

[4498] $$\delta v^{v} = 47^{s},115141 \,.\, \sin.(2\, n^{iv}\, t - 3\, n^{v}\, t + 2\, \varepsilon^{iv} - 3\, \varepsilon^{v} + 148^{d}\, 08^{m}\, 08^{s});$$

and in 1950,

[4499] $$\delta v^{v} = 46^{s},307169 \,.\, \sin.(2\, n^{iv} t - 3\, n^{v}\, t + 2\, \varepsilon^{iv} - 3\, \varepsilon^{v} + 149^{d}\, 41^{m}\, 16^{s}).$$

Therefore its value for any time whatever t, is,

[4500] $$\delta v^{v} = (47^{s},115141 - t\,.\,0^{s},0010399)\,.\,\sin.(2\, n^{iv} t - 3\, n^{v} t + 2\, \varepsilon^{iv} - 3\, \varepsilon^{v} + 148^{d} 08^{m} 08^{s} + t.27^{s},94).$$

Connecting this expression with the following, obtained in [4466],

[4501] $$\begin{aligned}\delta v^{v} = & + (34^{s},341627 - t\,.\,0^{s},0019)\,.\,\sin.(3 n^{v} t - 2 n^{iv} t + 3\,\varepsilon^{v} - 2\,\varepsilon^{iv} - \varpi^{v}) \\ & - 17^{s},654164\,.\,\sin.(3\, n^{v}\, t - 2\, n^{iv}\, t + 3\, \varepsilon^{v} - 2\, \varepsilon^{iv} - \varpi^{iv});\end{aligned}$$

we shall obtain for their sum, the following inequality,†

[4502] $$\delta v^{v} = -(24^{s},571253 - t\,.\,0^{s},004392)\,.\sin.(2\, n^{iv} t - 3\, n^{v} t + 2\, \varepsilon^{iv} - 3\, \varepsilon^{v} + 144^{d} 48^{m} 19^{s} - t.12^{s},38).$$

We have found, in [3777], that Saturn's mean motion is subjected to a secular equation, corresponding to that of Jupiter in [4446], namely,

[4503] $$\delta v^{iv} = -t^{2}.\, 0^{s},00000065.$$

Secular equation. The corresponding secular equation of Saturn is represented, as in [3777], by,‡

[4504] $$\delta v^{v} = \frac{m^{iv} \sqrt{a^{iv}}}{m^{v} \sqrt{a^{v}}}\,.\, t^{2}.\, 0^{s},00000065\,;$$

and is therefore, in numbers,

[4505] $$\delta v^{v} = t^{2}.\, 0^{s},00000151\,;$$

which may be neglected without any sensible error.

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* (2689) If we retain the terms depending on t, in the values of K', B' [4495b, 4468], we shall have,

[4498a] $$K' = 669^{s},682372 - t\,.\,0^{s},015169\,; \qquad B' = -56^{d}\, 10^{m}\, 57^{s} - t\,.\,49^{s},5\,;$$
$$T = 1^{d}\, 21^{m}\, 20^{s} - t\,.\,77^{s},629 \quad [4492, 3926], \text{ \&c.}$$

[4498b] With these values, and those of e^{v}, ϖ^{v} [4407], we may compute the function [4497], for the years 1750, 1950, as in [4498, 4499]; hence we may deduce the general expression [4500], by the same method as in [4017—4021].

[4502a] † (2690) This reduction is made as in [4282h—l].

[4505a] ‡ (2691) The integral of [3777 or 3785], being divided by $m'\sqrt{a'}$, gives,

It now remains to consider the radius vector of Saturn. We have seen, in [3847], that the terms, depending on the third power or product of the excentricities, add to the expression of the radius vector of Saturn, the quantity,*

$$\begin{aligned}\delta r^{v} = &- H' a^{v}.e^{v}.\cos.(5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv} - \varpi^{v} + A') \\ &+ H' a^{v}.e^{v}.\cos.(3n^{v}t - 2n^{iv}t + 3\varepsilon^{v} - 2\varepsilon^{iv} + \varpi^{v} + A') \\ &- \frac{10m^{iv}.n^{v}.a^{v^2}}{5n^{v} - 2n^{iv}}.\left\{\begin{array}{l} P.\sin.(5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv}) \\ + P'.\cos.(5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv}) \end{array}\right\}.\end{aligned}$$ [4506]

Reducing this function to numbers, we obtain,

$$\delta r^{v} = (1+\mu^{iv}).\left\{\begin{array}{l} +0{,}0035199456 5.\cos.(5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv} + 13^{d}01^{m}49^{s}) \\ -0{,}0008553506.\cos.(2n^{iv}t - 3n^{v}t + 2\varepsilon^{iv} - 3\varepsilon^{v} + 35^{d}49^{m}08^{s}) \end{array}\right\}.$$ [4507]

Connecting the last of these two inequalities with those we have found in [4467], depending on the first power of the excentricities, namely, Inequalities in the radius vector.

$$\delta r^{v} = (1+\mu^{iv}).\left\{\begin{array}{l} +0{,}0011594872.\cos.(3n^{v}t - 2n^{iv}t + 3\varepsilon^{v} - 2\varepsilon^{iv} - \varpi^{v}) \\ -0{,}0006217670.\cos.(3n^{v}t - 2n^{iv}t + 3\varepsilon^{v} - 2\varepsilon^{iv} - \varpi^{iv}) \end{array}\right\},$$ [4508]

we get,†

$$\delta r^{v} = -(1+\mu^{iv}).0{,}0013806201.\cos.(2n^{iv}t - 3n^{v}t + 2\varepsilon^{iv} - 3\varepsilon^{v} - 23^{d}19^{m}18^{s}).$$ [4509]

$$\delta v^{v} = -\delta v^{iv}.\frac{m^{iv}\sqrt{a^{iv}}}{m^{v}\sqrt{a^{v}}};$$ [4505b]

the accents being increased as in [4388a]. Substituting δv^{iv} [4503], we get δv^{v} [4504], which is reduced to numbers as in [4505], by using the elements m^{iv}, m^{v}, a^{iv}, a^{v} [4505c] [4061, 4079]. This correction is only $1^{s}{,}5$, in 1000 years, which is hardly deserving of notice.

* (2692) The function [4506] is the same as the three last terms of [3847], multiplied by a', and increasing the accents [4388a]; the first term of [3847] being of the second order in e, e', γ, is included in [4170]. H' represents the part of $\frac{r^{v}\,\delta r^{v}}{a'^2}$ [3848], [4506a] depending on the angle $4n^{v}t - 2n^{iv}t$; P, P', are given in [4402, &c.]. Hence the expression [4506] becomes, in numbers, as in [4507].

† (2693) The function [4508] is the same as the fourth and fifth lines of [4467]. Connecting these with the similar terms [4507], and reducing the whole to one term, by the [4509a] method in [4282h—l], it becomes as in [4509].

The semi-major axis, which is used in calculating the elliptical part of the radius vector, must be increased as in [4058], by the quantity $\frac{1}{3}a^{\text{v}}.m^{\text{v}}$; and by adding it to the value of a^{v} [4079], we obtain,

[4510] $$a^{\text{v}} = 9{,}53881757.$$

Inequalities of Saturn's motion in latitude.

36. The formula [1030] gives,*

Inequalities in the latitude.

[4511]
$$\delta s^{\text{v}} = (1+\mu^{\text{iv}}) . \left\{ \begin{array}{l} +\,1^{s},787358 . \sin . (n^{\text{iv}}t + \varepsilon^{\text{iv}} - \Pi^{\text{v}}) \\ -\,0^{s},250180 . \sin . (2n^{\text{iv}}t - n^{\text{v}}t + 2\varepsilon^{\text{iv}} - \varepsilon^{\text{v}} - \Pi^{\text{v}}) \\ -\,0^{s},083946 . \sin . (3n^{\text{iv}}t - 2n^{\text{v}}t + 3\varepsilon^{\text{iv}} - 2\varepsilon^{\text{v}} - \Pi^{\text{v}}) \\ +\,3^{s},143523 . \sin . (2n^{\text{v}}t - n^{\text{iv}}t + 2\varepsilon^{\text{v}} - \varepsilon^{\text{iv}} - \Pi^{\text{v}}) \\ -\,0^{s},522865 . \sin . (3n^{\text{v}}t - 2n^{\text{iv}}t + 3\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}} - \Pi^{\text{v}}) \\ -\,0^{s},083182 . \sin . (4n^{\text{v}}t - 3n^{\text{iv}}t + 4\varepsilon^{\text{v}} - 3\varepsilon^{\text{iv}} - \Pi^{\text{v}}) \end{array} \right\}$$
$$+ (1+\mu^{\text{vi}}) . \left\{ \begin{array}{l} +\,0^{s},084871 . \sin . (n^{\text{vi}}t + \varepsilon^{\text{vi}} - \Pi^{\text{vi}}) \\ +\,0^{s},122203 . \sin . (2n^{\text{vi}}t - n^{\text{v}}t + 2\varepsilon^{\text{vi}} - \varepsilon^{\text{v}} - \Pi^{\text{vi}}) \\ +\,0^{s},662991 . \sin . (3n^{\text{vi}}t - 2n^{\text{v}}t + 3\varepsilon^{\text{vi}} - 2\varepsilon^{\text{v}} - \Pi^{\text{vi}}) \end{array} \right\} .$$

Π^{v}, *being the longitude of the node of Jupiter's orbit on that of Saturn,*
[4512] and Π^{vi}, *the longitude of the orbit of Uranus on that of Saturn.* Lastly, we have, in [3886], the inequality,†

[4513] $$\delta s^{\text{v}} = -\,9^{s},163599 . \sin . (2n^{\text{iv}}t - 4n^{\text{v}}t + 2\varepsilon^{\text{iv}} - 4\varepsilon^{\text{v}} + 59^{d}\,30^{m}\,35^{s}).$$

It follows, from [3932, 3932'], that the terms depending on the square of the disturbing force, add to the values of $\frac{d\varphi^{\text{v}}}{dt}$, $\frac{d\theta^{\text{v}}}{dt}$, the quantities,‡

[4511a] * (2694) The terms of δs^{v} [4511], are computed from [4295b], increasing the accents, so that m^{v} may be the attracted planet, and m^{iv} or m^{vi} the disturbing planet.

[4513a] † (2695) The inequality [4513] is the same as [3886], reduced to one term, as in [4282h—l].

[4514a] ‡ (2696) The values [4514, 4515], are deduced from [3932, 3932'], in the same manner as [4152, 4153], are derived from [3931, 3931']. We may also derive [4514] from [4452], and [4515] from [4453], by the following method. The expressions

$$\frac{d\varphi^{\text{v}}}{dt}=\frac{m^{\text{iv}}.\sqrt{a^{\text{iv}}}}{m^{\text{iv}}.\sqrt{a^{\text{iv}}}+m^{\text{v}}.\sqrt{a^{\text{v}}}}.\left\{\frac{\delta\gamma}{t}.\cos.(\Pi-\theta^{\text{v}})-\frac{\gamma\,\delta\Pi}{t}.\sin.(\Pi-\theta^{\text{v}})\right\}; \qquad [4514]$$

$$\frac{d\theta^{\text{v}}}{dt}=\frac{m^{\text{iv}}.\sqrt{a^{\text{iv}}}}{m^{\text{iv}}.\sqrt{a^{\text{iv}}}+m^{\text{v}}.\sqrt{a^{\text{v}}}}.\left\{\frac{\delta\gamma}{t}.\sin.(\Pi-\theta^{\text{v}})+\frac{\gamma\,\delta\Pi}{t}.\cos.(\Pi-\theta^{\text{v}})\right\}; \qquad [4515]$$

$\delta\gamma$, $\delta\Pi$, being determined as in [3935, 3936]. Reducing the functions [4514, 4515] to numbers, we get,

$$\frac{d\varphi^{\text{v}}}{dt}=+\,0^{s},000154; \qquad [4516]$$

$$\frac{d\theta^{\text{v}}}{dt}=-\,0^{s},001873. \qquad [4517]$$

The expression [4516] is to be added to the values of $\frac{d\varphi^{\text{v}}}{dt}$, $\frac{d\varphi_{\prime}^{\text{v}}}{dt}$ [4247]; and the expression [4517] is to be added to the values of $\frac{d\theta^{\text{v}}}{dt}$, $\frac{d\theta_{\prime}^{\text{v}}}{dt}$ [4247]. Hence we obtain,

$$\begin{aligned}\frac{d\varphi^{\text{v}}}{dt}&=+\,0^{s},099894;\\ \frac{d\varphi_{\prime}^{\text{v}}}{dt}&=-\,0^{s},155136;\\ \frac{d\theta^{\text{v}}}{dt}&=-\,9^{s},007165;\\ \frac{d\theta_{\prime}^{\text{v}}}{dt}&=-19^{s},043372.\end{aligned} \qquad [4518]$$

[3931, 3931′], become the same as [3932, 3932′], respectively, by changing, in the second members, θ into θ', and multiplying by $-\frac{m\sqrt{a}}{m'\sqrt{a'}}$. This is equivalent, in the present [4514b] notation, to the change of θ^{iv}, into θ^{v}, and then multiplying by the factor $-\frac{m^{\text{iv}}\sqrt{a^{\text{iv}}}}{m^{\text{v}}\sqrt{a^{\text{v}}}}$. Therefore, if we perform this operation on the formulas [4452, 4453], they become respectively, as in [4514, 4515]; in which we must compute $\delta\gamma$, $\delta\Pi$, as in [4452h]; and then, as in [4452h, &c.], we obtain the other quantities [4516, 4517, 4518]. [4514c]

We have already remarked, that the inequalities of the motion of this planet are again noticed by the author, in book x. chap. viii. [9037, &c.], and the subject is also resumed in the notes on this part of the work. [4514d]

CHAPTER XIV.

THEORY OF URANUS.

37. The equation [4460],

[4519] $$\delta r^{\text{v}} = \frac{r^{\text{v}\,2}}{r''} \cdot (1 - \alpha^2) \cdot \delta \mathrm{V}^{\text{v}},$$

corresponding to Saturn, becomes for Uranus,

[4520] $$\delta r^{\text{vi}} = \frac{r^{\text{vi}\,2}}{r''} \cdot (1 - \alpha^2) \cdot \delta \mathrm{V}^{\text{vi}}.$$

If we take the mean distances of the earth and Uranus from the sun, for r'', and r^{vi}, and suppose $\delta \mathrm{V}^{\text{vi}} = \pm 1'' = \pm 0^s{,}324$, we shall find,

[4521] $$\delta r^{\text{vi}} = \pm 0{,}00057648.$$

Terms which may be neglected. Therefore we may neglect the inequalities of δr^{vi}, below $\pm 0{,}00057$; and we shall also omit the inequalities of the motion of Uranus, in
[4522] longitude or latitude, below a quarter of a centesimal second, or $0^s{,}081$.

*Inequalities of Uranus, independent of the excentricities.**

[4523] $$\delta v^{\text{vi}} = (1+\mu^{\text{iv}}) \cdot \left\{ \begin{array}{l} +52^s{,}306055 \,.\, \sin. \;\, (n^{\text{iv}}t - n^{\text{vi}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{vi}}) \\ -\; 0^s{,}190366 \,.\, \sin. 2(n^{\text{iv}}t - n^{\text{vi}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{vi}}) \\ -\; 0^s{,}026023 \,.\, \sin. 3(n^{\text{iv}}t - n^{\text{vi}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{vi}}) \\ -\; 0^s{,}003593 \,.\, \sin. 4(n^{\text{iv}}t - n^{\text{vi}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{vi}}) \\ -\; 0^s{,}000768 \,.\, \sin. 5(n^{\text{iv}}t - n^{\text{vi}}t + \varepsilon^{\text{iv}} - \varepsilon^{\text{vi}}) \end{array} \right\}$$

[4523a] * (2697) Computed as in [4277a, &c.], changing the accents on a, n, n', &c. to conform to the case now under consideration.

$$+(1+\mu^{\mathrm{v}}).\left\{\begin{array}{l}+21^{\mathrm{s}},371379\,.\,\sin.\;(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\-\;4^{\mathrm{s}},220972\,.\,\sin.\,2(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\-\;0^{\mathrm{s}},862115\,.\,\sin.\,3(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\-\;0^{\mathrm{s}},244409\,.\,\sin.\,4(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\-\;0^{\mathrm{s}},080211\,.\,\sin.\,5(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\-\;0^{\mathrm{s}},028931\,.\,\sin.\,6(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\-\;0^{\mathrm{s}},010929\,.\,\sin.\,7(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\-\;0^{\mathrm{s}},004148\,.\,\sin.\,8(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\end{array}\right\}. \qquad [4523]$$

Inequalities independent of the excentricities.

$$\delta r^{\mathrm{vi}}=(1+\mu^{\mathrm{iv}}).\left\{\begin{array}{l}0,0063473160\\+0,0048914790.\cos.\;(n^{\mathrm{iv}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{iv}}-\varepsilon^{\mathrm{vi}})\\+0,0000236184.\cos.\,2\,(n^{\mathrm{iv}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{iv}}-\varepsilon^{\mathrm{vi}})\\+0,0000030669\,.\cos.\,3\,(n^{\mathrm{iv}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{iv}}-\varepsilon^{\mathrm{vi}})\\+0,0000005044.\cos.\,4\,(n^{\mathrm{iv}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{iv}}-\varepsilon^{\mathrm{vi}})\end{array}\right\}$$

[4524]

$$+(1+\mu^{\mathrm{v}}).\left\{\begin{array}{l}+\;0,0023641285\\+\;0,0035433901.\cos.\;(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\+\;0,0004061682\,.\cos.\,2\,(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\+\;0,0000889425\,.\cos.\,3\,(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\\+\;0,0000255870\,.\cos.\,4\,(n^{\mathrm{v}}t-n^{\mathrm{vi}}t+\varepsilon^{\mathrm{v}}-\varepsilon^{\mathrm{vi}})\end{array}\right\}.$$

*Inequalities depending on the first power of the excentricities.**

$$\delta v^{\mathrm{vi}}=(1+\mu^{\mathrm{iv}}).\left\{\begin{array}{l}-\;1^{\mathrm{s}},233612\,.\,\sin.\,(n^{\mathrm{iv}}t+\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{vi}})\\+\;1^{\mathrm{s}},259548\,.\,\sin.\,(2\,n^{\mathrm{iv}}t-n^{\mathrm{vi}}t+2\,\varepsilon^{\mathrm{iv}}-\varepsilon^{\mathrm{vi}}-\varpi^{\mathrm{iv}})\\-\;3^{\mathrm{s}},636663\,.\,\sin.\,(2\,n^{\mathrm{vi}}t-n^{\mathrm{iv}}t+2\,\varepsilon^{\mathrm{vi}}-\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{vi}})\\-\;0^{\mathrm{s}},221997\,.\,\sin.\,(2\,n^{\mathrm{vi}}t-n^{\mathrm{iv}}t+2\,\varepsilon^{\mathrm{vi}}-\varepsilon^{\mathrm{iv}}-\varpi^{\mathrm{iv}})\end{array}\right\} \qquad [4525]$$

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* (2698) These inequalties were computed in the same manner as those for Jupiter in [4375a]. [4525a]

Inequalities depending on the first power of the excentricities.

[4525]
$$+(1+\mu^{v})\cdot\left\{\begin{array}{l}
-\ 1^{s},402359\ .\ \sin.\,(n^{v}t+\varepsilon^{v}-\varpi^{vi}) \\
+\ 0^{s},214857\ .\ \sin.\,(n^{v}t+\varepsilon^{v}-\varpi^{v}) \\
-\ 0^{s},219788\ .\ \sin.\,(2\,n^{v}t-n^{vi}t+2\,\varepsilon^{v}-\varepsilon^{vi}-\varpi^{vi}) \\
+\ 0^{s},878763\ .\ \sin.\,(2\,n^{v}t-n^{vi}t+2\,\varepsilon^{v}-\varepsilon^{vi}-\varpi^{v}) \\
-\ (44^{s},051575-t\,.\,0^{s},000247)\,.\sin.\left(\begin{array}{c}2n^{vi}t-n^{v}t \\ +2\varepsilon^{vi}-\varepsilon^{v}-\varpi^{vi}\end{array}\right) \\
+\ (149^{s},807764-t\,.\,0^{s},008306)\,.\sin.\left(\begin{array}{c}2n^{vi}t-n^{v}t \\ +2\varepsilon^{vi}-\varepsilon^{v}-\varpi^{v}\end{array}\right) \\
+\ 2^{s},486191\ .\ \sin.\,(3\,n^{vi}t-2\,n^{v}t+3\,\varepsilon^{vi}-2\,\varepsilon^{v}-\varpi^{vi}) \\
-\ 1^{s},642451\ .\ \sin.\,(3\,n^{vi}t-2\,n^{v}t+3\,\varepsilon^{vi}-2\,\varepsilon^{v}-\varpi^{v}) \\
+\ 0^{s},422729\ .\ \sin.\,(4\,n^{vi}t-3\,n^{v}t+4\,\varepsilon^{vi}-3\,\varepsilon^{v}-\varpi^{vi}) \\
-\ 0^{s},281800\ .\ \sin.\,(4\,n^{vi}t-3\,n^{v}t+4\,\varepsilon^{vi}-3\,\varepsilon^{v}-\varpi^{v}) \\
+\ 0^{s},126493\ .\ \sin.\,(5\,n^{vi}t-4\,n^{v}t+5\,\varepsilon^{vi}-4\,\varepsilon^{v}-\varpi^{vi})
\end{array}\right\}.$$

[4526]
$$\delta r^{vi}=(1+\mu^{v})\,.\left\{\begin{array}{l}
-\,0{,}0016092001\,.\,\cos.\,(2\,n^{vi}t-n^{v}t+2\,\varepsilon^{vi}-\varepsilon^{v}-\varpi^{vi}) \\
+\,0{,}0061835858\,.\,\cos.\,(2\,n^{vi}t-n^{v}t+2\,\varepsilon^{vi}-\varepsilon^{v}-\varpi^{v})
\end{array}\right\}.$$

*Inequalities depending on the squares and products of the excentricities and inclinations of the orbits.**

[4527]
$$\delta v^{vi}=(1+\mu^{v})\,.\left\{\begin{array}{l}
-(132^{s},508872-t.0^{s},0145205).\sin.\left(\begin{array}{c}3\,n^{vi}t-n^{v}t+3\,\varepsilon^{vi}-\varepsilon^{v} \\ -88^{d}19^{m}05^{s}-t.17^{s},3\end{array}\right) \\
+\quad 1^{s},713455\,.\sin.\,(4\,n^{vi}t-2\,n^{v}t+4\varepsilon^{vi}-2\,\varepsilon^{v}-38^{d}\,34^{m}\,54^{s}) \\
+\quad 8^{s},380157\,.\sin.\,(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi}+88^{d}29^{m}\,40^{s})
\end{array}\right\}.$$

Inequalities of the second order.

The first of these inequalities must be applied to the mean motion of the planet, on account of the length of its period. The last of these inequalities, being connected with the corresponding one in [4523], which is independent of the excentricities, gives the following,†

[4528]
$$\delta v^{vi}=(1+\mu^{v})\,.\,23^{s},156281\ .\ \sin.(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi}+21^{d}\,11^{m}\,05^{s}).$$

[4527a] * (2699) Computed as in [4377a,&c.], for Jupiter.

† (2700) The term $+(1+\mu^{v})\,.\,21^{s},371379\,.\sin.\,(n^{v}t-n^{vi}t+\varepsilon^{v}-\varepsilon^{vi})$ **[4523],**

[4528a] being connected with the last term of [4527], by the method used in [4282h—l], becomes as in [4528].

Then we have,*

$$\delta r^{\text{vi}} = -(1+\mu^{\text{v}}) \,.\, 0{,}0007553840 \,.\, \cos.(3\,n^{\text{vi}}t - n^{\text{v}}t + 3\,\varepsilon^{\text{vi}} - \varepsilon^{\text{v}} + 75^d\,00^m\,42^s). \qquad [4529]$$

Inequalities depending on the powers and products of three dimensions of the excentricities and inclinations of the orbits.† Inequalities of the third order.

$$\delta v^{\text{vi}} = -(1+\mu^{\text{v}}) \,.\, 0^s{,}964688 \,.\, \sin.(5\,n^{\text{v}}\,t - 2\,n^{\text{v}}t + 5\,\varepsilon^{\text{vi}} - 2\,\varepsilon^{\text{v}} + 68^d\,23^m\,31^s). \qquad [4530]$$

Inequalities of the motion of Uranus in latitude.

38. From the formula [1030], we obtain,‡ Inequalities in the latitude.

$$\delta s^{\text{vi}} = (1+\mu^{\text{iv}}) \,.\, 0^s{,}638393 \,.\, \sin.(n^{\text{iv}}\,t + \varepsilon^{\text{iv}} - \Pi^{\text{iv}})$$
$$+ (1+\mu^{\text{v}}) \,.\, \left\{ \begin{array}{l} 0^s{,}915741 \,.\, \sin.(n^{\text{v}}t + \varepsilon^{\text{v}} - \Pi^{\text{v}}) \\ + 2^s{,}921052 \,.\, \sin.(2\,n^{\text{vi}}t - n^{\text{v}}t + 2\,\varepsilon^{\text{vi}} - \varepsilon^{\text{v}} - \Pi^{\text{v}}) \end{array} \right\}. \qquad [4531]$$

Π^{iv} being here the longitude of the ascending node of Jupiter's orbit upon that of Uranus, and Π^{v} the longitude of the ascending node of Saturn's orbit upon that of Uranus. [4531']

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* (2701) Computed as in [4394*a*, &c.] for Jupiter. [4529*a*]

† (2702) This computation is made as in [4417, &c.] for Jupiter; changing the accents to conform to the present notation. [4530*a*]

‡ (2703) The terms [4531] are computed from the formula [4295*b*], altering the accents to conform to the present case. [4531*a*]

CHAPTER XV.

ON SOME EQUATIONS OF CONDITION BETWEEN THE INEQUALITIES OF THE PLANETS, WHICH MAY BE USED IN VERIFYING THEIR NUMERICAL VALUES.

39. The inequalities of a long period, produced by the reciprocal action of two planets m, and m', are nearly in the ratio of $m'\sqrt{a'}$ to $-m\sqrt{a}$ [1208]; so that *to obtain the perturbations of this kind, corresponding, in the motion of* m', *to those in the motion of* m, *we need only to multiply the last*

[4532] *by* $-\frac{m\sqrt{a}}{m'\sqrt{a'}}$. This result is most to be relied upon, in those cases, in which the ratio of the mean motions of the two planets is such, as to render the period of these inequalities great, in comparison with the times of their revolutions. We shall now, by means of this theorem, verify several of the preceding inequalities.

The action of the earth on Venus produces, in [4291], the two following inequalities, whose period is about four years,

[4533]
$$\begin{aligned}\delta v' = &-1^s{,}549550\,.\sin.(3n''t-2n't+3\varepsilon''-2\varepsilon'-\varpi')\\&+4^s{,}766332\,.\sin.(3n''t-2n't+3\varepsilon''-2\varepsilon'-\varpi'').\end{aligned}$$

Venus and the Earth.

By multiplying them by $-\frac{m'\sqrt{a'}}{m''\sqrt{a''}}$, we have, for the corresponding inequality of the earth,

[4534]
$$\begin{aligned}\delta v'' = &\quad 1^s{,}133838\,.\sin.(3n''t-2n't+3\varepsilon''-2\varepsilon'-\varpi')\\&-3^s{,}487666\,.\sin.(3n''t-2n't+3\varepsilon''-2\varepsilon'-\varpi'').\end{aligned}$$

We have found, by a direct calculation, in [4307], that these inequalities are,

[4535]
$$\begin{aligned}\delta v'' = &\quad 1^s{,}186390\,.\sin.(3n''t-2n't+3\varepsilon''-2\varepsilon'-\varpi')\\&-3^s{,}667112\,.\sin.(3n''t-2n't+3\varepsilon''-2\varepsilon'-\varpi'');\end{aligned}$$

which differs but little from the preceding expression [4534].

The action of the earth upon Venus, produces also, in [4293], the following inequality, whose period is about eight years,

$$\delta v' = -1^s{,}505036 . \sin.(5n''t - 3n't + 5\varepsilon'' - 3\varepsilon' + 20^d\, 54^m\, 26^s). \qquad [4536]$$

Multiplying it by, $-\frac{m'\sqrt{a'}}{m''\sqrt{a''}}$, we obtain, for the corresponding inequality of the earth,

$$\delta v'' = 1^s{,}101277 . \sin.(5n''t - 3n't + 5\varepsilon'' - 3\varepsilon' + 20^d\, 54^m\, 26^s); \qquad [4537]$$

and, by a direct calculation, we have, in [4309],

$$\delta v'' = 1^s{,}125575 . \sin.(5n''t - 3n't + 5\varepsilon'' - 3\varepsilon' + 21^d\, 02^m\, 18^s). \qquad [4538]$$

Mars suffers, by the action of Venus, as we have seen in [4377], the following inequality of a long period,

$$\delta v''' = -6^s{,}899619 . \sin.(3n'''t - n't + 3\varepsilon''' - \varepsilon' + 65^d\, 26^m\, 15^s). \qquad [4539]$$

Multiplying it by $-\frac{m'''\sqrt{a'''}}{m'\sqrt{a'}}$, we obtain, Venus and Mars.

$$\delta v'' = 2^s{,}078266 . \sin.(3n'''t - n't + 3\varepsilon''' - \varepsilon' + 65^d\, 26^m\, 15^s); \qquad [4540]$$

and the direct calculation [4293] gives,

$$\delta v' = 2^s{,}009677 . \sin.(3n'''t - n't + 3\varepsilon''' - \varepsilon' + 65^d\, 53^m\, 09^s); \qquad [4541]$$

which differs but little from the preceding.

Mars suffers, from the action of the earth [4375], the two following inequalities, whose period is about sixteen years, The Earth and Mars.

$$\begin{aligned}\delta v''' = &-10^s{,}114699 . \sin.(2n'''t - n''t + 2\varepsilon''' - \varepsilon'' - \varpi''') \\ &+ 5^s{,}123062 . \sin.(2n'''t - n''t + 2\varepsilon''' - \varepsilon'' - \varpi'').\end{aligned} \qquad [4542]$$

Multiplying them by $-\frac{m'''\sqrt{a'''}}{m''\sqrt{a''}}$, we obtain, for the corresponding inequalities of the earth,

$$\begin{aligned}\delta v'' = &\ 2^s{,}2293 . \sin.(2n'''t - n''t + 2\varepsilon''' - \varepsilon'' - \varpi''') \\ &-1^s{,}1292 . \sin.(2n'''t - n''t + 2\varepsilon''' - \varepsilon'' - \varpi'');\end{aligned} \qquad [4543]$$

and the direct calculation gives, in [4307],

[4544]
$$\delta v'' = 2^s{,}137658 \,.\, \sin.(2n'''t - n''t + 2\varepsilon''' - \varepsilon'' - \varpi''')$$
$$-1^s{,}095603 \,.\, \sin.(2n'''t - n''t + 2\varepsilon''' - \varepsilon'' - \varpi'') ;$$

which differ but little from the preceding.

Mars also suffers, on the part of the earth, in [4377], the following inequality of a long period,

[4545]
$$\delta v''' = -4^s{,}370903 \sin.(4n'''t - 2n''t + 4\varepsilon''' - 2\varepsilon'' + 67^d\, 49^m\, 00^s).$$

Multiplying it by $-\frac{m'''\sqrt{a'''}}{m''\sqrt{a''}}$, we obtain, for the corresponding inequality of the earth,

[4546]
$$\delta v'' = 0^s{,}9634 \,.\, \sin.(4n'''t - 2n''t + 4\varepsilon''' - 2\varepsilon'' + 67^d\, 49^m\, 00^s) ;$$

which differs but little from the expression, given in [4309],

[4547]
$$\delta v'' = 0^s{,}993935 .\, \sin.(4n'''t - 2n''t + 4\varepsilon''' - 2\varepsilon'' + 67^d\, 48^m\, 56^s).$$

Jupiter and Saturn.

The two great inequalities of Jupiter and Saturn, are also to each other, nearly in the ratio of $-m^{v}\sqrt{a^{v}}$ to $m^{iv}\sqrt{a^{iv}}$, as is evident by comparing
[4548] [4434, 4492].

Saturn and Uranus.

Lastly, Uranus suffers, from the action of Saturn, the following inequality of a long period [4527],

[4549]
$$\delta v^{vi} = -132^s{,}508872 \,.\, \sin.(3n^{vi}t - n^{v}t + 3\varepsilon^{vi} - \varepsilon^{v} - 88^d\, 19^m\, 05^s).$$

Multiplying it by $-\frac{m^{vi}\sqrt{a^{vi}}}{m^{v}\sqrt{a^{v}}}$, we obtain, in the motion of Saturn, the inequality,

[4550]
$$\delta v^{v} = 32^s{,}368 \,.\, \sin.(3n^{vi}t - n^{v}t + 3\varepsilon^{vi} - \varepsilon^{v} - 88^d\, 19^m\, 05^s) ;$$

which differs but little from the inequality, given in [4468],*

[4551]
$$\delta v^{v} = 30^s{,}888288 \,.\, \sin.(3n^{vi}t - n^{v}t + 3\varepsilon^{vi} - \varepsilon^{v} - 87^d\, 25^m\, 07^s) ;$$

40. We shall now consider, in the development of R, the term of the form [3745'],

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* (2704) The term here referred to is the last one of the expression [4468]; which
[4550a] differs, however, a little; the coefficient being $31^s{,}025379$, instead of $30^s{,}888288$; and the constant angle $85^d\, 34^m\, 12^s$, instead of $87^d\, 25^m\, 07^s$.

$$R = m'.M^{(1)}.ee'.\cos.\{i.(n't - nt + \varepsilon' - \varepsilon) + 2nt + 2\varepsilon - \varpi - \varpi'\};$$ [4552]

supposing $i.(n-n')+2n$ *to be very small in comparison with* n or n'. We find, in [1286, &c.], that this term produces, in the excentricity e, of the orbit of the planet m, considered as a variable ellipsis, the following inequality, which we shall represent by,*

$$\delta e = -\frac{m'.an}{i.(n'-n)+2n}.M^{(1)}.e'.\cos.\{i.(n't - nt + \varepsilon' - \varepsilon) + 2nt + 2\varepsilon - \varpi - \varpi'\};$$ [4553]

and in the position of the perihelion ϖ, an inequality [1294, &c.], which we shall represent by,

$$\delta\varpi = -\frac{m'.an}{i.(n'-n)+2n}.M^{(1)}.\frac{e'}{e}.\sin.\{i.(n't - nt + \varepsilon' - \varepsilon) + 2nt + 2\varepsilon - \varpi - \varpi'\}.$$ [4554]

The expression of v contains the term $2e.\sin.(nt+\varepsilon-\varpi)$; and the variation of the elliptical elements, produces, in this quantity, the following expression,† [4555]

$$\delta v = 2\delta e.\sin.(nt+\varepsilon-\varpi) - 2e\delta\varpi.\cos.(nt+\varepsilon-\varpi);$$ [4556]

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* (2705) If we take the partial differential of R [1281], relative to e, and multiply it by $-\frac{.an}{\mu.(i'n'-in).de}$, it will produce the corresponding term of e, represented by δe [4553a] [1286]. Now, if we perform the same operation on the assumed value of R [4552], and put $\mu=1$ [3709]; changing also i', i, into i, $i-2$, respectively, we shall get δe [4553b] [4553]. Again, if we multiply the same partial differential of R [1281], relative to e, by $-\ .andt$, putting $\mu=1$, it becomes like the expression of $e\,d\varpi$ [1294]; and by the same process we deduce, from R [4552], the expression,

$$e\,d\varpi = -m'.andt.M^{(1)}.e'.\cos.\{i.(n't - nt + \varepsilon' - \varepsilon) + 2nt + 2\varepsilon - \varpi - \varpi'\}.$$ [4553c]

Dividing this by e, and integrating, we get the part of ϖ, represented by $\delta\varpi$ [4554]; observing that we may consider the terms M, e, e', of the second member, as constant quantities, in taking this integral; always neglecting quantities of a higher order than those which are retained, and such as depend on different angles.

† (2706) Since v [3834] contains the term $2e.\sin.(nt+\varepsilon-\varpi)$, it is evident that the variation of v, corresponding to the increments δe, $\delta\varpi$, in e, ϖ, respectively, is as in [4556]; and by using the symbol $W = nt+\varepsilon-\varpi$ [3702a], it becomes,

$$\delta v = 2\delta e.\sin.W - 2e\delta\varpi.\cos.W.$$ [4557a]

Now, if we put, for brevity,

which gives in v the inequality,

[4557] $$\delta v = \frac{2\,m'.\,a\,n}{i.(n'-n)+2n}\,.\,M^{(1)}.\,e'.\,\sin.\,\{(i-1)\,.\,(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi'\}.$$

It follows, from § 65 of the second book, that *in the case of* $i.(n'-n)+2n$
[4557'] *being very small*, the expression of R', relative to the action of m upon m', contains also a term, of the following form and value, very nearly,*

[4558] $$R' = m\,.\,M^{(1)}.\,e\,e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'\}:$$

since, by noticing only the two terms of this kind, in R, and R', we have, as in [1202], very nearly,

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[4557b] $$\mathrm{T}_1 = i.(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon-\varpi-\varpi'; \qquad M_1 = \frac{m'.\,a\,n}{i.(n'-n)+2n}.M^{(1)}.\,e';$$

the expressions [4553, 4554] become,

[4557c] $$\delta e = -M_1\,.\cos.\mathrm{T}_1; \qquad e\,\delta\varpi = -M_1\,.\sin.\mathrm{T}_1:$$

substituting these in [4557a], we get,

[4557d] $$\delta v = 2\,M_1.\{-\cos.\mathrm{T}_1.\sin.W+\sin.\mathrm{T}_1.\cos.W\} = 2M_1.\sin.(\mathrm{T}_1-W)$$
$$= 2\,M_1.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}$$
[4557e] $$= 2\,M_1.\sin.\{(i-1).(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi'\}, \text{ as in [4557]}.$$

* (2707) Using the symbol T_1 [4557b], we get, from [4552],

[4558a] $$R = m'.\,M^{(1)}.\,e\,e'\,.\cos.\mathrm{T}_1.$$

Its differential, relative to d [3705b—c], is,

[4558b] $$\mathrm{d}R = m'.\,M^{(1)}.\,e\,e'.\,(i-2)\,.\,n\,dt\,.\sin.\mathrm{T}_1;$$

substituting this in the differential of [4559], which gives $m'.\mathrm{d}'R' = -m\,.\,\mathrm{d}R$, and dividing by m', we obtain,

[4558c] $$\mathrm{d}'R' = -m\,.\,M^{(1)}.\,e\,e'.(i-2)\,.\,n\,dt\,.\sin.\mathrm{T}_1.$$

Now, $i.(n'-n)+2n$, being very small [4557'], we have, very nearly,

[4558d] $$(i-2)\,.\,n\,dt = i\,n'dt;$$

hence,

[4558e] $$\mathrm{d}'R' = -m\,.\,M^{(1)}.\,e\,e'.\,i\,n'\,dt\,.\sin.\mathrm{T}_1.$$

[4558f] Integrating this, relative to the characteristic d', which does not affect nt [3982a], we obtain, as in [4558],

[4558g] $$R' = m\,.\,M^{(1)}.\,e\,e'.\cos.\mathrm{T}_1.$$

$$m\,.\!\int \mathrm{d}R + m'.\!\int \mathrm{d}'R' = 0\,;$$ [4559]

therefore we have, in v', the inequality,*

$$\delta v' = \frac{2m\,.\,a'n'}{i.(n'-n)+2n}.M^{(1)}.\,e\,.\sin.\{(i-1).(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}.$$ [4560]

These two inequalities of v and v' [4557, 4560], are in the ratio of $m'.e'.\sqrt{a'}$ [4560′]
to $m.e.\sqrt{a}$; so that the second may be deduced from the first, by multiplying
the coefficient of the first by $\frac{m.\sqrt{a}}{m'.\sqrt{a'}}.\frac{e}{e'}$ [4560g]. [4560″]

The quantity $5n''-3n'$ being small, in comparison with n' or n'', we have, in v' [4557], by supposing $i=5$, an inequality depending on the argument $5n''t-4n't+5\varepsilon''-4\varepsilon'-\varpi''$; and in v'' [4560], an inequality [4560‴] depending on the argument $4n''t-3n't+4\varepsilon''-3\varepsilon'-\varpi'$. The first of these inequalities is, by [4291],

$$\delta v' = 2^{s}{,}196527\,.\sin.(5n''t-4n't+5\varepsilon''-4\varepsilon'-\varpi'').$$ [4561]

Multiplying its coefficient by $\frac{m'\sqrt{a'}}{m''\sqrt{a''}}.\frac{e'}{e''}$, we have, for the earth, the inequality, Venus and the Earth.

$$\delta v'' = 0^{s}{,}6580\,.\sin.(4n''t-3n't+4\varepsilon''-3\varepsilon'-\varpi').$$ [4562]

By a direct calculation, we have found, in [4307], this inequality to be,

$$\delta v'' = 0^{s}{,}722424\,.\sin.(4n''t-3n't+4\varepsilon''-3\varepsilon'-\varpi')\,;$$ [4563]

which differs but little from the preceding.

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* (2708) We may obtain $\delta v'$ from R', by a similar process to that used in the two preceding notes; or, more simply, by derivation, in the following manner. If we [4560a] change, in [4552], i, m, a, n, e, v, &c. into $-i+2$, m', a', n', e', v' &c. respectively, without altering $M^{(1)}$, R changes into R' [4558a,g], and the factor $i.(n'-n)+2n$, becomes,

$$(-i+2).(n-n')+2n'\,;$$ [4560b]

which, by reduction, is easily reduced to its original form; so that the angle T_1 [4557b] remains unaltered. The factor M_1 [4557b], changes into

$$M_2 = \frac{m\,.\,a'n'}{i.(n'-n)+2n}.\,M^{(1)}.\,e\,;$$ [4560c]

W' changes into W^2 [3726a]; and the second expression of δv [4557d], becomes as in the first of the following expressions of $\delta v'$, which, by successive operations, is reduced to the form [4560e], as in [4560]:

In like manner, $4n''' - 2n''$ is rather small, in comparison with n'' or n''' [4076h]; and if we suppose $i = 4$, we obtain in v'' [4557], an inequality depending on the argument

$$4n'''t - 3n''t + 4\varepsilon''' - 3\varepsilon'' - \varpi''';$$

[4564] and in v''' [4560], an inequality depending on the argument

$$3n'''t - 2n''t + 3\varepsilon''' - 2\varepsilon'' - \varpi''.$$

The first of these inequalities is, by [4307],

[4565] $$\delta v'' = 0^s{,}807111 \,.\, \sin.(4n'''t - 3n''t + 4\varepsilon''' - 3\varepsilon'' - \varpi'').$$

The Earth and Mars. Multiplying its coefficient by $\frac{m''\sqrt{a''}}{m'''\sqrt{a'''}} \cdot \frac{e''}{e'''}$ [4560''], we get, for Mars, the inequality,

[4566] $$\delta v''' = 0^s{,}661446 \,.\, \sin.(3n'''t - 2n''t + 3\varepsilon''' - 2\varepsilon'' - \varpi'');$$

and by direct calculation we have, in [4375],

[4567] $$\delta v''' = 0^s{,}846004 \,.\, \sin.(3n'''t - 2n''t + 3\varepsilon''' - 2\varepsilon'' - \varpi'');$$

the difference is within the limits of the error which may be supposed to
[4568] exist, taking into consideration, that the ratio $4n''' - 2n''$ to n''', instead of being very small, is nearly equal to $\frac{1}{4}$.

41. It also follows, from § 71, of the second book, that *if* $i.(n'-n)+2n$
[4569] *be very small in comparison with* n', the inequality of m, in latitude, depending on $(i-1).(n't - nt + \varepsilon' - \varepsilon) + n't + \varepsilon'$, is to the inequality of m', in
[4569'] latitude, depending on $(i-1).(n't - nt + \varepsilon' - \varepsilon) + nt + \varepsilon$, in the ratio of $m'\sqrt{a'}$ to $-m\sqrt{a}$.*

[4560d] $$\delta v' = 2M_2 \,.\, \sin.(\mathrm{T}_1 - W') = 2M_2 \,.\, \sin.\{i.(n't - nt + \varepsilon' - \varepsilon) + 2nt + 2\varepsilon - n't - \varepsilon' - \varpi\}$$

[4560e] $$= 2M_2 \,.\, \sin.\{(i-1).(n't - nt + \varepsilon' - \varepsilon) + nt + \varepsilon - \varpi\}.$$

[4560f] Dividing the value of $\delta v'$ [4560] by that of δv [4557], we get, successively, by using $an = a^{-\frac{1}{2}}$, $a'n' = a'^{-\frac{1}{2}}$ [3709'],

[4560g] $$\frac{\delta v'}{\delta v} = \frac{m\,.\,a'n'}{m'.\,an} \cdot \frac{e}{e'} = \frac{m\,.\,a'^{-\frac{1}{2}}}{m'.\,a^{-\frac{1}{2}}} \cdot \frac{e}{e'} = \frac{m\,.\,a^{\frac{1}{2}}}{m'.\,a'^{\frac{1}{2}}} \cdot \frac{e}{e'}, \text{ as in [4560'']}.$$

[4560h] In applying this formula to numbers, we must vary the accents in the elements, so as to conform to the notation used in this book, as is done in [4560''', &c.].

[4569a] * (2709) The inequality of s, here referred to, is given in [1342]; that of s', depending

If we suppose $i=5$, we shall have, in the motion of Venus in latitude [4569g, 4295j], the inequality [4295], Venus and the Earth.

$$\delta s' = -0^s{,}312535 . \sin. (5n''t - 4n't + 5\varepsilon'' - 4\varepsilon' - \theta'). \qquad [4570]$$

Multiplying the coefficient of this inequality by $-\frac{m'\sqrt{a'}}{m''\sqrt{a''}}$ [4569'], we get, in the motion of the earth in latitude, the inequality [4569i], [4570']

$$\delta s'' = 0^s{,}228691 . \sin. (4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \theta'); \qquad [4571]$$

and, by direct calculation, we have found, in [4312], the inequality,

$$\delta s'' = 0^s{,}234256 . \sin. (4n''t - 3n't + 4\varepsilon'' - 3\varepsilon' - \theta'); \qquad [4572]$$

which differs but little from the preceding.

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on the same angle, is similar, the accents being changed so as to adapt them to the value of s'. Instead of this formula, we may use [4295b], observing that the second line of this expression is used in computing the inequalities which are taken into consideration in [4569—4576]. The expression of δs, deduced from this part of [4295b], may be simplified; because the divisor $n^2 - \{n - i.(n-n')\}^2$, may be reduced to the form $i.(n-n').\{i.(n'-n)+2n\}$. Hence this part of δs becomes, [4569b] [4569c]

$$\delta s = \tfrac{1}{2}m'.n^2.a^2a'.\frac{B^{(i-1)}}{i.(n-n').\{i.(n'-n)+2n\}}.\gamma.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\Pi\}; \qquad [4569d]$$

γ being the inclination, and Π the longitude of the ascending node of m', upon the orbit of m. This expression may be simplified, from the circumstance, that, in the terms here taken into consideration, the divisor $i.(n-n')$ is very nearly equal to $2n$ [4569]. Substituting this, and $n a^{\frac{3}{2}} = 1$, in [4569d]; making also a slight reduction in the arrangement of the terms depending on i, we get, [4569e] [4569f]

$$\delta s = +\tfrac{1}{4}m'.\sqrt{a'}.(aa')^{\frac{1}{2}}.\frac{B^{(i-1)}}{(i-1).(n'-n)+n'+n}.\gamma.\sin.\{(i-1).(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\Pi\}. \qquad [4569g]$$

Changing the elements m, a, n, s, Π, &c. into m', a', n', s', $\Pi+180^d$, &c. respectively, and altering the sign of $i-1$, which does not affect $B^{(i-1)}$ [956, 956'], we get, [4569h]

$$\delta s' = -\tfrac{1}{4}m.\sqrt{a}.(aa')^{\frac{1}{2}}.\frac{B^{(i-1)}}{(i-1).(n'-n)+n'+n}.\gamma.\sin.\{(i-1).(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\Pi\}. \qquad [4569i]$$

Hence we evidently perceive, that δs is to $\delta s'$ as $m'\sqrt{a'}$ to $-m\sqrt{a}$, as in [4569']. [4569k]

Now, the values n', n'' [4076h] make $5n'' - 3n'$ quite small, in comparison with n'. This corresponds with the value assumed for $i.(n'-n)+2n$ [4569], supposing $i=5$; hence we get [4570—4572]. In like manner, $3n^{vi} - n^{v}$ [4076h], is very small, in [4569l]

[4573] The quantity $3n^{vi} - n^{v}$ is small in comparison with n^{vi}; therefore,
[4573'] by making $i = 3$ [4569g, i], we obtain in δs^{v}, an inequality depending on

$$3n^{vi}t - 2n^{v}t + 3\varepsilon^{vi} - 2\varepsilon^{v};$$

and in δs^{vi}, an inequality depending on

$$2n^{vi}t - n^{v}t + 2\varepsilon^{vi} - \varepsilon^{v}.$$

The first of these inequalities is, by [4511],

[4574] $$\delta s^{v} = 0^{s},662991 \,.\, \sin.(3n^{vi}t - 2n^{v}t + 3\varepsilon^{vi} - 2\varepsilon^{v} - \Pi^{vi}).$$

Π^{vi} being the longitude of the ascending node of the orbit of Uranus upon
[4574'] that of Saturn. Multiplying the coefficient of this inequality by $-\frac{m^{v}\sqrt{a^{v}}}{m^{vi}\sqrt{a^{vi}}}$,
Saturn and Uranus. we obtain in δs^{vi}, the inequality,

[4575] $$\delta s^{vi} = -2^{s},714213 \,.\, \sin.(2n^{vi}t - n^{v}t + 2\varepsilon^{vi} - \varepsilon^{v} - \Pi^{vi});$$

and by [4531], this inequality becomes, by putting $\Pi^{v} = \Pi^{vi} + 180^{d}$ [4531', 4574'],

[4576] $$\delta s^{vi} = -2^{s},921052 \,.\, \sin.(2n^{vi}t - n^{v}t + 2\varepsilon^{vi} - \varepsilon^{v} - \Pi^{vi});$$

which differs but little from the preceding.

42. It follows, from § 69, of the second book, that if we suppose
[4576'] $i'n' - in$ to be very small relatively to n and n', and represent by,*

[4577] $$R = m'.P.\sin.(i'n't - int + i'\varepsilon' - i\varepsilon) + m'.P'.\cos.(i'n't - int + i'\varepsilon' - i\varepsilon),$$

the part of the development of R, depending on the angle

$$i'n't - int + i'\varepsilon' - i\varepsilon;$$

it will produce, in δv, the inequality,

[4569m] comparison with n^{v} or n^{vi}; and this comes under the form [4569], by putting $i = 3$; hence we get [4574—4576]; observing in [4576], that $\Pi^{v} = \Pi^{vi} + 180^{d}$.

[4577a] * (2710) Using the value $T_9 = i'n't - int + i'\varepsilon' - i\varepsilon$ [4019a], and $\mu = 1$ [3709], we find that the terms of R, δe, $e\delta\varpi$, which correspond to each other in [1287, 1288, 1297], become,

[4577b] $$R = m'.P.\sin.T_9 + m'.P'.\cos.T_9;$$

[4577c] $$\delta e = \frac{m'.an}{i'n' - in} \cdot \left\{ -\left(\frac{dP}{de}\right).\sin.T_9 - \left(\frac{dP'}{de}\right).\cos.T_9 \right\};$$

$$\delta v = \frac{2m'.an}{i'n'-in} \cdot \left\{ \begin{array}{l} -\left(\frac{dP}{de}\right) . \cos. (i'n't - int + i'\varepsilon' - i\varepsilon - nt - \varepsilon + \varpi) \\ + \left(\frac{dP'}{de}\right) . \sin. (i'n't - int + i'\varepsilon' - i\varepsilon - nt - \varepsilon + \varpi) \end{array} \right\}; \qquad [4578]$$

and in $\delta v'$, the inequality,*

$$\delta v' = \frac{2m.a'n'}{i'n'-in} \cdot \left\{ \begin{array}{l} -\left(\frac{dP}{de'}\right) . \cos. (i'n't - int + i'\varepsilon' - i\varepsilon - n't - \varepsilon' + \varpi') \\ + \left(\frac{dP'}{de'}\right) . \sin. (i'n't - int + i'\varepsilon' - i\varepsilon - n't - \varepsilon' + \varpi') \end{array} \right\}. \qquad [4579]$$

$$e\delta\varpi = \frac{m'.an}{i'n'-in} \cdot \left\{ \left(\frac{dP}{de}\right) . \cos. T_9 - \left(\frac{dP'}{d\varepsilon}\right) . \sin. T_9 \right\}. \qquad [4577d]$$

Substituting these in δv [4556], using for brevity, $W = nt + \varepsilon - \varpi$ [3702a], and reducing, by [22, 24] Int. we get, as in [4578],

$$\delta v = \frac{2m'.an}{i'n'-in} \cdot \left\{ \begin{array}{l} -\left(\frac{dP}{de}\right) . (\sin. T_9 . \sin. W + \cos. T_9 . \cos. W) \\ + \left(\frac{dP'}{de}\right) . (\sin. T_9 . \cos. W - \cos. T_9 . \sin. W) \end{array} \right\}$$

$$= \frac{2m'.an}{i'n'-in} \cdot \left\{ -\left(\frac{dP}{de}\right) . \cos. (T_9 - W) + \left(\frac{dP'}{de}\right) . \sin. (T_9 - W) \right\}. \qquad [4577e]$$

* (2711) Proceeding in the same manner as in [4558a—c], and using T_9 [4577a], we have,

$$\mathrm{d}T_9 = -in\,dt, \qquad \mathrm{d}'T_9 = i'n'\,dt; \qquad [4578a]$$

hence the differential of R [4577b], relative to the characteristic d, becomes,

$$\mathrm{d}R = -m'.in.\{P.\cos.T_9 - P'.\sin.T_9\}. \qquad [4578b]$$

Substituting this in $m'.\mathrm{d}'R' = -m.\mathrm{d}R$ [4558b—c], we get,

$$\mathrm{d}'R' = m.in.\{P.\cos.T_9 - P'.\sin.T_9\}. \qquad [4578c]$$

Integrating this, relatively to d′, and observing that the divisor $i'n'$ is, by hypothesis, very nearly equal to in [4576′], we get, for the corresponding terms of R', depending on the angle T_9, the following expression;

$$R' = m.\{P.\sin.T_9 + P'.\cos.T_9\}. \qquad [4578d]$$

From this value of R' we may compute $\delta v'$, in the same manner as we have found δv [4578], from R [4577b]. It will, however, be rather more simple to use the principle of derivation, by observing, that if we take the differential coefficient of R [4577b], relative to e, multiply it by $2andt$, then take its integral relative to t, and change T_9 into $T_9 - W$, it will become equal to δv [4577e]. In like manner, if we take the differential coefficient of R' [4578d], relative to e', multiply it by $2a'n'dt$, take its integral relative [4578e] [4578f]

It follows also, from § 71, book ii. that the same terms of R [4577], produce, in δs, the inequality,*

[4580] $$\delta s = \frac{m'.an}{i'n'-in} \cdot \left\{ \begin{array}{l} \left(\frac{dP}{d\gamma}\right) . \cos .(i'n't - int + i'\varepsilon' - i\varepsilon - nt - \varepsilon + \Pi) \\ -\left(\frac{dP'}{d\gamma}\right) . \sin. (i'n't - int + i'\varepsilon' - i\varepsilon - nt - \varepsilon + \Pi) \end{array} \right\};$$

[4580'] *γ being the tangent of the respective inclinations of the orbits of m and m', and Π the longitude of the ascending node of the orbit of m' upon that of m [4295b—e].*

[4580''] If we increase the argument of the inequality of δv [4578], by $nt + \varepsilon - \varpi$, and multiply its coefficient by e; also, if we increase the argument of the inequality of $\delta v'$ [4579], by $n't + \varepsilon' - \varpi'$, and multiply

[4581] its coefficient by $\frac{m'\sqrt{a'}}{m\sqrt{a}} . e'$;† lastly, if we increase the argument of the inequality of δs [4580], by $nt + \varepsilon - \Pi$, and multiply its coefficient by -2γ, the sum of these three inequalities will be,

[4582] $$\frac{2m'.an}{i'n'-in} \cdot \left\{ \begin{array}{l} -\left\{ e . \left(\frac{dP}{de}\right) + e' . \left(\frac{dP}{de'}\right) + \gamma . \left(\frac{dP}{d\gamma}\right) \right\} . \cos. (i'n't - int + i'\varepsilon' - i\varepsilon) \\ + \left\{ e . \left(\frac{dP'}{de}\right) + e' . \left(\frac{dP'}{de'}\right) + \gamma . \left(\frac{dP'}{d\gamma}\right) \right\} . \sin. (i'n't - int + i'\varepsilon' - i\varepsilon) \end{array} \right\}.$$

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to t, and afterwards change T_9 into $T_9 - W'$ [3726a], it will produce the following expression of $\delta v'$, which is equivalent to [4579];

[4579g] $$\delta v' = \frac{2m.a'n'}{i'n'-in} . \left\{ -\left(\frac{dP}{de'}\right) . \cos. (T_9 - W') + \left(\frac{dP'}{de'}\right) . \sin. (T_9 - W') \right\}.$$

[4580a] * (2712) If we put, for brevity, $T_2 = i'n't - int + A - g\theta_{\prime}'$, also $\gamma = \text{tang.}\, \varphi_{\prime}'$ [1337', 3739]; the assumed value of R [1337''] becomes, $R = m'.k.\gamma^g. \cos. T_2$.

[4580b] Substituting this in the expression $-\int\left(\frac{dR}{d\gamma}\right) . a n d t$, we find that it becomes equal to the expression of s or δs [1342]; provided the angle T_2 be decreased, after the integration,

[4580c] by the quantity $v - \theta_{\prime}'$, or by the angular distance of the body m from the ascending node of the orbit of m' upon that of m [1337']. In the present notation $v - \theta_{\prime}'$ is represented

[4580d] by the quantity $nt + \varepsilon - \Pi$, neglecting terms of the order e [4295b—c]. The same process being performed upon the assumed value of R [4577], produces the expression of δs [4580].

[4581a] † (2713) This factor is equal to $\frac{m'.an}{m.a'n'} . e'$ [4560f].

Now, P and P' are homogeneous functions of e, e', γ, of the dimension $i'-i$, and i' is supposed greater than i; therefore the preceding function is equal to,*

$$\frac{2m'.an.(i'-i)}{i'n'-in}.\{-P.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)+P'.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)\}. \quad [4583]$$

Now we have, in δv, the inequality, [1304],

$$\delta v=\frac{3m'.an^2.i}{(i'n'-in)^2}.\{P.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)-P'.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)\}; \quad [4584]$$

hence it follows, that if we represent by

$$\delta v=K.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-nt-\varepsilon+O), \quad [4585]$$

the inequality of δv, depending on the angle $i'n't-int+i'\varepsilon'-i\varepsilon-nt-\varepsilon$; and by

$$\delta v'=K'.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-n't-\varepsilon'+O'),$$

the inequality of $\delta v'$, depending on the angle $i'n't-int+i'\varepsilon'-i\varepsilon-n't-\varepsilon'$: [4586]
lastly, if we represent by

$$\delta s=K''.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-nt-\varepsilon+O''), \quad [4587]$$

the inequality of δs, depending on the angle $i'n't-int+i'\varepsilon'-i\varepsilon-nt-\varepsilon$, we shall have,†

$$\begin{aligned} &Ke.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-\varpi+O) \\ &+\frac{m'\sqrt{a'}}{m\sqrt{a}}.K'e'.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-\varpi'+O') \\ &-2K''\gamma.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-\Pi+O'') \\ &=-\frac{2.(i'-i)}{3i}.H.\frac{(i'n'-in)}{n}.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon+Q); \end{aligned} \quad [4588]$$

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* (2714) From [957ix] it appears, that any part of R, depending on angles of the form $i'n't-int$, must be composed of terms in e, e', γ, of the orders $i'-i$, [4583a]
$i'-i+2$, &c.; and by neglecting all, except the first, on account of their smallness, they must be of the order $i'-i$; and therefore homogeneous in these quantities. Now, if we [4583b]
put, in [1001a], $a=e$, $a'=e'$, $a''=\gamma$, $m=i'-i$, and then, successively, $A^{(i)}=P$, $A^{(i)}=P'$, we get,

$$e.\left(\frac{dP}{de}\right)+e'.\left(\frac{dP}{de'}\right)+\gamma.\left(\frac{dP}{d\gamma}\right)=(i'-i).P;$$
$$e.\left(\frac{dP'}{de}\right)+e'.\left(\frac{dP'}{de'}\right)+\gamma.\left(\frac{dP'}{d\gamma}\right)=(i'-i).P'. \quad [4583c]$$

Substituting these in [4582], we obtain [4583].

† (2715) The first member of [4588] is equal to the sum of the inequalities δv, $\delta v'$,

[4588′] $\delta v = H.\sin.(i'n't - int + i'\varepsilon' - i\varepsilon + Q)$ being the inequality of δv depending on the angle $i'n't - int + i'\varepsilon' - i\varepsilon$.

[4588″] The quantity $5n' - 2n$ [4076h] is very small in comparison with n'; and we have, in δv [4282], the inequality,

[4589] $$\delta v = 1^s{,}690443\,.\sin.(5n't - 3nt + 5\varepsilon' - 3\varepsilon + 43^d\,18^m\,32^s).$$

Mercury and Venus. The inequality δs [4283′], depending on $5n't - 3nt + 5\varepsilon' - 3\varepsilon$, is insensible; and we have, in $\delta v'$ [4293], the inequality,

[4590] $$\delta v' = -0^s{,}333596\,.\sin.(4n't - 2nt + 4\varepsilon' - 2\varepsilon - 39^d\,30^m\,30^s).$$

Lastly, we have, in δv [4283], the inequality,

[4591] $$\delta v = 8^s{,}483765\,.\sin.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - 30^d\,13^m\,36^s).$$

In this case $i' = 5$, $i = 2$ [4584, 4591]; and we have, by what precedes [4585—4591], the following equation of condition;

[4592] $$\begin{aligned} &1^s{,}690443\,.e\,.\sin.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - \varpi + 43^d\,18^m\,32^s) \\ &- 0^s{,}333596\,.e'.\frac{m'\sqrt{a'}}{m\sqrt{a}}\,.\sin.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - \varpi' - 39^d\,30^m\,30^s) \\ &= -8^s{,}483765\,.\frac{(5n'-2n)}{n}.\sin.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - 30^d\,13^m\,36^s). \end{aligned}$$

The first member of this equation is,*

[4593] $$0^s{,}359753\,.\sin.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - 28^d\,27^m\,33^s);$$

the second member is,

[4594] $$0^s{,}3605\,.\sin.(5n't - 2nt + 5\varepsilon' - 2\varepsilon - 30^d\,13^m\,36^s);$$

and their difference is insensible.

[4588a] δs, [4585, 4586, 4587]; multiplied respectively by e, $\frac{m'\sqrt{a'}}{m\sqrt{a}}.e'$, and -2γ; the arguments being also increased by $nt + \varepsilon - \varpi$, $n't + \varepsilon' - \varpi'$, $nt + \varepsilon - \Pi$, respectively, according to the directions in [4580″—4581]. Now, it is shown, in [4580″—4583], that this sum is equal to the expression [4583], which is the same as that of δv [4584], multiplied by $-\frac{2(i'-i)}{3i}.\left(\frac{i'n'-in}{n}\right)$; and if we suppose this expression of δv to be reduced to the
[4588b] form [4588′], this product will be represented by the second member of [4588].

[4593a] * (2716) This is easily obtained, by reducing the two terms of the first member of [4592] into one, by the method [4282h—l], after substituting the values m, m', a, a', &c. [4061, 4079, 4080].

We may verify, by the preceding theorems, many of the corresponding inequalities of Jupiter and Saturn; but as all the inequalities of these two planets have been verified several times, with much care, by different computers, this last verification is unnecessary. [4594′]

43. The inequality of m, produced by the action of m', and depending on the argument $n't+\varepsilon'-\varpi'$, is expressed as in book ii. § 50, 55, by,*

$$\delta v=\frac{-4n^2}{n'.(n^2-n'^2)}\cdot(0,1)\,.\,e'.\sin.(n't+\varepsilon'-\varpi'). \qquad [4595]$$

The inequality of m', produced by the action of m, and depending on the argument $nt+\varepsilon-\varpi$, is,

$$\delta v'=\frac{4n'^2}{n.(n^2-n'^2)}\cdot(1,0)\,.\,e\,.\sin.(nt+\varepsilon-\varpi). \qquad [4596]$$

The coefficients of these two inequalities are, therefore, in the ratio of $-(0,1)\,.\,n^3.e'$ to $(1,0)\,.\,n'^3.\,e$; now we have, in [1093], [4596′]

$$(1,0)=(0,1)\,.\,\frac{m\sqrt{a}}{m'\sqrt{a'}}; \qquad [4597]$$

therefore, if we put Q for the coefficient of the inequality δv [4595], we shall find, that the coefficient of the inequality $\delta v'$ [4596], will be represented by,

$$-\frac{m\,.\,a^5}{m'.\,a'^5}\cdot\frac{e}{e'}\cdot Q \quad [4595f]. \qquad [4598]$$

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* (2717) The term of δv depending on $n't+\varepsilon'-\varpi'$, is deduced from that in [1021], depending on $G^{(i)}$, by putting $i=1$; whence we obtain,

$$\delta v=\frac{m'n}{n'}\,.\,G^{(1)}.\,e'.\sin.(n't+\varepsilon'-\varpi'). \qquad [4595a]$$

Now, from [1018, 1019, 1073], we have, in the case of $i=1$,

$$D^{(1)}=-a^2.\left(\frac{d.A^{(0)}}{da}\right)-\tfrac{1}{2}a^3.\left(\frac{d^2.A^{(0)}}{da^2}\right)=\frac{2}{m'.n}\cdot(0,1); \qquad [4595b]$$

$$G^{(1)}=-\frac{2n^2}{n^2-n'^2}\cdot D^{(1)}=-\frac{4n}{m'.(n^2-n'^2)}\cdot(0,1). \qquad [4595c]$$

Substituting this value of $G^{(1)}$, in δv [4595a], it becomes as in [4595]. The value of $\delta v'$ [4596] may be directly computed in a similar manner; or it may be obtained more simply by derivation from [4595]; changing m, a, n, e, &c. into m', a', n', e', &c.; and the contrary; observing, that by these changes, $(0,1)$ becomes $(1,0)$, according to the [4595d]

The inequalities of this kind have been verified, either by means of this equation of condition, or by that of the preceding expression of Q. Thus, the action of Jupiter produces, in the earth, the sensible inequality [4307],

[4599] $$\delta v'' = -2^s{,}539884 . \sin . (n^{iv} t + \varepsilon^{iv} - \varpi^{iv}).$$

This inequality, by what precedes, is represented by [4595],

[4600] $$\delta v'' = \frac{-4\, n''^2}{n^{iv} . (n''^2 - n^{iv 2})} . (2{,}4) . e^{iv} . \sin . (n^{iv} t + \varepsilon^{iv} - \varpi^{iv});$$

The Earth and Jupiter. and we have $(2{,}4) = 6^s{,}947861$ [4233]. If we substitute this, in [4600], also the values of n'', n^{iv}, e^{iv} [4077, 4080]; then multiply the result by the expression of the radius in seconds, we shall obtain,

[4601] $$\delta v'' = -2^s{,}5401 . \sin . (n^{iv} t + \varepsilon^{iv} - \varpi^{iv}).$$

The action of Uranus upon Saturn, produces, in the motion of Saturn, the inequality [4466],

[4602] $$\delta v^{v} = -1^s{,}011647 . \sin . (n^{vi} t + \varepsilon^{vi} - \varpi^{vi}).$$

Saturn and Uranus. Multiplying its coefficient by $-\frac{m^{v} . a^{v5}}{m^{vi} . a^{vi5}} . \frac{e^{v}}{e^{vi}}$ [4598], we obtain, in Uranus, the inequality,

[4603] $$\delta v^{vi} = 0^s{,}214852 . \sin . (n^{v} t + \varepsilon^{v} - \varpi^{v});$$

and the direct calculation has given, in [4525],

[4604] $$\delta v^{vi} = 0^s{,}214857 . \sin . (n^{v} t + \varepsilon^{v} - \varpi^{v}).$$

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notation in [1085, &c.]. Comparing the values of δv, $\delta v'$ [4595, 4596], we get the first

[4595e] expression of [4595f]; and by substituting the value of $(1{,}0)$ [4597]; also $n^3 = a^{-\frac{9}{2}}$, $n'^3 = a'^{-\frac{9}{2}}$ [3709'], we get successively the last expression [4595f], which is equivalent to [4598];

[4595f] $$\delta v' = -\frac{(1{,}0)}{(0{,}1)} . \frac{n'^3 e}{n^3 e'} . \delta v = -\frac{m \sqrt{a}}{m' \sqrt{a'}} . \frac{n'^3 e}{n^3 e'} . \delta v = -\frac{m . a^5}{m' . a'^5} . \frac{e}{e'} . \delta v.$$

[4600a] * (2718) The expression [4600] is similar to [4595], changing m, m', &c. into m'', m^{iv}, &c.

CHAPTER XVI.

ON THE MASSES OF THE PLANETS AND MOON.

44. One of the most important objects in the theory of the planets is the determination of their masses; and we have pointed out, in [4062—4076'], the imperfections of our present estimation of these values. The most sure method of obtaining a more accurate result, is that which depends on the
development of the secular inequalities of the motions of the planets; but [4604]
until future ages shall make known these inequalities with greater precision, we may use the periodical inequalities, deduced from a great number of observations. For this purpose, Delambre has discussed the numerous observations of the sun, by Bradley and Maskelyne; from which he has
obtained the maximum of the inequalities produced by the actions of Venus, [4604']
Mars and the moon. The whole collection of these observations of Bradley and Maskelyne, makes the *maximum* of the action of Venus greater than
that which corresponds to the mass we have assumed for Venus [4061], in [4604'']
the ratio of 1,0743 to 1; hence the mass of Venus is $\frac{1}{356632}$ of that of [4605]
the sun. The observations of Bradley and Maskelyne, when we take them Mass of Venus.
separately into consideration, give nearly the same results; therefore, it is probable, that this estimate of the mass of Venus is not liable to an error of
a fifteenth part of its value. [4605']

Hence it follows, incontestably, that the secular diminution of the obliquity of the ecliptic approaches very near to $154''=49^s,9$. To reduce
it, as some astronomers have done, to $105''=34^s$, we must decrease the [4606]
mass of Venus one half;* and this is evidently incompatible with the [4606']

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* (2719) This appears, by substituting $q''=-34^s$, $t=100$ [4606], in [4074r]; whence we get, very nearly, $-34^s=-50^s-31^s\mu'$; consequently, $\mu'=-\frac{1}{2}$ nearly. [4606a]

observations of the periodical inequalities, produced by Venus, in the motion
[4606''] of the earth. The best modern observations of the obliquity of the ecliptic are too near to each other, to determine this element with accuracy. The observations of the Arabs appear to have been taken with much care. They
[4607] made no alteration in the system of Ptolemy; but directed their attention particularly to the perfection of the instruments, and to the accuracy of their observations. These observations give a secular diminution of the obliquity
[4607'] of the ecliptic, which differs but very little from $154'' = 49^s,9$. This diminution is also confirmed by the observations of Cocheouking, made in China, by means of a high gnomon; and it appears to me, that these observations may be relied upon for their accuracy.

Delambre has also determined, by a great number of observations, the
[4607''] maximum of the action of Mars upon the motion of the earth. He has
Mass of Mars. found this action to be less than that which corresponds to the mass we have assumed for Mars [4061], in the ratio of 0,725 to 1; making the
[4608] mass of Mars $\frac{1}{2546320}$ of that of the sun. This value is probably not quite so accurate as that of the mass of Venus, because its effect is less; but, as the data [4076], from which we have determined the mass of Mars,
[4608'] in [4075, &c.], are very hypothetical, it is important to ascertain the error which might result from this cause, in the theory of the sun's apparent motion. Now, the observations of Bradley and Maskelyne, combined together, or taken separately, concur in indicating a diminution in the mass of Mars; therefore, we shall decrease the preceding inequalities, produced
[4609] by Mars, in the earth's motion, in the ratio of 0,725 to unity.

These changes, in the masses of Venus and Mars, produce sensible alterations in the secular variations of the elements of the earth's orbit.
Longitude of the earth's perihelion. We find the longitude of the earth's perihelion to be represented by the following expression;*

[4610] $$\text{Long. perihelion } \oplus = \varpi'' + t\,.\,11^s,807719 + t^2\,.\,0^s,0000816482;$$

the coefficient of the *equation of the centre* of the earth's orbit is represented by,

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* (2720) The expression [4610] is computed as in [4331], changing the masses of
[4610a] Venus and Mars, as in [4605—4608]. The formulas [4611, 4612] are computed in like manner as [4330, 4332], respectively.

$$\text{Coeff. equat. centre} \oplus = 2E - t.0^s{,}171793 - t^2.0^s{,}0000068194. \qquad [4611]$$

Lastly, the values of p'' and q'' [4332], become,

$$p'' = \quad t.0^s{,}080543 + t^2.\,0^s{,}0000231134;$$
$$q'' = -t.0^s{,}521142 + t^2.\,0^s{,}0000071196. \qquad [4612]$$

Hence it follows, from [4074c, 4613a], that the secular diminution of the obliquity of the ecliptic, in this century, is equal to $52^s{,}1142$.* Using these *data*, we find, by the formulas of § 31,† [4613]

$$\begin{aligned}\psi &= t.\,155'',5927 + 3^\circ{,}11019 + 42556'',2.\sin.(t.155'',5927 + 95^\circ{,}0733)\\ &\qquad - 73530'',8.\cos.(t.99'',1227) - 17572'',4.\sin.(t.43'',0446)\\ &= t.\,50^s{,}412 + 2^d\,47^m\,57^s + 13788^s{,}2.\sin.(t.50^s{,}412 + 85^d\,33^m\,57^s)\\ &\qquad - 23823^s{,}98.\cos.(t.32^s{,}1158) - 5693^s{,}5.\sin.(t.13^s{,}9465);\end{aligned} \qquad [4614]$$

[Fixed orbit.]

$$\begin{aligned}V &= 26^\circ{,}0796 - 3676'',6 - 18187'',6.\cos.(t.155'',5927 + 95^\circ{,}0733)\\ &\qquad + 5082'',7.\cos(t.43'',0446) - 28463'',6.\sin.(t.99'',1227)\\ &= 23^d\,28^m\,17^s{,}9 - 1191^s{,}2 - 5892^s{,}8.\cos.(t.50^s{,}412 + 85^d\,33^m\,57^s)\\ &\qquad + 1646^s{,}8.\cos.(t.13^s{,}9465) - 9222^s{,}2.\sin.(t.32^s{,}1158);\end{aligned} \qquad [4615]$$

Corrected values of the precession and obliquity of the ecliptic for the year t, after the epoch 1750.

$$\begin{aligned}\psi' &= t.155'',5927 + 3^\circ{,}11019 - 3^\circ{,}11019.\cos.(t.99'',1227)\\ &\qquad - 14282'',3.\sin.(t.43'',0446)\\ &= t.\,50^s{,}4120 + 2^d\,47^m\,57^s - 2^d\,47^m\,57^s.\cos.(t.32^s{,}1158)\\ &\qquad - 4627^s{,}5.\sin.(t.13^s{,}9465);\end{aligned} \qquad [4616]$$

[Apparent orbit.]

$$\begin{aligned}V' &= 26^\circ{,}0796 - 3676'',6.\{1 - \cos.(t.43'',0446)\}\\ &\qquad - 10330'',4.\sin.(t.99'',1227)\\ &= 23^d\,28^m\,17^s{,}9 - 1191^s{,}2.\{1 - \cos.(t.13^s{,}9465)\}\\ &\qquad - 3347^s{,}05.\sin.(t.32^s{,}1158).\end{aligned} \qquad [4617]$$

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* (2721) The chief term of the value of q'' [4612] is $-t.0^s{,}521142$, and by putting $t = 100$, it becomes $q'' = -52^s{,}1142$. This represents, by [4074a—c], the secular variation of the obliquity of the ecliptic, corresponding to the second formula [4612]; in the original work it is printed $160'',85 = 52^s{,}1154$, and it is thus quoted in [3380n]. [4613a]

† (2722) The formulas [4614—4618], are computed in precisely the same manner as

Increment of the year. The increment of the tropical year, counted from 1750, is, then, represented by,

[4618] $$\text{Increment of the year} = -0^{\text{day}},000086354\,.\{1-\cos.(t\,.\,13^{s},9465)\} \\ -0^{\text{day}},000442198\,.\,\sin.(t\,.\,32^{s},1158).$$

Hence it follows, that, *at the time of Hipparchus, the tropical year was*
[4618′] $10^{s},9528$ *sexagesimal seconds longer than in* 1750. *The obliquity of the ecliptic was then greater by* $955^{s},2168$. *Lastly, the greater axis of the sun's*
[4618″] *orbit coincided with the line of equinoxes, in the year* 4089 *before our era; it was perpendicular to that line in* 1248.

The mass of the moon has been determined by the observations of the tides in the port of Brest; and, although these observations are
[4619] far from being so complete as we could wish, yet they give, with considerable precision, the ratio of the action of the moon, to that of the sun, upon the tides of that port. But, it has been observed, in [2435—2437′], that local circumstances may have a very sensible influence on this ratio, and also on the resulting value of the moon's mass. Several methods have been pointed out, in the second book, to ascertain this influence; but they require very exact observations of the tides. The observations which have been made at Brest, leave, in their results, such a degree of uncertainty, as makes us fear that there may be an error of at least an eighth part, in the value of the moon's mass. Indeed, the observations of the equinoctial and solsticial tides,
[4620] seem to indicate, that the action of the moon upon these tides is augmented one tenth part, by the local circumstances of the port. This will decrease,
[4621] by one tenth, the assumed value of the moon's mass; and, in fact, it appears, by several astronomical phenomena, that the assumed value [4321] is rather too great.

The first of these phenomena is the lunar equation, in the tables of the
[4622] sun's motion. We have found, in [4324], $8^{s},8298$ for the coefficient of this inequality, supposing the sun's parallax to be $8^{s},8$ [4322]. It will be

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[4357—4360, 4362], altering the masses of Venus and Mars, as in [4605, 4608]. We have previously spoken of this change of the masses of these two planets, in [3380n,&c.],
[4614a] and have also given the formulas of Poisson and Bessel [3380p,q], for the determination of the precession and the obliquity of the ecliptic.

$8^s,5767$,* if the sun's parallax be $8^s,56$, which is the value deduced [4622′]
from the lunar theory, as will be seen in the following book. Delambre has determined the coefficient of this lunar equation, by the comparison of a very great number of observations of the moon, and has found it equal to $7^s,5$. [4623]
If we adopt this value, and also the second of the above estimates of the sun's parallax, which several astronomers have deduced from the last transit of Venus over the sun's disc, we find the mass of the moon to be $\frac{1}{69,2}$ of the earth's mass [4622b]. [4624]

Moon's mass.

The second astronomical phenomenon is the nutation of the earth's axis. We have found, in [3378a], the coefficient of the inequality of the nutation to be equal to $10^s,0556$;† supposing the mass of the moon, divided by the cube of its mean distance from the earth, to be equal to triple the mass of the sun, divided by the cube of the mean distance of the earth from the sun [2706]. This makes the mass of the moon equal to $\frac{1}{58,6}$ of the earth's mass [4321]. Maskelyne has found, by the comparison of all Bradley's observations on the nutation, that the coefficient of this inequality is equal [4625] [4626]

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* (2723) The coefficient of this inequality, neglecting its sign, is $\frac{m}{M}\cdot\frac{R}{r''}$, multiplied by the radius in seconds 206265^s [4314]; and by substituting $\frac{m}{M}=\frac{1}{58,6}$, and $\frac{R}{r''}=\frac{\odot\text{'s par.}}{3454^s}$ [4321, 4323], it becomes $\frac{1}{58,6}\times\frac{\odot\text{'s par.}}{3454^s}\times 206265^s$. Putting this parallax equal to $8^s,8$, the coefficient becomes nearly equal to $8^s,8298$ [4324]; and by using the value of the parallax $8^s,56$ [5589], the coefficient becomes $8^s,58$ nearly, as in [4622′]. To reduce this to $7^s,5$, the value obtained by Delambre, we must decrease the moon's mass in the ratio of the numbers $7^s,5$ to $8^s,58$, so that it will be equal to $\frac{7,5}{8,58}\times\frac{1}{58,6}=\frac{1}{67}$, instead of $\frac{1}{69,2}$, given by the author in [4624]. [4622a] [4622b]

† (2724) The coefficient $31'',036=10^s,0556$ is computed, in [3376e], from the formula $\frac{\lambda}{1+\lambda}\cdot\frac{lc'}{f'}=10^s,0556$; in which $\lambda=3$ [3376, 3079] represents the assumed ratio of the lunar to the solar force on the tide. This value of λ is used, in [4319], in computing the value of m [4321, 4626]. Now, substituting $\lambda=3$, in [4625a], we obtain, [4625a]

$$\frac{lc'}{f'}=\frac{4}{3}\times 10^s,0556=13^s,4074;$$

[4627] to $9^s,55$; and this result makes the moon's mass equal to $\frac{1}{71}$ of the earth's mass.

Lastly, the third astronomical phenomenon is the moon's parallax. We shall see, in [5605], that the constant term contained in the expression of this parallax, when developed in a function of the moon's true longitude, is [4628] $3427^s,93$; supposing the moon's mass to be $\frac{1}{58,6}$ of the earth's mass. Burg has computed this constant term, by means of a very great number of [4629] observations of the moon. He finds it equal to $3432^s,04$ [5605]; and, by the formulas given in the next book, this result will be found to correspond [4629′] with a mass of the moon, which is equal to $\frac{1}{74,2}$ of that of the earth.*

[4625b] substituting this value in the first member of the equation [4625a], we get $\frac{\lambda}{1+\lambda}\cdot 13^s,4074$, for the nutation, corresponding to any assumed value of λ. If we put this equal to the value $9^s,55$, obtained by Maskelyne [4627], we get,

[4625c] $$\frac{\lambda}{1+\lambda}=\frac{9,5500}{13,4074};\quad \text{hence}\quad \lambda=\frac{9,5500}{3,8574}=2,476,\quad \text{instead of}\quad \lambda=3,\quad \text{used above};$$

and as the mass of the moon is proportional to λ [3079], it will be reduced, from $\frac{1}{58,6}$ [4321], to $\frac{1}{58,6}\times\frac{2,476}{3,000}=\frac{1}{71}$; as in [4627].

* (2725) The constant term of the parallax is $\frac{D}{a}\cdot(1+ee)$ [5311]; and by substituting [4629a] the value of $\frac{D}{a}$ [5324], it becomes of the form $A\cdot\left(\frac{M}{M+m}\right)^{\frac{1}{3}}$; A being a function of the known quantities a, e, &c., which are independent of M, m. Now, by using the value of $\frac{m}{M}=\frac{1}{58,6}$ [4628], we obtain the constant term [5330′], corresponding to the latitude whose sine is $\sqrt{\tfrac{1}{3}}$; also the constant term $3427^s,93$ [5605] of the *horizontal* parallax; hence we have,

[4629b] $$A\cdot\left(\frac{58,6}{59,6}\right)^{\frac{1}{3}}=3427^s,93,\quad \text{and}\quad A=3447^s,32;$$

so that the constant term of the horizontal parallax is,

[4629c] $$3447^s,32\cdot\left(\frac{M}{M+m}\right)^{\frac{1}{3}}.$$

Putting this equal to the constant term of Burg's tables $3442^s,44-10^s,40=3432^s,04$ [5605], we get,

[4629d] $$\frac{M+m}{M}=\left(\frac{3447,32}{3432,04}\right)^3=1,01341=1+\frac{1}{74}\quad \text{nearly, as in [4629′].}$$

Hence it appears, from all three of these phenomena, that we must decrease a little the mass of the moon, deduced from the observations of the tides at Brest; therefore, the action of the moon on the tides in that port, is [4630] sensibly increased by local circumstances. For the numerous observations, both of the heights and intervals of the tides, do not permit us to suppose this action to be less than triple the action of the sun.

The most probable value of the moon's mass, which appears to result from these various phenomena, is $\frac{1}{68,5}$ of the earth's mass.* By using this [4631] value, we find $7^s,572$,† for the coefficient of the lunar equation of the solar [4632] tables, and $3430^s,88$,‡ for the constant term of the expression of the [4633] moon's parallax. We also find $9^s,648 . \cos.$(longitude of the moon's node), [4634] for the inequality of the nutation, and $-18^s,03 . \sin.$(long. moon's node),§ [4635]

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* (2726) Subsequent observations of the tides at Brest, induced the author to reduce this value of λ [3079], from $\lambda=3$ to $\lambda=2,35333$ [11905]; making the mass of the [4631a] moon equal to $\frac{1}{74,946}$ of that of the earth [11906]; as we have already remarked in [3380b', &c.]. We may observe, that the value of $\lambda=3$ [4318, 4319] corresponds with [4631b] $\frac{m}{M}=\frac{1}{58,6}$ [4321], and that λ is proportional to m; hence we get, in the case of $\frac{m}{M}=\frac{1}{68,5}$ [4631c] [4631], the value $\lambda=3.\frac{58,6}{68,5}=2,566$, as in [4637].

† (2727) This equation of the earth's motion is proportional to $\frac{m}{M}$ [4314]; and if we suppose $\frac{m}{M}=\frac{1}{58,6}$ [4321], it becomes $8^s,58$ nearly, as in [4622']; but if we use [4632a] $\frac{m}{M}=\frac{1}{68,5}$ [4631], this equation becomes $8^s,58\times\frac{58,6}{68,5}=7^s,34$; which differs a little [4632b] from [4632].

‡ (2728) Substituting $M=68,5.m$ [4631c], in the constant term of the moon's parallax [4629c], it becomes $3447^s,32.\left(\frac{68,5}{69,5}\right)^{\frac{1}{3}}=3430^s,8$, as in [4633]. Moreover, by [4633a] substituting $\lambda=2,566$ [4631c], in the coefficient of the nutation [4625b], it becomes,

$$\frac{\lambda}{1+\lambda}.13^s,4074=\frac{2,566}{3,566}.13^s,4074=9^s,648, \text{ as in [4634].} \qquad [4633b]$$

§ (2729) The coefficients of the inequalities in the nutation and precession are represented, in [3376e, f, 3378, 3380], by $\frac{l\lambda c'}{(1+\lambda).f'}$, $-\frac{2l\lambda c'}{(1+\lambda).f'}.\cot.2h$; which are to [4635a]

for the inequality of the precession of the equinoxes. The ratio of the
[4636] moon's action on the tides to that of the sun is then 2,566 [4631c]; and as the observations of the tides in the port of Brest make this ratio equal to 3 [4631b], it appears evident that it is increased, by local circumstances,
[4637] in the ratio of 3 to 2,566. Future observations, made with great exactness, will enable us to determine, with precision, these points, in which there remains, at present, some slight degree of uncertainty.

[4638] Jupiter's mass appears to be well determined; Saturn's has still some degree of uncertainty [4635c], and it is a desirable object to correct it. This may be done by observing the greatest elongations of the two outer
[4638'] satellites, in opposite points of their orbits, in order to have regard to the ellipticity of the orbits. We may also use, for this purpose, the great inequality of Jupiter [4417], when the mean motions of Jupiter and Saturn shall be accurately determined; for these mean motions have a very sensible
[4639] influence upon the divisor $(5n^{\text{v}} - 2n^{\text{iv}})^2$, which affects this inequality. It appears probable, that the mean annual motion we have assigned to Jupiter, must be increased, one or two centesimal seconds; and that of Saturn, decreased, by nearly the same quantity. The periodical inequalities of Jupiter and Uranus, produced by the action of Saturn, afford also a tolerably
[4640] accurate method of determining the mass of Uranus.

The value we have assigned to the mass of Uranus, depends on the
[4641] greatest elongation of its satellites, which were observed by Herschel. These elongations should be verified with great care.

With respect to Mercury's mass, we may use, in ascertaining its value, the inequalities it produces in the motion of Venus. Fortunately, the influence
[4642] of Mercury on the planetary system is very small; so that the error, depending on any inaccuracy in this estimate of its mass, must be nearly insensible.

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each other as 1 to $-2.\cot.2h$. Hence, if we suppose the inequality of the nutation to
[4635b] be $9^s,648$, as in [4634], that of the precession will be $-2\times 9^s,648.\cot.2h$; and by using $2h = 52^\circ,1592 = 46^d\,56^m\,35^s,8$, it becomes $-18^s,03$, as in [4635].

[4635c] Before concluding this note we may observe, that the late estimates of these masses, by different astronomers, have already been given in [4061d—m].

CHAPTER XVII.

ON THE FORMATION OF ASTRONOMICAL TABLES, AND ON THE INVARIABLE PLANE OF THE PLANETARY SYSTEM.

45. We shall now proceed to explain the method which must be used in constructing astronomical tables. We have given the inequalities, in longitude and in latitude, to a quarter of a centesimal second; but the most perfect observations do not attain to that degree of accuracy; so that we may simplify the calculations, by *neglecting the inequalities which are less than a centesimal second.* We must form, by means of a great number of [4643] observations, selected and combined in the most advantageous manner, the same number of equations of condition, between the corrections of the elliptical elements of each planet. These elements being already known, to a considerable degree of accuracy, their corrections must be so small that we may neglect their squares and higher powers; and by this means the equations of condition become linear.* We must add together all the equations in which the coefficients of the same unknown quantity are considerable; so that from these sums we can form *the same number of fundamental equations as there are unknown quantities;* and then, by [4644] elimination, we may obtain each of the unknown quantities. We can also find, by the same method, the corrections which may be necessary in the assumed masses of the planets. If the numerical values of the planetary inequalities be accurately calculated, which may be ascertained by a careful verification of the preceding results; we may, with each new observation,

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* (2730) We have given the form of an equation of this kind, in [849*d*]; and have shown, in [849*a*—*r*], how to combine any number of them together, by the method of the [4644*a*] least squares; which process is now generally used, in preference to that in [4644].

form another equation of condition. Then if we determine, every ten years, the corrections resulting from the combination of these equations with all the preceding ones, we may, from time to time, correct the elements of the orbits; and by this means obtain more accurate tables of the motions;
[4645] supposing that the comets do not produce any alteration in the elements; and there is every reason to believe that their action on the planetary system is insensible.

46. We have determined, in [1162'], the invariable plane, in which the sum of the products of the mass of each planet, by the area its radius vector
[4646] describes about the sun, when projected upon this plane, is a maximum. If we put γ for the inclination of this plane to the fixed ecliptic of 1750, and Π for the longitude of its ascending node upon that plane, we shall have, as in [1162'],

[4647]
$$\tan g.\gamma \,.\, \sin.\Pi = \frac{\Sigma.m.\sqrt{a.(1-ee)}.\sin.\varphi.\sin.\theta}{\Sigma.m.\sqrt{a.(1-ee)}.\cos.\varphi};$$
$$\tan g.\gamma \,.\, \cos.\Pi = \frac{\Sigma.m.\sqrt{a.(1-ee)}.\sin.\varphi.\cos.\theta}{\Sigma.m.\sqrt{a.(1-ee)}.\cos.\varphi}.$$

The integral sign of finite differences Σ includes all the similar terms relative to each planet. If we use the values of m, a, e, φ, and θ, given for each of these bodies, in [4061—4083], we shall find, by these formulas,

[4648]
$$\gamma = 1^d\, 35^m\, 31^s;$$
$$\Pi = 102^d\, 57^m\, 29^s.$$

Then, by substituting for e, φ, θ, their values, relative to the epoch 1950 [4081—4083, 4242, &c.], we shall obtain,

[4649]
$$\gamma = 1^d\, 35^m\, 31^s;$$
$$\Pi = 102^d\, 57^m\, 15^s;$$

which differ but very little from the preceding values [4648]. This serves as a confirmation of the variations we have previously computed in the inclinations and in the nodes of the planetary orbits.

CHAPTER XVIII.

ON THE ACTION OF THE FIXED STARS UPON THE PLANETARY SYSTEM.

47. *To complete the theory of the perturbations of the planetary system, there yet remains to be noticed those, which this system suffers, from the action of the comets and fixed stars.* Now, if we take into consideration, that we do not accurately know the elements of the orbits of most of the comets; and, that there may be some, which are always invisible to us, by reason of their great perihelion distance, though they may act on the remote planets; it must be evident, that it is impossible to determine their action. Fortunately, there are many reasons for believing, that the masses of the comets are very small; consequently, their action must be nearly insensible. We shall, therefore, restrict ourselves, in this article, to the consideration of the action of the fixed stars. [4649′] [4650]

For this purpose, we shall resume the formulas [930, 931, 932],

General expressions of δr, δv, δs.

$$\delta r=\frac{\left\{\begin{array}{l} a.\cos.v.\int n\,dt.r.\sin.v.\left\{2\int \mathrm{d}R+r.\left(\frac{dR}{dr}\right)\right\} \\ -a.\sin.v.\int n\,dt.r.\cos.v.\left\{2\int \mathrm{d}R+r.\left(\frac{dR}{dr}\right)\right\}\end{array}\right\}}{\mu.\sqrt{1-ee}}; \qquad (X) \qquad [4651]$$

$$\delta v=\frac{\frac{2r.d.\delta r+dr.\delta r}{a^2.\,n\,dt}+\frac{3a}{\mu}.\int\int n\,dt.\mathrm{d}R+\frac{2a}{\mu}.\int n\,dt.r.\left(\frac{dR}{dr}\right)}{\sqrt{1-ee}}; \qquad (Y) \qquad [4652]$$

$$\delta s=\frac{a.\cos.v.\int n\,dt.r.\sin.v.\left(\frac{dR}{dz}\right)-a.\sin.v.\int n\,dt.r.\cos.v.\left(\frac{dR}{dz}\right)}{\mu.\sqrt{1-ee}}. \qquad (Z) \qquad [4653]$$

We shall put m' for the mass of the star; x', y', z', its three rectangular
[4654] co-ordinates, referred to the sun's centre of gravity; r', its distance from that centre; x, y, z, the three co-ordinates of the planet m; and r, its distance from the sun. We shall have, as in [3736],

[4655] $$R = \frac{m'.(xx'+yy'+zz')}{r'^3} - \frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}.$$

Developing the second member of this equation, according to the descending powers of r', we shall have,*

[4656] $$R = -\frac{m'}{r'} + \frac{m'.r^2}{2r'^3} - \tfrac{3}{2}\,m'.\frac{(xx'+yy'+zz'-\frac{1}{2}r^2)^2}{r'^5} - \&c.$$

[4656′] We shall take, for the fixed plane, that of the primitive orbit of the planet; and we shall have, by neglecting the square of z,†

[4657] $$x = r.\cos.v; \qquad y = r.\sin.v; \qquad z = rs.$$

[4658] Putting l for the latitude of the star m', and U for its longitude, we obtain,‡

[4659] $$x' = r'.\cos.l.\cos.U; \qquad y' = r'.\cos.l.\sin.U; \qquad z' = r'.\sin.l.$$

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* (2731) Putting, for brevity, $xx'+yy'+zz'=rr'.f$; and, as in [914′],

[4655a] $$x^2+y^2+z^2=r^2, \qquad x'^2+y'^2+z'^2=r'^2,$$

we find, that the last term of [4655] becomes, by successive reductions, as in [4655c];

[4655b] $$-m'.\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{-\frac{1}{2}} = -m'.\{r'^2-2r'rf+r^2\}^{-\frac{1}{2}} = -\frac{m'}{r'}.\left\{1-2\left(\frac{r'rf-\frac{1}{2}r^2}{r'^2}\right)\right\}^{-\frac{1}{2}}$$

[4655c] $$= -\frac{m'}{r'} - \frac{m'}{r'}.\left(\frac{r'rf-\frac{1}{2}r^2}{r'^2}\right) - \frac{3}{2}.\frac{m'}{r'}.\left(\frac{r'rf-\frac{1}{2}r^2}{r'^2}\right)^2 - \&c.$$

Substituting this in [4655], we find that the first term of [4655] is destroyed by the second term of [4655c], and the whole expression of R becomes, by a slight reduction, as in [4656].

† (2732) The values of x, y [4657], correspond with those found in [926′—927].
[4657a] The value of $z=rs$ [4657] is the same as that in [931″], changing δs into s, to conform to the present notation.

‡ (2733) The radius vector of the body m' is r', and its latitude above the fixed
[4659a] plane l. Hence it is evident, from the principles of the orthographic projection, that the projection of r', upon the fixed plane, is $r'.\cos.l$; and the perpendicular z', let fall from m',

Hence we deduce, by neglecting the descending powers of r', below r'^{-3},* [4659']

$$R=-\frac{m'}{r'}+\frac{m'.r^2}{4r'^3}.\{2-3.\cos.^2l-3.\cos.^2l.\cos.(2v-2U)-6s.\sin.2l.\cos.(v-U)\}. \quad [4660]$$

Now, r', l, and U, vary nearly by insensible degrees; hence, if we put $R_{\prime}$ [4661] for the part of R, divided by r'^3, and neglect the square of the excentricity of the orbit of m; also, the term depending on s, which is of the order of the disturbing forces, that m suffers by the action of the planets; we shall have,† [4661']

$$\int dR=R_{\prime}-\frac{7m'.a^2}{12r'^3}.(2-3.\cos.^2l); \quad [4662]$$

$$r.\left(\frac{dR}{dr}\right)=2R_{\prime}. \quad [4662']$$

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upon the fixed plane, is equal to $r'.\sin.l$, as in [4659]. Now, this projected radius $r'.\cos.l$, makes the angle U with the axis of x' [4658, &c.], and 90^d-U with the axis of y'. [4659b] Hence we easily obtain expressions of x', y', similar to those of x, y [4657], and which may be deduced from them, by changing r into $r'.\cos.l$, and v into U, as in [4659]. [4659c]

* (2734) Substituting the values of x, y, &c. [4657, 4659], in the first member of [4660a], reducing, developing and neglecting terms of the order s^2, we get, by using [24, 6, 31] Int. the following expressions,

$$\{xx'+yy'+zz'\}^2=r^2r'^2.\{\cos.l.(\cos.v.\cos.U+\sin.v.\sin.U)+s.\sin.l\}^2 \quad [4660a]$$
$$=r^2r'^2.\{\cos.l.\cos.(v-U)+s.\sin.l\}^2$$
$$=r^2r'^2.\{\cos.^2l.\cos.^2(v-U)+2s.\sin.l.\cos.l.\cos.(v-U)\}$$
$$=r^2r'^2.\{\cos.^2l.[\tfrac{1}{2}+\tfrac{1}{2}\cos.(2v-2U)]+s.\sin.2l.\cos.(v-U)\}. \quad [4660b]$$

Now, the first and second terms of [4656], are the same as the first and second terms of [4660] respectively; so that if we neglect terms of the order mentioned in [4659'], we shall find, that the remaining part of [4656] becomes,

$$-\frac{3m'}{2r'^5}.\{xx'+yy'+zz'\}^2. \quad [4660c]$$

Substituting in this the expression [4660b], it produces the three last terms of R [4660].

† (2735) If we use the symbol $R_{\prime}$, we shall have, from [4660, 4661],

$$R_{\prime}=\frac{m'.r^2}{4r'^3}.\{2-3\cos.^2l-3\cos.^2l.\cos.(2v-2U)-6s.\sin.2l.\cos.(v-U)\}; \quad [4662a]$$

$$R=-\frac{m'}{r'}+R_{\prime}. \quad [4662b]$$

[4662''] Then, if we put $\mu=1$, which is nearly equivalent to the supposition, that the sun's mass is equal to unity [3709], we shall obtain from the formula [4651],*

The characteristic d affects the elements of the orbit of the body m, namely, r, v, s, &c.; but does not affect those of the body m', as r', l, U, &c. hence the differential of [4662b]
[4662c] becomes, $\mathrm{d}R=\mathrm{d}R_{,}$. Integrating this, and adding, as in [1012'], the constant quantity
[4662d] $m'g$, to complete the integral, we get $\int \mathrm{d}R=\int \mathrm{d}R_{,}+m'g$. Now, as r', l, U, are nearly constant, we may neglect their variations, and then the quantity $\mathrm{d}R_{,}$ will be the complete
[4662d'] differential of $R_{,}$; so that we may write $R_{,}$ for $\int \mathrm{d}R_{,}$; hence the expression [4662d]
[4662e] becomes $\int \mathrm{d}R=R_{,}+m'g$. If we neglect terms of the order e^2, in the expression of
[4662f] r [1256], it becomes as in [4664]; and if we substitute this in the expression of $r^2.dv$ [1256], we easily obtain the expression of $n\,dt$ [4664]. By inadvertence, *the author has*
[4662g] *given a wrong sign to the term depending on e, in the value of r* [4664], which in the
[4662h] original work is $r=a.\{1+e.\cos.(v-\varpi)\}$. This affects the numerical coefficients of the formulas [4666, 4666', &c.], but does not alter the general results [4669', 4673, &c.]. Putting,
[4662i] for brevity, h equal to the coefficient of r^2, in the expression of $R_{,}$ [4662a], we have,

[4662k] $$h=\frac{m'}{4r'^3}.\{2-3.\cos.^2 l-3.\cos.^2 l.\cos.(2v-2U)-6s.\sin.2l.\cos.(v-U)\};$$

[4662l] $$R_{,}=h.r^2; \qquad \text{whence} \qquad \left(\frac{dR_{,}}{dr}\right)=2hr=\frac{2R_{,}}{r}.$$

Substituting this in the partial differential of R [4662b], relatively to r, we obtain the following expression,

[4662m] $$\left(\frac{dR}{dr}\right)=\left(\frac{dR_{,}}{dr}\right)=\frac{2R_{,}}{r};$$

multiplying this by r, we get [4662']. If we determine the constant quantity g, as in
[4662n] [1016'', &c.], by making the coefficient of t vanish from the expression of δv, we shall find, by putting $\mu=1$, and neglecting e^2, that the terms of δv [4652], necessary to be noticed in finding the constant quantity, are,

[4662o] $$a.\int\{3\int \mathrm{d}R+2r.\left(\frac{dR}{dr}\right)\}.n\,dt.$$

[4662o'] Substituting the values [4662e, 4662'], it becomes, $a.\int(7R_{,}+3m'g).n\,dt$; and if we retain only the constant part of $R_{,}$, the preceding expression will vanish, and we shall have
[4662p] the constant part of δv equal to nothing, by putting $7R_{,}+3m'g=0$; or $m'g=-\frac{7}{3}.R_{,}$. Now, the constant part of $R_{,}$ is evidently obtained, by putting $r=a$, and retaining only the two first terms of [4662a]. Hence we get,

[4662q] $$m'g=-\frac{7m'.a^2}{12r'^3}.(2-3.\cos.^2 l);$$

and $\int \mathrm{d}R$ [4662e] becomes as in [4662]. In the original work the numerical coefficient is $-\frac{1}{4}$, instead of $-\frac{7}{12}$.

* (2736) From [4662e, 4662'], we get,

$$\delta r = 4\,a\,.\cos.v\,.\int n\,dt\,.\,r\,R_{\prime}\,.\sin.v - 4\,a\,.\sin.v\,.\int n\,dt\,.\,r\,R_{\prime}\,.\cos.v. \qquad [4663]$$

Substituting the following expressions [1256, 4662*f*, &c.],

$$r = a\,.\{1 - e\,.\cos.(v - \varpi)\}; \qquad n\,dt = dv\,.\{1 - 2\,e\,.\cos.(v - \varpi)\}; \qquad [4664]$$

and neglecting under the sign $\int$, the periodical terms, affected with the angle v, we shall have,* [4665]

$$ndt.r.R_{\prime}.\sin.v = -\frac{5m'.a^3.dv}{4r'^3}.\{(1-\tfrac{3}{2}.\cos.^2 l).e.\sin.\varpi + \tfrac{3}{4}.\cos.^2 l.e.\sin.(\varpi - 2U)\}; \qquad [4666]$$

$$ndt.r.R_{\prime}.\cos.v = -\frac{5m'.a^3.dv}{4r'^3}.\{(1-\tfrac{3}{2}.\cos.^2 l).e.\cos.\varpi - \tfrac{3}{4}.\cos.^2 l.e.\cos.(\varpi - 2U)\}; \qquad [4666']$$

$$2\int dR + r.\left(\frac{dR}{dr}\right) = 4R_{\prime} + 2m'g. \qquad [4663a]$$

Substituting this in [4651], also $\mu = 1$, and neglecting e^2, we get,

$$\frac{\delta r}{a} = 4.\cos.v.\int ndt.rR_{\prime}.\sin.v - 4.\sin.v.\int ndt.rR_{\prime}.\cos.v$$
$$+ 2m'g.\cos.v.\int ndt.r.\sin.v - 2m'g.\sin.v.\int ndt.r.\cos.v. \qquad [4663a']$$

This differs from [4663], in the terms multiplied by g. The two expressions would agree, if we were to take the arbitrary constant quantity g [4662*d*] equal to nothing; but this would be inconsistent with [4662*n*, 4668]. [4663*b*]

* (2737) From [4662*l*], we obtain $ndt.rR_{\prime} = h.ndt.r^3$. Now we have, by neglecting e^2, $r^3 = a^3.\{1 - 3\,e.\cos.(v - \varpi)\}$ [4664]; multiplying this by ndt [4664], we get, [4666*a*]

$$ndt.r^3 = a^3.dv.\{1 - 5e.\cos.(v - \varpi)\}; \text{ hence, } ndt.rR_{\prime} = h.a^3.dv.\{1 - 5e.\cos.(v - \varpi)\}. \qquad [4666b]$$

Multiplying this successively, by $\sin.v$, and $\cos.v$, we get, by reduction,

$$ndt.rR_{\prime}.\sin.v = h.a^3.dv.\{\sin.v - \tfrac{5}{2}e.\sin.\varpi - \tfrac{5}{2}e.\sin.(2v - \varpi)\}; \qquad [4666c]$$

$$ndt.rR_{\prime}.\cos.v = h.a^3.dv.\{\cos.v - \tfrac{5}{2}e.\cos.\varpi - \tfrac{5}{2}e.\cos.(2v - \varpi)\}. \qquad [4666d]$$

The second of these expressions may be derived from the first, by augmenting each of the angles v, ϖ, U, by 90^d; as appears, by making this change in the second members; no alteration being made in r', l, &c.; so that h [4662*k*] may remain the same. If we suppose the plane of xy, to be the primitive orbit of m, the latitude s will be of the order of the disturbing forces of the planets, which is neglected in [4661′]; and then h [4662*k*] is composed of the two terms, [4666*e*] [4666*f*]

$$\frac{m'}{4r'^3}.(2 - 3.\cos.^2 l), \qquad -\frac{m'}{4r'^3}.3.\cos.^2 l.\cos.(2v - 2U). \qquad [4666g]$$

These are to be substituted in [4666*c*], and those terms retained, which do not contain the

[4666''] which gives, by considering ϖ, l, r', U, as very nearly constant,*

[4667] $$\frac{\delta r}{a}=\frac{3m'.a^3.v}{2r'^3}.\{(1-\tfrac{3}{2}.\cos.^2l).e.\sin.(v-\varpi)-\tfrac{5}{2}.\cos.^2l.e.\sin.(v+\varpi-2U)\}.$$

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angle v, or its multiples [4665]; consequently, the first of these terms of h must be combined with the second of [4666c]; and the second of these terms of h, with the third of [4666c]; hence we shall have,

[4666h] $$n\,dt.rR_{,}.\sin.v=\frac{m'.a^3.dv}{4r'^3}.\{-(2-3.\cos.^2l).\tfrac{5}{2}e.\sin.\varpi-\tfrac{15}{4}e.\cos.^2l.\sin.(\varpi-2U)\};$$

which is easily reduced to the form [4666]. In like manner we may compute [4666']; or,
[4666i] we may obtain it much more easily, by derivation from [4666], by increasing the angles v, ϖ, U, by 90^d, as in [4666c]. These results are free from the error in the value of r [4662g]; and if we compare them with those given by the author, in the original work, we
[4666k] find, that we must multiply his expressions by -5, to obtain those in [4666, 4666']; or, in other words, we must change e into $-5e$, in his formulas.

* (2738) Putting, for brevity,

[4667a] $$A=\frac{5m'.a^3}{4r'^3}.(1-\tfrac{3}{2}.\cos.^2l).e; \qquad B=\frac{15m'.a^3}{16r'^3}.\cos.^2l.e;$$

we find, that the integrals of [4666, 4666'] become, very nearly,

[4667b] $$\textstyle\int n\,dt.rR_{,}.\sin.v=-Av.\sin.\varpi-Bv.\sin.(\varpi-2U);$$

[4667c] $$\textstyle\int n\,dt.rR_{,}.\cos.v=-Av.\cos.\varpi+Bv.\cos.(\varpi-2U).$$

Multiplying the first of these expressions by $4.\cos.v$, the second by $-4.\sin.v$, and taking the sum of the products; putting

[4667d] $$-\sin.\varpi.\cos.v+\cos.\varpi.\sin.v=\sin.(v-\varpi);$$
$$-\sin.(\varpi-2U).\cos.v-\cos.(\varpi-2U).\sin.v=-\sin.(v+\varpi-2U);$$

we get, for the terms in the first line of [4663d'], the following expression,

[4667e] $$4.\cos.v.\textstyle\int n\,dt.rR_{,}.\sin.v-4.\sin.v.\int n\,dt.rR_{,}.\cos.v$$
$$=4.A.v.\sin.(v-\varpi)-4.B.v.\sin.(v+\varpi-2U).$$

Again, if we multiply together the expressions of r and $n\,dt$ [4664], neglecting e^2, we obtain,

[4667f] $$n\,dt.r=a\,dv.\{1-3e.\cos.(v-\varpi)\}.$$

Multiplying this, successively, by $\sin.v$, $\cos.v$; reducing and retaining only the terms, which are independent of the angle v, we get,

[4667g] $$n\,dt.r.\sin.v=-a\,dv.\tfrac{3}{2}e.\sin.\varpi; \qquad n\,dt.r.\cos.v=-a\,dv.\tfrac{3}{2}e.\cos.\varpi.$$

[4667h] $$\textstyle\int n\,dt.r.\sin.v=-av.\frac{3}{2}e.\sin.\varpi; \qquad \int n\,dt.r.\cos.v=-av.\frac{3}{2}e.\cos.\varpi.$$

Multiplying these integrals, respectively, by $2m'g.\cos.v$, $-2m'g.\sin.v$; taking the sum of the products, and reducing, by means of [4667d]; then substituting the value of

Now we have,*

$$\frac{\delta r}{a} = -\delta e \,.\, \cos.(v-\varpi) - e\,\delta\varpi \,.\, \sin.(v-\varpi). \qquad [4668]$$

Secular variations of the excentricity and perihelion.

Comparing together the two expressions [4667, 4668], we obtain,†

$$\delta e = \quad \frac{15\,m'.\,a^3 v}{4\,r'^3} \,.\, \cos.^2 l \,.\, e \,.\, \sin.(2\varpi - 2U)\,; \qquad [4669]$$

$$\delta\varpi = -\frac{3\,m'.\,a^3 v}{2\,r'^3} \,.\, \{1 - \tfrac{3}{2} \,.\, \cos.^2 l - \tfrac{5}{2} \,.\, \cos.^2 l \,.\, \cos.(2\varpi - 2U)\}. \qquad [4669']$$

Thus the action of the star m' produces secular variations in the excentricity and in the longitude of the perihelion of the orbit of the planet m; but these variations are incomparably smaller than those arising from the action of the other planets. For, if we suppose m to be the earth, r' cannot, by observation, [4669'']

$m'g$ [4662q], we finally get, for the second line of [4663a'],

$$2m'g.\cos.v.\textstyle\int n\,dt.r.\sin.v - 2\,m'g.\sin.v.\int n\,dt.r.\cos.v$$

$$= 2m'g.\tfrac{3}{2}.ave.\{-\sin.\varpi.\cos.v + \cos.\varpi.\sin.v\} \qquad [4667i]$$

$$= m'g.3ave.\sin.(v-\varpi) = -\frac{7\,m'.\,a^3 v}{2\,r'^3}.(1-\tfrac{3}{2}.\cos.^2 l).e.\sin.(v-\varpi). \qquad [4667k]$$

Adding together the expressions [4667e, k]; re-substituting the values of A, B [4667a], we get the complete value of $\frac{\delta r}{a}$ [4663a'], as in [4667]. In the original work, the author makes the factor, which is without the braces, equal to $-\frac{m'.\,a^3 v}{r'^3}$, instead of $\frac{3\,m'.\,a^3 v}{2\,r'^3}$; and the numerical coefficient of the second term within the braces is erroneously printed $-\frac{3}{4}$ instead of $-\frac{5}{2}$. These mistakes are the consequences of using erroneous values of g and r [4662g, p]. [4667l] [4667m]

* (2739) In finding the variation of r [4664], we must neglect that of r, arising from the constant quantity g' [4662n], and the expression becomes as in [4668]; which is similar to [3876]. The signs of the terms in the second member of [4668], in the original work, are incorrect, by reason of the mistake mentioned in [4662g]. [4668a]

† (2740) From [21] Int. we have,

$$\sin.\{v+\varpi-2U\} = \sin.\{(v-\varpi)+(2\varpi-2U)\}$$
$$= \sin.(v-\varpi).\cos.(2\varpi-2U)+\cos.(v-\varpi).\sin.(2\varpi-2U). \qquad [4669a]$$

Substituting this in the last term of [4667], and then comparing separately, the coefficients of $\sin.(v-\varpi)$ and $\cos.(v-\varpi)$, in the two expressions [4667, 4668]; we get, by a slight reduction, the values of δe, $\delta\varpi$ [4669, 4669']. These expressions agree with those given [4669b]

[4670] be supposed less than $100000a$, and then the term $\frac{m'.a^3 v}{r'^3}$, does not exceed,*

[4671] $$m't\,.\,0^s{,}000000001\,;$$

t denoting the number of Julian years. This is incomparably less than the
[4671′] secular variation of the excentricity of the earth's orbit, resulting from the action of the planets, which, by [4244], is equal to,

The action of the stars has no sensible effect on the excentricities and perihelia of the planets.

[4672] $$-t\,.\,0^s{,}093819,$$

unless we suppose, that m' has a value which is wholly improbable. Hence we may conclude, *that the action of the stars has no sensible influence on the secular variations of the excentricities and perihelia of the planetary orbits.*
[4673] In like manner, it is evident, from the development of the formula [4653], that their action has not any sensible influence on the position of these orbits.†

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by Mr. Plana, in the *Memoirs of the Astronomical Society of London*, vol. ii. p. 354; which he deduced from the formulas [1258a]. Hence we see, that the method here
[4669c] proposed by La Place, to find δe, $\delta\varpi$, when it is correctly followed, leads to an accurate result; and is not liable to the objection made by Mr. Plana, in the same page of that volume, namely; that it is nowise fit for the intended purpose, without taking into view other circumstances, which render the calculation more complicated. We may remark, that in
[4669d] the original work, the factor $\frac{15}{4}$ [4669], is printed $\frac{3}{4}$; and, in [4669′], the factors $-\frac{3m'}{2}$,
[4669e] $-\frac{5}{2}.\cos.^2 l$, are changed into $-\frac{m'}{e}$, $-\frac{3}{4}.\cos.^2 l$, respectively.

* (2741) The value of $r=100000a$ [4670], corresponds to an annual parallax of
[4671a] about 2^s; and we have nearly $v=1295977^s.t$ [4077]; substituting these in $\frac{m'.a^3 v}{r'^3}$ [4670], it becomes as in [4671]; or simply, by supposing m'= the sun's mass $=1$,

$$t\,.\,0^s{,}000000001.$$

The secular variation of e'' [4330a], is nearly represented by,

[4671b] $$\frac{de''}{dt}.t=-\tfrac{1}{2}.(0^s{,}187638).t=-0^s{,}093819.t \quad [4244, 4672]\,;$$

which is much greater than the expression [4671].

† (2742) If we substitute $rs=z$ [4657], in $R_{,}$, R [4662b, a], and retain only
[4673a] the terms of R, containing z, we find,

[4673b] $$R=-\frac{6m'.rz}{4r'^3}.\sin.2l.\cos.(v-U), \qquad \text{and} \qquad \left(\frac{dR}{dz}\right)=-\frac{6m'.r}{4r'^3}.\sin.2l.\cos.(v-U).$$

We shall now examine into the influence of the attraction of the stars on the mean motion of the planets. For this purpose, we shall observe, that the formula [4652] gives, in $d.\delta v$, the term* $d.\delta v = 4\,a\,n\,d\,t\,.\,R_{\prime}$; from [4674] which we deduce the following expression,†

$$d.\delta v = \frac{m'.a^3}{r'^3}.n\,dt.\{2-3.\cos.^2 l\}. \qquad [4675]$$

We shall put

$$r' = r'_{\prime}.(1-\alpha\,t)\,; \qquad l = l_{\prime}.(1-\beta\,t)\,; \qquad [4676]$$

$r'_{\prime}$ and $l_{\prime}$ being the values of r' and l, in 1750, or when $t=0$; we shall [4676'] have, in δv, the variation,‡

$$\delta v = \frac{3\,m'.a^3}{r'^3_{\prime}}.(1-\tfrac{3}{2}.\cos.^2 l_{\prime}).\alpha.n\,t^2 - \frac{3\,m'.a^3}{2\,r'^3_{\prime}}.\sin.2\,l_{\prime}.\beta.n\,t^2. \qquad [4677]$$

Substituting this in δs [4653], we find that the terms are multiplied by the very small factor of the order [4670, 4671], which renders them insensible [4671'].

* (2743) This expression arises from the last term of δv [4652], which, by neglecting quantities of the order e^2, and putting $\mu=1$ [3709], becomes,

$$2af\,n\,dt.r.\left(\frac{dR}{dr}\right) = 2af\,n\,dt.2R_{\prime} \quad [4662']. \qquad [4674a]$$

Its differential gives, in $d.\delta v$, the term $4\,a\,n\,dt.R_{\prime}$, as in [4674]. This would be increased to $7\,a\,n\,dt.R_{\prime}$, by noticing the term depending on $f dR$ [4652], as we have [4674b] seen in [4662o']. This increases the terms [4675, 4677] in the ratio of 7 to 4.

† (2744) The two first and chief terms of $R_{\prime}$ [4662a], are $\frac{m'.r^2}{4\,r'^3}.(2-3.\cos.^2 l)$.

Substituting the value of r [4664], we obtain the part $\frac{m'.a^2}{4\,r'^3}.(2-3.\cos.^2 l)$, which [4675a] does not contain v; hence, the term of $d.\delta v$ [4674], becomes as in [4675].

‡ (2745) The value of l [4676] gives $\cos.l = \cos.(l_{\prime}-\beta t.l_{\prime}) = \cos.l_{\prime}+\beta t.\sin.l_{\prime}$, [4676a] by using [61] Int. Squaring this, neglecting t^2, and putting $2.\sin.l_{\prime}.\cos.l_{\prime} = \sin.2l_{\prime}$ [31] Int., we get $\cos.^2 l = \cos.^2 l_{\prime} + \beta t.\sin.2l_{\prime}$; whence,

$$2-3.\cos.^2 l = 2.(1-\tfrac{3}{2}.\cos.^2 l_{\prime}) - 3\,\beta t.\sin.2l_{\prime}. \qquad [4676b]$$

If we now substitute the value of r' [4676], in the first member of the following expression, and then develop it according to the powers of α, neglecting α^2, we get,

$$\frac{m'.a^3}{r'^3}.n\,dt = \frac{m'.a^3}{r'^3_{\prime}}.n\,dt.(1+3\,\alpha\,t). \qquad [4676c]$$

We cannot ascertain, by observation, the value of αt, but may determine that of βt. Now, if we suppose, relatively to the earth, $\beta = 1'' = 0^s,324$, and

[4678] $r_{,}' = 100000\, a$; the quantity $\frac{m'.a^3}{r_{,}'^3}.\beta n t^2$ becomes, very nearly,*

[4679] $$\frac{m'.a^3}{r_{,}'^3}.\beta\, n\, t^2 = \frac{m'.t^2.2^s,0357}{10^{15}};$$

[4679'] which is insensible, from the time of the most early observations on record.

The expression of $d.\delta v$, contains also, by what precedes, the terms,†

[4680] $$d.\delta v = -\tfrac{9}{2}.m'.a^3.ndt.\textstyle\int d.\left\{\frac{s.\sin.2l}{r'^3}.\cos.(v-U)\right\} - 6m'.a^3.ndt.\left\{\frac{s.\sin.2l}{r'^3}.\cos.(v-U)\right\}.$$

Multiplying together the expressions [4676b, c], we get the value of $d.\delta v$ [4675], nearly,

[4676d] $$d.\delta v = \frac{2\,m'.a^3}{r_{,}'^3}.n\,dt.(1-\tfrac{3}{2}.\cos.^2 l_{,}) + \frac{6\,m'.a^3}{r_{,}'^3}.(1-\tfrac{3}{2}.\cos.^2 l_{,}).\alpha n t dt - \frac{3\,m'.a^3}{r_{,}'^3}.\sin.2l_{,}.\beta n t dt.$$

We may neglect the first term of this formula, because we have taken the constant quantity

[4676e] g' so as to make the coefficient of t vanish from the expression of δv [4662n]. Integrating the other two terms of [4676d], we get the value of δv [4677].

* (2746) The assumed values of β, $r_{,}'$, are taken within reasonable limits; since the value of β corresponds to an annual variation in the latitude of the star, of about a third of a

[4679a] sexagesimal second; and the value of $r_{,}'$ to an annual parallax of nearly two sexagesimal seconds. To reduce the expression [4678] to numbers, we have, in the case of $t=1$, $n t =$ circumference of the circle $= 6,2831$; hence, generally,

[4679b] $$n\,t = 6,2831.t; \qquad \text{also,} \qquad \beta\,t = 0^s,324.t.$$

The product of these two expressions is,

[4679c] $$\beta\, n\, t^2 = 2^s,0357.t^2.$$

Substituting this, and $r_{,}' = 10^5.a$, in the first member of [4679], it becomes as in the

[4679d] second member of that equation. This is wholly insensible in observations made 3000 years ago; since, by putting $t = -3000$, and $m' = 1$, it becomes less than $0^s,00000002$.

† (2747) If we now notice only the terms of R, $R_{,}$ [4662a, b], depending on s, we obtain,

[4680a] $$R = -\tfrac{3}{2}.\frac{m'.r^2}{r'^3}.s.\sin.2l.\cos.(v-U); \quad \text{whence,} \quad r.\left(\frac{dR}{dr}\right) = -3.\frac{m'.r^2}{r'^3}.s.\sin.2l.\cos.(v-U).$$

If we substitute the value of r [4664], and neglect terms of the order es, we get,

[4680b] $$R = -\tfrac{3}{2}.m'.a^2.\left\{\frac{s.\sin.2l}{r'^3}.\cos.(v-U)\right\}; \qquad r.\left(\frac{dR}{dr}\right) = -3.\frac{m'.a^2}{r'^3}.s.\sin.2l.\cos.(v-U).$$

Now, if we put $\mu = 1$, and neglect e^2; noticing only the terms of [4652], where R

Now we have,*

$$s = t \cdot \frac{dq}{dt} \cdot \sin. v - t \cdot \frac{dp}{dt} \cdot \cos. v\,;$$ [4681]

which gives, by neglecting the quantities multiplied by the sine or cosine of the angle v,†

$$\frac{s \cdot \sin. 2l}{r'^3} \cdot \cos. (v - U) = t \cdot \frac{\sin. 2l}{2r'^3} \cdot \left\{ \frac{dq}{dt} \cdot \sin. U - \frac{dp}{dt} \cdot \cos. U \right\};$$ [4682]

consequently,‡

$$\int \mathrm{d} \cdot \frac{s \cdot \sin. 2l}{r'^3} \cdot \cos. (v - U) = t \cdot \frac{\sin. 2l}{2r'^3} \cdot \left\{ \frac{dq}{dt} \cdot \sin. U - \frac{dp}{dt} \cdot \cos. U \right\}.$$ [4683]

Hence we obtain, in $d \cdot \delta v$, the term,§

$$d \cdot \delta v = -\frac{21}{4} \cdot \frac{m' \cdot a^3}{r'^3} \cdot n\,t\,dt \cdot \sin. 2l \cdot \left\{ \frac{dq}{dt} \cdot \sin. U - \frac{dp}{dt} \cdot \cos. U \right\};$$ [4684]

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explicitly occurs, we get, for its differential,

$$d \cdot \delta v = 3a \cdot n\,dt \cdot \int \mathrm{d} R + 2a \cdot n\,dt \cdot r \cdot \left(\frac{dR}{dr}\right).$$ [4680c]

Substituting, in the first term of this expression, the value of R [4680b], we get the first term of [4680]; and we obtain the last term of [4680], by the substitution of the second expression [4680b] in the last term of [4680c]. [4680d]

* (2748) This expression is similar to that in [3802, &c.]. We may remark, that the author, in this article, has interchanged the usual signification of the symbols p, q [3802]. We have rectified this, by changing p into q, and q into p, in all the formulas [4681—4685] of the original work. [4681a]

† (2749) If we multiply the expression [4681] by $\frac{\sin. 2l}{r'^3} \cdot \cos. (v - U)$, and reduce the products by [19, 20] Int., we shall obtain the equation [4682], by retaining only the terms which are independent of v; or in other words, by retaining only the terms $\frac{1}{2} \sin. U$, $\frac{1}{2} \cos. U$, of the expressions $\sin. v \cdot \cos. (v - U)$, and $\cos. v \cdot \cos. (v - U)$, respectively. [4682a]

‡ (2750) If we neglect the variations of r', l, U, in the second member of [4682], the sign d may be considered as the complete differential, and then the signs $\int \mathrm{d}$, mutually counteract each other, and they may be prefixed to the first member of [4682], without altering its second member; hence we get [4683] from [4682]. [4683a]

§ (2751) Multiplying [4683] by $-\frac{9}{2} \cdot m' \cdot a^3 \cdot n\,dt$, and [4682] by $-6 \cdot m' \cdot a^3 \cdot n\,dt$, we find, that the sum of the products, or the second member of [4680], is as in [4684]. Integrating this, we get, [4685]. [4684a]

consequently, we have, in δv, the secular inequality,

[4685] $$\delta v = -\frac{21}{8} \cdot \frac{m'.a^3}{r'^3} \cdot n\,t^2 . \sin. 2\,l. \left\{ \frac{dq}{dt} . \sin. U - \frac{dp}{dt} . \cos. U \right\}.$$

We have given the values of $\frac{dp''}{dt}$, $\frac{dq''}{dt}$, [4332], relatively to the earth. If we substitute them in the preceding term of δv [4685], we shall find that it is insensible,* even in the most ancient observations.

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* (2752) From [4332] it appears, that $\frac{dp''}{dt}$, $\frac{dq''}{dt}$, are each less than 1^s, and $\sin.2l$,

[4685a] $\sin.U$, $\cos.U$, do not exceed unity; therefore, $\sin.2l. \left\{ \frac{dq''}{dt} . \sin.U - \frac{dp''}{dt} . \cos. U \right\}$, may be considered as less than 1^s; and then, the expression [4685], neglecting its sign, becomes less than $\frac{21}{8} \cdot \frac{m'.a^3}{r'^3} . nt^2 . 1^s$; which is found to be insensible, in [4679'].

Other terms of the like nature with those which have been particularly examined, in this

[4685b] chapter, may be deduced from the formulas [4651—4653]; but it is evident, from what we have seen, that they must be excessively small; so that it is hardly worth the labor of a

[4685c] more thorough examination. The author himself, seems to have considered the subject as not deserving much attention, and has been quite negligent in the numerical details of this article; so that it has been found necessary to correct the text in several places, as we have

[4685d] already remarked. In writing the notes on this volume, soon after its first publication by the author, I pointed out the mistakes in this chapter. It has since been done by Mr. Plana, in vol. ii. p. 351 of the *Memoirs of the Astronomical Society of London*, for 1826; and

[4685e] subsequently by La Place, in the *Connaissance des Tems*, for the year 1829, page 250. The method used by Mr. Plana is more direct and simple than that of the author. It consists in

[4685f] substituting the value of R [4660], in the formulas [5787—5791], and making the necessary reductions; but, as the process is simple, it is unnecessary to enter minutely upon it.

Mr. Plana remarks, in page 355 of the work above-mentioned, that the action of the fixed stars affects the mathematical accuracy of the equation [1114],

[4685g] $$e^2 . m . \sqrt{a} + e'^2 . m' . \sqrt{a'} + \&c. = \text{constant};$$

as we have already remarked in [1114b]. This is evident; for, if we increase the quantity e, in the first member, by the expression δe [4669], the second member will be increased by the quantity,

[4685h] $$2e\delta e = \frac{15\, m\, m'. a^{\frac{7}{2}} . v}{2\, r'^3} . \cos.^2 l . e^2 . \sin. (2\varpi - 2U), \text{ nearly};$$

which destroys the constancy of the second member. The same defect exists in the equation [1134 or 1155].

It is easy also, to satisfy ourselves, that the preceding results hold good, relatively to those planets which are the most distant from the sun. Hence it appears, that the action of the stars upon the planetary system, is so much [4686] decreased, by reason of their great distance, that it is wholly insensible.

It now remains to compare with observations, the formulas of the planetary perturbations, given in this book, and particularly those of the two great inequalities of Jupiter and Saturn. This comparison requires too much detail for the limits of the present work; we shall, therefore, merely remark, that before the discovery of these great inequalities, the errors of the best tables sometimes amounted to thirty-five or forty minutes; and now they [4687] do not exceed a minute. Halley had concluded, by the comparison of modern observations, the one with the other; and also, by comparing the modern with the ancient observations, that Saturn's motion is retarded, and Jupiter's accelerated, from age to age. On the other hand, Lambert ascertained, from the comparison of modern observations alone, that Saturn's motion was then [4688] accelerated, and Jupiter's motion retarded. These phenomena, apparently opposed to each other, indicated, in the motions of the two planets, great inequalities of a long period, of which it was important to know the laws and the cause. By submitting to analysis their mutual perturbations, I discovered the two principal inequalities [4431, 4492]; and perceived, that the [4689] phenomena, observed by Halley and Lambert, naturally arise from them; and, that they represent, with remarkable accuracy, both ancient and modern observations. The magnitude of these inequalities, and the great length of the period of revolution, to complete which requires more than nine hundred years, depend, as we have seen, on the nearly commensurable ratio which [4690] obtains between the mean motions of Jupiter and Saturn. This ratio gives rise to several other important inequalities, which I have determined, and these inequalities have given to the tables the precision they now have. The same analysis, being applied to all the other planets, has enabled me to discover, in their motions, some very sensible inequalities, which have been confirmed by [4691] observation. I have reason to believe, that the preceding formulas, computed with particular care, will give a still greater degree of precision to the tables of the motions of the planetary bodies.

SEVENTH BOOK.

THEORY OF THE MOON.

The theory of the moon has difficulties peculiar to itself, arising from the magnitude of its numerous inequalities, and from the slow convergency of the series by which they are determined. If the body were nearer to the earth, the inequalities of its motion would be less, and their approximations more converging, But, in its present distance, these approximations depend on a very complicated analysis; and it is only by a very particular attention, and a nice discrimination, that we can determine the influence of the successive integrations, upon the various terms of the expression of the disturbing force. The selection of co-ordinates is not unimportant for the
[4692] success of the approximations. The sun's disturbing force depends on the sines and cosines of the moon's elongation from the sun, and on its multiples. Their reduction to sines and cosines of angles, depending on the mean motions of the sun and moon, is troublesome, and has but little convergency, on account of the moon's great inequalities. It is, therefore, advantageous to avoid this reduction, and to determine the moon's mean longitude in a function of the true longitude, which may be useful on several occasions. We may, then, if it be required, determine accurately, by inverting the series, the true longitude, in a function of the mean longitude. It is in this way we shall consider the lunar theory.

Terms of different orders.

To arrange conveniently these approximations, we shall divide the inequalities, and the terms which compose them, into several orders. We shall consider as quantities of the first order, the ratio of the sun's mean motion to that of the moon, the excentricity of the orbit of the moon or earth, and the inclination of the moon's orbit to the ecliptic. Thus, in the expression of the
[4693] mean longitude, in a function of the true longitude [5574—5578], the principal term of the moon's equation of the centre is of the first order [5574]. The second order includes the second term of that equation; the

reduction to the ecliptic; and the three great inequalities, known under the names of *variation*, *evection*, and *annual equation* [5575]. The [4694]
inequalities of the third order are fifteen in number [5576]. The present tables contain all these inequalities, together with the most important ones of the fourth order; and it is on this account, that they correspond with the observations made on the moon, with a degree of accuracy that it will be difficult to surpass; and to this great correctness we are indebted for the important improvements in geography and nautical astronomy.

The object of this book is to show, in the first place, that the law of universal gravity is the only source of all the inequalities of the lunar motions; and then, to use this law as a method of discovery, to perfect the theory of these inequalities, and to deduce from them several important elements of the system of the world; such as the secular equations of the moon, the parallaxes of the moon and sun, and the oblateness of the earth. A judicious choice of the co-ordinates, and well conducted approximations, with calculations made carefully, and verified several times, ought to give the same results as those derived from observation; if the law of gravity, inversely as the square of the distance, be the law of nature. We have, therefore, endeavored to satisfy these conditions; which require [4695]
the consideration of some very intricate points; the neglect of which is the cause of the discrepances, that have been observed in the previously known theories of the moon. It is in these points, that the main difficulty of the problem consists. We may easily conceive of a great many different and new methods of expressing the problem by equations; but it is the discussion of all those terms, which are of themselves very small, and acquire a sensible value, by the successive integrations, which constitutes the important and difficult part of the process, when we endeavor to make the theory agree with observation; which should be the chief object of the analysis. We have determined all the inequalities of the first, second and third orders, and the most important ones of the fourth order, continuing the approximation to quantities of the fourth order inclusively; and retaining those of the fifth order, which arise in the calculation. For the purpose of comparing this analysis with observation, we may observe, that the coefficients of Mason's lunar tables are the result of the comparison of the theory of gravity with eleven hundred and thirty-seven observations [4696]
of Bradley, made between the years 1750 and 1760; that the eminent

astronomer Burg has rectified these tables, by means of more than three thousand of Maskelyne's observations, from 1765 to 1793; and, that the corrections he has made are small; with the addition of nine equations,
[4697] indicated by the theory. The tables of both these astronomers are arranged in the same form as those of Mayer, of which they are successive improvements: and we ought, in justice to this celebrated astronomer, to observe, that he was not only the first, who constructed lunar tables, sufficiently correct to be used in the solution of the problem of finding the longitude at sea, but also, that Mason and Burg have deduced, from his theory, the methods of improving their tables. The arguments are made to depend on each other, in order to decrease the number of them. We have reduced them, with particular care, to the form which is adopted in the
[4698] present theory; that is, to sines and cosines of angles, increasing in proportion to the moon's true longitude. By comparing these results with the coefficients of the present theory, we have the satisfaction of perceiving, that the greatest difference, which, in Mayer's theory, one of the most accurate heretofore published, amounts to nearly one hundred centesimal seconds [$=32'',4$], is here reduced to thirty [$9'',8$], relative to the tables of Mason, and to less than twenty-six centesimal seconds [$=8'',3$], relative to the still more accurate tables of Burg. We could diminish this difference, by noticing quantities of the fifth order, which have some influence, as may be known by inspecting the terms of this kind already calculated. This is proved by the computation of the two inequalities [5286''', &c.], in which we have carried on the approximation to quantities of the fifth order. The
[4699] present theory agrees yet better with the tables, relative to the motion in latitude. The approximations of this motion are more simple and converging than those of the motions in longitude; and the greatest difference between the coefficients of my analysis and those of the tables, is only six centesimal seconds [$= 1'',9$], so that we may consider this part of the tables as being founded upon the theory itself. As to the third co-ordinate of the moon, or
[4700] its parallax, we have preferred, without hesitation, to form the tables by the theory alone, which, on account of the smallness of the inequalities of the lunar parallax, must give them more accurately than they can be obtained by observation. The differences between the results of the present theory and those of the tables, express, therefore, the differences between this theory and that of Mayer, which has been adopted by Mason and Burg. These differences are so small that they are hardly deserving of notice; but, as the

present theory agrees better with observation than Mayer's, in the motion in longitude, there is also reason to believe, that it possesses the same advantage relative to the inequalities in the parallax. [4701]

The motions of the perigee and nodes of the lunar orbit, afford also a method of verifying the law of gravity. In the first approximation to the value of the motion of the perigee, by the theory of gravity, it was found, by mathematicians, only one half of what it was known to be, by observation; and Clairaut inferred, from this circumstance, that we must modify the law of gravity, by adding to it a second term. But he afterwards made the important remark, that by continuing the approximations to terms of a higher order, the theory would be found to agree nearly with observation. The motion, deduced from the present analysis, differs from the actual motion only a four hundredth part [5231]; the difference is not a three hundred and fiftieth part in the motion of the nodes [5233']. [4702] [4703]

Hence it incontestably follows, that the law of universal gravitation is the sole cause of the lunar inequalities. Now, if we consider the great number and extent of these inequalities, and the proximity of the moon to the earth, we must be satisfied, that it is, of all the heavenly bodies, the best adapted to confirm this great law of nature, as well as to show the power of analysis, that wonderful instrument, without the aid of which it would be impossible for the human mind to penetrate into so complicated a theory, and that can be used, as a means of discovery, as sure as by direct observation. [4704]

Among the periodical inequalities of the moon's motion in longitude, that which depends on the simple angular distance of the moon from the sun is important, on account of the great light it throws on the sun's parallax. It has been determined by the theory; noticing quantities of the fifth order, and also the perturbation of the earth by the moon, which are indispensable in this laborious research. Burg found this inequality to be $122^s,38$, by the comparison of a very great number of observations. If we put this equal to the result by the theory, we obtain $8^s,56$, for the sun's mean parallax; being the same as several astronomers have found, from the last transit of Venus over the sun [5586]. [4705] [4706] [4707]

An inequality, which is not less important, is that which depends on the longitude of the moon's node. Mayer discovered it by observation, and Mason fixed it at $7^s,7$; but, as it did not appear to depend on the theory [4708]

of gravity, it was neglected by most astronomers. A more thorough examination of this theory led me to the discovery, that its cause is the oblateness of the earth. Burg found it, by a great number of Maskelyne's
[4709] observations, to be $6^s,8$; which corresponds to an oblateness of $\frac{1}{305,05}$ [5593].

We may also determine this oblateness, by means of an inequality in the moon's motion in latitude; which I discovered also by the theory; and
[4710] which depends on the sine of the moon's true longitude. It is the result of a nutation in the lunar orbit, produced by the action of the terrestrial spheroid, and corresponds to that produced by the moon in our equator; so that the one of these nutations is the reaction of the other: and, if all the particles of the earth and moon were firmly connected together, by inflexible right lines, void of mass, the whole system would be in equilibrium about the centre of gravity of the earth, in virtue of the forces producing
[4711] these two nutations: the force, acting on the moon, compensating for its smallness, by the length of the lever to which it is attached. We may represent this inequality in latitude, by supposing the lunar orbit, instead of moving uniformly on the ecliptic, with a constant inclination, to move, with the same conditions, upon a plane but little inclined to the ecliptic, and which always passes through the equinoxes, between the ecliptic and equator: a phenomenon which occurs in the theory of Jupiter's satellites, in a still more striking manner. Thus, this inequality decreases the
[4712] inclination of the moon's orbit to the ecliptic, when the ascending node of that orbit coincides with the vernal equinox. This inclination is increased, when the ascending node coincides with the autumnal equinox, which was the case in 1755; in consequence of which, the inclination, as it was found by Mason, from 1750 to 1760, is too great. This point has been determined by Burg, by observations made during a much longer interval, noticing the preceding inequality; and he has found the inclination
[4713] to be less, by $3^s,7$. At my request, this astronomer has undertaken to determine the coefficient of this inequality, by a very great number of observations; and he has found it to be equal to -8^s. The oblateness of
[4714] the earth, deduced from it, is $\frac{1}{304,6}$ [5602], being very nearly the same as that which is computed from the preceding inequality of longitude. Thus, the moon, by the observation of her motions, renders sensible to modern astronomy the ellipticity of the earth, whose roundness was made

known to the early astronomers by her eclipses. The experiments on the pendulum seem to indicate a less oblateness,* as we have seen in the third book. This difference may depend on the terms by which the earth varies from an elliptical figure; which may have some small effect in the expression of the length of the pendulum, but is wholly insensible, at the distance of the moon.

The two preceding inequalities deserve every attention of observers; because they have the advantage over geodetical measures, in giving the oblateness of the earth, in a manner which is less dependant on the irregularities of its figure. If the earth were homogeneous, these inequalities would be much greater than they are found to be by observation. They [4715] concur, therefore, with the phenomena of the precession of the equinoxes, and the variation of gravity at the surface of the earth, to exclude its homogeneity. *It results also, that the moon's gravity towards the earth, is composed of the attractions of all the particles of the earth; which furnishes another proof of the attraction of all the particles of matter.*

Theory combined with experiments on the pendulum, the geodetical measures, and the phenomena of the tides, make the constant term of the expression of the moon's parallax less than by Mason's tables. It differs but [4716] very little from that which Burg computed from a great number of observations of the moon, of eclipses of the sun, and of occultations of stars by the moon. It is only necessary to decrease a little the mass of the moon, which was determined by the phenomena of the tides, to make this constant term coincide with the result of that skilful astronomer. This diminution is also [4717] indicated by the observations of the lunar equation of the solar tables, and by the nutation of the earth's axis. This seems to prove, that in the port of Brest, the ratio of the moon's action on the tides to that of the sun, is [4718] sensibly increased by local circumstances. Future observations of all these phenomena will remove this slight degree of uncertainty.

One of the most interesting results of the theory of gravity, is the knowledge of the secular inequalities of the moon. Ancient eclipses

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* (2753) Later and more accurate observations give a different result, as may be seen, [4715a] by referring to [2017n, 2056i, &c.].

indicated, in the moon's mean motion, an acceleration; the cause of which was sought for a long time in vain. Finally, I discovered, by the theory,
[4719] that it depends on the secular variations of the excentricity of the earth's orbit. The same cause decreases the mean motions of the perigee and nodes of the moon, while her mean motion is increased; so that the secular equations of the mean motions of the moon, the perigee and the nodes,
[4720] are always in the ratio of the numbers 1, 3 and 0,74 [5235]. *Future ages will develop these great inequalities, which are periodical, like the variations of the excentricity of the earth's orbit, upon which they depend.* These will finally produce variations which amount, at the least estimate, to a fortieth part of the circumference [9^d], in the moon's secular motion;
[4721] and to a twelfth of the circumference [30^d], in that of the perigee. Observations have already confirmed these secular inequalities in a remarkable manner. The discovery of them induced me to believe, that we must diminish, by fifteen or sixteen centesimal minutes, the present secular motion of the moon's perigee, which astronomers had determined,
[4722] by comparing modern observations with ancient ones. All the observations, which have been made during the last century, have put beyond doubt, this result of analysis. We see, in this, an example of the manner in which the phenomena, as they are developed, throw light upon their true causes. When
[4723] the acceleration of the moon's mean motion only was known, it could be attributed to the resistance of the ether, or to the successive transmission of gravity; but analysis shows us, that both these causes produce no sensible alteration, either in the mean motion of the nodes, or in that of the lunar perigee: this is a sufficient reason for rejecting them, even if we were
[4724] ignorant of the true cause. The agreement of the theory with observations, proves, that if the moon's mean motion is affected by any causes, besides the action of gravity, their influence is very small, and is not yet perceptible.

This agreement establishes, with certainty, the constancy of the duration of a day; which is an essential element in all astronomical theories. If this duration were now one hundredth part of a centesimal second [or $0^s,864$]
[4725] more than in the time of Hipparchus, the duration of the present century would be greater than in his time, by $365\frac{1}{4}$ centesimal seconds [or $315^s,576$].
[4725′] In this interval, the moon would describe an arch of $173^s,2$, and the present mean secular motion of the moon, would appear to be augmented by the

same quantity. This would add $4^s,4$ * to the secular equation, which is found, by the theory, to be $10^s,181621$ [5543], in the first century after the year 1750. This augmentation is incompatible with the best observations, which do not permit us to suppose, that the secular equation can exceed, by $1^s,62$, the result of the analysis [5543]. We may, therefore, conclude, that the duration of the day has not varied a hundredth part of a centesimal second, since the time of Hipparchus; which confirms what has been found *a priori*, in book v. § 12 [3347, &c.], by the discussion of all the causes which could alter it. [4726] [4727]

To omit nothing which can have an influence on the moon's motion, we have considered the direct action of the planets upon this satellite, and have found, that it is of very little importance. But the sun, by transmitting to the moon the action of the planets on the elements of the earth's orbit, renders their influence on the lunar motions very remarkable, and makes it much greater than on the elements themselves; so that the secular variation of the excentricity of the earth's orbit is much more sensible, in the moon's motion, than in the earth's. It is in this manner, that the moon's action on the earth, which produces, in the earth's motion, the inequality known by the name of the *lunar equation*, is, if it may be so expressed, reflected back to the moon, by means of the sun, but decreased in nearly the ratio of five to nine [5226]. This new consideration adds some terms to the action of the planets on the moon, which are of more importance than those depending on their direct action. We have investigated the principal lunar inequalities, resulting from the direct and indirect actions of the planets upon the moon; [4728] [4729] [4730]

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* (2754) If we neglect the term of the secular equation [5543], depending on i^3, and put $a = 10^s,181621$, we may represent the moon's mean motion, in i centuries after 1750, by $ni + ai^2$. If we substitute in this successively, $i = -\frac{1}{2}$, $i = +\frac{1}{2}$, and take the difference of the two results, it will be found equal to n, which must, therefore, represent the motion between 1700 and 1800. In like manner, by putting successively $i = -20$, $i = -19$, and taking the difference of the two results, we get $n - 39a$, for the motion in the century included between the years 250 and 150 before the Christian era. The difference of these two results $39a$, represents the augmentation of the secular motion between these two epochs; and, if this quantity were increased $173^s,2$, as in [4725′], we must increase the value of a by $\frac{1}{39} \times 173^s,2 = 4^s,4$, as in [4726]. [4726a] [4726b] [4726c]

and, if we take into view the accuracy to which the lunar tables have been carried, it must be considered useful to introduce these inequalities.

The moon's parallax, the excentricity and the inclination of the lunar orbit to the apparent ecliptic, and, in general, the coefficients of all the lunar inequalities, are likewise subjected to secular variations; but, up to the
[4731] present period, they are hardly sensible. This is the reason why we find now, the same inclination, that Ptolemy obtained from his observations; although the obliquity of the ecliptic to the equator has sensibly decreased since the time of that astronomer; so that the secular variation of the obliquity affects only the moon's declination. However, the coefficient of the annual
[4732] equation, having for a factor, the excentricity of the earth's orbit, its variation is sufficiently great to be noticed, in computing ancient eclipses.

The numerous comparisons, which Burg and Bouvard have made, of Mason's tables, with the observations of the moon; at the end of the seventeenth century, by LaHire and Flamsteed; in the middle of the eighteenth century, by Bradley; and the uninterrupted series of observations of Maskelyne, from the time of Bradley to the year 1800, give a result which was wholly
[4733] unexpected. The observations of La Hire and Flamsteed, being compared with those of Bradley, indicate a secular motion, exceeding by fifteen or twenty centesimal seconds, that which is inserted in the third edition of La Lande's astronomy; which, in a hundred Julian years, exceeds a whole number of
[4734] revolutions, by $307^{d}\,53^{m}\,12^{s}$. Bradley's observations, being compared with the last ones of Maskelyne, give, on the contrary, a smaller secular motion, by at least one hundred and fifty centesimal seconds. Lastly, the observations
[4735] made within fifteen or twenty years, prove, that the diminution of the moon's motion is now decreasing. Hence, it becomes necessary to vary incessantly the epochs of the tables; and it is an object of importance to correct this imperfection. This evidently indicates the existence of one or more unknown
[4736] inequalities of a long period, which the theory alone can point out. By a careful examination, I have not been able to discover any such inequality, depending on the action of the planets. If there were one in the rotation of the earth, it could be perceived in the moon's mean motion, and might introduce the observed anomalies: but an attentive examination of all the causes which can alter the rotation of the earth, has more fully convinced
[4737] me, that its variations are insensible. Returning back, therefore, to the

examination of the sun's action on the moon; I have discovered, that this action produces an inequality, whose argument is double the longitude of the node of the lunar orbit, *plus* the longitude of its perigee, *minus* three times the longitude of the sun's perigee. This inequality, whose period is 184 years, depends on the products of these four quantities, namely; the square [4738]
of the inclination of the moon's orbit to the ecliptic; the excentricity of that orbit; the cube of the excentricity of the sun's orbit, and the ratio of the sun's parallax to that of the moon. Hence it would seem, that it ought to [4739]
be insensible; but the small divisors it acquires by integration, may render it sensible, especially, if the most important terms, of which it is composed, are affected with the same sign. It is very difficult to obtain its coefficient by the theory, on account of the great number of terms, and the extreme [4740]
difficulty of appreciating them; the difficulty being much greater in this than in the other inequalities of the moon. This coefficient has, therefore, been ascertained by means of the observations made during the last century; and I have found it to be nearly equal to $15^{s},39$. Its introduction in the tables [4741]
must change the epoch and mean motion; and I have also found, that we must decrease, by $31^{s},964$, the mean secular motion, in the third edition of [4742]
LaLande's astronomy, and have determined the following formula, which must be applied to the mean longitude given by these tables, the epoch [4743]
of which, in 1750, is $188^{d}\,17^{m}\,14^{s},6$;

Equation of 184 years.

$$\text{Correction of moon's mean long.} = -12^{s},78 - 31^{s},964\,.\,i + 15^{s},39\,.\,\sin.E; \qquad [4744]$$

i being the number of centuries elapsed since 1750, and E the double of the longitude of the node of the lunar orbit, *plus* the longitude of its perigee, [4745]
minus three times the longitude of the sun's perigee. This formula represents, with remarkable precision, the corrections of the epochs of those tables, which have been determined, by a very great number of observations, for the six epochs of 1691, 1756, 1766, 1779, 1789 and 1801. By a most scrupulous examination of the theory, I have not been able to discover any other lunar inequality with a long period; hence, it appears to me certain, that the [4746]
anomalies observed in the mean motion of the moon, depend on the preceding inequality; and I do not hesitate, therefore, to propose it to astronomers, as the only means of correcting these anomalies.*

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* (2755) It has not been found necessary to introduce this equation in the new tables of Damoiseau, published in 1824; since the elements he has used, give very nearly the [4746a]

We see, by this exposition, how many interesting and delicate elements have been deduced, by analysis, from observations of the moon, and how
[4747] important it is to multiply and improve them. Since, by the greatness of their number, and by their correctness, we may more and more confirm the various results of analysis.

The error of the tables formed from the theory, which is given in this book, does not exceed a hundred centesimal seconds, except in very rare cases;
[4748] therefore, these tables will give, with sufficient accuracy, the longitude at sea. It is very easy to reduce them to the form of Mayer's tables; but, as in the problem of the longitude, it is proposed to find the time corresponding
[4749] to an observed longitude of the moon, there is some advantage in reducing into tables, the expression of the time in a function of the apparent longitude. Considering the extreme complication of the successive approximations, and the correctness of modern observations, the greatest part of the moon's inequalities have heretofore been better determined by observations than by analysis. Thus, by deriving from the theory those coefficients which it gives with accuracy, and also the forms of all the
[4750] arguments; then rectifying, by the comparison of a great number of observations, the coefficients which it gives by approximations, with some degree of uncertainty; we must finally obtain very accurate tables. This is the method which has been used with success by Mayer and Mason, and lately by Burg, who, by pursuing it, and profiting by the late improvements in the lunar theory, has constructed tables, whose greatest errors fall short of forty centesimal seconds. However, it would be useful, for the perfection of astronomical theories, if all the tables
[4751] could be derived solely from the principle of universal gravity; without borrowing from observation any, except the indispensable data. I am induced to believe, that the following analysis leaves but little wanting to procure this advantage to the lunar tables; and that, by carrying on farther the approximations, we may soon obtain the required degree of correctness, at least, as it respects the periodical inequalities; for, however great the accuracy of the calculations may be, the motions of the nodes and

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same mean longitudes, at the epochs 1756, 1770, 1801 and 1812, as Burckhardt has deduced from the observations made in that interval.

perigee will always be best determined by observation.* [4752]

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* (2756) Since the publication of this volume, two very important works on the lunar theory have been published; the one by Baron Damoiseau, in the first volume of the *Mémoires présentés par divers savans à l'Académie Royale des Sciences;* the other by Messrs. Plana and Carlini. We shall have occasion to speak of these works in the notes [4752a] on this book, and shall now merely remark, that the object of them is to carry on the approximation to such a degree of accuracy, as to be able to deduce all the inequalities from the theory alone.

CHAPTER I.

INTEGRATION OF THE DIFFERENTIAL EQUATIONS OF THE MOON'S MOTION.

1. Resuming the differential equations [525], we shall put them under the following forms,*

[4753] $$dt=\frac{dv}{hu^2.\sqrt{1+\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}}};$$

[4754] $$0=\left(\frac{ddu}{dv^2}+u\right).\left\{1+\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}\right\}+\frac{du}{h^2u^2.dv}.\left(\frac{dQ}{dv}\right)$$

General equations.

$$-\frac{1}{h^2}.\left(\frac{dQ}{du}\right)-\frac{s}{h^2u}.\left(\frac{dQ}{ds}\right); \qquad (L)$$

[4755] $$0=\left(\frac{dds}{dv^2}+s\right).\left\{1+\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}\right\}+\frac{1}{h^2u^2}.\frac{ds}{dv}.\left(\frac{dQ}{dv}\right)$$

$$-\frac{s}{h^2u}.\left(\frac{dQ}{du}\right)-\frac{(1+ss)}{h^2u^2}.\left(\frac{dQ}{ds}\right).$$

In these equations, t denotes the time, and we have, as in [499′, 397];

[4756] $$Q=\frac{M+m}{r}-\frac{m'.(xx'+yy'+zz')}{r'^3}+\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}.$$

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* (2757) The equation [4753] is the same as the first of [525], and if we multiply the other two equations [525] by

[4754a] $$1+\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}, \quad \text{or,} \quad \frac{1}{h^2}.\left\{h^2+2\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}\right\},$$

they will become as in [4754, 4755].

M is the mass of the earth; [4757]

m the mass of the moon;* [4757']

m' the mass of the sun; [4757'']

x, y, z, the rectangular co-ordinates of the moon, referred to the centre of gravity of the earth, and to the ecliptic of a given epoch, taken as a fixed plane; [4758] Symbols.

x', y', z', the rectangular co-ordinates of the sun, referred to the same centre and plane; [4758']

r the radius vector of the moon; [4759]

r' the radius vector of the sun; [4759']

s the tangent of the moon's latitude above the fixed plane; [4759'']

$\frac{1}{u}$ the projection of the moon's radius vector r, upon the fixed plane; [4760]

v the angle formed by this projection of r and the axis of x; [4760']

h^2 a constant quantity [518—519], depending chiefly on the moon's distance from the earth [4825, &c.]. [4760'']

In the value of Q [4756], the earth and moon are supposed to be spherical. To obtain the true value, corresponding to the actual forms of these bodies, we shall observe, that, by the properties of the centre of gravity, we must transfer to the moon's centre of gravity the following forces; *first*, all the forces by which each of its particles is urged by the action of the particles of the earth, and divide the sum by the whole of the moon's mass; *second*, the force by which the centre of gravity of the earth is urged, by the moon's action, taking it in a contrary direction. This being premised, it is evident, that dM being a particle of the earth, and dm a particle of the moon, whose distance from the particle dM is f, we shall have the forces by which the moon's centre of gravity is urged, in its relative motion about the earth, by means of the partial differentials of the double integral,† [4761] [4762] [4762']

$$\frac{(M+m)}{Mm}\cdot\iint\frac{dM.dm}{f}, \qquad [4763]$$

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* (2758) This value of m is used in the two first sections of this book; but its signification is changed in [4793], so that, in the rest of the book, mt represents the sun's mean motion. [4757a]

† (2759) If we substitute, in [455], the value of dM [452], also

taken relatively to the co-ordinates of the moon's centre. Therefore, we
[4764] must substitute this function for $\frac{M+m}{r}$, in the expression of Q [4756].
If the moon were spherical, we might suppose the whole mass to be collected in the centre of gravity [470‴]; and then, by putting V *equal to the sum of*
[4765] *the quotients, formed by dividing each particle of the earth by its distance from the moon's centre,* we shall have [4767a],

[4766] $$\iint \frac{dM \,.\, dm}{f} = m \,.\, V.$$

[4763a] $f=\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}$ [455a], it becomes, $V=\int \frac{dM}{f}$;
and then, the corresponding force of the body M on the particle dm, in the direction $-x$, will be represented by $\left(\frac{dV}{dx}\right)$ [455′]. This accelerative force, acting on the single particle dm, is to be decreased in the ratio of dm to m, to obtain the corresponding effect
[4763b] on the whole body m, of which it forms a part; by which means it becomes $\frac{dm}{m} \int \frac{dM}{f}$.
Integrating this, so as to include all the particles dm, of which the body m is composed, it becomes,

[4763b′] $$\int \frac{dm}{m} \int \frac{dM}{f}, \qquad \text{or,} \qquad \frac{1}{m} \iint \frac{dM \,.\, dm}{f},$$

which represents the value of V, to be used in finding the accelerative force of the body m, from the attraction of the body M. If we change m, M into M, m respectively, we shall get $\frac{1}{M} \iint \frac{dM \,.\, dm}{f}$, for the value of V, to be used in finding the accelerative force of the body M, from the attraction of the body m. Adding these two parts together, we
[4763c] obtain the complete value of $V=\left(\frac{1}{m}+\frac{1}{M}\right) . \iint \frac{dM \,.\, dm}{f}$, corresponding to the whole accelerative force of m towards M, supposing M to be at rest. This is easily reduced to the form [4763]; and its partial differentials, relative to the co-ordinates x, y, z, give the
[4763d] accelerative forces parallel to those co-ordinates respectively. Now, when the bodies M, m are spherical, these accelerative forces $\frac{ddx}{dt^2}$, $\frac{ddy}{dt^2}$, $\frac{ddz}{dt^2}$, are represented by the
[4763d′] partial differentials of Q, taken relatively to x, y, z [199], retaining in Q [4756] only the term $Q=\frac{M+m}{r}$, which is independent of the disturbing mass m'. Therefore,
[4763e] to notice the non-spherical forms of the bodies M, m, we have only to substitute the expression [4763], in the place of $\frac{M+m}{r}$, in the function Q [4756].

* V would be equal to $\frac{M}{r}$ if the earth were spherical; hence, if we put

$$\delta V = V - \frac{M}{r}; \qquad [4767]$$

$m.\delta V$ will be the part of the integral $\iint \frac{dM.dm}{f}$, depending on the non-sphericity of the earth. In like manner, if the earth be supposed spherical, and we put V' *equal to the sum of the quotients, formed by dividing each particle of the moon by its distance from the centre of gravity of the earth,* we shall have, [4768] [4769]

$$\iint \frac{dM.dm}{f} = M.V'; \qquad [4770]$$

and if we put

$$\delta V' = V' - \frac{m}{r}, \qquad [4770']$$

$M.\delta V'$ will be the part of the integral $\iint \frac{dM.dm}{f}$, depending on the non-sphericity of the moon; hence we shall have, very nearly,† [4771]

$$\frac{M+m}{Mm} \cdot \iint \frac{dM.dm}{f} = \frac{M+m}{r} + (M+m) \cdot \left\{ \frac{\delta V}{M} + \frac{\delta V'}{m} \right\}. \qquad [4772]$$

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* (2760) If the mass m were collected in its centre of gravity, the integral $\iint \frac{dM.dm}{f}$ would become $m \int \frac{dM}{f}$; and, by putting $\int \frac{dM}{f} = V$ [4765], it changes into $m.V$, as in [4766]. The expression [4770] is found in a similar manner. [4767a]

† (2761) If we suppose m to be spherical, we shall have

$$\iint \frac{dM.dm}{f} = m \int \frac{dM}{f}, \quad \text{as in [4767}a\text{]};$$

and if M also be spherical, [4772a]

$$\int \frac{dM}{f} = \frac{M}{r}; \quad \text{hence,} \quad \iint \frac{dM.dm}{f} = \frac{mM}{r}.$$

Adding to this the parts $m.\delta V$, $M.\delta V'$ [4768, 4771], depending on the non-sphericity, we obtain the complete value of

$$\iint \frac{dM.dm}{f} = \frac{mM}{r} + m.\delta V + M.\delta V'. \qquad [4772b]$$

Multiplying this by $\frac{M+m}{Mm}$, we obtain the value of $\frac{M+m}{Mm} \cdot \iint \frac{dM.dm}{f}$ [4772]; which

Therefore, in the preceding expression of Q [4756], we must augment the term $\frac{M+m}{r}$, by the quantity,

[4773] $$(M+m).\left\{\frac{\delta V}{M}+\frac{\delta V'}{m}\right\}= \text{increment of } Q \text{ [4756]},$$

Increment of Q, from the oblate forms of the earth and moon.

in order to notice the effect of the non-sphericity of the earth and moon.

2. *We shall, in the first place, suppose both bodies to be spherical*, and shall develop the expression of Q in a series. Now, we have,*

[4774] $$\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}=\frac{1}{\sqrt{r'^2+r^2-2xx'-2yy'-2zz'}}.$$

If we develop the second member of this expression, according to the descending powers of r', it becomes,

[4775] $$\frac{1}{r'}+\frac{(xx'+yy'+zz'-\frac{1}{2}r^2)}{r'^3}+\tfrac{3}{2}\cdot\frac{(xx'+yy'+zz'-\frac{1}{2}r^2)^2}{r'^5}$$
$$+\tfrac{5}{2}\cdot\frac{(xx'+yy'+zz'-\frac{1}{2}r^2)^3}{r'^7}+\&c.$$

[4775′] *Taking for the unit of mass the sum $M+m$ of the masses of the earth and moon*, we shall have,†

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is to be substituted for $\frac{M+m}{r}$ in the function Q [4763c,4756]; and by this means the general value of Q [4756] will be increased by the function [4773].

* (2762) The development [4774,4775], is the same as in [4655b,c], rejecting the factor $-m'$, which is common to all the terms. We may remark, that if we use the values

[4774a] of R, $M+m$ [4655,4775″], the expression of Q [4756] becomes $Q=\frac{1}{r}-R$, which will be of use hereafter.

[4776a] † (2763) If we put l for the latitude of the moon, we shall have, as in [4759″], [31′,31‴] Int.,

[4776b] $$\text{tang.}\,l=s;\qquad \sin.l=\frac{s}{\sqrt{1+ss}};\qquad \cos.l=\frac{1}{\sqrt{1+ss}}.$$

If we proceed, as in [4659,&c.], changing r' into r, and U into v, we get,

[4776c] $$x=r.\cos.l.\cos.v;\qquad y=r.\cos.l.\sin.v;\qquad z=r.\sin.l=rs.\cos.l.$$

[4776d] Now, the projection of r, upon the plane of xy, is represented by $r.\cos.l=\frac{1}{u}$ [4659a,4760];

$$1 = M + m = \mu ;$$ [4775'']

$$r = \frac{\sqrt{1+ss}}{u} ; \qquad u = \frac{\sqrt{1+ss}}{r} ;$$ [4776]

Lunar co-ordinates.

$$x = \frac{\cos. v}{u} ;$$ [4777]

$$y = \frac{\sin. v}{u} ;$$ [4778]

$$z = \frac{s}{u} .$$ [4779]

We shall *mark with one accent, for the sun, the quantities u, s and v, relative to the earth.** Then we have,† [4779']

$$Q = \frac{u}{(1+ss)^{\frac{1}{2}}} + \frac{m'.u'}{(1+s'^2)^{\frac{1}{2}}} \cdot \left\{ \begin{array}{l} 1 + \frac{3}{2} \cdot \frac{\{u u'.\cos.(v'-v) + u u'.s s' - \frac{1}{2} u'^2.(1+ss)\}^2}{(1+s'^2)^2.u^4} \\ + \frac{5}{2} \cdot \frac{\{u u'.\cos.(v'-v) + u u'.s s' - \frac{1}{2} u'^2.(1+ss)\}^3}{(1+s'^2)^3.u^6} + \&c. \\ - \frac{(1+s^2).u'^2}{2.(1+s'^2).u^2} \end{array} \right\} .$$ Value of Q. [4780]

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substituting in this the value of $\cos. l$ [4776b], we get [4776]; moreover, by substituting the value of $r.\cos. l$ [4776d] in the expressions of x, y, z [4776c], they become as in [4777—4779].

* (2764) By this means the solar co-ordinates become,

r' the radius vector of the sun; [4777a]

s' the tangent of the sun's latitude above the fixed plane; [4777b]

$\frac{1}{u'}$ the projection of the sun's radius vector upon the fixed plane; [4777c]

v' the angle formed by the projection of r' and the axis of x, or x'; [4777d]

$$r' = \frac{\sqrt{1+s's'}}{u'} ;$$ [4777e]

Solar co-ordinates.

$$x' = \frac{\cos. v'}{u'} ;$$ [4777f]

$$y' = \frac{\sin. v'}{u'} ;$$ [4777g]

$$z' = \frac{s'}{u'} .$$ [4777h]

† (2765) Substituting the value of R [4656], in [4774a], we get,

[4781] The sun's distance from the earth is nearly four hundred times as great as that of the moon; so that u' is very small, in comparison with u; and we
[4782] may, therefore, neglect terms of the order u'^5, in the lunar theory. *We may also simplify the calculations, by taking the ecliptic for the plane of projection. It is true, that this last plane is not fixed; but, in its secular motion, it carries the moon's orbit with it; so that the mean inclination of the moon's orbit,*
[4783] *upon the variable ecliptic, remains constant*, and the phenomena, depending on their respective inclinations, are always the same.

3. To prove this, we shall observe, that, from § 59, book ii., s' is equal to
[4784] a series of terms of the form $k \,.\, \sin.(v' + it + \varepsilon)$; we shall represent it by*

[4780a] $$Q = \frac{1}{r} + \frac{m'}{r'} - \frac{m'.r^2}{2r'^3} + \tfrac{3}{2}.m'.\frac{(xx'+yy'+zz'-\frac{1}{2}r^2)^2}{r'^5} + \tfrac{5}{2}.m'.\frac{(xx'+yy'+zz'-\frac{1}{2}r^2)^3}{r'^7} + \&c.$$

Now, if we substitute the values [4776—4779, 4777e—h], in the first members of [4780b,c], they become, by slight reductions and using [24] Int., the same as in the second members of those expressions,

[4780b] $$xx'+yy'+zz' = \frac{1}{uu'}.\{\cos.v.\cos.v' + \sin.v.\sin.v' + ss'\} = \frac{1}{uu'}.\{\cos.(v'-v) + ss'\};$$

[4780c] $$xx'+yy'+zz'-\tfrac{1}{2}r^2 = \frac{\cos.(v'-v)+ss'}{uu'} - \frac{\frac{1}{2}.(1+s^2)}{u^2} = \frac{uu'.\cos.(v'-v)+uu'.ss'-\frac{1}{2}u'^2.(1+ss)}{u^2u'^2}.$$

By means of these values the expression of Q [4780a] becomes as in [4780]. For the first and second terms of [4780a] correspond, respectively, to the first and second of [4780];
[4780d] the third of [4780a] gives the last of [4780]; finally, the terms of [4780a], connected with the factors $\frac{3}{2}m'$, $\frac{5}{2}m'$, by the substitution of [4780c], become respectively equal to the terms connected with the factors $\frac{3}{2}$, $\frac{5}{2}$, in [4780].

* (2766) Using the same notation as in [4230], we shall have, for the earth's latitude s'', above the fixed ecliptic, the expression,

[4785a] $$s'' = q''.\sin.v'' - p''.\cos.v'' \quad [1335'].$$

Substituting in this the values of p'', q'' [4334], and observing, that

[4785a'] $$\sin.v''.\cos.(gt+\beta) - \cos.v''.\sin.(gt+\beta) = \sin.(v''-gt-\beta),$$

we get the earth's latitude,

[4785b] $$s'' = \Sigma.c.\sin.(v''-gt-\beta).$$

Changing v'' into the sun's longitude v' [4777d], we get the sun's latitude,

[4785c] $$s' = \Sigma.c.\sin.(v'-gt-\beta).$$

This is of the same form as [4785], the constant quantities c, g, β, being changed into
[4785c'] k, $-i$, $-\varepsilon$, respectively. Hence, the coefficient i is of the same order as the quantities

$$s' = \Sigma \,.\, k \,.\, \sin.. (v' + i\,t + \varepsilon);$$ [4785]

i being a very small coefficient [4785d], whose product, by $m'u'^3$ we shall neglect. The value of s, neglecting quantities of the order s^3, may be represented by* [4785']

$$s = s_{,} + \Sigma \,.\, k \,.\, \sin. (v + i\,t + \varepsilon);$$ [4786]

$s_{,}$ being the tangent of the moon's latitude, above the apparent ecliptic. This being premised, we have,† [4786']

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g, g', &c., which are very small [4339, 3113q]. The values [4339] are nearly $g = -36^s$, $g' = -18^s$; these quantities may serve to give an idea of the magnitude of g, g', &c., though they are not computed strictly by the method given in [1098, &c.]. [4785d]

* (2767) If the moon were to move in the apparent ecliptic, her latitude above the fixed plane, or its tangent, corresponding to the longitude v, would be $\Sigma.k.\sin.(v + it + \varepsilon)$ [4785]. Adding to this the quantity $s_{,}$ [4786'], we get, very nearly, the tangent of the moon's latitude s, above the fixed plane, as in [4786]. [4786a]

† (2768) The quantity Q occurs in the first member of [4787], under a linear form only; therefore, we may take each term of Q [4780] separately, and compute its effect. In making the substitution of any term of Q, we may consider the quantity $u.(1+ss)^{-\frac{1}{2}}$, and its powers, as constant. For, if we put $Q = A.\{u.(1+ss)^{-\frac{1}{2}}\}^b$, for any term of Q, neglecting, for a moment, the variable parts contained in A, and taking the differential of $\log. Q$, we shall get, [4787a] [4787b]

$$\frac{dQ}{Q} = b \,.\, \frac{du}{u} - b \,.\, \frac{s\,ds}{1+ss};$$ [4787c]

hence,

$$\left(\frac{dQ}{dv}\right) = 0; \qquad \left(\frac{dQ}{du}\right) = \frac{1}{u} \,.\, b\,Q; \qquad \left(\frac{dQ}{ds}\right) = -\frac{s}{1+ss} \,.\, b\,Q.$$ [4787d]

Substituting these in the first member of [4787], we find, that the terms mutually destroy each other. Hence, it is evident, that we may neglect the first term of Q [4780], which corresponds to $b=1$, $A=1$; the second term, which corresponds to $b=0$, $A = \frac{m'.u'}{(1+s's')^{\frac{1}{2}}}$; and the last term, which corresponds to $b = -2$, $A = -\frac{m'.u'^3}{2.(1+s's')^{\frac{3}{2}}}$. [4787e]

Then using, for brevity, the following abridged symbol B, we get from [4780],

$$B = \frac{\{u\,u'.\cos.(v-v') + u\,u'.s\,s' - \frac{1}{2}u'^2.(1+ss)\}}{(1+s's').u^2};$$ [4787f]

$$Q = \frac{m'.u'}{(1+s's')^{\frac{1}{2}}} \,.\, \{\tfrac{3}{2}B^2 + \tfrac{5}{2}B^3 + \&c.\};$$ [4787g]

$$dQ = \frac{3\,m'.u'}{(1+s's')^{\frac{1}{2}}} \,.\, \{B + \tfrac{5}{2}B^2 + \&c.\}.dB + \&c.$$ [4787h]

[4787]
$$\frac{ds}{dv}\cdot\left(\frac{dQ}{dv}\right)-us.\left(\frac{dQ}{du}\right)-(1+ss).\left(\frac{dQ}{ds}\right)$$
$$=\frac{3m'.u'^3}{u^2}.\left\{\begin{array}{l}\cos.(v-v')-\frac{u'}{2u}\\ +\frac{5u'}{2u}.\cos.^2(v-v')+\&c.\end{array}\right\}.\left\{\begin{array}{l}s.\cos.(v-v')\\ -\frac{ds}{dv}.\sin.(v-v')-s'\end{array}\right\}.$$

[4788] Substituting, in the second member of this equation, the values of s', s, [4785, 4786], we get,*

Substituting the partial differentials of Q, in the first member of [4787], it becomes,

[4787h'] $$\frac{3m'.u'}{(1+s's')^{\frac{1}{2}}}.\{B+\tfrac{5}{2}B^2\}.\left\{\frac{ds}{dv}.\left(\frac{dB}{dv}\right)-us.\left(\frac{dB}{du}\right)-(1+ss).\left(\frac{dB}{ds}\right)\right\}.$$

The part of this expression depending on dB, in the last factor, is of the same form as the first member of [4787], changing Q into B; therefore, it has the property mentioned in
[4787i] [4787b]; that is to say, we may consider the powers of $u.(1+ss)^{-\frac{1}{2}}$ as constant. Now, the last term of B [4787f] corresponds to the power -2 of that quantity; therefore, we may neglect its partial differentials, and, in finding dB, may use the remaining terms as in the following expression;

[4787k] $$B=\frac{1}{(1+s's')}.\{u^{-1}u'.\cos.(v-v')+u^{-1}u'.ss'\}.$$

The partial differentials of this expression give,

[4787l] $$\frac{ds}{dv}.\left(\frac{dB}{dv}\right)=\frac{u'}{(1+s's').u}.\left\{-\frac{ds}{dv}.\sin.(v-v')\right\};$$

[4787m] $$-us.\left(\frac{dB}{du}\right)=\frac{u'}{(1+s's').u}.\{s.\cos.(v-v')+s^2s'\};$$

[4787n] $$-(1+ss).\left(\frac{dB}{ds}\right)=\frac{u'}{(1+s's').u}.\{-s'-s^2s'\}.$$

Adding these three expressions together, we find, that the terms depending on s^2s' destroy each other, and we get,

[4787o] $$\frac{ds}{dv}.\left(\frac{dB}{dv}\right)-us.\left(\frac{dB}{du}\right)-(1+ss).\left(\frac{dB}{ds}\right)=\frac{u'}{(1+s's')\,u}.\left\{s.\cos.(v-v')-\frac{ds}{dv}.\sin.(v-v')-s'\right\}.$$

Now, if we retain, explicitly, the terms of B [4787f], which do not contain s, s', we obtain,

[4787p] $$B+\tfrac{5}{2}B^2=\frac{u'}{u}.\left\{\cos.(v-v')-\frac{u'}{2u}+\frac{5u'}{2u}.\cos.^2(v-v')+\&c.\right\}.$$

Substituting the expressions [4787o,p] in [4787h'], and neglecting terms of the third order in s, s', it becomes as in the second member of [4787].

* (2769) If we substitute the values of s', s, [4785, 4786], in the last factor of [4787],

$$\frac{3m'.u'^3}{u^2}.\left\{\cos.(v-v')-\frac{u'}{2u}+\frac{5u'}{2u}.\cos.^2(v-v')+\&c.\right\}.\left\{s_{,}.\cos.(v-v')-\frac{ds_{,}}{dv}.\sin.(v-v')\right\}. \quad [4789]$$

Hence the equation [4755] becomes,*

$$0=\frac{dds}{dv^2}+s+\frac{\frac{3}{2}.m'.u'^3s_{,}+\&c.}{u^4.\left\{h^2+2\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}\right\}}; \quad [4790]$$

or,

$$0=\frac{dds}{dv^2}+s+\frac{\frac{3}{2}.m'.u'^3s_{,}}{h^2.u^4}+\&c. \quad [4790']$$

If we neglect the excentricities and inclinations of the orbits, we shall have $u=\frac{1}{a}$, $u'=\frac{1}{a'}$ [4826, 4833]; a' and a being the mean distances of the [4791]
sun and moon from the earth. We shall see, in the following article [4825], that $h^2=a$, very nearly; therefore, we shall have [4791d], [4791']

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we shall find, that the terms depending on k mutually destroy each other. For these terms produce, without reduction, the following expression, neglecting quantities of the order mentioned in [4785'];

$$\Sigma.k.\{\sin.(v+it+\varepsilon).\cos.(v-v')-\cos.(v+it+\varepsilon).\sin.(v-v')-\sin.(v'+it+\varepsilon)\}. \quad [4789a]$$

The two first terms, between the braces, are reduced by [22] Int. to

$$\sin.\{(v+it+\varepsilon)-(v-v')\}=\sin.(v'+it+\varepsilon); \quad [4789b]$$

which is destroyed by the third term. The remaining terms of [4785, 4786] are $s'=0$, $s=s_{,}$; substituting these in the last factor of [4787], we obtain the expression [4789]. [4789c]

* (2770) Multiplying together the two factors of [4789], we find, that the product of the term $\cos.(v-v')$ by $s_{,}.\cos.(v-v')$, produces $\frac{1}{2}s_{,}$, disconnected from the periodical [4791a]
angle $v-v'$; so that we may put the expression under the form $\frac{\frac{3}{2}m'.u'^3s_{,}+\&c.}{u^2}$; as we [4791b]
shall soon see, that it is not necessary for the present object to mention particularly the parts included in the general term $+\&c.$ This represents the value of the function in the first member of [4787], and if we divide it by h^2u^2, it produces the three last terms of [4755]; which will, therefore, be represented by $\frac{\frac{3}{2}m'.u'^3s_{,}+\&c.}{h^2u^4}$. Substituting this in [4755], [4791c]
and dividing by $1+\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$, we get [4790]. Reducing the denominator of the last term of this expression into a series; neglecting m'^2, and observing, that $\left(\frac{dQ}{dv}\right)$ [4809] is of the order $m'u'^3$, it becomes as in [4790']. Finally, substituting in [4791d]
this the values of u, u', h^2 [4791, 4791'], we get [4792].

[4792] $$0=\frac{dds}{dv^2}+s+\tfrac{3}{2}\cdot m'\cdot\frac{a^3}{a'^3}\cdot s_{,}+\&c.$$

[4793] *We shall put mt for the sun's mean motion;* so that m will no longer denote

Change in m.

[4794] the moon's mass; we shall have, by § 16 of the second book,* $m^2=\frac{m'}{a'^3}$. Then, if we suppose the time t to be represented by the moon's mean

[4795] motion, which can always be done, we shall have $\frac{1}{a^3}=1$; therefore,

[4796] $$0=\frac{dds}{dv^2}+s+\tfrac{3}{2}\cdot m^2\cdot s_{,}+\&c.$$

[4797] Substituting, in this equation, the value of s [4786], and observing, that we may, in this case, change it into iv, we shall have,†

[4798] $$0=\frac{dds_{,}}{dv^2}+(1+\tfrac{3}{2}\cdot m^2)\cdot s_{,}+\Sigma\cdot k\cdot\{1-(i+1)^2\}\cdot\sin.(v+iv+\varepsilon)+\&c.;$$

which gives, for the part of s, relative to the secular motion of the ecliptic,‡

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* (2771) If we change, in the equation [605′ or 3700], a into a', and n into m, to

[4794a] conform to the notation [4791, 4793], we get $m^2=\mu\cdot a'^{-3}$; μ being the sum of the masses of the sun and earth. If we neglect the mass of the earth, in comparison with that of the sun, we have $\mu=m'$ [4757″], and the preceding expression becomes as in [4794]. In

[4794b] the moon's motion about the earth, the equation [605′] becomes $n^2=(M+m)\cdot a^{-3}$ [4757, 4757′]; and, as the moon's mean motion nt, is here represented by t [4794], we

[4794c] have $n=1$; substituting this, and $M+m=1$ [4775″], in the preceding value of n^2, we obtain $1=a^{-3}$, as in [4795]. Dividing the value of m^2 [4794] by this last expression,

[4794d] we get $m^2=m'\cdot\frac{a^3}{a'^3}$; substituting this in [4792], it becomes as in [4796].

† (2772) The terms neglected, by writing iv for it, are of the order of the

[4798a] excentricities and inclinations, multiplied by the very small quantity i, and connected with terms containing $\sin.cv$, $\sin.gv$, and their multiples, as is evident from [4828, 4794c].

[4798b] All the neglected terms are considered as being included in the general expression $+\&c.$ Now we have,

[4798c] $s=s_{,}+\Sigma.k.\sin.(v+iv+\varepsilon)$ [4786, 4797]; hence $\frac{dds}{dv^2}=\frac{dds_{,}}{dv^2}-\Sigma.k.(i+1)^2.\sin.(v+iv+\varepsilon)$; substituting these in [4796], we get [4798].

‡ (2773) This equation is of the same form as [865], which is solved in [871]; changing y, a^2, t, m into $s_{,}$, $1+\frac{3}{2}m^2$, v, $1+i$, respectively; and putting for aQ, or

[4799a] aK, the terms under the sign Σ [4798]. These changes being made in [871], it becomes as in [4799], by a slight reduction, and changing the signs in the numerator and denominator.

$$s_{,}=\frac{\Sigma.(2i+i^2).k.\sin.(v+iv+\varepsilon)}{\frac{3}{2}.m^2-2i-i^2}.$$ [4799]

This last quantity is insensible; for iv, at the most, does not exceed fifty centesimal seconds $[=16^s,2]$ in a year;* and $\frac{3}{4}m^2v$ *expresses very nearly, as we shall hereafter see* [4800d], *the retrograde motion of the nodes*, which exceeds 19^d [3373]; therefore $\frac{3}{4}m^2$ is at least four thousand times as great as i; so that we may neglect the term, [4800] [4801]

$$\Sigma.k.\{1-(i+1)^2\}.\sin.(v+iv+\varepsilon),$$ [4802]

in the differential equation [4798]; and then this equation becomes independent of every thing connected with the secular motion of the ecliptic. The mean inclination of the moon's orbit to the apparent ecliptic, is one of the arbitrary quantities of the integral of this equation; hence we perceive, that *on account of the rapidity of the motion of the moon's nodes, this inclination is constant; and the latitude $s_{,}$ of the moon, above the apparent ecliptic, is the same as if the ecliptic were immoveable.* We may, therefore, suppose $s'=0$, in the following investigations; which will simplify the calculations. [4803] [4804]

Inclination of the lunar orbit to the apparent ecliptic.

Therefore, we have, by neglecting quantities of the order $m'u'^3s^4$, $m'u'^5$,† [4805]

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* (2774) This agrees nearly with the remarks made in [4785d], relative to the value of i. Moreover, the retrograde motion of the nodes is expressed by $(g-1).v$ [4817], and the values of m, g [5117], give $g-1=\frac{3}{4}m^2$ nearly; therefore, the retrograde motion of the nodes is nearly equal to $\frac{3}{4}m^2.v$, as in [4800]. The same result may be obtained analytically; for, if we neglect terms of the order p''^2, e'^2, the motion of the nodes [5059] becomes $\frac{1}{2}p''.v$. Now, by comparing the coefficients of $\sin.(gv-\theta)$, in [5053, 5049], and retaining only the first term of each of them, we get, [4800a] [4800b]

$$p''=\tfrac{3}{2}\bar{m}^2.\frac{a}{a_{,}}=\tfrac{3}{2}m^2 \quad [5094];$$ [4800c]

whence, the motion of the nodes becomes $\frac{1}{2}p''.v=\frac{3}{4}m^2.v$. This exceeds 19^d in a year [3373]; which is more than 4000 times the value of iv, assumed in [4785d]; hence the term of $s_{,}$ [4799] must be insensible, and we may, therefore, neglect the corresponding terms of [4798], which are given in [4802]. Then all the remaining terms of [4798], which are included in the expression $+$&c. [4798b], may be considered as independent of the secular terms arising from i. [4800d] [4800e]

† (2775) Substituting $s'=0$ [4804] in the value of Q [4780], it becomes, without any reduction, as in [4806a]. Developing the powers, and neglecting terms of the orders mentioned in [4805], it becomes as in [4806b]. This is reduced to the form [4806c] by [4800f]

[4806]
$$Q=\frac{u}{(1+s^2)^{\frac{1}{2}}}+m'u'+\frac{m'.u'^3}{4u^2}.\{1+3.\cos.(2v-2v')-2s^2\}$$
$$+\frac{m'.u'^4}{8u^3}.\{3.(1-4s^2).\cos.(v-v')+5.\cos.(3v-3v')\}.$$

[4807] Hence we get, by neglecting quantities of the order $m'u'^4s^3$,*

[4808]
$$\left(\frac{dQ}{du}\right)+\frac{s}{u}.\left(\frac{dQ}{ds}\right)=\frac{1}{(1+s^2)^{\frac{3}{2}}}-\frac{m'.u'^3}{2u^3}.\{1+3.\cos.(2v-2v')\}$$
$$-\frac{3m'.u'^4}{8u^4}.\{(3-4s^2).\cos.(v-v')+5.\cos.(3v-3v')\};$$

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using [6, 7] Int.; and if we connect the terms depending on the same powers of u' it becomes as in [4806];

[4806a]
$$Q=\frac{u}{(1+s^2)^{\frac{1}{2}}}+m'u'.\left\{\begin{array}{l}1+\frac{3}{2u^4}.[uu'.\cos.(v-v')-\tfrac{1}{2}u'^2.(1+ss)]^2\\ +\frac{5}{2u^6}.[uu'.\cos.(v-v')-\tfrac{1}{2}u'^2.(1+ss)]^3+\&c.-\frac{(1+ss).u'^2}{2u^2}\end{array}\right\}$$

[4806b]
$$=\frac{u}{(1+s^2)^{\frac{1}{2}}}+m'u'.\left\{\begin{array}{l}1+\frac{3u'^2}{2u^2}.\cos.^2(v-v')-\frac{3u'^3}{2u^3}.(1+ss).\cos.(v-v')\\ +\frac{5u'^3}{2u^3}.\cos.^3(v-v')-\frac{(1+ss).u'^2}{2u^2}\end{array}\right\}$$

[4806c]
$$=\frac{u}{(1+s^2)^{\frac{1}{2}}}+m'u'.\left\{\begin{array}{l}1+\frac{3u'^2}{2u^2}.[\tfrac{1}{2}+\tfrac{1}{2}\cos.2(v-v')]-\frac{3u'^3}{2u^3}.(1+ss).\cos.(v-v')\\ +\frac{5u'^3}{2u^3}.[\tfrac{3}{4}\cos.(v-v')+\tfrac{1}{4}\cos.3(v-v')]-\frac{(1+ss).u'^2}{2u^2}\end{array}\right\}.$$

* (2776) The partial differentials of Q [4806], taken relatively to v, s, u, become, without any reduction, as in [4809, 4810, 4810a], respectively. Multiplying [4810] by $\frac{s}{u}$, we get [4810b]; adding together the expressions [4810a, b], and making some slight reductions, we get [4808];

[4810a]
$$\left(\frac{dQ}{du}\right)=\left\{\begin{array}{l}\frac{1}{(1+ss)^{\frac{1}{2}}}-\frac{m'.u'^3}{2u^3}.[1+3.\cos.(2v-2v')-2s^2]\\ -\frac{3m'.u'^4}{8u^4}.[(3-12s^2).\cos.(v-v')+5\cos.(3v-3v')]\end{array}\right\};$$

[4810b]
$$\frac{s}{u}.\left(\frac{dQ}{ds}\right)=-\frac{ss}{(1+ss)^{\frac{3}{2}}}-\frac{m'.u'^3s^2}{u^3}-\frac{3m'.u'^4s^2}{u^4}.\cos.(v-v').$$

$$\left(\frac{dQ}{dv}\right)=-\frac{3m'.u'^3}{2u^2}.\sin.(2v-2v')$$
$$-\frac{m'.u'^4}{8u^3}.\{3.(1-4s^2).\sin.(v-v')+15.\sin.(3v-3v')\};$$ [4809]

$$\left(\frac{dQ}{ds}\right)=-\frac{us}{(1+ss)^{\frac{3}{2}}}-\frac{m'.u'^3s}{u^2}-\frac{3m'.u'^4s}{u^3}.\cos.(v-v').$$ [4810]

4. *To integrate the equations* [4753—4755], we shall observe, that, by excluding the sun's disturbing force, the moon will describe an ellipsis, in which the earth occupies one of the foci. We shall then have, as in [532,533], [4810′]

$$s=\gamma.\sin.(v-\theta);$$ [4811]
$$u=\frac{1}{h^2.(1+\gamma^2)}.\{(1+ss)^{\frac{1}{2}}+e.\cos.(v-\varpi)\}.$$ [4812]

Values of s, u, in an invariable ellipsis.

In these equations, γ *is the tangent of the inclination of the lunar orbit;* [4813] θ *the longitude of its ascending node* [533″]; *e and* ϖ *are two arbitrary quantities, depending chiefly, on the excentricity of the orbit, and on the* [4814] *position of the perihelion* [534′]. γ and e are very small quantities. If we [4815] neglect the fourth power of γ, we shall have,*

$$u=\frac{1}{h^2.(1+\gamma^2)}.\{1+\tfrac{1}{4}\gamma^2+e.\cos.(v-\varpi)-\tfrac{1}{4}\gamma^2.\cos.(2v-2\theta)\}.$$ [4816]

In this value of u the ellipse is supposed to be immoveable; but *we shall soon see, that in consequence of the sun's action, the nodes and perigee of this ellipsis are in motion.* Then putting,

$(1-c).v=$ the direct motion of the perigee;
$(g-1).v=$ the retrograde motion of the nodes; [4817]

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* (2777) Developing $(1+ss)^{\frac{1}{2}}$, according to the powers of s, substituting [4811], neglecting s^6, and reducing, by means of [1, 3] Int., we get, successively,

$$(1+ss)^{\frac{1}{2}}=1+\tfrac{1}{2}s^2-\tfrac{1}{8}s^4$$
$$=1+\frac{\gamma^2}{2}.\{\tfrac{1}{2}-\tfrac{1}{2}\cos.(2v-2\theta)\}-\frac{\gamma^4}{8}.\{\tfrac{3}{8}-\tfrac{4}{8}.\cos.(2v-2\theta)+\tfrac{1}{8}\cos.(4v-4\theta)\}$$ [4812a]
$$=(1+\tfrac{1}{4}\gamma^2-\tfrac{3}{64}\gamma^4)-(\tfrac{1}{4}\gamma^2-\tfrac{1}{16}\gamma^4).\cos.(2v-2\theta)-\tfrac{1}{64}\gamma^4.\cos.(4v-4\theta).$$

Substituting this in [4812], and neglecting γ^4, it becomes as in [4816]. We have retained the terms of the order γ^4, in [4812a], because they are required hereafter. [4812b]

we shall have, from [4811,4816],*

[4818] $$s=\gamma.\sin.(gv-\theta);$$

[4819] $$u=\frac{1}{h^2.(1+\gamma^2)}.\{1+\tfrac{1}{4}\gamma^2+e.\cos.(cv-\varpi)-\tfrac{1}{4}\gamma^2.\cos.(2gv-2\theta)\}.$$

Assumed forms of s, u, in a moveable ellipsis.

If we substitute this value of u, in the expression of dt [4753], observing,

[4820] that, if we neglect the solar attraction, $\left(\frac{dQ}{dv}\right)$ vanishes; we shall have,

[4821] $$dt=h^3.dv.\left\{\begin{array}{l}1+\frac{3}{2}.(e^2+\gamma^2)-2e.(1+\frac{3}{2}e^2+\frac{5}{4}\gamma^2).\cos.(cv-\varpi)\\ +\frac{3}{2}.e^2.\cos.(2cv-2\varpi)-e^3.\cos.(3cv-3\varpi)+\frac{1}{2}\gamma^2.\cos.(2gv-2\theta)\\ -\frac{3}{4}.e\gamma^2.\{\cos.(2gv+cv-2\theta-\varpi)+\cos.(2gv-cv-2\theta+\varpi)\}\end{array}\right\}.$$

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* (2778) The object of this article is to obtain approximate values of u, u', s, v',

[4821a] expressed in terms of v; for the purpose of substituting them in Q, and in its differentials; as is observed in [4838']. Now, s, u [4818, 4819], are the approximate values of s, u, corresponding to the equations [4755,4754], noticing two of the most important perturbations,

[4821b] namely; the mean motions of the perigee and nodes. Substituting these in [4753], we get the approximate values of dt, t [4821, 4822], which are afterwards corrected in [5081,5095]. In finding the approximate value of dt [4821], from [4753], the term

[4821c] $\left(\frac{dQ}{dv}\right)$ is neglected, and then [4753] becomes $dt=\frac{dv}{hu^2}$; in which we must substitute the

[4821d] value of u [4819]. In making these substitutions, we shall put for a moment, for brevity,

[4821e] $f=\frac{1}{4}\gamma^2-\frac{1}{4}\gamma^2.\cos.(2gv-2\theta)$; and, *during the process of the calculation, we shall omit the symbols* θ, ϖ, ϖ', *which are connected respectively with the angles* $gv-\theta$, $cv-\varpi$,

[4821f] $c'v'-\varpi'$, $c'mv-\varpi'$; *taking care to re-substitute them at the end of the operation. This abridged form of writing the angles, will be used frequently, in the notes which follow;* it saves considerable labor, renders the formulas more simple, and cannot be attended with any inconvenience. Hence, the preceding expression of dt [4821d] becomes as in [4821h];

[4821g] developing the factors, and neglecting f^2, fe^2, e^4, γ^4, &c., we get successively [4821i,k,l]. Substituting the value of f [4821e], and reducing, by means of [6, 7, 20] Int., we get [4821m]: connecting together the terms depending on the same angles, we obtain [4821]; whose integral is as in [4822]:

[4821h] $$dt=h^3.(1+\gamma^2)^2.dv.\{1+(f+e.\cos.cv)\}^{-2}$$

[4821i] $$=h^3.(1+2\gamma^2).dv.\{1-2(f+e.\cos.cv)+3(f+e.\cos.cv)^2-4(f+e.\cos.cv)^3\}$$

[4821k] $$=h^3.(1+2\gamma^2).dv.\{1-2e.\cos.cv+3e^2.\cos.^2cv-4e^3.\cos.^3cv-2f+6fe.\cos.cv\}$$

[4821l] $$=h^3.dv.\{(1+2\gamma^2)-2e(1+2\gamma^2).\cos.cv+3e^2.\cos.^2cv-4e^3.\cos.^3cv-2f+6fe.\cos.cv\}$$

[4821m] $$=h^3.dv.\left\{\begin{array}{l}(1+2\gamma^2)-2e(1+2\gamma^2).\cos.cv+\frac{3}{2}e^2.(1+\cos.2cv)-e^3.(3\cos.cv+\cos.3cv)\\ -\frac{1}{2}\gamma^2+\frac{1}{2}\gamma^2.\cos.2gv+\frac{3}{2}e\gamma^2.\cos.cv-\frac{3}{4}e\gamma^2.[\cos.(2gv+cv)+\cos.(2gv-cv)]\end{array}\right\}.$$

This gives, by integration,

$$t = \text{constant} + h^3 v.(1+\tfrac{3}{2}e^2+\tfrac{3}{2}\gamma^2) - \frac{2h^3 e}{c}.(1+\tfrac{3}{2}e^2+\tfrac{5}{4}\gamma^2).\sin.(cv-\varpi)$$
$$+\frac{3h^3.e^2}{4c}.\sin.(2cv-2\varpi) - \frac{h^3.e^3}{3c}.\sin.(3cv-3\varpi) + \frac{h^3.\gamma^2}{4g}.\sin.(2gv-2\theta) \qquad [4822]$$
$$-\frac{3h^3.e\gamma^2}{4.(2g+c)}.\sin.(2gv+cv-2\theta-\varpi) - \frac{3h^3.e\gamma^2}{4.(2g-c)}.\sin.(2gv-cv-2\theta+\varpi);$$

the coefficients of this equation are modified a little by the sun's action, as we shall hereafter see [5081, 5095].

In the elliptical hypothesis, the coefficient of v, in this expression, is, by [4822′]
[541′—543], equal to $a^{\frac{3}{2}}$; which gives,*

$$h^3.(1+\tfrac{3}{2}e^2+\tfrac{3}{2}\gamma^2) = a^{\frac{3}{2}}; \qquad [4823]$$

a being the semi-major axis of the ellipsis; hence we have, [4824]

$$h = a^{\frac{1}{2}}.(1-\tfrac{1}{2}e^2-\tfrac{1}{2}\gamma^2); \qquad [4825]$$

consequently,

$$u = \frac{1}{a}.\{1+e^2+\tfrac{1}{4}\gamma^2+e.(1+ee).\cos.(cv-\varpi)-\tfrac{1}{4}\gamma^2.\cos.(2gv-2\theta)\}. \qquad [4826]$$

Then, by putting $n = a^{-\frac{3}{2}}$ [4823a], we get,† [4827]

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* (2779) Substituting $\mu=1$ [4775″], in [541′], we get $n=a^{-\frac{3}{2}}$; hence [543] gives $t+a^{\frac{3}{2}}\varepsilon = a^{\frac{3}{2}}v+\&c.$; in which the coefficient of v is $a^{\frac{3}{2}}$. To make this conform to the [4823a] result of the elliptical theory [4822], we must put the coefficients of v equal to each other; hence we get [4823]. Dividing this equation by the coefficient of h^3, and taking the [4823b] cube root, we obtain h [4825], neglecting terms of the fourth order in e, γ. This value of h gives $h^2.(1+\gamma^2) = a.(1-e^2)$; whence, $\frac{1}{h^2.(1+\gamma^2)} = \frac{1}{a}.(1+e^2)$; substituting [4823c] this in [4819], we get [4826].

† (2780) Multiplying [4823] by $1-\frac{1}{4}\gamma^2$, and neglecting γ^4, we get

$$h^3.(1+\tfrac{3}{2}e^2+\tfrac{5}{4}\gamma^2) = a^{\frac{3}{2}}.(1-\tfrac{1}{4}\gamma^2); \qquad [4828a]$$

substituting this in the third term of the second member of [4822]; also [4823], in the second term, and putting the constant quantity equal to $-a^{\frac{3}{2}}\varepsilon$; we shall obtain for these

[4828] $$nt+\varepsilon = v - \frac{2e}{c}\cdot(1-\tfrac{1}{4}\gamma^2)\cdot\sin.(cv-\varpi) + \frac{3ee}{4c}\cdot\sin.(2cv-2\varpi)$$

Approximate value of $nt+\varepsilon$.

$$-\frac{e^3}{3c}\cdot\sin.(3cv-3\varpi) + \frac{\gamma^2}{4g}\cdot\sin.(2gv-2\theta)$$

$$-\frac{3e\gamma^2}{4.(2g+c)}\cdot\sin.(2gv+cv-2\theta-\varpi) - \frac{3e\gamma^2}{4.(2g-c)}\cdot\sin.(2gv-cv-2\theta+\varpi);$$

[4829] ε being an arbitrary constant quantity. *In substituting* $nt+\varepsilon$, *we may suppose* c *and* g *to be equal to unity* [5117], and neglect quantities of the order e^3, or $e\gamma^2$, in the coefficients of the sines. Thus we shall have, by retaining the term depending on $\sin.(2gv-cv-2\theta+\varpi)$, which will be useful hereafter [4828d];

[4830] $$nt+\varepsilon = v-2e\,.\sin.(cv-\varpi)+\tfrac{3}{4}e^2.\sin.(2cv-2\varpi)+\tfrac{1}{4}\gamma^2.\sin.(2gv-2\theta)$$

$$-\tfrac{3}{4}e\gamma^2.\sin.(2gv-cv-2\theta+\varpi).$$

[4831] If we mark with one accent for the sun, the symbols relative to the moon [4779′], and observe, that $\gamma'=0$ [4804], we shall have,*

Approximate values of t, u'.

[4832] $$n't+\varepsilon' = v'-2e'.\sin.(c'v'-\varpi')+\tfrac{3}{4}e'^2.\sin.(2c'v'-2\varpi');$$

[4833] $$u' = \frac{1}{a'}\cdot\{1+e'^2+e'.(1+e'^2).\cos.(c'v'-\varpi')\}.$$

[4834] The origin of the time t being arbitrary, we may suppose ε and ε' nothing,

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[4828b] three terms, the expression $-a^{\frac{3}{2}}\varepsilon+a^{\frac{3}{2}}v-\frac{2e}{c}.a^{\frac{3}{2}}.(1-\frac{1}{4}\gamma^2).\sin.(cv-\varpi)$. Substituting this in [4822], then multiplying the first member by n, and the second by its equivalent expression $a^{-\frac{3}{2}}$ [4823a], it becomes, by slight reductions, as in [4828]; observing, that, in the second

[4828c] and third lines of [4822], we may put $h^3a^{-\frac{3}{2}}=1$ [4823], since these terms are of the second or third orders in e, γ. Now, putting c and g equal to unity, in the coefficients of [4828], and retaining terms of the second order in e, γ, also the term depending on the

[4828d] angle $2gv-cv$, we get [4830]. The reason for retaining this term, is on account of the smallness of the divisors introduced by it, in consequence of $2g-c$ being very nearly equal to unity. For the values of c, g, m [5117], give very nearly,

[4828e] $$c=1-\tfrac{3}{2}m^2,\qquad g=1+\tfrac{3}{4}m^2,\qquad 2g-c=1+3m^2.$$

* (2781) The values [4832, 4833], relative to the sun, are deduced from those of the

[4832a] moon [4830, 4826], by merely accenting the symbols, as in [4779′]; observing also, that $s'=0$ [4804], corresponds to $\gamma'=0$ [4818].

and then putting $\frac{n'}{n}=m$, the comparison of the values of nt and $n't$ will [4835]
give,*

$$\begin{aligned} & v'-2e'.\sin.(c'v'-\varpi')+\tfrac{3}{4}e'^2.\sin.2(c'v'-\varpi') \\ &= mv-2mc.\sin.(cv-\varpi)+\tfrac{3}{4}me^2.\sin.(2cv-2\varpi) \\ &\quad +\tfrac{1}{4}m.\gamma^2.\sin.(2gv-2\theta)-\tfrac{3}{4}mc\gamma^2.\sin.(2gv-cv-2\theta+\varpi). \end{aligned} \quad [4836]$$

Hence we deduce, by observing, that c' varies but very little from unity,† [4836']

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* (2782) If we take, for the origin of t, the moment when the bodies are in their mean conjunction, or $nt+\varepsilon$ equal to $n't+\varepsilon'$, we shall have $\varepsilon=\varepsilon'=0$. Substituting these in [4830, 4832], we get the values of nt, $n't$. Multiplying the former by m, and substituting $mn=n'$ [4835], we get an expression of $n't$, which is to be put equal to that in [4832]; hence we get [4836]. [4834a] [4834b]

† (2783) We may obtain v' from [4836], by means of the theorem of La Grange [629c], which, by changing ψx into x, then x into v' and t into t, becomes,

$$v'-F(v')=\mathrm{t}; \quad [4837a]$$

$$v'=\mathrm{t}+F(\mathrm{t})+\tfrac{1}{2}.\frac{d.F(\mathrm{t})^2}{d\mathrm{t}}+\tfrac{1}{6}.\frac{d^2.F(\mathrm{t})^3}{d\mathrm{t}^2}+\&c. \quad [4837b]$$

Comparing the equations [4836, 4837a], we find, that t represents the second member of the equation [4836], and, that

$$F(v')=2e'.\sin.(c'v'-\varpi')-\tfrac{3}{4}e'^2.\sin.(2c'v'-2\varpi'). \quad [4837c]$$

Changing v' into t, we get $F(\mathrm{t})$ [4837e], its powers [4837f], and the differentials [4837g], omitting, for brevity, the symbol $-\varpi'$, which is connected with $c'\mathrm{t}$; the reductions being made by means of [1, 2, 17] Int. Substituting these in the second member of [4837b], we get v' [4837h]; [4837d]

$$F(\mathrm{t})=2e'.\sin.(c'\mathrm{t}-\varpi')-\tfrac{3}{4}e'^2.\sin.(2c'\mathrm{t}-2\varpi')+\&c.; \quad [4837e]$$

$$F(\mathrm{t})^2=2e'^2.(1-\cos.2c'\mathrm{t})-\tfrac{3}{2}e'^3.\cos.c'\mathrm{t}+\&c.; \qquad F(\mathrm{t})^3=6e'^3.\sin.e'\mathrm{t}+\&c.; \quad [4837f]$$

$$\tfrac{1}{2}.\frac{d.F(\mathrm{t})^2}{d\mathrm{t}}=2e'^2.\sin.2c'\mathrm{t}+\tfrac{3}{4}e'^3.\sin.c'\mathrm{t}+\&c.; \qquad \tfrac{1}{6}.\frac{d^2.F(\mathrm{t})^3}{d\mathrm{t}^2}=-e'^3.\sin.c'\mathrm{t}+\&c.; \quad [4837g]$$

$$v'=\mathrm{t}+(2e'-\tfrac{1}{4}e'^3).\sin.(c'\mathrm{t}-\varpi')+\tfrac{5}{4}e'^2.\sin.(2c'\mathrm{t}-2\varpi'). \quad [4837h]$$

Now, t represents the second member of [4836], and the substitution of this value in the first term of [4837h] produces the four first terms, or the two first lines of the second member of [4837]. The last term of [4837h] produces the last term of [4837], by putting for t the first term mv of the second member of [4836]; it being unnecessary to take any other term of t, because m is of the same order as e, or e'. To obtain the value of the second term of v' [4837h], we must have the expression of $\sin.(c'\mathrm{t}-\varpi')$. Now, as this [4837i] [4837k]

[4837] Approximate values of v', u'.

$$v'=mv-2me.\sin.(cv-\varpi)+\tfrac{3}{4}me^2.\sin.(2cv-2\varpi)$$
$$+\tfrac{1}{4}m\gamma^2.\sin.(2gv-2\theta)-\tfrac{3}{4}me\gamma^2.\sin.(2gv-cv-2\theta+\varpi)$$
$$+2e'.(1-\tfrac{1}{8}e'^2).\sin.(c'mv-\varpi')-2mce'.\sin.(cv+c'mv-\varpi-\varpi')$$
$$-2mce'.\sin.(cv-c'mv-\varpi+\varpi')+\tfrac{5}{4}e'^2.\sin.(2c'mv-2\varpi')$$

[4838] $$u'=\frac{1}{a'}.\left\{\begin{array}{l}1+e'.(1-\frac{1}{8}e'^2).\cos.(c'mv-\varpi')+e'^2.\cos.(2c'mv-2\varpi')\\ +mee'.\cos.(cv-c'mv-\varpi+\varpi')-mee'.\cos.(cv+c'mv-\varpi-\varpi')\end{array}\right\}.\text{*}$$

5. *We must substitute these values of* u, u', s and v', *in the expression*
[4838'] *of* Q [4806], and of its partial differentials [4808—4810], which will, by this means, be developed in sines and cosines of angles proportional to v; but it is necessary, for this development, to establish some principles relative to

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term is of the order e', it will be sufficient to take the two first terms of [4836], namely;

[4837l] $$\mathrm{t}=mv-2me.\sin.(cv-\varpi);\ \text{whence,}\ c'\mathrm{t}-\varpi'=(c'mv-\varpi')-2c'me.\sin.(cv-\varpi).$$

Developing the sine of this expression, by means of [60, 18] Int., neglecting e^2, we get, successively,

[4837m] $$\sin.(c'\mathrm{t}-\varpi')=\sin.(c'mv-\varpi')-2c'me.\sin.(cv-\varpi).\cos.(c'mv-\varpi')$$

[4837n] $$=\sin.(c'mv-\varpi')-c'me.\sin.(cv+c'mv-\varpi-\varpi')-c'me.\sin.(cv-c'mv-\varpi+\varpi').$$

Multiplying this by its coefficient $2e'-\frac{1}{4}e'^3$, or $2e'.(1-\frac{1}{8}e'^2)$, neglecting terms of the fourth order, and putting $c'=1$, we get the sixth, seventh and eighth terms of [4837].

* (2784) To obtain u', we must substitute the value of v' [4837] in [4833]; and, as we retain terms of the third order in e, e', γ', m, in [4838], it is necessary to retain those of the second order in v' [4837]. Hence, if we put for a moment, for brevity,

[4838a] $$z=2e'.\sin.(c'mv-\varpi')+\tfrac{5}{4}e'^2.\sin.(2c'mv-2\varpi')-2me.\sin.(cv-\varpi);$$

and observe, that c' is very nearly equal to unity, we shall have, from [4837],

[4838b] $$v'=mv+z,\quad\text{and}\quad c'v'-\varpi'=(c'mv-\varpi')+z.$$

Its cosine, reduced by formulas [23, 43, 44] Int., becomes, by neglecting z^3,

[4838c] $$\cos.(c'v'-\varpi')=\cos.z.\cos.(c'mv-\varpi')-\sin.z.\sin.(c'mv-\varpi')$$

[4838d] $$=(1-\tfrac{1}{2}z^2).\cos.(c'mv-\varpi')-z.\sin.(c'mv-\varpi');$$

hence,

$$e'.(1+e'^2).\cos.(c'v-\varpi')$$

[4838e] $$=e'.(1+e'^2).\cos.(c'mv-\varpi')-\tfrac{1}{2}e'z^2.\cos.(c'mv-\varpi')-e'z.\sin.(c'mv-\varpi').$$

Now, substituting the value of z [4838a], in the first members of [4838g, h], neglecting

the magnitudes of the quantities which enter into these functions, and on the influence of the successive integrations upon the different terms. [4839]

The value of m [5117] *is very nearly equal to the fraction* $\frac{1}{13}$; *we shall consider it as a very small quantity of the first order. The excentricities of the orbits of the sun and moon, and the inclination of the lunar orbit to the ecliptic, are nearly of the same degree of smallness* [5117, 5194]. *Thus, we shall regard the squares and products of these quantities, as very small quantities of the second order; their cubes and products of three dimensions, as very small quantities of the third order; and so on for others.* The sun's disturbing force is of the order* $\frac{m'.u'^3}{u^3}$, and we have seen, in § 3, that this quantity is of the order m^2, or of the second order. The fraction $\frac{a}{a'}$, being very nearly equal to $\frac{1}{400}$, may be considered as of the second order. *We shall carry on the approximation to quantities of the third order inclusively;* [4840] Orders of the terms. [4841] [4842] [4843]

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terms of the fourth order, also those depending on the angle $3c'mv-3\varpi'$, we get, successively, by using [31, 17, 2] Int., the following expressions; omitting, for brevity, the symbols ϖ, ϖ', as in [4821f]; [4838f]

$$\begin{aligned}-\tfrac{1}{2}e'z^2.\cos.(c'mv-\varpi') &= -e'^3.(2\sin.c'mv.\cos.c'mv).\sin.c'mv \\ &= -e'^3.\sin.2c'mv.\sin.c'mv = -\tfrac{1}{2}e'^3.\cos.c'mv;\end{aligned} \qquad [4838g]$$

$$\begin{aligned}-e'z.\sin.(c'mv-\varpi') &= -e'^2.(1-\cos.2c'mv)-\tfrac{5}{8}e'^3.\cos.c'mv \\ &\quad +mce'.\cos.(cv-c'mv)-mce'.\cos.(cv+c'mv).\end{aligned} \qquad [4838h]$$

Substituting [4838g, h] in [4838e], we get, by connecting the terms,

$$\begin{aligned}e'.(1+e'^2).\cos.(c'v-\varpi') &= -e'^2+e'.(1-\tfrac{1}{8}e'^2).\cos.c'mv+e'^2.\cos.2c'mv \\ &\quad +mce'.\cos.(cv-c'mv)-mce'.\cos.(cv+c'mv).\end{aligned} \qquad [4838i]$$

Finally, by the substitution of this, in [4833], we get [4838].

* (2785) The accelerative forces [4763d'], are represented by the partial differentials of Q, relative to the co-ordinates. These partial differentials occur in the general equations [4753—4755], and are computed in [4807—4810]. Now, if we compare the part of [4808 or 4810], which does not contain the disturbing mass m', with the chief term of the same equation, depending on this disturbing mass, we shall find, that it is of the order $\frac{m'.u'^3}{u^3}$, or $\frac{m'a^3}{a'^3}$ [4791]; which, by means of [4794, 4795], is of the order m^2. [4842a] [4842b]

and in the calculation of these inequalities, we shall take notice of quantities
[4844] *of the fourth order;** but we must take particular care not to omit any quantities of that order in the integrals.

The equation [4754] becomes, by development, of the following form,†

[4845] $$0=\frac{ddu}{dv^2}+N^2.u+\Pi;$$

[4845′] N^2 differs from unity but by a quantity of the order m^2 [4845c], and Π is a
[4846] series of cosines, of the form $k.\cos.(iv+\varepsilon)$ [4961]. The part of u. relative to this cosine, is represented, as in [870′, 871], by

[4847] $$u=\frac{k}{i^2-N^2}.\cos.(iv+\varepsilon).$$

Now, it is evident, that if i^2 differs from unity by a quantity of the order m,
[4848] the term $k.\cos.(iv+\varepsilon)$ acquires, by integration, a divisor of that order: which increases the term considerably; so that it will become of the order
[4849] $r-1$, if it be of the order r, in the differential equation. We shall see
[4850] hereafter, that the greatness of the inequality named the *evection*, arises from this cause.‡

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* (2786) The angles connected with coefficients, as far as the third order inclusively,
[4844a] are retained; and, in computing the coefficients of these terms, the approximation is carried on, so as to include terms of the fourth order.

† (2787) The chief inequality of u [4819], is that depending on $\cos.(cv-\varpi)$, which we shall represent by $\mathrm{e}.\cos.(cv-\varpi)$; putting the other terms equal to δu, so that
[4845a] $u=\mathrm{e}.\cos.(cv-\varpi)+\delta u$. Its differential gives $\frac{ddu}{dt^2}=-c^2\,\mathrm{e}.\cos.(cv-\varpi)+\frac{d^2.\delta u}{dt^2}$.
Multiplying the first equation by c^2, and adding the product to the second equation, we get,

[4845b] $$\frac{ddu}{dt^2}+c^2u=\frac{d^2.\delta u}{dt^2}+c^2.\delta u.$$

Putting the second member of this last equation equal to $-\Pi$, we get,

[4845c] $$\frac{ddu}{dt^2}+c^2u=-\Pi;$$

and this is of the same form as [4845]; N^2 being changed into c^2, which differs from unity by a quantity of the order $3m^2$ [4828e].

‡ (2788) The evection depends on the angle $2v-2mv-cv+\varpi$, and its cosine is multiplied by $A_1^{(1)}e$, in the expression of δu [4901]. Now, in finding $A_1^{(1)}$, from the
[4850a] equation [4999], we must divide by the factor $1-(2-2m-c)^2$, which is of the order m; and by this division its value is very much increased.

The terms where i is very small, and which depend only on the sun's motion, do not increase, by integration, in the value of u;* but, it is evident, from the equation [4753], that these terms acquire, by integration, the divisor i, in the expression of t;† we must, therefore, pay great attention to these terms. It is on them, that the magnitude of the *annual equation* depends. [4850′] [4850″] [4851]

The terms of the form $k.dv.\sin(iv+\varepsilon)$, in the expression of $\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$ [4753, 4754] acquire, by the integration of that differential expression, a divisor of the order i, in the value of u. Hence, it would seem, that in the expression of the time t, these terms ought to acquire a divisor of the order i^2, which would render them very great when i is very small; but, it is essential to observe, that this is not the case, and that, *if we only notice the first power of the disturbing force*, these terms will not have the divisor i^2, in the expression of the time. To prove this, we shall observe, that by [1195, &c.], the expression of r, in a function of the time, cannot acquire a divisor of the order i^2, except by means of the function $-3a.\int ndt.\int \mathrm{d}Q$:‡ in which the [4852] [4853] [4853′] [4854]

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* (2789) When i is very small, the divisor i^2-N^2 [4817] becomes nearly equal to $-N^2$, which is of the order -1 [4815′]; consequently, the term [4817] is not increased by this division. [4850b]

† (2790) If the development of the denominator of dt [4753] contain a term of the form $k.\cos.(iv+\varepsilon)$, arising from u^2, it would introduce in dt a term of the form $k.dv.\cos.(iv+\varepsilon)$; whose integral would introduce in t a term of the form $\frac{k}{i}.\sin.(iv+\varepsilon)$, having the small divisor i, as in [4851]. [4851a]

‡ (2791) The differential of Q [1774a], relative to the characteristic d, gives,

$$\mathrm{d}R=-\frac{dr}{r^2}-\mathrm{d}Q;\quad \text{hence}\quad \int \mathrm{d}R=\frac{1}{r}-\int \mathrm{d}Q. \qquad [4854a]$$

Substituting this, and $\mu=1$ [1775″] in ζ [1195], we get,

$$\zeta=3a.\int ndt.\frac{1}{r}-3a.\int ndt.\int \mathrm{d}Q. \qquad [4854b]$$

Now, the first term of this expression has only *one* sign of integration, and can, therefore, introduce only the first power of the divisor i [1196′, &c.]; and, if we neglect this term, we shall have,

$$\zeta=-3a.\int ndt.\int \mathrm{d}Q,\quad \text{as in [4854]}. \qquad [4854c]$$

differential dQ refers only to the co-ordinates of the moon. If Q contain a
[4855] term of the form $k.\cos.(it+\varepsilon)$, i being very small; this term cannot acquire a divisor of the order i^2, except dQ does not acquire a multiplicator of the order i. The part of this angle it, relative to the moon, must depend solely on
[4855'] the mean motions of the moon, and on those of her perigee and nodes, when we neglect the square of the disturbing force. If i be very small, this part of i does not depend on the moon's mean motion; it must, therefore, depend
[4856] only on the motions of the perigee and nodes. In this case, dQ acquires a factor of the same order as the motions of the perigee and nodes, that is, of
[4856'] the second order [4817,4828e]; which causes the term in question to lose its divisor of the order i^2. Therefore, the angles increasing slowly have, in the
[4856''] expression of the true longitude in a function of the time, a divisor of the order i only; and it is evident, that this likewise holds good, in the expression
[4857] of the time in a function of the true longitude. But, if we notice the square of the disturbing force, the part of the angle it, relative to the moon's co-ordinates, may contain the sun's mean motion; and then, the differential
[4857'] dQ acquires only a factor of the first order, or of the order m. *From these principles we can judge of the order, to which the several terms of the differential equations are reduced, in the finite expressions of the co-ordinates.*

6. *Upon these considerations we shall develop the different terms of the equation* [4754]. In the elliptical hypothesis, the constant part of u is represented by,*

[4858] $$\frac{1}{a}.\{1+e^2+\tfrac{1}{4}\gamma^2+\beta\}=\text{constant part of } u\,;$$

[4858'] β being a function of the fourth dimension in e, γ, we also have,

[4859] $$h^2=a.\{1-e^2-\gamma^2+\beta'\};$$

[4859'] β' being likewise a function of the fourth dimension in e and γ. The sun's
[4860] action alters this constant part of u [4858, 4964]; but a being arbitrary,

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* (2792) Neglecting terms of the fourth order, we have, in [4826], the constant part of
[4858a] u equal to $\frac{1}{a}.\{1+e^2+\frac{1}{4}\gamma^2\}$; and, from [4825], $h^2=a.\{1-e^2-\gamma^2\}$. Adding to these the functions of the fourth order, depending on β, β', they become respectively, as in [4858, 4859].

we may suppose, that $\frac{1}{a}\cdot\{1+e^2+\frac{1}{4}\gamma^2+\beta\}$ [4858] always represents the constant part of u. In this case, we shall no longer have [4861]

$$h^2=a.(1-e^2-\gamma^2+\beta')\ \ [4859]\,; \qquad [4862]$$

and we shall then put, $a_{,}$.

$$h^2=a_{,}.(1-e^2-\gamma^2+\beta')\,; \qquad [4863]$$

$a_{,}$ being an arbitrary quantity which becomes equal to a, if we exclude the sun's action. We shall then put, [4864] $\overline{m}$.

$$\frac{m'.a^3}{a'^3}=\overline{m}^2\,. \qquad [4865]$$

This being premised, the term

$\frac{m'.u'^3}{2h^2.u^3}$, of the expression $-\frac{1}{h^2}\cdot\left(\frac{dQ}{du}\right)-\frac{s}{h^2u}\cdot\left(\frac{dQ}{ds}\right)$ [4808], [4865']

becomes, by development, as follows;*

$$\frac{m'.u'^3}{2h^2.u^3}=\frac{\overline{m}^2}{2a_{,}}\cdot\left\{\begin{array}{ll}
1+e^2+\frac{1}{4}\gamma^2+\frac{3}{2}e'^2 & 1\\
-3e.(1+\frac{1}{2}e^2+\frac{3}{2}e'^2).\cos.(cv-\varpi) & 2\\
+3e'.(1+e^2+\frac{1}{4}\gamma^2+\frac{9}{8}e'^2).\cos.(c'mv-\varpi') & 3\\
-\frac{3}{2}.(3+2m).ee'.\cos.(cv+c'mv-\varpi-\varpi') & 4\\
-\frac{3}{2}.(3-2m).ee'.\cos.(cv-c'mv-\varpi+\varpi') & 5\\
+3e^2.\cos.(2cv-2\varpi) & 6\\
+\frac{3}{4}\gamma^2.\cos.(2gv-2\theta) & 7\\
+\frac{9}{2}e'^2.\cos.(2c'mv-2\varpi') & 8\\
-\frac{3}{2}e\gamma^2.\cos.(2gv-cv-2\theta+\varpi) & 9
\end{array}\right\}. \qquad [4866]$$

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* (2793) If we separate the terms of the expression of $\frac{1}{u}$ [4826], into different classes; using the abridged symbols x_1, x_2, x_3 [4866b], whose *indices represent respectively the orders of the terms*, we shall have u [4866c], from which we obtain $\frac{1}{u^3}$ [4866d], neglecting terms of the fourth order in e, γ; [4866a]

$$x_1=e.\cos.(cv-\varpi)\,;\quad x_2=e^2+\tfrac{1}{4}\gamma^2-\tfrac{1}{4}\gamma^2.\cos.(2gv-2\theta)\,;\quad x_3=e^3.\cos.(cv-\varpi)\,; \qquad [4866b]$$

$$u=a^{-1}.\{1+x_1+x_2+x_3\}\,; \qquad [4866c]$$

$$u^{-3}=a^3.\{1-3.(x_1+x_2+x_3)+6.(x_1^2+2x_1x_2)-10x_1^3\}. \qquad [4866d]$$

To develop the term

[4866′] $\frac{3m'.u'^3}{2h^2.u^3}.\cos.(2v-2v')$, of the expression of $-\frac{1}{h^2}.\left(\frac{dQ}{du}\right)-\frac{s}{h^2u}.\left(\frac{dQ}{ds}\right)$ [4808];

Now, substituting the values of x_1, x_2, x_3 [4866b], in the first members of [4866f—i], and reducing the products, by means of [6, 20, 7] Int., we obtain the second members of these
[4866e] expressions respectively; always neglecting terms of the fourth order, and those depending on the angles $2gv+cv$, $3cv$, which are not retained in [4866]; and using the abridged notation [4821f];

[4866f] $$1-3.(x_1+x_2+x_3)=1-3e^2-\tfrac{3}{4}\gamma^2-3e.(1+e^2).\cos.cv+\tfrac{3}{4}\gamma^2.\cos.2gv;$$

[4866g] $$+\quad 6x_1^2=\ +3e^2+3e^2.\cos.2cv;$$

[4866h] $$+12x_1x_2=\qquad -3e.(-4e^2-\gamma^2).\cos.cv-\tfrac{3}{2}e\gamma^2.\cos.(2gv-cv);$$

[4866i] $$-\quad 10x_1^3=\qquad -3e.(\tfrac{5}{2}e^2).\cos.cv.$$

The sum of these four expressions being multiplied by a^3, gives the value of u^{-3} [4866d,k]. Moreover, from [4863], we get $\frac{1}{2}h^{-2}$ [4866l]; the product of the two expressions [4866k,l] gives [4866m], neglecting terms of the fourth order;

[4866k] $$u^{-3}=a^3.\{1-\tfrac{3}{4}\gamma^2-3e.(1-\tfrac{1}{2}e^2-\gamma^2).\cos.cv+3e^2.\cos.2cv+\tfrac{3}{4}\gamma^2.\cos.2gv-\tfrac{3}{2}e\gamma^2.\cos.(2gv-cv)\};$$

[4866l] $$\tfrac{1}{2}h^{-2}=\tfrac{1}{2}a_{\prime}^{-1}.\{1+e^2+\gamma^2\};$$

[4866m] $$\frac{1}{2h^2u^3}=\frac{a^3}{2a_{\prime}}.\{1+e^2+\tfrac{1}{4}\gamma^2-3e.(1+\tfrac{1}{2}e^2).\cos.cv+3e^2.\cos.2cv+\tfrac{3}{4}\gamma^2.\cos.2gv-\tfrac{3}{2}e\gamma^2.\cos.(2gv-cv)\}$$

[4866n] $$=\frac{a^3}{2a_{\prime}}.\{1+X\};$$

[4866o] X being put, for brevity, to denote all the terms between the braces in [4866m], except the first, or unity.

We may proceed, in the same manner, to find u'^{-3}. For, by using the symbols y_1, y_2, y_3 [4866q], the expression of u' [4838] becomes as in [4866r]; omitting, as above, the angles ϖ, ϖ', in the rest of the calculation. From this value of u' we get u'^{-3} [4866s].
[4866p] The terms, composing the factor of this expression, are found in [4866t—w]; whose sum, multiplied by a'^{-3}, gives u'^3 [4866s], as in [4866x]; neglecting the terms depending on the angle $3c'mv-3\varpi'$;

$$y_1=e'.\cos.c'mv;\qquad y_2=e'^2.\cos.2c'mv;$$

[4866q] $$y_3=-\tfrac{1}{8}e'^3.\cos.c'mv+mce'.\cos.(cv-c'mv)-mce'.\cos.(cv+c'mv);$$

[4866r] $$u'=a'^{-1}.\{1+y_1+y_2+y_3\};$$

[4866s] $$u'^3=a'^{-3}.\{1+3.(y_1+y_2+y_3)+3.(y_1^2+2y_1y_2)+y_1^3\};$$

we shall first give the development of

$$3\,m'\,.\,u'^3.\cos.(2\,v-2\,v'). \qquad [4866'']$$

This term, being developed, becomes,

$$1+3.(y_1+y_2+y_3)=1+3\,m\,e\,e'.\cos.(c\,v-c'\,m\,v)-3\,m\,e\,e'.\cos.(cv+c'mv) \qquad [4866t]$$
$$+(3\,e'-\tfrac{3}{8}e'^3).\cos.c'\,m\,v+3\,e'^2.\cos.2\,c'\,m\,v;$$

$$3\,y_1{}^2=\;+\tfrac{3}{2}e'^2 \qquad +\tfrac{3}{2}e'^2.\cos.2\,c'\,m\,v; \qquad [4866u]$$

$$6\,y_1\,y_2= \qquad +\tfrac{24}{8}e'^3.\cos.\;c'\,m\,v; \qquad [4866v]$$

$$y_1{}^3= \qquad +\tfrac{6}{8}\,e'^3.\cos.\;c'\,m\,v; \qquad [4866w]$$

$$u'^3=a'^{-3}.\left\{\begin{array}{l}1+\tfrac{3}{2}e'^2+3\,e'.(1+\tfrac{9}{8}e'^2).\cos.c'\,m\,v+\tfrac{9}{2}e'^2.\cos.2\,c'\,m\,v \\ \quad+3\,m\,c\,e'.\cos.(c\,v-c'\,m\,v)-3\,m\,e\,e'.\cos.(c\,v+c'\,m\,v)\end{array}\right\} \qquad [4866x]$$

$$=a'^{-3}.\{1+Y\}; \qquad [4866y]$$

$1+Y$ being used, for brevity, to denote all the terms between the braces, in [4866x]. [4866z]
Multiplying together the expressions [4866n, y], and their product by m'; then substituting $\overline{m}^2$ [4865], we get,

$$\frac{m'.u'^3}{2\,h^2.u^3}=\frac{\overline{m}^2}{2a_{\prime}}.\{1+X+Y+X\,Y\}. \qquad [4866\alpha]$$

Now, $X\,Y$ is of the second order; and, in finding its value, retaining the same angles and terms as in [4866], we may use the following expressions, which comprise the chief terms of X, Y [4866n, y];

$$X=e^2+\tfrac{1}{4}\gamma^2-3\,e.\cos.c\,v; \qquad Y=\tfrac{3}{2}e'^2+3\,e'.\cos.c'\,m\,v. \qquad [4866\beta]$$

Now, taking the terms of Y, and multiplying them separately by X, we get,

$$\tfrac{3}{2}e'^2.\,X=-\tfrac{9}{2}e\,e'^2.\cos.c\,v; \qquad [4866\gamma]$$

$$3e'.\cos.c'm\,v.\,X=3e'.(e^2+\tfrac{1}{4}\gamma^2).\cos.c'\,m\,v-\tfrac{9}{2}e\,e'.\cos.(c\,v+c'\,m\,v)-\tfrac{9}{2}e\,e'.\cos.(cv-c'm\,v). \qquad [4866\delta]$$

The sum of the expressions [4866γ, δ] is equal to the value of $X\,Y$, which is to be substituted in [4866α]; *moreover, the sum of the terms between the braces in* [4866m, x], *decreased by unity, is equal to the value of* $1+X+Y$. Hence we find, *that the terms of* [4866α, or 4866], *between the braces, are equal to the sum of the terms between the braces in* [4866m, x], *added to the second members of* [4866γ, δ], *and decreased by unity.* [4866ε]
Connecting the similar terms, we find the result of this calculation to be the same as in [4866].

[4867] $$3m'.u'^3.\cos.(2v-2v')=\frac{3m'}{a'^3}.\left\{\begin{array}{ll}
(1-\tfrac{5}{2}e'^2-4m^2e^2).\cos.(2v-2mv) & 1\\
+\tfrac{7}{2}e'.\cos.(2v-2mv-c'mv+\varpi') & 2\\
-\tfrac{1}{2}e'.\cos.(2v-2mv+c'mv-\varpi') & 3\\
+2me.\cos.(2v-2mv+cv-\varpi) & 4\\
-2me.\cos.(2v-2mv-cv+\varpi) & 5\\
+\tfrac{17}{2}e'^2.\cos.(2v-2mv-2c'mv+2\varpi') & 6\\
-\tfrac{21}{2}mee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') & 7\\
+\tfrac{21}{2}mee'.\cos.(2v-2mv+cv-c'mv-\varpi+\varpi') & 8\\
+\tfrac{1}{2}mee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') & 9\\
-\tfrac{1}{2}mee'.\cos.(2v-2mv+cv+c'mv-\varpi-\varpi') & 10\\
+\tfrac{1}{4}m.(3+8m).e^2.\cos.(2cv-2v+2mv-2\varpi) & 11\\
-\tfrac{1}{4}m.(3-8m).e^2.\cos.(2cv+2v-2mv-2\varpi) & 12\\
+\tfrac{1}{4}m\gamma^2.\cos.(2gv-2v+2mv-2\theta) & 13\\
-\tfrac{1}{4}m\gamma^2.\cos.(2gv+2v-2mv-2\theta) & 14\\
-\tfrac{3}{4}me\gamma^2.\cos.(2v-2mv-2gv+cv+2\theta-\varpi) & 15
\end{array}\right\}.$$

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* (2794) Using, for brevity, the value of v_1 [4867e], putting also v_2 equal to all the

[4867a] remaining terms of the second member of [4837], except the first mv, we shall have v', as in [4867f]; always omitting, for brevity, the symbols ϖ, ϖ', as in [4821f]. Substituting this value of v' in the first member of [4867g], and developing by means of [24, 43, 44] Int., it

[4867b] becomes as in [4867h]; observing, that v_1 is of the first order, v_2 of the second order, and, that some terms of the third order are neglected. Substituting in [4867h] the value

[4867c] $$-2v_1^2=-8m^2.e^2.\sin.^2cv+16mee'.\sin.cv.\sin.c'mv-8e'^2.\sin.^2c'mv \quad [4867e],$$

[4867d] and reducing it, by means of [1, 17] Int.; also, $2v_1+2v_2=2v'-2mv$ [4867f], it becomes as in [4867i];

[4867e] $$v_1=-2me.\sin.cv+2e'.\sin.c'mv;$$

[4867f] $$v'=mv+v_1+v_2;$$

[4867g] $$\begin{aligned}\cos.(2v-2v')&=\cos.\{(2v-2mv)-(2v_1+2v_2)\}\\
&=\cos.(2v_1+2v_2).\cos.(2v-2mv)+\sin.(2v_1+2v_2).\cos.(2v-2mv)\end{aligned}$$

[4867h] $$=(1-2v_1^2).\cos.(2v-2mv)+(2v_1+2v_2).\sin.(2v-2mv)$$

[4867i] $$\begin{aligned}&=\left\{\begin{array}{l}(1-4m^2e^2-4e'^2)+4m^2e^2.\cos.2cv+4e'^2.\cos.2c'mv\\ \quad+8mee'.\cos.(cv-c'mv)-8mee'.\cos.(cv+c'mv)\end{array}\right\}.\cos.(2v-2mv)\\
&\quad+\{2v'-2mv\}.\sin.(2v-2mv).\end{aligned}$$

We must multiply this function by $\frac{1}{2h^2u^3}$; and we have this factor, by [4868]

We must substitute, in the last line of this expression, the value of $2v'-2mv$, which is easily deduced from the second member of [4837], by neglecting the first term mv, and doubling the remaining eight terms. We must then reduce the products of the sines and cosines of this function, by means of [17, 20] Int., as in the following table; in which, the terms of column 1, corresponding to the different angles, are taken in the same order as in [4867i], namely; the first five terms in the same order as in the first and second lines of [4867i]; and the remaining eight lines as in $2v'-2mv$ [4837, 4867k]. We may observe, that a term is neglected in line 9, depending on the angle $2v-2mv+2gv-cv$, which is not expressly retained in [4867]; also a term, of the order e'^3, in line 10, &c.; [4867k] [4867l]

(Col. 1.)	(Col. 2.)	
1	$(1-4m^2e^2-4e'^2).\cos.(2v-2mv)$	
2	$+\frac{8}{4}.m^2e^2.\cos.(2cv-2v+2mv)+\frac{8}{4}.m^2e^2.\cos.(2cv+2v-2mv)$	
3	$+2e'^2.\cos.(2v-2mv-2c'mv)+2e'^2.\cos.(2v-2mv+2c'mv)$	
4	$+4mee'.\cos.(2v-2mv-cv+c'mv)+4mee'.\cos.(2v-2mv+cv-c'mv)$	
5	$-4mee'.\cos.(2v-2mv-cv-c'mv)-4mee'.\cos.(2v-2mv+cv+c'mv)$	
6	$+2me.\cos.(2v-2mv+cv)-2me.\cos.(2v-2mv-cv)$	
7	$+\frac{3}{4}me^2.\cos.(2cv-2v+2mv)-\frac{3}{4}me^2.\cos.(2cv+2v-2mv)$	[Terms of $\cos.(2v-2v')$.] [4867m]
8	$+\frac{1}{4}m\gamma^2.\cos.(2gv-2v+2mv)-\frac{1}{4}m\gamma^2.\cos.(2gv+2v-2mv)$	
9	$-\frac{3}{4}me\gamma^2.\cos.(2v-2mv-2gv+cv)+$ &c.	
10	$+2e'.\cos.(2v-2mv-c'mv)-2e'.\cos.(2v-2mv+c'mv)+$ &c.	
11	$-2mee'.\cos.(2v-2mv-cv-c'mv)+2mee'.\cos.(2v-2mv+cv+c'mv)$	
12	$-2mee'.\cos.(2v-2mv-cv+c'mv)+2mee'.\cos.(2v-2mv+cv-c'mv)$	
13	$+\frac{5}{4}e'^2.\cos.(2v-2mv-2c'mv)-\frac{5}{4}e'^2.\cos.(2v-2mv+2c'mv)$.	

To obtain the expression [4867], we must multiply this value of $\cos.(2v-2v')$ [4867m], by $3m'.u'^3$, or $3m'.a'^{-3}.(1+Y)$ [4866y]; by this means all the terms will have the common factor $\frac{3m'}{a'^3}$, like that without the braces in [4867]; and the terms of this expression within the braces will be obtained, by multiplying the function [4867m] by $1+Y$; or, in other words, by multiplying the functions [4867m] by Y, [4866x, y], and reducing the products as in [4867r], then adding together the two functions [4867m, r]. In the first column of [4867r], we have given the terms of Y [4866x, y]; and, in the second column, the terms of [4867m], by which they are multiplied: the third column contains their products, respectively. The numbers in column 2, refer to the numbers in the margin of the lines of [4867m], *putting one accent to denote the first term of any line, two accents for the* [4867n] [4867o] [4867p]

putting e' equal to nothing, in the preceding development of $\frac{m'.u'^3}{2h^2.u^3}$ [4866],
[4869] and by multiplying this last quantity by $\frac{a'^3}{m'}$. We shall thus have, very
[4869'] nearly, by neglecting quantities which remain of the order m^5 after the integration,*

[4867q] *second term of the same line, &c.* Thus, 6′ denotes the term $2me.\cos.(2v-2mv+cv)$; and 6″, the term $-2me.\cos.(2v-2mv-cv)$. *This method of distinguishing the terms will be frequently used.*

[4867r]

(Col. 1.) Terms of Y [4866r].	(Col. 2.) Terms of [4867m]	(Col. 3.) Products of these terms.
$+\frac{3}{2}e'^2$	1′	$+\frac{3}{2}e'^2.\cos.(2v-2mv)$
$+3e'.\cos.c'mv$	1′	$+\frac{3}{2}e'.\cos.(2v-2mv-c'mv)+\frac{3}{2}e'.\cos.(2v-2mv+c'mv)$
	6′	$+3mee'.\cos.(2v-2mv+cv-c'mv)+3mee'.\cos.(2v-2mv+cv+c'mv)$
	6″	$-3mee'.\cos.(2v-2mv-cv-c'mv)-3mee'.\cos.(2v-2mv-cv+c'mv)$
	10′	$+3e'^2.\cos.(2v-2mv-2c'mv)+3e'^2.\cos.(2v-2mv)$
	10″	$-3e'^2.\cos.(2v-2mv+2c'mv)-3e'^2.\cos.(2v-2mv)$
$\frac{9}{2}e'^2.\cos.2c'mv$	1′	$+\frac{9}{4}e'^2.\cos.(2v-2mv-2c'mv)+\frac{9}{4}e'^2.\cos.(2v-2mv+2c'mv)$
$+3mee'.\cos.(cv-c'mv)$	1′	$+\frac{3}{2}mee'.\cos.(2v-2mv-cv+c'mv)+\frac{3}{2}mee'.\cos.(2v-2mv+cv-c'mv)$
$-3mee'.\cos.(cv+c'mv)$	1′	$-\frac{3}{2}mee'.\cos.(2v-2mv-cv-c'mv)-\frac{3}{2}mee'.\cos.(2v-2mv+cv+c'mv)$.

Connecting together the terms of [4867m,r], depending on the same angles, we find, that the coefficient of $\cos.(2v-2mv+2c'mv)$ vanishes, and the rest become equal to the function between the braces in [4867], conformable to [4867o].

* (2795) The method given by the author, in [4869], is evidently correct. For, if we
[4869a] put $e'=0$, in [4838], we get $u'=\frac{1}{a'}$, whence, $\frac{1}{m'.u'^3}=\frac{a'^3}{m'}$; multiplying this by $\frac{m'.u'^3}{2h^2.u^3}$ gives $\frac{1}{2h^2.u^3}$. We shall not, however, be under the necessity of using this process,
[4869b] because we have already given the value of $\frac{1}{2h^2.u^3}=\frac{a^3}{2a_{,}}.(1+X)$ [4866m,n]; and, if we multiply this by the function [4867], we shall obtain [4870]. In the first place, the factors without the braces $\frac{3m'}{a'^3}$, $\frac{a^3}{2a_{,}}$, being multiplied together, produce,

[4869c]
$$\frac{3}{2a_{,}}\cdot\frac{m'.a^3}{a'^3}=\frac{3}{2a_{,}}\cdot\bar{m}^2 \quad [4865];$$

$$\frac{3m'.u'^3}{2h^2.u^3}.\cos.(2v-2v')=\frac{3.\overline{m}^2}{2a_{\prime}}.\left\{\begin{array}{ll}
(1+e^2+\frac{1}{4}\gamma^2-\frac{5}{2}e'^2).\cos.(2v-2mv) & 1 \\
-\frac{1}{2}(3+4m).e.(1+\frac{1}{2}e^2-\frac{5}{2}e'^2).\cos.(2v-2mv-cv+\varpi) & 2 \\
-\frac{1}{2}(3-4m).e.\cos(2v-2mv+cv-\varpi) & 3 \\
+\frac{7}{2}e'.\cos.(2v-2mv-c'mv+\varpi') & 4 \\
-\frac{1}{2}e'.\cos.(2v-2mv+c'mv-\varpi') & 5 \\
-\frac{21}{4}(1+2m).ee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') & 6 \\
-\frac{21}{4}(1-2m).ee'.\cos.(2v-2mv+cv-c'mv-\varpi+\varpi') & 7 \\
+\frac{1}{4}(3+2m).ee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') & .8 \\
+\frac{1}{4}(3-2m).ee'.\cos.(2v-2mv+cv+c'mv-\varpi-\varpi') & 9 \\
+\frac{17}{2}.e'^2.\cos.(2v-2mv-2c'mv+2\varpi') & 10 \\
+\frac{1}{4}(6+15m+8m^2).e^2.\cos.(2cv-2v+2mv-2\varpi) & 11 \\
+\frac{1}{4}(6-15m+8m^2).e^2.\cos.(2cv+2v-2mv-2\varpi) & 12 \\
+\frac{1}{8}(3+2m).\gamma^2.\cos.(2gv-2v+2mv-2\theta) & 13 \\
+\frac{1}{8}(3-2m).\gamma^2.\cos.(2gv+2v-2mv-2\theta) & 14 \\
-\frac{3}{8}(2+m).e\gamma^2.\cos.(2v-2mv-2gv+cv+2\theta-\varpi) & 15
\end{array}\right\} \quad [4870]$$

The term

$$\frac{9m'.u'^4}{8h^2.u^4}.\cos.(v-v'), \text{ of the expression } -\frac{1}{h^2}.\left(\frac{dQ}{du}\right)-\frac{s}{h^2u}.\left(\frac{dQ}{ds}\right) \text{ [4808]}, \quad [4871]$$

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which is the same as the common factor of [4870]. Moreover, the terms between the braces in [4870], are represented by the product of the terms between the braces in [4867], by $1+X$ [4866n]; or, in other words, this product is equal to the terms between the braces in [4867], added to the function [4869c]. This last function being the result of the product of these terms of [4867] by the the quantity X; and it is obtained in the following table, which is similar to [4867r]. The first column contains the terms of X; the second, the terms of [4867], and the third, the corresponding products, reduced in the usual manner, and using the accented number 1′, to denote the first term of the first line of [4867], as in [4867q]; [4869d] [4869d']

[4871'] gives the following ;*

	(Col. 1.) Terms of X [4866m,n].	(Col. 2.) Terms of [4867].	(Col. 3.) Products of these terms.
	$+e^2$	1′	$+e^2.\cos.(2v-2mv)$
	$+\frac{1}{4}\gamma^2$	1′	$+\frac{1}{4}\gamma^2.\cos.(2v-2mv)$
	$-3e.\cos.cv$	1′	$-\frac{3}{2}e.\cos.(2v-2mv+cv)-\frac{3}{2}e.(1-\frac{5}{2}e'^2).\cos.(2v-2mv-cv)$
		2	$-\frac{21}{4}ee'.\cos.(2v-2mv-cv-c'mv)-\frac{21}{4}ee'.\cos.(2v-2mv+cv-c'mv)$
		3	$+\frac{3}{4}ee'.\cos.(2v-2mv-cv+c'mv)+\frac{3}{4}ee'.\cos.(2v-2mv+cv+c'mv)$
[4869e]		4	$-3me^2.\cos.(2v-2mv)-3me^2.\cos.(2cv+2v-2mv)$
		5	$+3me^2.\cos.(2v-2mv)+3me^2.\cos.(2cv-2v+2mv)$
		13	$-\frac{3}{8}me\gamma^2.\cos.(2v-2mv-2gv+cv)$
	$-\frac{3}{2}e^3.\cos.cv$	1′	$-\frac{3}{4}e^3.\cos.(2v-2mv-cv)$
	$+3e^2.\cos.2cv$	1′	$+\frac{3}{2}e^2.\cos.(2cv-2v+2mv)+\frac{3}{2}e^2.\cos.(2cv+2v-2mv)$
	$+\frac{3}{4}\gamma^2.\cos.2gv$	1′	$+\frac{3}{8}\gamma^2.\cos.(2gv-2v+2mv)+\frac{3}{8}\gamma^2.\cos.(2gv+2v-2mv)$
		4	$+\frac{3}{4}me\gamma^2.\cos.(2v-2mv-2gv+cv)$
	$-\frac{3}{2}e\gamma^2.\cos.(2gv-cv)$	1′	$-\frac{3}{4}e\gamma^2.\cos.(2v-2mv-2gv+cv)$.

Now, adding the function [4869e] to the terms between the braces in [4867], we get very
[4869f] nearly, the expression between the braces [4870]. There are some slight differences, of the same order as that of the terms which we have usually neglected. Thus, the term $-4m^2e^2$, in the coefficient of line 1 [4867], is neglected in [4870]. The term $-2me$,
[4869g] in line 5 [4867], is connected with the factor $(1+\frac{1}{2}e^2-\frac{5}{2}e'^2)$ in line 2 [4870], which arises from the chief terms of this coefficient in [4869e]; but this merely introduces terms of the sixth order. Finally, we may observe, that a similar factor might be introduced in the coefficient of line 3 [4870].

* (2796) Proceeding in the same manner as in note 2793, and retaining terms of the
[4870a] second order only, we get, from [4866c] $u^{-4}=a^4.\{1-4.(x_1+x_2)+10x_1^2\}$; substituting in this the value of $10x_1^2=10e^2.\cos.^2cv=5e^2+5e^2.\cos.2cv$; also the value of x_1+x_2 [4866b], we get,

[4870b] $$u^{-4}=a^4.\{1+e^2-\gamma^2-4e.\cos.cv+5e^2.\cos.2cv+\gamma^2.\cos.2gv\}.$$

Multiplying this by $\frac{9m'}{8h^2}=\frac{9m'}{8a_{\prime}}.\{1+e^2+\gamma^2)$ [4863], we obtain,

[4870c] $$\frac{9m'}{8h^2.u^4}=\frac{9m'.a^4}{8a_{\prime}}.\{1+2e^2-4e.\cos.cv+5e^2.\cos.2cv+\gamma^2.\cos.2gv\}.$$

[4870d] Again, from [4866q,r], we have successively, $6y_1^2=3e'^2+3e'^2.\cos.2c'mv$;

$$u'^4=a'^{-4}.\{1+4.(y_1+y_2)+6y_1^2\}$$

[4870e] $$=a'^{-4}.\{1+3e'^2+4e'.\cos.c'mv+7e'^2.\cos.2c'mv\}.$$

$$\frac{9m'.u'^4}{8\,h^2.\,u^4}.\cos.(v-v')=\left\{\begin{array}{l} \frac{9\,\overline{m}^2}{8\,a_{,}}.(1+2e^2+2e'^2).\frac{a}{a'}.\cos.(v-mv) \\ +\frac{9\,\overline{m}^2}{8\,a_{,}}.\frac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi') \\ +\frac{27\,\overline{m}^2}{8\,a_{,}}.\frac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi') \end{array}\right\}. \quad \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \qquad [4872]$$

If we denote the factors between the braces in [4870c, e] by $1+X_1$, $1+Y_1$, respectively, their product will be $1+X_1+Y_1+X_1Y_1$; by noticing only the chief terms of X_1, Y_1, we have, [4870f]

$$X_1 Y_1 = (-4\,e.\cos.c\,v).(4\,e'.\cos.c'm\,v) = -8\,e\,e'.\cos.(cv-c'mv) - 8\,c\,e'.\cos.(cv+c'mv). \qquad [4870g]$$

Adding these terms of $X_1 Y_1$ to those of $1+X_1$, $1+Y_1$ [4870c, e], and decreasing the sum by unity, we get the expression of $1+X_1+Y_1+X_1Y_1$, to be used in the product of the functions [4870e, e], which becomes,

$$\frac{9m'.u'^4}{8h^2.u^4} = \frac{9m'.a^4}{8a_{,}.a'^4}.\left\{\begin{array}{l} 1+2e^2+3e'^2+4\,e'.\cos.c'm\,v-4\,e.\cos.c\,v \\ +5e^2.\cos.2e\,v+\gamma^2.\cos.2g\,v+7\,e'^2.\cos.2c'm\,v \\ -8\,e\,e'.\cos.(c\,v-c'm\,v)-8\,e\,e'.\cos.(c\,v+c'm\,v) \end{array}\right\}. \qquad [4870h]$$

Substituting the value of $\frac{m'.a^3}{a'^3}$ [4865], in the first factor of this expression, it becomes,

$$\frac{9m'.a^4}{8a_{,}.a'^4} = \tfrac{9}{8}\frac{\overline{m}^2}{a_{,}}.\frac{a}{a'}; \qquad [4870i]$$

which is of the fourth order [4842, 4843]; therefore, in finding the value of $\cos.(v-v')$, we need only to retain, in general, the terms of the *first* order; except in those depending on the angle $v-mv$; in which greater accuracy is required [4874]. Hence we may neglect [4870k] v_2 [4867f], and we shall have the value of $\cos.(v-v')$ [4870m], by proceeding as in [4867g, h]. Substituting in this the value of $v_1 = 2e'.\sin.c'm\,v$ [4867e], it becomes as in [4870n]. It being unnecessary to notice other terms of a higher order, or such as depend on [4870l] angles which differ from those in [4872];

$$\cos.(v-v') = (1-\tfrac{1}{2}v_1^2).\cos.(v-mv)+v_1.\sin.(v-mv) \qquad [4870m]$$

$$= (1-e'^2).\cos.(v-mv)-e'.\cos.(v-mv+c'mv)+e'.\cos.(v-mv-c'mv). \qquad [4870n]$$

The four terms of which this expression is composed, being multiplied by the terms between the braces in the function [4870h], produce respectively the terms in the four lines [4870o—r]. Their sum is given in [4870s]; to which we must annex the common factor [4870i], and we shall obtain the corresponding terms of $\frac{9m'.u^4}{8a_{,}.u'^4}.\cos.(v-v')$, as in [4872]. We shall hereafter, in [4870t—u], see, that the neglected terms have much less effect, in the value of u, than those we have explicitly retained;

[4872'] $\frac{a}{a'}$ being, by the preceding article [4843], of the order m^2; the two first of
[4873] these terms become of the order m^3 by the integrations. *The inequality,*
[4874] *depending on the angle $v-mv$, is remarkably well adapted to the determination of the sun's parallax, by means of the ratio* $\frac{a}{a'}$. It is, therefore, important

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(Col. 1.) (Col. 2.)

[4870o] 1 $(1+2e^2+3e'^2).\cos.(v-mv)+2e'.\cos.(v-mv+c'mv)+2e'.\cos.(v-mv-c'mv)$
[4870p] 2 $-e'^2.\cos.(v-mv)$
[4870q] 3 $-e'.\cos.(v-mv+c'mv)$
[4870r] 4 $+e'.\cos.(v-mv-c'mv).$

[4870s] $$(1+2e^2+2e'^2).\cos.(v-mv)+e'.\cos.(v-mv+c'mv)+3e'.\cos.(v-mv-c'mv).$$

If we compare the terms [4872] with the assumed form [4846], we find the values of i, corresponding to them respectively, are $i=1-m$, $i=1-m+c'm$, $i=1-m-c'm$; and,
[4870t] as c' hardly differs from unity, they are very nearly represented by $i=1-m$, $i=1$, $i=1-2m$. The corresponding divisors, in the value of u [4817], are of the orders
[4870u] $(1-m)^2-N^2$, $1-N^2$, $(1-2m)^2-N^2$; and, as N^2 differs from unity by quantities of the order m^2 [4815'], these divisors will be respectively of the orders m, m^2, m. In consequence of these divisors, the part of the first term [4872] which is independent of e, e',
[4870v] is reduced from the fourth to the third order; the second term is reduced from the fifth to the third order; and the third term is reduced from the fifth to the fourth order. Several terms of the function [4870v, or 4872], are not increased so sensibly in the value of u, and they are
[4870w] therefore neglected. Thus, the term $-4e.\cos.cv$ [4870h], being multiplied by the first term of [4870n], produces, in the function [4872], the following expression,

[4870x] $$\tfrac{9}{8}.\frac{\overline{m}^2}{a_{\prime}}.\frac{a}{a'}.(-4e.\cos.cv).\cos.(v-mv)=-\tfrac{9}{8}.\frac{\overline{m}^2}{a_{\prime}}.\frac{a}{a'}.2e.\{\cos.(cv-v+mv)+\cos.(cv+v-mv)\},$$

[4870y] corresponding, in [4846], to $i=c-1+m$, $i=c+1-m$, and as $c=1-\frac{3}{4}m^2$ nearly [4828e], these terms will not render the divisor i^2-N^2 small [4817].

We may observe, that the term treated of in [4871], occurs in [4808], under the form
[4870y'] $-\frac{3m'.u'^4}{8u^4}.(3-4s^2).\cos.(v-v')$, and in [4754], with a different sign, and under the form $\frac{3m'.u'^4}{8u^4}.(3-4s^2)\cos.(v-v')$, or, $\frac{9m'.u'^4}{8u^4}.(1-\frac{4}{3}s^2).\cos.(v-v')$; which, by neglecting s^2, becomes as in [4871]. Now, by [4818], we have,

[4870z] $$-\tfrac{4}{3}s^2=-\tfrac{4}{3}\gamma^2.\cos.^2(gv-\theta)=-\tfrac{2}{3}\gamma^2-\tfrac{2}{3}\gamma^2.\cos.(2gv-2\theta);$$

which contains the constant quantity $-\frac{2}{3}\gamma^2$; so that we might multiply the function [4871]
[4870z'] by $1-\frac{2}{3}\gamma^2$, which would change the factor $(1+2e^2+2e'^2)$ [4872] into $1+2e^2+2e'^2-\frac{2}{3}\gamma^2$.

to determine this inequality with particular care; and, for this purpose, we shall carry on the approximation so as to include terms of the order m^5. [4875]

We shall now develop the term $\left(\frac{dQ}{dv}\right).\frac{du}{h^2.u^2dv}$, of the equation [4754].

In the first place, this term contains the following,* $-\frac{3m'.u'^3}{2h^2.u^4}.\frac{du}{dv}.\sin.(2v-2v')$. [4876]

We shall have $-\frac{3m'.u'^3}{2h^2.u^3}.\sin.(2v-2v')$, by increasing $2v$ by a right angle,† [4876']

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* (2797) This is produced by the first term of [4809]. [4875a]

† (2798) We may change $2v$ into any other angle, as $2\mathrm{v}$ in [4867g—r, 4867, 4870], without altering the angles mv, gv, cv, $c'mv$, as is evident by the mere inspection of the process of calculation in [4767g, &c.]. This change being made in [4870], and then putting $2\mathrm{v}=2v+90^d$, its first member becomes, [4876a] [4876b]

$$-\frac{3m'.u'^3}{2h^2.u^3}.\sin.(2v-2v'),\quad \text{as in [4876']}. \qquad [4876c]$$

In the second member of [4870], we must, by the same process, change any term of the form $\cos.(2v+\beta)$ into $-\sin.(2v+\beta)$; and any one of the form $\cos.(\beta-2v)$ into $+\sin.(\beta-2v)$. Hence we get, by changing the signs of all the terms of [4870], and neglecting the symbols δ, ϖ, ϖ', as in [4821f], [4876d]

$$\frac{3m'.u'^3}{2h^2.u^3}.\sin.(2v-2v')=\frac{3.\overline{m}^2}{2a_{\prime}}.\left\{\begin{array}{ll}
(1+e^2+\frac{1}{4}\gamma^2-\frac{5}{2}e'^2).\sin.(2v-2mv) & 1\\
-\frac{1}{2}(3+4m).e.(1+\frac{1}{2}e^2-\frac{5}{2}e'^2).\sin.(2v-2mv-cv) & 2\\
-\frac{1}{2}(3-4m).e.\sin(2v-2mv+cv) & 3\\
+\frac{7}{2}e'.\sin.(2v-2mv-c'mv) & 4\\
-\frac{1}{2}e'.\sin.(2v-2mv+c'mv) & 5\\
-\frac{21}{4}(1+2m).ee'.\sin.(2v-2mv-cv-c'mv) & 6\\
-\frac{21}{4}(1-2m).ee'.\sin.(2v-2mv+cv-c'mv) & 7\\
+\frac{1}{4}(3+2m).ee'.\sin.(2v-2mv-cv+c'mv) & 8\\
+\frac{1}{4}(3-2m).ee'.\sin.(2v-2mv+cv+c'mv) & 9\\
+\frac{17}{2}.e'^2.\sin.(2v-2mv-2c'mv) & 10\\
-\frac{1}{4}(6+15m+8m^2).e^2.\sin.(2cv-2v+2mv) & 11\\
+\frac{1}{4}(6-15m+8m^2).e^2.\sin.(2cv+2v-2mv) & 12\\
-\frac{1}{8}(3+2m).\gamma^2.\sin.(2gv-2v+2mv) & 13\\
+\frac{1}{8}(3-2m).\gamma^2.\sin.(2gv+2v-2mv) & 14\\
-\frac{1}{8}(6+3m).e\gamma^2.\sin.(2v-2mv-2gv+cv) & 15
\end{array}\right\}. \qquad [4876e]$$

[4877] in the preceding development of $\frac{3m'.u'^3}{2h^2.u^3}.\cos.(2v-2v')$ [4870]. We must then mutiply this development by,*

[4878]
$$\frac{du}{u\,dv}=\left\{\begin{array}{ll}-ce.(1+\tfrac{1}{4}e^2-\tfrac{1}{4}\gamma^2).\sin.(cv-\varpi) & 1\\ +\tfrac{1}{2}ce^2.\sin.(2cv-2\varpi) & 2\\ -\tfrac{1}{4}ce^3.\sin.(3cv-3\varpi) & 3\\ +\tfrac{1}{2}g\gamma^2.\sin.(2gv-2\theta) & 4\\ -\tfrac{1}{8}e\gamma^2.\sin.(2gv-cv-2\theta+\varpi) & 5\end{array}\right\}.$$

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* (2799) The differential of [4826], relative to v, gives, by neglecting ϖ, θ, as in [4821f],

[4878a]
$$\frac{du}{dv}=a^{-1}.\{-ec.(1+e^2).\sin.cv+\tfrac{1}{2}g\gamma^2.\sin.2gv\};$$

and if we neglect terms of the third order in all the coefficients, except those which are connected with the angle $2gv-cv$, we obtain from u [4866c], the following value

[4878b] of $\frac{1}{u}$ [4878c, d], by observing, that $x_1=e^2.\cos.^2cv=\frac{1}{2}e^2+\frac{1}{2}e^2.\cos.2cv$ [4866b]. We may remark, that the author has retained, in the coefficient of $\cos.cv$, a term of the third order e^3, but has neglected others of the same order, as will be seen in [4884b];

[4878c]
$$\frac{1}{u}=a.\{1-(x_1+x_2+x_3)+(x_1+x_2+x_3)^2-(x_1+x_2+x_3)^3+\&c.\}$$

[4878d]
$$=a.\{(1-\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2)-e.(1+e^2).\cos.cv+\tfrac{1}{2}e^2.\cos.2cv+\tfrac{1}{4}\gamma^2.\cos.2gv\}.$$

Multiplying together the two expressions [4878a, d], we find, that the factor without the braces becomes $a^{-1}.a=1$; so, that we have only to notice the product of the factors between the

[4878e] braces. This is done in the following table; in which is given, in column 1, each of the four terms of the function [4878d]; and the corresponding products, by the function [4878a], are given in column 2, on the same lines respectively;

	(Col. 1.)	(Col. 2.)
[4878f]	$1-\frac{1}{2}e^2-\frac{1}{4}\gamma^2$	$-ce.(1+\frac{1}{2}e^2-\frac{1}{4}\gamma^2).\sin.cv+\frac{1}{2}g\gamma^2.\sin.2gv$
[4878g]	$-e.(1+e^2).\cos.cv$	$+\frac{1}{2}ce^2.\sin.2cv-\frac{1}{4}gc\gamma^2.\sin.(2gv-cv)-$&c.
[4878h]	$+\frac{1}{2}e^2.\cos.2cv$	$+\frac{1}{4}ce^3.\sin.cv-\frac{1}{4}ce^3.\sin.3cv$
[4878i]	$+\frac{1}{4}\gamma^2.\cos.2gv$	$+\frac{1}{8}ce\gamma^2.\sin.(2gv-cv)+$&c.

Connecting together the similar terms, and putting $c=1$, $g=1$, in those of the order $e\gamma^2$, it becomes as in [4878].

Then we shall have,*

$$-\frac{3m'.u'^3}{2h^2.u^4}.\frac{du}{dv}.\sin.(2v-2v')=\frac{3\overline{m}^2}{4a_{\prime}}.\left\{\begin{array}{ll}
ce.(1+\tfrac{1}{4}.[2-19m].e^2-\tfrac{5}{2}e'^2).\cos.(2v-2mv-cv+\varpi) & 1\\
-ce.\cos.(2v-2mv+cv-\varpi) & 2\\
+\tfrac{7}{2}.cee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') & 3\\
-\tfrac{7}{2}.cee'.\cos.(2v-2mv+cv-c'mv-\varpi+\varpi') & 4\\
-\tfrac{1}{2}.cee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') & 5\\
+\tfrac{1}{2}.cee'.\cos.(2v-2mv+cv+c'mv-\varpi-\varpi') & 6\\
-2c.(1+m).e^2.\cos.(2cv-2v+2mv-2\varpi) & 7\\
+2c.(1-m).e^2.\cos.(2cv+2v-2mv-2\varpi) & 8\\
+4m\,c.e^2.\cos.(2v-2mv) & 9\\
-\tfrac{1}{2}g\gamma^2.\cos.(2gv-2v+2mv-2\theta) & 10\\
+\tfrac{1}{2}g\gamma^2.\cos.(2gv+2v-2mv+2\theta) & 11\\
+\tfrac{1}{4}.(2-5m).e\gamma^2.\cos.(2v-2mv-2gv+cv+2\theta-\varpi) & 12
\end{array}\right\}. \qquad [4879]$$

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* (2800) If any term of [4876e], be represented by

$$\frac{3\overline{m}^2}{2a_{\prime}}..A.\sin.V, \qquad [4879a]$$

and any term of [4878], by $A'.\sin.V'$, the product of these two terms, changing its sign, will represent the corresponding part of $-\frac{3m'.u'^3}{2h^2.u^4}.\frac{du}{dv}.\sin.(2v-2v')$ [4879], which, by reduction, becomes, [4879b]

$$\frac{3\overline{m}^2}{4a_{\prime}}.\{A A'.\cos.(V+V')-A A'.\cos.(V\backsim V')\}. \qquad [4879c]$$

The factor of this expression, without the braces, is the same as in [4879]; consequently, the terms within the braces, must arise from the terms

$$A A'.\cos.(V+V')-A A'.\cos.(V\backsim V'). \qquad [4879d]$$

These terms are computed in the following table, neglecting quantities of the third order in e, e', γ, except they depend on the angles

$$2v-2mv\pm cv+\varpi, \qquad 2v-2mv-2gv+cv+2\theta-\varpi. \qquad [4879e]$$

The numbers in the first column refer, respectively, to the five terms or lines of [4878]; and those in the second column, to the terms or lines of [4876e]; in the third column are the corresponding terms of the function [4879f]; and the sum of all of them represents the terms between the braces in [4879]:

The terms,*

[4880] $$-\frac{m'.u'^4}{8h^2.u^5}.\{3.\sin.(v-v')+15.\sin.(3v-3v')\}.\frac{du}{dv},$$

	(Col. 1.)	(Col. 2.)	(Col. 3.)	
	A' [4878].	A [4876e]	Function [4879d].	
	$-ce(1+\frac{1}{4}e^2-\frac{1}{4}\gamma^2).\sin.cv$	1	$-ce.\cos.(2v-2mv+cv)+ce.(1+\frac{5}{4}e^2-\frac{5}{2}e'^2).\cos.(2v-2mv-cv)$	1
		2	$+\frac{1}{2}(3+4m).ce^2.\cos.(2v-2mv)-\frac{1}{2}(3+4m).ce^2.\cos.(2cv-2v+2mv)$	2
		3	$-\frac{1}{2}(3-4m).ce^2.\cos.(2v-2mv)+\frac{1}{2}(3-4m).ce^2.\cos.(2cv+2v-2mv)$	3
		4	$-\frac{7}{2}cee'.\cos.(2v-2mv+cv-c'mv)+\frac{7}{2}cee'.\cos.(2v-2mv-cv-c'mv)$	4
		5	$+\frac{1}{2}cee'.\cos.(2v-2mv+cv+c'mv)-\frac{1}{2}cee'.\cos.(2v-2mv-cv+c'mv)$	5
		11	$-\frac{1}{4}(6+15m).ce^3.\cos.(2v-2mv-cv)+$&c.	6
[4879f]		12	$+\frac{1}{4}(6-15m).ce^3.\cos.(2v-2mv+cv)+$&c.	7
		13	$-\frac{1}{8}(3+2m).c\gamma^2.\cos.(2v-2mv-2gv+cv)+$&c.	8
	$+\frac{1}{2}ce^2.\sin.2cv$	1	$+\frac{1}{2}ce^2.\cos(2cv+2v-2mv)-\frac{1}{2}ce^2.\cos.(2cv-2v+2mv)$	9
		2	$-\frac{1}{4}(3+4m).ce^3.\cos.(2v-2mv+cv)+$&c.	10
		3	$+\frac{1}{4}(3-4m).ce^3.\cos.(2v-2mv-cv)+$&c.	11
	$-\frac{1}{4}ce^3.\sin.3cv$	..	.... neglected.	12
	$+\frac{1}{2}g\gamma^2.\sin.2gv$	1	$+\frac{1}{2}g\gamma^2.\cos.(2gv+2v-2mv)-\frac{1}{2}g\gamma^2.\cos.(2gv-2v+2mv)$	13
		3	$+\frac{1}{4}(3-4m).e\gamma^2.\cos.(2v-2mv-2gv+cv)+$&c.	14
	$-\frac{1}{8}e\gamma^2.\sin.(2gv-cv)$	1	$+\frac{1}{8}e\gamma^2.\cos.(2v-2mv-2gv+cv)+$&c.	15

Connecting the terms of this expression, we obtain the factors between the braces in [4879],
[4879g] neglecting terms of the third order, connected with the angle $2v-2mv+cv$, or with other angles differing considerably from v. To estimate roughly one of these neglected terms, we shall observe, that $\gamma > e > e'$ [5117, 5120]; therefore, the greatest product of the third order, which can be made of these three quantities, and can occur in the above function, is $e\gamma^2$; and, if this be multiplied by the factor $\frac{3\bar{m}^2}{4a_{\prime}}$ [4879], or its equivalent expression $\frac{3}{4}m^2$, it becomes $\frac{3}{4}m^2.e\gamma^2$. Substituting the values [5117, 5120], and multiplying by the
[4879h] radius in seconds 206265^s, we get $\frac{3}{4}m^2.e\gamma^2=0^s,38$; which represents the order of the greatest neglected term in [4879]. This may be somewhat increased by integration in this
[4879i] value of u [4847], by means of the divisor $i^2-N'^2$; for which reason the author has retained the last term of the function [4879], which depends on the factor $e\gamma^2$. We may
[4879k] observe, that the factor $1+\frac{5}{2}e^2-\frac{5}{4}e'^2$, which occurs in the second term of the first line of [4879f], might also be connected with the first term in that line.

* (2801) Substituting, in $\left(\frac{dQ}{dv}\right).\frac{du}{h^2u^2.dv}$ [4754], the term of [4809], depending on u'^4,
[4880a] it becomes as in [4880]; neglecting the very small term depending on s^2. We have, in

in the expression of $\left(\frac{dQ}{dv}\right).\frac{du}{h^2u^2dv}$, produce no inequality of the third order [4881]
in the integrals.

Lastly, we shall develop $\frac{2}{h^2}.\int\frac{dQ}{dv}.\frac{dv}{u^2}$ [4754]. This function contains [4881']
the following term,* $-\frac{3m'}{h^2}.\int\frac{u'^3.dv}{u^4}.\sin.(2v-2v')$. The development of [4882]

$\frac{3m'.u'^3}{2h^2.u^3}.\cos.(2v-2v')$ [4870], gives that of $-\frac{3m'.u'^3}{h^2.u^4}.\sin.(2v-2v')$, [4883]

by increasing the angle $2v$ by a right angle [4883a], and multiplying it by† [4883']

[4872], the expression of $\frac{9m'.u'^4}{8h^2.u^4}.\cos.(v-v')$; in which we may change v into $v+90^d$, as in [4876b, c], without altering mv, $c'mv$; and we shall obtain the expression of

$$-\frac{9m'.u'^4}{8h^2.u^4}.\sin.(v-v'). \qquad [4880b]$$

This being multiplied by one third part of the expression [4878], gives the value of

$$-\frac{m'.u'^4}{8h^2.u^5}.3.\sin.(v-v').\frac{du}{dv} \quad [4880]. \qquad [4880c]$$

Now, the chief term of [4872] has the factor $\frac{3}{8}.m^2.\frac{a}{a'}$ [5094]; and that of [4878] is ce, or e, nearly, neglecting its sign. Hence, the greatest coefficient of this product, is,

$$\frac{3}{8}.m^2.\frac{a}{a'}.e = 0,0000004 \quad [5117, 5120]; \qquad [4880d]$$

which, in seconds, is less than $0^s,09$. This is insensible, and it is not increased by integration in [4847]. The same may be inferred, relative to the term of [4880], depending [4880e]
on the angle $3v-3v'$. Hence, we may conclude, that the expression [4880] may be neglected, as in [4881].

* (2802) The first term of $\left(\frac{dQ}{dv}\right)$ [4809], being substituted in [4881'], produces the expression [4882]; and we have already seen, that the expression [4870] gives that in [4883a]
[4876e]; by changing $2v$ into $2v+90^d$, according to the method proposed in [4876'] or [4883'].

† (2803) Retaining terms of the third order in [4878c], and multiplying by 2, we get,

$$\frac{2}{u} = 2a.\{1-(x_1+x_2+x_3)+x_1^2+2x_1x_2-x_1^3\}. \qquad [4884a]$$

Substituting the values [4866b], we obtain,

[4884] $$\frac{2}{u}=2a.\left\{\begin{array}{l} 1-\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2 \\ -e.(1-\tfrac{1}{4}e^2-\tfrac{1}{2}\gamma^2).\cos.(cv-\varpi) \\ +\tfrac{1}{2}e^2.\cos.(2cv-2\varpi) \\ +\tfrac{1}{4}\gamma^2.\cos.(2gv-2\theta) \\ -\tfrac{1}{4}e\gamma^2.\cos.(2gv-cv-2\theta+\varpi) \end{array}\right\}.$$

Hence we shall have,*

[4884b] $$\begin{array}{rlll} 1-(x_1+x_2+x_3)= & 1-e^2-\tfrac{1}{4}\gamma^2-e.(1+e^2).\cos.cv & & +\tfrac{1}{4}\gamma^2.\cos.2gv \\ x_{\prime}^2= & +\tfrac{1}{2}e^2 & +\tfrac{1}{2}e^2.\cos.2cv & \\ 2x_{\underset{2}{\prime}}x_1= & \qquad -e.(-2e^2-\tfrac{1}{2}\gamma^2).\cos.cv & & -\tfrac{1}{4}e\gamma^2.\cos.(2gv-cv) \\ -x_{\prime}^3= & \qquad -e.(\ \tfrac{3}{4}e^2\qquad).\cos.cv. & & \end{array}$$

The sum of these, gives the terms between the braces in [4881a, 4884].

* (2804) Multiplying together the second members of [1876e, 4884], we obtain the

[4885a] expression of $\frac{3m'.u'^3}{h^2.u^4}.\sin.(2v-2v')$; and the factor without the braces becomes $3\overset{2}{m}.\frac{a}{a_{\prime}}$, as in [4885]. The products of the terms between the braces, are found in the following table;

[4885b] in which the first column contains the terms of [4884]; the second column, the terms of [4876e]; and the third column, their respective products, reduced by [18, 19] Int.; using the abridged notation [4821f];

(Col. 1.) [4884].	(Col. 2.) [4876e]	(Col. 3.) Corresponding terms of $\frac{3m'.u'^3}{h^2.u^4}.\sin.(2v-2v')$.	
1	{ All the terms. }	the whole function [4876e] between the braces	1
$-\frac{1}{2}e^2-\frac{1}{4}\gamma^2$	1	$(-\frac{1}{2}e^2-\frac{1}{4}\gamma^2).\sin.(2v-2mv)$	2
	2	$+\frac{1}{2}(3+4m).e.(\frac{1}{2}e^2+\frac{1}{4}\gamma^2).\sin.(2v-2mv-cv)$	3
$-e.\cos.cv$	1	$+\frac{1}{2}e.(1+e^2+\frac{1}{4}\gamma^2-\frac{5}{2}e'^2).\{-\sin.(2v-2mv+cv)-\sin.(2v-2mv-cv)\}$	4
	2	$+\frac{1}{4}(3+4m).e^2.\{\sin.(2v-2mv)-\sin.(2cv-2v+2mv)\}$	5
	3	$+\frac{1}{4}(3-4m).e^2.\{\sin.(2v-2mv)+\sin.(2cv+2v-2mv)\}$	6
[4885c]	4	$+\frac{7}{4}ee'.\{-\sin.(2v-2mv+cv-c'mv)-\sin.(2v-2mv-cv-c'mv)\}$	7
	5	$+\frac{1}{4}ee'.\{+\sin.(2v-2mv+cv+c'mv)+\sin.(2v-2mv-cv+c'mv)\}$	8
	11	$-\frac{1}{8}(6+15m+8m^2).e^3.\sin.(2v-2mv-cv)$	9
	13	$-\frac{1}{16}(3+2m).e\gamma^2.\sin.(2v-2mv-2gv+cv)$	10
$+e(\frac{1}{4}e^2+\frac{1}{2}\gamma^2)\cos.cv$	1	$+(\frac{1}{8}e^3+\frac{1}{4}e\gamma^2).\sin.(2v-2mv-cv\qquad)$	11
$+\frac{1}{2}e^2.\cos.2cv$	1	$-\frac{1}{4}e^2.\sin.(2cv-2v+2mv)+\frac{1}{4}e^2.\sin.(2cv+2v-2mv)$	12
	3	$-\frac{1}{8}(3-4m).e^3.\sin.(2v-2mv-cv)$	13
$+\frac{1}{4}\gamma^2.\cos.2gv.$	1	$+\frac{1}{8}\gamma^2.\sin.(2gv+2v-2mv)-\frac{1}{8}\gamma^2.\sin.(2gv-2v+2mv)$	14
	3	$-\frac{1}{16}(3-4m).e\gamma^2.\sin.(2v-2mv-2gv+cv)$	15
$-\frac{1}{4}e\gamma^2.\cos.(2gv-cv)$	1	$-\frac{1}{8}e\gamma^2.\sin.(2v-2mv-2gv+cv).$	16

$$
-\frac{3m'}{h^2}\cdot\int\frac{u'^3.dv}{u^4}\,.\sin.(2v-2v')
$$

$$
=3.\overline{m}^2.\frac{a}{a_{\prime}}.\left\{
\begin{array}{lr}
\frac{(1+2e^2-\frac{5}{2}e'^2)}{2-2m}.\cos.(2v-2mv) & 1 \\
-\frac{2.(1+m)}{2-2m-c}.\{1+\frac{3}{4}e^2-\frac{1}{4}\gamma^2-\frac{5}{2}e'^2\}.e.\cos.(2v-2mv-cv+\varpi) & 2 \\
-\frac{2.(1-m)}{2-2m+c}.e.\cos.(2v-2mv+cv-\varpi) & 3 \\
+\frac{7e'}{2.(2-3m)}.\cos.(2v-2mv-c'mv+\varpi') & 4 \\
-\frac{e'}{2.(2-m)}.\cos.(2v-2mv+c'mv-\varpi') & 5 \\
-\frac{7.(2+3m).ee'}{2.(2-3m-c)}.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') & 6 \\
-\frac{7.(2-3m).ee'}{2.(2-3m+c)}.\cos.(2v-2mv+cv-c'mv-\varpi+\varpi') & 7 \\
+\frac{(2+m).ee'}{2.(2-m-c)}.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') & 8 \\
+\frac{(2-m).ee'}{2.(2-m+c)}.\cos.(2v-2mv+cv+c'mv-\varpi-\varpi') & 9 \\
-\frac{(10+19m+8m^2)}{4.(2c-2+2m)}.e^2.\cos.(2cv-2v+2mv-2\varpi) & 10 \\
+\frac{(10-19m+8m^2)}{4.(2c+2-2m)}.e^2.\cos.(2cv+2v-2mv-2\varpi) & 11 \\
-\frac{(2+m)}{4.(2g-2+2m)}.\gamma^2.\cos.(2gv-2v+2mv-2\theta) & 12 \\
+\frac{(2-m)}{4.(2g+2-2m)}.\gamma^2.\cos.(2gv+2v-2mv-2\theta) & 13 \\
+\frac{17e'^2}{2.(2-4m)}.\cos.(2v-2mv-2c'mv+2\varpi') & 14 \\
-\frac{(5+m)}{4.(2-2m-2g+c)}.e\gamma^2.\cos.(2v-2mv-2gv+cv+2\theta-\varpi) & 15
\end{array}
\right\} \qquad [4885]
$$

The first line of this table includes the terms of the function [4876e], and by adding them to the remaining terms of [4885c], we get the terms of $\frac{3m'.u^3}{h^2.u^4}.\sin.(2v-2v')$; which ought to be [4885d]

The terms of this formula, depending on the angles $2cv-2v+2mv-2\varpi$ and
[4886] $2gv-2v+2mv-2\varpi$, have divisors of the order m; and they again acquire these divisors, by integration, in the expression of the moon's mean longitude; which reduces them to the second order; and this, it would
[4886'] seem, ought to make the inequalities relative to these angles become great. But we must observe, that, by [4853, &c.], the terms having for a divisor the square of the coefficient of v, in these angles, nearly destroy each other,
[4886''] in the expression of the mean longitude; so, that the inequalities in question, become of the third order, conformably to the result of observations, as will be seen hereafter [5576]. We may, therefore, for this reason, dispense
[4887] with the calculation of the terms multiplied by e^4, $e^2e'^2$, γ^4; because the

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equal to the differential of [4885] divided by $-dv$; or, in other words, it ought to be equal
[4885e] to the terms between the braces in [4885], changing *cos.* into *sin.*, and neglecting the divisors $2-2m$, $2-2m-c$, &c., which are introduced in [4885], by the integration. The comparison of the sums of the terms of [4876e, 4885e], with those of [4885], may be made, in most cases, by inspection, or by very slight reductions; and they will be found to agree,
[4885f] neglecting some terms of the third order, depending on angles which are not expressly included in [4885]; or, on angles, whose coefficients are not much increased by integration; as $2v-2mv+cv$, $2v-2mv+c'mv$, &c. The reductions, relative to the terms depending
[4885g] on the angle $2v-2mv-cv$, are rather more complicated than the others, on account of the great number of its terms. We have, therefore, placed these terms in the following table [4885l], in the order in which they occur in the functions [4876e, 4885e]; and have found
[4885h] their sum in [4885m]. Comparing this sum with the corresponding coefficient

$$-2.(1+m).(1+\tfrac{3}{4}e^2-\tfrac{1}{4}\gamma^2-\tfrac{5}{2}e'^2).e,$$

in the second line of [4885], we find that they nearly agree; their difference being equal to
[4885i] the very small quantity $2me.\tfrac{1}{16}e^2$, which may be considered as of the fifth order; and, as this is to be multiplied by the factor without the braces, which is of the order m^2, or of
[4885k] the second order, it becomes of the seventh order, which is usually neglected in this coefficient:

[4885l]

[4876e], line 2	$-2e.(\tfrac{3}{4}+\tfrac{6}{16}e^2\qquad-\tfrac{15}{8}e'^2)-2me.(1+\tfrac{1}{2}e^2\qquad-\tfrac{5}{2}e'^2)$
[4885e], line 3	$-2e.(\;-\tfrac{6}{16}e^2-\tfrac{3}{16}\gamma^2\qquad)-2me.(\;-\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2\qquad)$
4	$-2e.(\tfrac{1}{4}+\tfrac{4}{16}e^2+\tfrac{1}{16}\gamma^2-\tfrac{5}{8}e'^2)$
9	$-2e.(\;+\tfrac{6}{16}e^2\qquad)-2me.(\;+\tfrac{15}{16}e^2\qquad)$
11	$-2e.(\;-\tfrac{1}{16}e^2-\tfrac{2}{16}\gamma^2\qquad)$
13	$-2e.(\;+\tfrac{3}{16}e^2\qquad)-2me.(\;-\tfrac{4}{16}e^2\qquad).$

[4885m] Sum is $=-2e.(1+\tfrac{3}{4}e^2-\tfrac{1}{4}\gamma^2-\tfrac{5}{2}e'^2)-2me.(1+\tfrac{11}{16}e^2-\tfrac{1}{4}\gamma^2-\tfrac{5}{2}e'^2).$

quantities of the fourth order, which result, after integration, nearly destroy each other.

The integral $\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$ [4754], contains also the following term,* [4887]

$$-\frac{3m'}{4h^2}.\int\frac{u'^4.dv}{u^5}.\sin.(v-v'). \qquad [4888]$$

This quantity, by development, produces the following expression,†

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* (2805) The second term of $\left(\frac{dQ}{dv}\right)$ [4809], namely, $-\frac{3m'.u'^4}{8u^3}.\sin.(v-v')$, being multiplied by $\frac{2dv}{h^2.u^2}$, produces, in $\frac{2}{h^2}.\left(\frac{dQ}{dv}\right)\frac{dv}{u^2}$, the term, $-\frac{3m'}{4h^2}.\frac{u'^4.dv}{u^5}.\sin.(v-v')$; [4887a]
whose integral is as in [4888].

† (2806) We may change v into $v+90^d$, in [4872], in the parts which are not connected with mv, or $c'mv$, upon the same principles as in [4876a, &c.]. By this means, [4889a]
the expression [4872], with the addition of the two terms [4870i], becomes as in [4889b].
Multiplying [4884] by $\frac{1}{3}$, we get [4889c]; always using the abridged notation [4821f], [4889a']
which will frequently be done, in the commentary on this book, without any particular notice, that the angles ϖ, ϖ', θ, are omitted;

$$-\frac{9m'.u'^4}{8h^2.u^4}.\sin.(v-v')=-\frac{9\overline{m}^2}{8a_{\prime}}.\frac{a}{a'}.\left\{\begin{array}{l}(1+2e^2+2e'^2).\sin.(v-mv)\\ \quad+2e.\sin.(cv-v+mv)-2e.\sin.(cv+v-mv)\\ \quad+e'.\sin.(v-mv+c'mv)+3e'.\sin.(v-mv-c'mv)\end{array}\right\}; \qquad [4889b]$$

$$\frac{2}{3u}=\tfrac{2}{3}a.\{(1-\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2)-e.\cos.cv+\&c.\}. \qquad [4889c]$$

The product of these two expressions, retaining terms of the same form and order as in [4889], becomes as in [4889h]. For the product of the two factors without the braces, is evidently equal to $\frac{3\overline{m}^2}{4}.\frac{a}{a_{\prime}}.\frac{a}{a'}$, as in [4889$h$]. We shall now multiply the terms between [4889d]
the braces in [4889b], by those in [4889c]. The first line of [4889b], being multiplied by the factor $(1-\frac{1}{2}e^2-\frac{1}{4}\gamma^2)$ [4889c], produces the expression,

$$(1+\tfrac{3}{2}e^2-\tfrac{1}{4}\gamma^2+2e'^2).\sin.(v-mv); \qquad [4889e]$$

and the term $-e.\cos cv$ [4889c], being multiplied by *each* of the terms depending on e, in the second line of [4889b], produces a term of the form $e^2.\sin.(v-mv)$; adding these [4889f]
two terms to those in [4889e], we get,

$$(1+\tfrac{7}{2}e^2-\tfrac{1}{4}\gamma^2+2e'^2).\sin.(v-mv), \quad \text{as in [4889}h]. \qquad [4889g]$$

[4889] $$-\frac{3m'}{4h^2}.\int\frac{u'^4.dv}{u^5}.\sin.(v-v')=\frac{3\overline{m}^2}{4}.\frac{a}{a_{,}}.\frac{a}{a'}.\left\{\begin{array}{ll}\frac{1+\frac{7}{2}e^2-\frac{1}{4}\gamma^2+2e'^2}{1-m}.\cos.(v-mv) & 1\\ +e'.\cos.(v-mv+c'mv-\varpi') & 2\\ +\frac{3e'}{1-2m}.\cos.(v-mv-c'mv+\varpi') & 3\end{array}\right\};$$

the other terms of the integral [4887′] may, in this part, be neglected. This being premised, if we observe, that the expression of u [4826] gives,*

[4890] $$\frac{ddu}{dv^2}+u=\frac{1}{a}.\left\{\begin{array}{ll}1+e^2+\frac{1}{4}\gamma^2 & 1\\ +(1-e^2).e.\cos.(cv-\varpi) & 2\\ +\frac{(4g^2-1)}{4}.\gamma^2.\cos.(2gv-2\theta) & 3\end{array}\right\};$$

[4891] the term $\left(\frac{ddu}{dv^2}+u\right).\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$, of the equation [4754], will produce, by its development,†

Lastly, the first term, or unity [4889e], being multiplied by the terms in the third line of [4889b], produces those depending on e', in [4889h];

[4889h] $$-\frac{3m'}{4h^2}.\frac{u'^4}{u^5}.\sin(v-v')=-\tfrac{3}{4}.\overline{m}^2.\frac{a}{a_{,}}.\frac{a}{a'}.\left\{\begin{array}{l}(1+\frac{7}{2}e^2-\frac{1}{4}\gamma^2+2e'^2).\sin.(v-mv)\\ +e'.\sin.(v-mv+c'mv-\varpi')\\ +3e'.\sin.(v-mv-c'mv+\varpi')\end{array}\right\}.$$

Multiplying this by dv, integrating, and putting in the divisors $c'=1$, it becomes as in
[4889i] [4889]. We may remark, that the term $-\frac{3}{2}\gamma^2$, which we have connected with the factor
[4889k] $(1+2e^2+2e'^2)$, in [4870z′, 4872], ought also to be connected with that in [4889h, 4889]; so that, instead of $1+\frac{7}{2}e^2-\frac{1}{4}\gamma^2+2e'^2$, we may write $1+\frac{7}{2}e^2-1\frac{1}{2}\gamma^2+2e'^2$.

* (2807) The second differential of u [4826], taken relatively to v, and divided by dv^2, gives,

[4890a] $$\frac{ddu}{dv^2}=\frac{1}{a}.\{-c^2e.(1+e^2).\cos.(cv-\varpi)+\frac{4g^2}{4}.\gamma^2.\cos.(2gv-2\theta)\}.$$

[4890b] Adding this to the expression [4826], and neglecting terms of the fifth order $(1-c^2).e^3$ [4828e], we get [4890].

† (2808) The terms of the integral $\frac{2}{h^2}.\int\frac{dQ}{dv}.\frac{dv}{u^2}$, are contained in [4885, 4889].
[4892a] These two functions must be multiplied by the expression of $\frac{ddu}{dv^2}+u$ [4890]; and the

$$\left(\frac{ddu}{dv^2}+u\right)\cdot\frac{2}{h^2}\cdot\int\frac{dQ}{dv}\cdot\frac{dv}{u^2}$$

$$
=\frac{3\bar{m}^2}{a_{,}}\cdot\left\{\begin{array}{ll}
\frac{(1+3e^2+\frac{1}{4}\gamma^2-\frac{5}{2}e'^2)}{2-2m}\cdot\cos.(2v-2mv) & 1\\
+\left\{\frac{(1-e^2)}{4.(1-m)}-\frac{2.(1+m)}{2-2m-c}\cdot(1+\frac{7}{4}e^2-\frac{5}{2}e'^2)\right\}\cdot e.\cos.(2v-2mv-cv+\varpi) & 2\\
-\frac{2.(1-m)}{2-2m+c}\cdot e.\cos.(2v-2mv+cv-\varpi) & 3\\
+\frac{7e'}{2.(2-3m)}\cdot\cos.(2v-2mv-c'mv+\varpi') & 4\\
-\frac{e'}{2.(2-m)}\cdot\cos.(2v-2mv+c'mv-\varpi') & 5\\
-\frac{7.(2+3m)}{2.(2-3m-c)}\cdot ee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') & 6\\
-\frac{7.(2-3m)}{2.(2-3m+c)}\cdot ee'.\cos.(2v-2mv+cv-c'mv-\varpi+\varpi') & 7\\
+\frac{(2+m)}{2.(2-m-c)}\cdot ee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') & 8\\
+\frac{(2-m)}{2.(2-m+c)}\cdot ee'.\cos.(2v-2mv+cv+c'mv-\varpi-\varpi') & 9\\
-\frac{(10+19m+8m^2)}{4.(2c-2+2m)}\cdot e^2.\cos.(2cv-2v+2mv-2\varpi) & 10\\
+\frac{(10-19m+8m^2)}{4.(2c+2-2m)}\cdot e^2.\cos.(2cv+2v-2mv-2\varpi) & 11\\
+\left\{\frac{(4g^2-1)}{16.(1-m)}-\frac{(2+m)}{4.(2g-2+2m)}\right\}\cdot\gamma^2.\cos.(2gv-2v+2mv-2\theta) & 12\\
+\left\{\frac{(4g^2-1)}{16.(1-m)}+\frac{(2-m)}{4.(2g+2-2m)}\right\}\cdot\gamma^2.\cos.(2gv+2v-2mv-2\theta) & 13\\
+\frac{17e'^2}{2.(2-4m)}\cdot\cos.(2v-2mv-2c'mv+2\varpi') & 14\\
-\left\{\frac{(5+m)}{4.(2-2m-2g+c)}+\frac{3.(1-m)}{4(2-2m+c)}\right\}\cdot e\gamma^2.\cos.(2v-2mv-2gv+cv+2\theta-\varpi) & 15\\
+\frac{(1+\frac{9}{2}e^2+2e'^2)}{4.(1-m)}\cdot\frac{a}{a'}\cdot\cos.(v-mv) & 16\\
+\frac{1}{4}\cdot\frac{a}{a'}\cdot e'.\cos.(v-mv+c'mv-\varpi') & 17\\
+\frac{3}{4.(1-2m)}\cdot\frac{a}{a'}\cdot e'.\cos.(v-mv-c'mv+\varpi') & 18
\end{array}\right\}. \qquad [4892]
$$

sum of the products will be equal to the function [4892]. In finding the products of the [4892a]

[4893] 7. The term $-\frac{1}{h^2 . (1+s^2)^{\frac{3}{2}}}$, of the expression

[4894] $$-\frac{1}{h^2} . \left(\frac{dQ}{du}\right) - \frac{s}{h^2 u} . \left(\frac{dQ}{ds}\right) \quad [4808],$$

[4892b] functions [1889, 4890], we may neglect the second and third lines of [4890]; for $(1-c^2).e$ is of the *third* order, γ^2 is of the *second* order; and these are to be multiplied by the factor

[4892c] $\overline{m}^2 . \frac{a}{a'}$ [4889], which is of the *fourth* order; by this means, these terms become so small, that they may be neglected, and the function [4890] is reduced to its first term

[4892d] $\frac{1}{a}.(1+e^2+\frac{1}{4}\gamma^2)$. Multiplying this by the terms in [4889], lines 1, 2, 3, we obtain respectively the terms in [4892], lines 16, 17, 18. In the term depending on $\cos.(v-mv)$, in line 16, we

[4892e] may, for greater accuracy, decrease the factor $1+\frac{3}{2}e^2+2e'^2$, by $\frac{8}{3}\gamma^2$, as in [4889*i*].

We shall now compute the product of the functions [4885, 4890]. In the first place, the product of the factors, without the braces, is

[4892f] $$3\,\overline{m}^2 . \frac{a}{a_{\prime}} \times \frac{1}{a} = \frac{3\,\overline{m}^2}{a_{\prime}}; \quad \text{as in [1892].}$$

[4892g] The multiplication of the factors, between the braces, is made, term by term, as in the following table; in which, the first column contains the terms of [4890], the second column the terms of [4885], and the third column the corresponding products of the terms between the braces, in these lines of the two functions respectively: observing, that $4g^2-1=3$, nearly:

	(Col. 1.) Terms of [4890].	(Col. 2.) Terms of [4885].	(Col. 3.) Products of these terms.	
	1	whole of [4885]	whole function [4885] between the braces	1
	$e^2+\frac{1}{4}\gamma^2$	1	$+\frac{(e^2+\frac{1}{4}\gamma^2)}{2-2m}.\cos.(2v-2mv)$	2
[4892h]		2	$-\frac{2.(1+m)}{2-2m-c}.(e^2+\frac{1}{4}\gamma^2).e.\cos.(2v-2mv-cv)$	3
	$(1-c^2).e.\cos.cv$	1	$+\frac{(1-c^2)}{4.(1-m)}.e.\cos.(2v-2mv-cv)+\&c.$	4
	$\frac{(4g^2-1)}{4}.\gamma^2.\cos.2gv$	1	$+\frac{(4g^2-1)}{16(1-m)}.\gamma^2.\{\cos.(2gv-2v+2mv)+\cos.(2gv+2v-2mv)\}$	5
		3	$-\frac{3.(1-m)}{4.(2-2m+c)}.e\gamma^2.\cos.(2v-2mv-2gv+cv)+\&c.$	6

Connecting the terms from lines 2 to 6 of this table, with those in line 1, or the lines between the braces of [1885]; we get the corresponding terms between the braces, of the function [4892].

becomes, by neglecting quantities of the fourth order,*

$$-\frac{1}{a_{\prime}}\cdot\left\{1+e^2+\frac{\gamma^2}{4}+\tfrac{3}{4}\gamma^2.(1+e^2-\tfrac{1}{4}\gamma^2).\cos.(2gv-2\theta)+\beta''\right\}+\frac{3s\delta s}{h^2}; \qquad [4895]$$

β'' being a function of the fourth dimension in e, γ; and *δs the part of s arising from the disturbing force.* We shall see, in [5596], that δs is of the following form;† [4896]

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* (2809) Developing the expression [4893], according to the powers of s, it becomes $-h^{-2}.(1-\frac{3}{2}s^2+\frac{15}{8}s^4-\&c.)$. If we substitute in this the value of s [4818], augmented by the term δs, and neglect terms of the order δs^2, which are noticed in [4958, &c.], we shall find, that the part of the function [4893], depending on δs, is equal to the differential of the expression [4893a], relative to δ, which is $-h^{-2}.(-3s\delta s+\frac{15}{2}s^3\delta s-\&c.)$. Neglecting terms of the order $s^3\delta s$, it becomes $3h^{-2}.s\delta s$, as in the last term of [4895]. Now, the value of s [4818] gives, by means of [1, 3] Int., [4893a] [4893b]

$$1-\tfrac{3}{2}s^2=(1-\tfrac{3}{4}\gamma^2)+\tfrac{3}{4}\gamma^2.\cos.2gv; \qquad \tfrac{15}{8}s^4=\tfrac{45}{64}\gamma^4-\tfrac{15}{16}\gamma^4.\cos.2g\theta+\&c.; \qquad [4893c]$$

$$1-\tfrac{3}{2}s^2+\tfrac{15}{8}s^4-\&c.=(1-\tfrac{3}{4}\gamma^2)+\tfrac{3}{4}\gamma^2.(1-\tfrac{5}{4}\gamma^2).\cos.2gv+\text{ terms of the 4th order.} \qquad [4893d]$$

And, from h^2 [4863], we get,

$$-h^{-2}.=-\frac{1}{a_{\prime}}.\{(1+e^2+\gamma^2)+\text{terms of the 4th order}\}. \qquad [4893e]$$

Multiplying together the two expressions [4893d,e], we get the part of the function [4893a], which is independent of δs, as in [4895].

† (2810) The form here assumed for δs is easily obtained from a comparison of the equations [4754, 4755], by which u, s, are determined, with the preceding development of the terms of u. For the equation [4754] contains the function $-\frac{1}{h^2}.\left(\frac{dQ}{du}\right)-\frac{s}{h^2u}.\left(\frac{dQ}{ds}\right)$, whose terms have been developed in [4866, 4870, 4872, &c.]; and the equation [4755], by which s is determined, contains *the same function, multiplied by* $\frac{s}{u}$. Now, the chief term of the factor $\frac{s}{u}$ is equal to $a\gamma.\sin.(gv-\theta)$, as is evident from [4818, 4791]; and, if we multiply the terms we have just mentioned [4866, 4870, 4872, &c.] by $a\gamma.\sin.(gv-\theta)$, we shall obtain the most important terms of [4755], depending on the function [4897c]. Thus, the first term of [4866] produces a term depending on $\sin.(gv-\theta)$, which may be considered as being included in the form [4818]. The second term of [4866] produces the angles $gv\pm cv$ [4897], lines 3, 4. The third term of [4866] produces the angles $gv\pm c'mv$ [4897], lines 8, 9. The first term of [4870] produces the angles $2v-2mv\pm gv$ [4897], lines 1, 2. The second term of [4870] produces the angles $2v-2mv\pm gv-cv$ [4897], lines 6, 7. The third line of [4870] produces the fifth line of [4897]; and so on, [4897a] [4897b] [4897c] [4897d] [4897e] [4897f] [4897g]

$$\delta s = B_1^{(0)}.\gamma.\sin.(2v-2mv-gv+\theta)$$ 1

$$+B_2^{(1)}.\gamma.\sin.(2v-2mv+gv-\theta)$$ 2

$$+B_2^{(2)}.c\gamma.\sin.(gv+cv-\theta-\varpi)$$ 3

$$+B_2^{(3)}.c\gamma.\sin.(gv-cv-\theta+\varpi)$$ 4

Assumed form of δs.

$$+B_2^{(4)}.e\gamma.\sin.(2v-2mv-gv+cv+\theta-\varpi)$$ 5

$$+B_2^{(5)}.e\gamma.\sin.(2v-2mv+gv-cv-\theta+\varpi)$$ 6

$$+B_2^{(6)}.c\gamma.\sin.(2v-2mv-gv-cv+\theta+\varpi)$$ 7

$$+B_1^{(7)}.e'\gamma.\sin.(gv+c'mv-\theta-\varpi')$$ 8

[4897] $$+B_1^{(8)}.e'\gamma.\sin.(gv-c'mv-\theta+\varpi')$$ 9

$$+B_1^{(9)}.e'\gamma.\sin.(2v-2mv-gv+c'mv+\theta-\varpi')$$ 10

$$+B_1^{(10)}.e'\gamma.\sin.(2v-2mv-gv-c'mv+\theta+\varpi')$$ 11

$$+B_0^{(11)}.e^2\gamma.\sin.(2cv-gv-2\varpi+\theta)$$ 12

$$+B_1^{(12)}.e^2\gamma.\sin.(2v-2mv-2cv+gv+2\varpi-\theta)$$ 13

$$+B_1^{(13)}.c^2\gamma.\sin.(2cv+gv-2v+2mv-2\varpi-\theta)$$ 14

$$+B_2^{(14)}.\frac{a}{a'}.\gamma.\sin.(gv-v+mv-\theta)$$ 15

$$+B_2^{(15)}.\frac{a}{a'}.\gamma.\sin.(gv+v-mv-\theta).$$ 16

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for other terms. Hence we see, that the forms of the angles in [4897], are given *a priori* [4897h] by the theory; and they agree with the results of observation [5596]. The differential equation in s [4755], is similar to that of u [4754], and may be reduced to the form [4897m], which is similar to [4845]. For the chief term of s is given in [4818], and if we [4897i] suppose the other terms of s to be represented by δs, we shall have $s=\gamma.\sin.(gv-\theta)+\delta s$. Its differential gives $\frac{dds}{dv^2}=-g^2.\gamma.\sin.(gv-\theta)+\frac{d^2.\delta s}{dv^2}$. Multiplying the first of these [4897k] expressions by g^2. and adding it to the second, we get $\frac{dds}{dv^2}+g^2.s=\frac{d^2.\delta s}{dv^2}+g^2.\delta s$; and if [4897l] we put the second member of this expression equal to $-\Pi'$, we shall get,

[4897m] $$\frac{dds}{dv^2}+g^2.s+\Pi'=0.$$

This is of the same form as [4845], g taking the place of N, and differing from unity by quantities of the order m^2 [4828c, 4845']. Moreover, Π' may be considered as a series of [4897n] terms, whose general form is $k'.\sin.(iv-\theta)$, like that in [4846]; and the part of s, relative to this sine, is represented as in [4847, &c.] by

The number placed below any one of the letters B, indicates the order of that letter. Thus, $B_2^{(2)}$ is of the second order; $B_1^{(0)}$ is of the first order; and [4898]
$B_0^{(11)}$ is finite. We may observe, that this takes place according as the number [4898′]
by which v is multiplied, in the corresponding sine, differs from unity, by a
finite number, by a quantity of the order m, or by a quantity of the order m^2, [4899]
respectively; because the integration [4897o] causes the terms to acquire a [4900]
divisor of the same order. This being premised, we shall have,*

$$s=\frac{k'}{i^2-N^2}\cdot\sin.(iv-\theta);$$ [4897o]

so that these terms may be much increased by this integration, when i is nearly equal to unity. From the similarity of the equations [4754, 4755] it is evident, that the terms of Π'
[4897m], depending on the disturbing force of the sun, must have the same factor $\overline{m}^2$, as [4897p]
the functions [4866, 4870, 4872, &c.]; and $\overline{m}^2$ is of the order m^2 [5094], or of the
second order. This factor is divided by i^2-N^2, in finding the value of s [4897o], or that [4897q]
of δs [4897]; and, as i^2-N^2 may be considered as of the same order as $i^2-g^2=i^2-1-\frac{3}{2}m^2$
[4828e]; *the order of the symbol B will be represented by* $\frac{m^2}{i^2-1-\frac{3}{2}m^2}$. Hence, it [4897$r$]
appears, that if i differs considerably from unity, the corresponding symbol B will be of the second order, as in [4897], lines 2, 3, 4, 5, &c.; using the values of c, g [4828e]. In the first term of [4897], the coefficient of v is $i=2-2m-g=1-2m$ nearly; hence,
$i^2-1-\frac{3}{2}m^2$ is of the order m, and the corresponding value of B [4897r] is of the order [4897s]
m, represented by $B_1^{(0)}$; and the same occurs in lines 8—11 [4897]. In line 12 we have,
$i=2c-g=1-\frac{15}{4}m^2$ [4828e]; hence, the divisor of the expression [4897r] becomes of
the order m^2, and the corresponding value of B is reduced to the order m^0, or a *finite* order, [4897t]
as it is called by the author in [4898′], and is represented by $B_0^{(11)}$. If we compare the indices of B [4897], with their values, computed in [5122—5214], we shall find they generally agree; but the term $B_2^{(3)}$ [5179] is nearly of the *first*, instead of the *second* order; $B_1^{(12)}$ is of the second order, &c. [4897u]

* (2811) Substituting in the first member of [4901], the values of h^{-2}, s [4893c, 4897i], and neglecting terms of the order δs^2, we get [4901a]. If we also neglect terms of the fifth order, it becomes as in [4901b];

$$\frac{3s.\delta s}{h^2}=\frac{3}{a_{\prime}}.\gamma\,\delta s.\sin.(gv-\theta)\times\{1+e^2+\gamma^2+\text{ terms of the fourth order}\}$$ [4901a]

$$=\frac{3}{a_{\prime}}.\gamma\,\delta s.\sin.(gv-\theta).$$ [4901b]

We must substitute in this last expression, the value of δs [4897], and we shall get [4901].
If any term of δs be represented by $C.\sin.V$, the two corresponding terms of [4901b] [4901c]

$$\frac{3s.\delta s}{h^2} = -\frac{3}{2a_{\prime}}.\{B_1^{(0)} - B_2^{(1)}\}.\gamma^2.\cos.(2v - 2mv) \qquad 1$$

$$+\frac{3}{2a_{\prime}}.B_1^{(0)}.\gamma^2.\cos.(2v - 2mv - 2gv + 2\theta) \qquad 2$$

$$+\frac{3}{2a_{\prime}}.\{B_2^{(2)} + B_2^{(3)}\}.e\gamma^2.\cos.(cv - \varpi) \qquad 3$$

$$-\frac{3}{2a_{\prime}}.B_2^{(3)}.e\gamma^2.\cos.(2gv - cv - 2\theta + \varpi) \qquad 4$$

$$+\frac{3}{2a_{\prime}}.B_2^{(4)}.e\gamma^2.\cos.(2v - 2mv - 2gv + cv + 2\theta - \varpi) \qquad 5$$

[4901]
$$+\frac{3}{2a_{\prime}}.\{B_2^{(5)} - B_2^{(6)}\}.e\gamma^2.\cos.(2v - 2mv - cv + \varpi) \qquad 6$$

$$+\frac{3}{2a_{\prime}}.\{B_1^{(7)} + B_1^{(8)}\}.e'\gamma^2.\cos.(c'mv - \varpi') \qquad 7$$

$$-\frac{3}{2a_{\prime}}.B_1^{(9)}.e'\gamma^2.\cos.(2v - 2mv + c'mv - \varpi') \qquad 8$$

$$-\frac{3}{2a_{\prime}}.B_1^{(10)}.e'\gamma^2.\cos.(2v - 2mv - c'mv + \varpi') \qquad 9$$

$$-\frac{3}{2a_{\prime}}.B_0^{(11)}.e^2\gamma^2.\cos.(2cv - 2\varpi) \qquad 10$$

$$+\frac{3}{2a_{\prime}}.\{B_2^{(14)} + B_2^{(15)}\}.\frac{a}{a_{\prime}}.\gamma^2.\cos.(v - mv). \qquad 11$$

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will be

[4901d]
$$\frac{3}{2a_{\prime}}.\gamma.C.\cos\{(gv - \theta) \backsim \Gamma\} - \frac{3}{2a_{\prime}}.\gamma.C.\cos.\{gv - \theta + \Gamma\};$$

but it is not, in general, found to be necessary to notice more than one of these terms. The
[4901e] only cases in which the author has noticed both terms, are those depending on $B_1^{(0)}$, $B_2^{(3)}$ [4897], lines 1—4. The neglected terms are generally smaller than those which are retained, or they are such as depend on angles that have not been usually noticed, because their coefficients do not increase by the integrations. For, the function [4901] forms part of
[4901f] the expression of Π [4902, or 4845]; and its coefficients may be increased by the divisor $i^2 - N^2$ [4847, &c.], when i differs but little from unity; as is the case in lines 3—6, 11 [4901]. To estimate roughly the order of the terms, which are not increased by the integrations, and are neglected as in [4901], we may observe, that they produce terms of a
[4901g] similar order in u [4847], and in the lunar parallax [5309, &c.]. Now, if we put $\frac{1}{a_{\prime}}$ equal

If we connect together the different terms which we have developed, we shall find, that the equation [4754] becomes of the following form,*

$$0=\frac{ddu}{dv^2}+u+\Pi\,;$$ [4902]

Π *being a rational and integral function of constant quantities, and of sines and cosines of angles proportional to* v; *but, as we propose to notice all the* [4903]

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to the constant term of the lunar parallax $3424^s,16$ [5331], and use the values of c, e', γ [5194, 5117], also $\frac{a}{a'}=\frac{1}{400}$ [5221], we shall get, very nearly,

$$\frac{3}{2a_{,}}\cdot\gamma^2=40^s;\qquad \frac{3}{2a_{,}}\cdot e\,\gamma^2=2^s,3;\qquad \frac{3}{2a_{,}}\cdot e'\gamma^2=0^s,7;$$
$$\frac{3}{2a_{,}}\cdot e^2\gamma^2=0^s,1;\qquad \frac{3}{2a_{,}}\cdot\frac{a}{a'}\cdot\gamma^2=0^s,1.$$ [4901h]

The first of these expressions, being multiplied by the very small quantity $B_2^{(1)}$ [5177], becomes insensible; and it is retained in [4901] line 1, merely because there is no inconvenience in doing it, since it is found necessary to notice the angle $2v-2mv$, in consequence of the magnitude of the other term $B_1^{(0)}$. In like manner, the term

$$\frac{3}{2a_{,}}\cdot e\gamma^2.\,B_2^{(2)}=-0^s,01\quad[5178, 4901h],$$ [4901i]

is nearly insensible; but it is retained in [4901] line 3, because the coefficient c, in the angle $cv-\varpi$, differs but very little from unity [4828e], and it is increased by integration; which is not the case with the coefficient depending on the other angle $2gv+cv-2\theta-\varpi$, with which $B_2^{(2)}$ is connected. One of the largest of the values of B, is that denoted by $B_1^{(7)}=0{,}07824$ [5183]; multiplying it by the coefficient $\frac{3}{2a_{,}}\cdot e'\gamma^2=0^s,7$, with which [4901k] it is connected in [4901] line 7, it becomes $0^s,05$; this is retained in the angle $e'mv-\varpi'$ [4901] line 7, because the divisor i^2-N^2 [4847] is nearly equal to unity; but it is neglected in the angle $2gv+c'mv-2\theta-\varpi'$; because it is considerably decreased by the [4901l] divisor i^2-N^2, which is nearly equal to 3. We may also observe, that it is of more importance to retain the terms depending on the angle $c'mv-\varpi'$, than those on $2gv+c'mv-2\theta-\varpi'$; because the terms introduced by the former, in the value of dt [4753], are increased by integration, in finding the value of t, in consequence of the smallness of the coefficient $e'm$ [4901m] of the angle v. Similar remarks may be made relative to the other terms, which are neglected or retained.

* (2812) Connecting together the terms [4866, 4870, 4872, 4892, 4895, 4901, &c.], depending on Q, and putting the sum equal to Π; then adding it to the terms of [4754], [4902a] which are independent of Q, it becomes as in [4902].

inequalities of the third order, and the quantities of the fourth order connected
[4903'] *with them, we must add to the preceding terms all those which depend on the square of the disturbing force, and become of these orders by integrations.* We shall now examine these new terms.

[4903''] 8. For this purpose *we shall suppose δu to be the part of u arising from the disturbing force;* and, that we have,*

Assumed form of δu.

[4904]

$$
\begin{aligned}
a\delta u = {} & A_2^{(0)}.\cos.(2v-2mv) && 1\\
& +A_1^{(1)}.e.\cos.(2v-2mv-cv+\varpi) && 2\\
& +A_2^{(2)}.e.\cos.(2v-2mv+cv-\varpi) && 3\\
& +A_2^{(3)}.e'.\cos.(2v-2mv+c'mv-\varpi') && 4\\
& +A_2^{(4)}.e'.\cos.(2v-2mv-c'mv+\varpi') && 5\\
& +A_2^{(5)}.e'.\cos.(c'mv-\varpi') && 6\\
& +A_1^{(6)}.ee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') && 7\\
& +A_1^{(7)}.ee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') && 8\\
& +A_1^{(8)}.ee'.\cos.(cv+c'mv-\varpi-\varpi') && 9\\
& +A_1^{(9)}.ee'.\cos.(cv-c'mv-\varpi+\varpi') && 10\\
& +A_2^{(10)}.e^2.\cos.(2cv-2\varpi) && 11\\
& +A_1^{(11)}.e^2.\cos.(2cv-2v+2mv-2\varpi) && 12\\
& +A_2^{(12)}.\gamma^2.\cos.(2gv-2\theta) && 13\\
& +A_1^{(13)}.\gamma^2.\cos.(2gv-2v+2mv-2\theta) && 14\\
& +A_2^{(14)}.e'^2.\cos.(2c'mv-2\varpi') && 15\\
& +A_0^{(15)}.e\gamma^2.\cos.(2gv-cv-2\theta+\varpi) && 16\\
& +A_1^{(16)}.e\gamma^2.\cos.(2v-2mv-2gv+cv+2\theta-\varpi) && 17\\
& +A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv) && 18\\
& +A_0^{(18)}.\frac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi') && 19\\
& +A_1^{(19)}.\frac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi') && 20
\end{aligned}
$$

[4904a] * (2813) The terms of $a\delta u$ [1904] are evidently of the same form as those of the function

The number 0, 1, or 2, placed below any one of the letters A, denotes, that it is of the order zero, or of the order m, or of the order m^2, respectively. [4905]
We shall here take into consideration the inequalities of the third order, and those of the fourth order, which can produce terms of the fourth order in the coefficients of the inequalities of the third order. [4905′] *We shall continue the approximation to a greater degree of accuracy, relative to the inequality depending on* $\cos.(v-m\,v)$. [4906] This being premised, we find, that the term $\frac{m'.u'^3}{2h^2.u^3}$ [4865′] gives, by its variation, the expression $-\frac{3m'.u'^3.\delta u}{2h^2.u^4}$; from [4907]
which we deduce the following function;*

Π [4902a]. The order of the coefficient A may be found by the formula $\frac{m^2}{i^2-1+3m^2}$, [4904b]
which is similar to that in [4897r], using for N^2 the value of $c^2=1-3m^2$, instead of g^2, which is used in [4897q, r]; i being the coefficient of v, in the angle corresponding to the coefficient A. [4904c] Thus, for $A^{0)}$ [4901], we have $i=2-2m$; hence A^0 is of the order m^2, or 2. For $A_1^{1)}$, we have $i=2-2m-c=1-m$, nearly; hence $A_1^{(1)}$ is of the order m, or 1; and so on, for the other coefficients of [4901]. If we compare these indices of A, with the values obtained by numerical calculation in [5122—5213], we shall find, that in general they are correctly marked. [4904d]

* (2814) The expression [4907], whose value is to be determined, may be put under the form

$$-\frac{3}{2a}\times\frac{2}{u}\cdot\frac{m'.u'^3}{2h^2.u^3}\times a\,\delta u; \qquad [4908a]$$

in which the second and third factors have been already computed in [4884, 4866]; we shall first find the product of these two factors, and then multiply it by $-\frac{3}{2a}$ and $a\,\delta u$. Now, if we multiply the factors without the braces, in [4884, 4866], by $-\frac{3}{2a}$, the product becomes [4908b]

$$-\frac{\overline{m}^2}{2a_{,}}.2a.\frac{3}{2a}=-\frac{3\overline{m}^2}{2a_{,}},$$

as in the second member of [4903f]. The products of the terms between the braces, in [4884, 4866], are found in the following table; in which the first column gives the terms of [4884]; the second column, the terms of [4866]; and the third column, the products of these terms respectively; using the abridged notation [4821f], and neglecting the same terms and angles as we have usually done; [4908c]

[4908]
$$-\frac{3m'.u'^3.\delta u}{2h^2.u^4}=-\frac{3\bar{m}^2.(1+\frac{3}{2}e'^2)}{2a_{\prime}}.\left\{\begin{array}{ll} a.\delta u \quad [4904] & 1 \\ -2A_2^{(0)}.e.\cos.(2v-2mv-cv+\varpi) & 2 \\ -2A_1^{(1)}.e^2.\cos.(2v-2mv-2cv+2\varpi) & 3 \\ +\frac{3}{2}A_1^{(1)}.ee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') & 4 \\ +\frac{3}{2}A_1^{(1)}.ee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') & 5 \\ +\frac{3}{2}\{A_1^{(8)}+A_1^{(9)}\}.ee'^2.\cos.(cv-\varpi) & 6 \\ +\frac{3}{2}A_1^{(17)}.\frac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi') & 7 \\ +\frac{3}{2}A_1^{(17)}.\frac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi') & 8 \\ +\frac{3}{2}A_0^{(18)}.\frac{a}{a'}.e'^2.\cos.(v-mv) & 9 \end{array}\right\}$$

[4908'] u' varies by means of the variation of v', which depends on the time t, and on its inequalities in functions of v [4822, or 4828]; but these inequalities
[4909] are multiplied by m, in the expression of v' [4837], and also, by e', in the expression of u' [4838]; we may, therefore, at first, neglect $\delta u'$, without

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(Col. 1.) Terms of [4884].	(Col. 2.) Terms of [4866].	(Col. 3.) Products of these terms.
1	whole of [4866]	whole of the function [4866]
$-\frac{1}{2}e^2-\frac{1}{4}\gamma^2$	1	$-\frac{1}{2}e^2-\frac{1}{4}\gamma^2$
	$-3e.\cos.cv$	$+(+\frac{3}{2}e^3+\frac{3}{4}e\gamma^2).\cos.cv$
	$+3e'.\cos.c'mv$	$+(-\frac{3}{2}e^2e'-\frac{3}{4}e'\gamma^2).\cos.c'mv$
$-e(1-\frac{1}{4}e^2-\frac{1}{2}\gamma^2)\cos.cv$	$1+e^2+\frac{1}{4}\gamma^2+\frac{3}{2}e'^2$	$-(1+\frac{3}{4}e^2-\frac{1}{4}\gamma^2+\frac{3}{2}e'^2).e.\cos.cv$
	$-3e.\cos.cv$	$+\frac{3}{2}e^2 \qquad +\frac{3}{2}e^2.\cos.2cv$
[4908d]	$+3e'.\cos.c'mv$	$-\frac{3}{2}ee'.\cos.(cv-c'mv)-\frac{3}{2}ee'.\cos.(cv+c'mv)$
	$-\frac{9}{2}ee'.\cos.(cv+c'mv)$	$+\frac{9}{4}e^2e'.\cos.c'mv+$&c.
	$-\frac{9}{2}ee'.\cos.(cv-c'mv)$	$+\frac{9}{4}e^2e'.\cos.c'mv+$&c.
	$+3e^2.\cos.2cv$	$-\frac{3}{2}e^3.\cos.cv+$&c.
	$+\frac{3}{4}\gamma^2.\cos.2gv$	$-\frac{3}{8}e\gamma^2.\cos.(2gv-cv)+$&c.
$+\frac{1}{2}e^2.\cos.2cv$	$1-3e.\cos.cv$	$-\frac{3}{4}e^3.\cos.cv \qquad +\frac{1}{2}e^2.\cos.2cv+$&c.
$+\frac{1}{4}\gamma^2.\cos.2gv.$	$1-3e.\cos.cv$	$\frac{1}{4}\gamma^2.\cos.2gv \qquad -\frac{3}{8}e\gamma^2.\cos.(2gv-cv)+$&c.

Connecting together the terms which are explicitly given in this table, with those between
[4908e] the braces in [4866], which are included in the first line of this table; the sum becomes equal to the expression between the braces in [4908f]; and the factor of $a\delta u$ [4908a] becomes as in the second member of [4908f]:

any sensible error. We shall hereafter [4947, &c.] notice the term of this variation, which depends upon the action of the moon upon the earth. [4909']

$$-\frac{3m'.u'^3}{2h^2.u^4}\cdot\frac{1}{a}=-\frac{3\overline{m}^2}{2a_{\prime}}\cdot\left\{\begin{array}{l}1+2e^2+\frac{3}{2}e'^2\\ +(-4e-3e^3-6ee'^2+e\gamma^2).\cos.cv\\ +3e'.(1+2e^2+\frac{9}{8}e'^2).\cos.c'mv\\ -3.(2+m).ee'.\cos.(cv+c'mv)\\ -3.(2-m).ee'.\cos.(cv-c'mv)\\ +5e^2.\cos.2cv\\ +\gamma^2.\cos.2gv\\ +\frac{9}{2}e'^2.\cos.2c'mv\\ -\frac{9}{4}e\gamma^2.\cos.(2gv-cv)\end{array}\right\}. \qquad [4908f]$$

Multiplying this by $a\delta u$, we obtain the value of the function [4908a, or 4907]. To reduce this to the form [4908], we may divide the terms, between the braces, by $1+\frac{3}{2}e'^2$, and connect this with the factor without the braces; and, by neglecting terms of the fourth order in e, e', γ, between the braces, we get,

$$-\frac{3m'.u'^3.\delta u}{2h^2.u^4}=-\frac{3\overline{m}^2.(1+\frac{3}{2}e'^2)}{2a_{\prime}}\cdot\left\{\begin{array}{l}1+2e^2\\ +(-4e-3e^3+e\gamma^2).\cos.cv\\ +3e'.(1+2e^2-\frac{3}{8}e'^2).\cos.c'mv\\ -3.(2+m).ee'.\cos.(cv+c'mv)\\ -3.(2-m).ee'.\cos(cv-c'mv)\\ +5e^2.\cos.2cv\\ +\gamma^2.\cos.2gv\\ +\frac{9}{2}e'^2.\cos.2c'mv\\ -\frac{9}{4}e\gamma^2.\cos.(2gv-cv)\end{array}\right\}.a\delta u. \qquad [4908g]$$

The factor $-\frac{3\overline{m}^2.(1+\frac{3}{2}e'^2)}{2a_{\prime}}$ is the same as in [4908]. The term 1, between the braces in [4908g], being multiplied by the external factor $a\delta u$, produces the term $a\delta u$ in the first line of [4908]. Now, if we neglect this term 1, between the braces in [4908g], and multiply the remaining terms by $a\delta u$ [4904], it will produce the terms of [4908], between the braces, which contain A explicitly. In performing this multiplication, it will only be necessary to retain the two following terms of [4908g]; namely, [4908h]

$$-4e.\cos.cv+3e'.\cos.c'mv. \qquad [4908i]$$

For, the other terms, between the braces, are of the *second* order; and these are multiplied

[4909''] The term $\frac{3m'.u'^3}{2h^2.u^3}.\cos.(2v-2v')$ [4870], has, for its variation,

[4910] $$-\frac{9m'.u'^3}{2h^2.u^4}.\delta u.\cos.(2v-2v')+\frac{3m'.u'^3}{h^2.u^3}.\delta v'.\sin.(2v-2v').$$

If we substitute the preceding value of δu, we shall find, that the first of these terms produces the function,*

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by $\overline{m}^2$, of the *second* order, and by $a\delta u$, of the *second* order; producing terms of the *sixth* order; some of which may be reduced to the *fifth* by integration [4847]. The terms,
[4908k] depending on the angle $v-mv$, of higher orders, are retained as in [4874, &c.]. The two terms [4908i] evidently produce those in [4908], which depend explicitly on the symbol A, neglecting the terms which have been usually rejected.

* (2815) If we take the differential of [4885], relative to dv, and multiply it by
[4910a] $\frac{3}{4a.dv}$, we shall obtain the expression of $-\frac{9m'.u'^3}{4h^2.u^4.a}.\sin.(2v-2v')$. The effect of this
[4910b] operation will be to change the factor $3\overline{m}^2.\frac{a}{a_{,}}$ [4885] into $-\frac{9\overline{m}^2}{4a_{,}}$, as in [4910k];
[4910c] moreover, it will take away the divisors $2-2m$, $2-2m-c$, &c., which were introduced
[4910d] by the integration, and will change, in the second member, *cos.* into *sin.* When the
[4910e] function is reduced to this form, we may change $2v$ into $2v+90^d$, as in [4876a—d]; and we shall obtain the expression of

[4910f] $$-\frac{9m'.u'^3}{4h^2.u^4.a}.\cos.(2v-2v') \quad [4910k].$$

If an angle, in the second member of [4885], be of the form $\cos.(2v+\beta)$, it becomes, in
[4910g] [4910d], $\sin.(2v+\beta)$; and, in [4910e], it changes into $\sin.(2v+\beta+90^d)$, or $\cos.(2v+\beta)$; which is the same as its original form in [4885]. But, if it be of the form $\cos.(\beta-2v)$, the successive changes are

[4910h] $$\sin.(\beta-2v), \qquad \sin.(\beta-2v-90^d), \quad \text{and} \quad -\cos.(\beta-2v);$$

this last being the same form as the original, but with a different sign. Hence we easily derive the expression [4910k] from [4885], by using the factor

[4910i] $$-\frac{9\overline{m}^2}{4a_{,}} \quad [4910b],$$

neglecting the denominators $2-2m$, &c. [4910c], and changing the signs of the terms depending on angles of the form $\cos.(\beta-2v)$;

$$-\frac{9m'.u'^3}{2h^2.u^4}.\delta u.\cos.(2v-2v')$$

$$=-\frac{9\overline{m}^2}{4a_{\prime}}.\left(\begin{array}{l}
A_2^{(0)}.(1-\tfrac{5}{2}e'^2) \\
+\{A_1^{(1)}-4A_2^{(0)}+A_2^{(2)}-\tfrac{1}{2}A_1^{(6)}.e'^2+\tfrac{7}{2}A_1^{(7)}.c'^2\}.e.(1-\tfrac{5}{2}c'^2).\cos.(cv-\varpi) \\
+\{3A_2^{(0)}+A_2^{(3)}+A_2^{(4)}\}.e'.\cos.(c'mv-\varpi') \\
+\{A_1^{(6)}+\tfrac{7}{2}A_1^{(1)}\}.ee'.\cos.(cv-c'mv-\varpi+\varpi') \\
+\{A_1^{(7)}-\tfrac{1}{2}A_1^{(1)}\}.ee'.\cos.(cv+c'mv-\varpi-\varpi') \\
+A_1^{(8)}.ee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') \\
+A_1^{(9)}.ee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') \\
+\{A_1^{(16)}+\tfrac{1}{4}(2+m).A_1^{1)}-2(1+m).A_1^{(13)}\}.e\gamma^2.\cos.(2gv-cv-2\vartheta+\varpi) \\
+A_0^{(15)}.e\gamma^2.\cos.(2v-2mv-2gv+cv+2\vartheta-\varpi) \\
+\{A_1^{(17)}-\tfrac{1}{2}A_0^{(18)}.e'^2\}.\dfrac{a}{a'}.\cos.(v-mv) \\
+\{A_1^{(19)}-\tfrac{1}{2}A_1^{(17)}\}.\dfrac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi') \\
+\{A_0^{(18)}+\tfrac{7}{2}A_1^{(17)}\}.\dfrac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi')
\end{array}\right).\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\end{array} \quad [4911]$$

$$-\frac{9m'.u'^3}{4h^2.u^4.a}.\cos.(2v-2v')=-\frac{9\overline{m}^2}{4a_{\prime}}.\left(\begin{array}{l}
(1+2e^2-\tfrac{5}{2}e'^2).\cos.(2v-2mv) \\
-2(1+m).(1+\tfrac{3}{4}e^2-\tfrac{1}{4}\gamma^2-\tfrac{5}{2}e'^2).e.\cos.(2v-2mv-cv) \\
-2(1-m).e.\cos.(2v-2mv+cv) \\
+\tfrac{7}{2}e'.\cos.(2v-2mv-c'mv) \\
-\tfrac{1}{2}e'.\cos.(2v-2mv+c'mv) \\
-\tfrac{7}{2}(2+3m).ee'.\cos.(2v-2mv-cv-c'mv) \\
-\tfrac{7}{2}(2-3m).ee'.\cos.(2v-2mv+cv-c'mv) \\
+\tfrac{1}{2}(2+m).ee'.\cos.(2v-2mv-cv+c'mv) \\
+\tfrac{1}{2}(2-m).ee'.\cos.(2v-2mv+cv+c'mv) \\
+\tfrac{1}{4}(10+19m+8m^2).e^2.\cos.(2cv-2v+2mv) \\
+\tfrac{1}{4}(10-19m+8m^2).e^2.\cos.(2cv+2v-2mv) \\
+\tfrac{1}{4}(2+m).\gamma^2.\cos.(2gv-2v+2mv) \\
+\tfrac{1}{4}(2-m).\gamma^2.\cos.(2gv+2v-2mv) \\
+\tfrac{17}{2}.e'^2.\cos.(2v-2mv-2c'mv) \\
-\tfrac{1}{4}(5+m).e\gamma^2.\cos.(2v-2mv-2gv+cv)
\end{array}\right).\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\end{array} \quad [4910k]$$

[4911′] $a\delta u$ contains a term, depending on $\cos.(3v-3mv)$, which we have

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Multiplying the first member of this expression by $2.a\delta u$, and the second by its equivalent
[4910l] expression [4904], we shall obtain, by making the usual reductions, the value of the first term of [4910], as in the second member of [4911]. For, the factor, without the braces,
[4910m] $-\frac{9\overline{m}^2}{4a_{,}}$, is the same in both these functions; we shall, therefore, neglect the consideration of it in the remainder of this note; and, in speaking of the functions [4910k, 4911], shall
[4910n] refer exclusively to the terms between the braces; and, shall separately investigate the results arising from *each line* of the function $2.a\delta u$ [4904], by the *whole* of the function [4910k].

First. We shall take into consideration the product of the term $2A_2^{(0)}.\cos.(2v-2mv)$, by the whole of the function [4910k]; and shall reduce the products by formula [20] Int., retaining the same angles as in [4911]. The first line of [4910k] produces the term $(1+2e^2-\frac{5}{2}e'^2).A_2^{(0)}$; the part depending on $\cos.(4v-4mv)$ being neglected. This corresponds to the first line of [4911], neglecting the part depending on $\overline{m}^2.e^2.A_2^{(0)}$, of
[4910o] the *sixth* order, as is done generally in the rest of this calculation; the term, depending on $\frac{5}{2}e'^2$, is retained, on account of its importance in the secular equations of the moon's motion [4982, 5059, 5087, &c.]. Again, if we neglect e^2, γ^2, in the factor [4910k] line 2, and introduce the factor $(1-\frac{5}{2}e'^2)$ in [4910k] line 3, according to the directions in [4869g, &c.], we shall find, that these terms, when multiplied by $2A_2^{(0)}.\cos.(2v-2mv)$, produce respectively the terms

$$-2.(1+m).(1-\tfrac{5}{2}e'^2).A_2^{(0)}.e.\cos.cv, \qquad -2.(1-m).(1-\tfrac{5}{2}e'^2).A_2^{(0)}.e.\cos.cv;$$

whose sum is

$$-4.(1-\tfrac{5}{2}e'^2).A_2^{(0)}e\cos.cv, \quad \text{as in [4911] line 2.}$$

In like manner, the terms in [4910k] lines 4, 5 being multiplied by $2A_2^{(0)}.\cos.(2v-2mv)$, produce respectively the terms

$$\tfrac{7}{2}A_2^{(0)}.e'.\cos.c'mv, \qquad -\tfrac{1}{2}A_2^{(0)}.e'.\cos.c'mv;$$

whose sum is

$$3A_2^{(0)}.e'.\cos.c'mv, \quad \text{as in [4911] line 3.}$$

the remaining terms of the function [4910k] may be neglected, on account of their smallness, and the forms of the angles.

Second. We shall now compute the terms produced by multiplying

$$2A_1^{(1)}.e.\cos.(2v-2mv-cv) \quad [4904],$$

by the terms of [4910k]. The first line of [4910k] produces $A_1^{(1)}.e.(1-\frac{5}{2}e'^2).\cos.cv$, as in [4911] line 2. The second and third lines of [4910k] depend on e^2, which is neglected.
[4910p] The fourth line of [4910k] gives $\frac{7}{2}ee'.A_1^{(1)}.\cos.(cv-c'mv)$, as in [4911] line 4; the fifth line, $-\frac{1}{2}ee'.A_1^{(1)}.\cos.(cv+c'mv)$, as in [4911] line 5; and the twelfth line

$$\tfrac{1}{4}(2+m).e\gamma^2.\cos.(2gv-cv), \quad \text{as in [4911] line 8.}$$

neglected,* on account of its smallness in [4904]; but, as it may have an influence in the term depending on $\cos.(v-mv)$, we shall take notice [4911'']

The other terms, depending on $A_1^{(1)}$, are neglected, on account of their smallness, &c.

Third. The product of $2A_2^{(2)}.e.\cos.(2v-2mv+ev)$ [4904], by the first term of [4910k], produces the term $A_2^{(2)}.e.(1-\frac{5}{2}e'^2).\cos.cv$, as in [4911] line 2. This is the only term depending on $A_2^{(2)}$, which requires attention; the other terms being small, or of forms which are unnoticed. [4910q]

Fourth. The product of $2A_2^{(3)}.e'.\cos.(2v-2mv+c'mv)$ [4904], by the first term of [4910k], produces the term $A_2^{(3)}.e'.\cos.c'mv$ [4911] line 3; the other terms may be neglected. In like manner, $2A_2^{(4)}.e'.\cos.(2v-2mv-c'mv)$ [4904], produces $A_2^{(4)}.e'\cos.c'mv$ [4911] line 3; and $2A_2^{(5)}.e'.\cos.c'mv$ [4904], gives nothing deserving of notice. [4910r]

Fifth. The term $2A_1^{(6)}ee'.\cos.(2v-2mv-cv+c'mv)$ [4904], being multiplied by the first term of [4910k], produces $A_1^{(6)}.ce'.\cos.(cv-c'mv)$ [4911] line 4; and the same term, being multiplied by the fifth term of [4910k], produces $-\frac{1}{2}ee'^2.A_1^{(6)}.\cos.cv$; which is nearly the same as in [4911] line 2. In like manner, the term [4910s]

$$2A_1^{(7)}.ce'.\cos.(2v-2mv-cv-c'mv),$$

being multiplied by the first and fourth terms of [4910k], produces the terms

$$A_1^{(7)}.ee'.\cos.(cv+c'mv), \quad \text{and} \quad +\tfrac{7}{2}A_1^{(7)}.ec'^2.\cos.ev; \quad \text{as in [4911] lines 5, 2.}$$

Sixth. The terms depending on $A_1^{(8)}$, $A_1^{(9)}$ [4904], being combined with the first term of [4910k], produce the terms [4911] lines 6, 7. Those depending on $A_2^{(10)}$, $A_1^{(11)}$, $A_2^{(12)}$, produce small terms, which are not noticed. The term [4910t]

$$2A_1^{(13)}.\gamma^2.\cos.(2gv-2v+2mv),$$

being combined with the term $-2.(1+m).e.\cos(2v-2mv-ev)$ [4910k] line 2, produces the term depending on $A_1^{(13)}$ [4911] line 8. The term depending on $A_2^{(14)}$ [4904], produces nothing of importance.

Seventh. The terms $2A_0^{(15)}.e\gamma^2.\cos.(2gv-ev)$, $2A_1^{(16)}.e\gamma^2.\cos.(2v-2mv-2gv+cv)$ [4904], being combined with $\cos.(2v-2mv)$ [4910k], produce respectively the terms in [4911] lines 9, 8, depending on $A_0^{(15)}$, $A_1^{(16)}$. [4910u]

Eighth. The term $2A_1^{(17)}\frac{a}{a'}\cos.(v-mv)$, being combined with the terms in [4910k] lines 1, 5, 4, produces the terms depending on $A_1^{(17)}$, in [1911] lines 10, 11, 12, respectively. [4910v]

Ninth. The first term of [4910k], being combined with the terms of $2.a\delta u$ [4904], depending on $A_0^{(18)}$, $A_0^{(19)}$, produces the corresponding terms of [4911] lines 12, 11. [4910w]

* (2816) This term occurs in [4808], and must, therefore, be found in the differential equation in u [4754], and in its integral δu, or $a\delta u$. [4911a]

of it. For this purpose, we shall put it under the following form;

[4912] $$\text{Term of } a\delta u = \lambda_2 \cdot \frac{a}{a'} \cdot \cos.(3v - 3v').$$

[4912'] Substituting this in the expression $-\frac{9m'.u'^3}{2h^2.u^4}.\delta u.\cos.(2v-2v')$ [4910], it produces the term,*

[4913] $$-\frac{9\,\overline{m}^2}{4a_{\prime}}\cdot\lambda_2\cdot\frac{a}{a'}\cdot\cos.(v-mv).$$

[4914] To develop the variation $\frac{3m'.u'^3}{h^2.u^3}.\delta v'.\sin.(2v-2v')$ [4910], we shall observe, that $\delta v'$ contains, in [4837], the same inequalities as the expression
[4914'] of the moon's mean longitude, in terms of the true longitude; but they are multiplied by the small quantity m. It is sufficient, in this case, to notice the
[4915] terms in which the coefficient of v differs but little from unity;† and it is evident that as the term $c.\cos.(cv-\varpi)$, of the expression of au [4826], gives, in v', the
[4916] term‡ $-2mc.\sin.(cv-\varpi)$; any term, whatever, of $a\delta u$, such as $k.\cos.(iv+\varepsilon)$,

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[4913a] * (2817) Substituting the values of u, u', [4791], and $h^2=a_{\prime}$ [4863], also $v'=mv$ [4837] nearly, in the expression [4912'], it becomes

[4913b] $$-\frac{9\,m'.a^3}{2a_{\prime}.a'^3}.a\,\delta u.\cos.(2v-2v') = -\frac{9\,\overline{m}^2}{2\,a_{\prime}}.a\,\delta\,u.\cos.(2v-2mv) \quad [4865].$$

If we now substitute the term of $a\delta u$ [4912], we obtain that in [4913], and also one depending on the angle $5v-5mv$, which may be neglected.

† (2818) We shall see, in [4918], that the terms of this form, in which the coefficients
[4914a] of v are nearly equal to unity, produce only small quantities of the fifth or sixth order. These terms are noticed, because they are much increased, by integration, in finding the
[4914b] value of u [4847]; but this does not happen with the terms in which the coefficient of v differs considerably from unity; and we may also observe, that, in this last case, the terms
[4914c] may also be decreased by the integration in [4822]. Hence, we see the propriety of noticing only the terms mentioned by the author in [4915].

‡ (2819) If we inspect the calculation in [4812—4837], we shall find, that the term
[4915a] $e.\cos.(cv-\varpi)$, which occurs in u [4812, 4816, 4819, 4826], is introduced into dt [4821], and by integration, produces in t [4822], or rather, in $nt+\varepsilon$ [4830], a term $-2e.\sin.(cv-\varpi)$.
[4915b] This is multiplied by m in the second member of the equation [4836]; and it finally produces in v' [4837], the term $-2\,m\,e.\sin.(cv-\varpi)$, as in [4916]. This may be derived
[4915c] from the preceding term of u, by changing *cos.* into *sin.* and multiplying the result by

in which i differs but little from unity, gives very nearly, in $\delta v'$, the term $-2mk.\sin.(iv+\varepsilon)$. Thus we find, that the preceding term [4914] gives, by its development, the function,* [4917]

$-2m$. The same method of derivation may be used with any other term of u, in which the coefficient of v differs but little from c, or from unity [4828c]; as is the case with the term $k.\cos.(iv+\varepsilon)$ of u [4916], which produces, in $\delta v'$, the term $-2mk.\sin.(iv+\varepsilon)$ [4917]. [4915d]

* (2820) Instead of the angle $iv+\varepsilon$ [4916, &c.], we shall, for brevity, use iv, omitting ε, as we have ϖ, ϖ', θ, in [4821f], and re-substituting it at the end of the calculation. [4918a]
Then, if we represent any term of $a\delta u$ [4904], in which i differs but little from unity, by $a\delta u = k.\cos.iv$ [4916], the corresponding term of $\delta v'$ will be very nearly represented by $\delta v' = -2m\,k.\sin.iv$ [4917]. [4918b] Moreover, if we represent any term between the braces of the second member of [4876e], by $A.\sin.V$; or, in other words, any term of the function [4918c] [4918d]

$$\frac{3m'.u'^3}{h^2.u^3}.\sin.(2v-2v') \quad \text{by} \quad \frac{3\overline{m}^2}{a_,}..A.\sin.V; \qquad [4918e]$$

and then multiply it by the preceding expression of $\delta v'$, we get, by using [17] Int.,

$$\frac{3m'.u'^3}{h^2.u^3}.\delta v'.\sin.(2v-2v') = -\frac{3\overline{m}^2}{a_,}.\{Amk.\cos.(iv \backsim V) - Amk.\cos.(iv+V)\}. \qquad [4918f]$$

The factor, without the braces, is the same as in [4918]; consequently, the terms, between the braces, in [4918], must arise from the other factor of [4918f]; namely,

$$Amk.\cos.(iv \backsim V) - Amk.\cos.(iv+V); \qquad [4918g]$$

in which we must substitute the terms of $a\delta u$ [4904], for $k.\cos.iv$; and, the terms between the braces in [4876e], for $A.\sin.V$; neglecting the terms which are insensible from their smallness, or those, where the coefficients of v, in the angles, vary much from unity [4915]. [4918h]

We shall, in the first place, compare the terms of the function [4918g], with the terms between the braces in [4918], taking successively, for k, the coefficients of the terms [4904], which are retained by the author. *First.* The term $A_1^{(1)}.e.\cos.(2v-2mv-cv)$ [4904], corresponds to $k = A_1^{(1)}.e$, $iv = 2v-2mv-cv$; combining this with the first line of [4876e], neglecting $e^2+\frac{1}{4}\gamma^2$, we find that this first term of [4918g] produces the first line of [4918]. If we combine the same term of [4904] with the first term in line 13 [4876e], we find, that the second term of [4918g] produces the second line of [4918]. It is unnecessary to notice the products of the other terms of [4876e], by the term [4918k]; because the coefficients are small, or the angles are different from those which are usually retained. *Second.* The term $A_0^{(15)}.e\gamma^2.\cos.(2gv-cv)$, being combined with the first term of [4876e], produces, by means of the first term of [1918g], the third line of [4918]. [4918i] [4918k] [4918l] [4918m] [4918n]

[4918]
$$\frac{3m'.u'^3}{h^2.u^3}.\delta v'.\sin.(2v-2v')=-\frac{3\,\overline{m}^2}{a_{\prime}}.\left\{\begin{array}{ll} m.A_1^{(1)}.e.(1-\tfrac{5}{2}e'^2).\cos.(cv-\varpi) & 1 \\ +\tfrac{3}{8}.m.A_1^{(1)}.e_{\prime}^2.\cos.(2gv-cv-2\theta+\varpi) & 2 \\ +m.A_0^{(15)}.e\gamma^2.\cos.\left(\begin{array}{c}2v-2mv-2gv \\ +cv+2\theta-\varpi\end{array}\right) & 3 \\ +m.A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv) & 4 \\ +m.A_0^{(18)}.\frac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi') & 5 \end{array}\right\}.$$

The other terms of this development are insensible.

The terms

[4919]
$$\frac{3\,m'.u'^4}{8h^2.u^4}.\{3.\cos.(v-v')+5.\cos.(3v-3v')\},$$

of the expression

[4920]
$$-\frac{1}{h^2}.\left\{\left(\frac{dQ}{du}\right)+\frac{s}{u}.\left(\frac{dQ}{ds}\right)\right\} \quad [4808],$$

[4918o] *Third.* The term $A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$, [4904], combined with the first term of [4876e], produces, in like manner, the fourth line of [4918]. *Fourth.* The term
[4918p] $A_0^{(18)}.\frac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi')$ [4904], combined with the same first term of [4876e], produces the fifth line of [4918].

It appears, from [4840, &c.], that the terms in the five lines of the function [4918], are of
[4918q] the orders 5, 7, 6, 6, 6, respectively. The integration [4847], introduces divisors of the order m^2 [4828e], in the first and second lines of [4918], and of the order m, in the other
[4918r] three lines. By this means, the first line of [4918] produces, in the value of u, a term of the third order, and the other lines produce terms of the fifth order; which are within the limits proposed in [4905′, &c.]. With respect to the order of the terms which have been neglected, we may observe, that, in calculating in [4918l] the quantity produced by one of
[4918s] the *greatest* terms of [4901]; namely, $A^{(1)}.e.\cos.(2v-2mv-cv)$, when combined with the *greatest* term of [4876e], contained in its first line, we have noticed only the first term
[4918t] of the function [4918g], and neglected its second. This second term has the same coefficient of the fifth order, as in the first line of [4918], but the quantity $\cos.cv$ is changed into $\cos.(4v-4mv-cv)$; making $i=4-4m-c=3$, nearly [4846]; and the divisor i^2-N^2
[4918u] [4847] becomes so large, that the corresponding term is much decreased, so that it may be neglected. Similar results will be obtained relative to the other neglected terms.

have, for variation,*

$$-\frac{3\overset{2}{\bar{m}}\,.\,a\,\delta u}{2\,a_{\prime}}\,.\,\frac{a}{a'}\,.\,\{3.\cos.(v-mv)+5.\cos.(3\,v-3\,m\,v)\}.$$ [4921]

Substituting $A_2^{(0)}.\cos.(2\,v-2\,m\,v)$, for $a\,\delta u$, we obtain the term,† [4921']

$$-\frac{6\,\overset{2}{\bar{m}}}{a_{\prime}}.A_2^{(0)}\,.\,\frac{a}{a_{\prime}}\,.\cos.(v-m\,v).$$ [4922]

The variation of the term [4876],

$$-\frac{3\,m'.\,u'^3}{2\,h^2.\,u^4}\,.\,\frac{du}{dv}\,.\sin.(2\,v-2\,v'),$$ [4923]

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* (2821) The variation of [4919], relative to u, which is the most important part of this expression, as we shall see in [4922e], is

$$-\frac{3\,m'.\,u'^4}{2\,h^2.\,u^5}.\delta u.\{3.\cos.(v-v')+5.\cos.3(v-v')\}.$$ [4921a]

If we neglect terms of the order e, we may substitute the values of u, u' [4791], $h^2=a_{\prime}$ [4863], and $\overset{2}{\bar{m}}$ [4865], in the factor, without the braces, and it will become,

$$-\frac{3\,m'.\,u'^4.\,\delta u}{2\,h^2.\,u^5}=-\tfrac{3}{2}.\frac{m'.a^3}{a'^3}\,.\,\frac{a\,\delta u}{a_{\prime}}\,.\,\frac{a}{a'}=-\frac{3\,\overset{2}{\bar{m}}\,.\,a\,\delta u}{2\,a_{\prime}}\,.\,\frac{a}{a'},$$ as in [4921]. [4921b]

Moreover, by putting $v'=m\,v$ [1837], in the term between the braces [4921a], it becomes as in [4921]. [4921c]

† (2822) Taking, for $a\,\delta u$, its first term [4904]; namely, $a\,\delta u=A_2^{(0)}.\cos.(2v-2mv)$, we get, by noticing only the angle $v-m\,v$, which requires particular attention, as is observed in [4874, &c.], we obtain, [4922a]

$a\,\delta u.3.\cos.(v-m\,v)=\tfrac{3}{2}.A_2^{(0)}.\cos\,(v-mv)$; $a\,\delta u.5.\cos.(3\,v-3\,mv)=\tfrac{5}{2}.A_2^{(0)}.\cos.(v-mv)$; [4922b]

whose sum is $4.A_2^{(0)}.\cos.(v-m\,v)$. Substituting this in [4921], it becomes as in [4922]. [4922c] The remaining terms of $a\,\delta u$ are of the second, third, &c. orders; and, when multiplied by the factor $\overset{2}{\bar{m}}\,.\,\frac{a}{a'}$, they become of the sixth, seventh, &c. orders, which are usually neglected. If we notice the variation of v', in [4919], it will produce terms of an order [4922d] equal to those in [4921], multiplied by the factor $\frac{\delta v'}{a\,\delta u}$, which factor is of the order m [4916, 4917]; so that, the terms produced by $\delta v'$, will be less than those retained in [4921, 4922], and may, therefore, be neglected. [4922e]

may be reduced to the following terms;*

[4924]
$$\frac{6m'.u'^3}{h^2.u^4}\cdot\frac{du}{dv}\cdot\frac{\delta u}{u}.\sin.(2v-2v')-\frac{3m'.u'^3}{2h^2.u^4}\cdot\frac{d\delta u}{dv}.\sin.(2v-2v')+\frac{3m'.u'^3.\delta v'}{h^2.u^4}\cdot\frac{du}{dv}.\cos.(2v-2v');$$

these terms, by development, produce the following expression;†

[4922f] * (2823) The term [4923], is the same as that whose approximate value is computed in [4876, 4879]. Its variation, considering u, du, v', as variable, and neglecting $\delta u'$, as in [4909], becomes as in [4924].

† (2824) Multiplying the equation [4881] by $-2\delta u$, we get, by using the abridged notation [4821f],

[4923a]
$$-\frac{4\delta u}{u}\text{ or }-\frac{4a\delta u}{au}=a\delta u.\{-4+4e.\cos.cv+\&c.\}.$$

[4923b] Multiplying this by the function [4879], we get the expression of the first term of [4924]. Now, the function [4879] is of the *third* order, and $a\delta u$ [4904] is of the *second* order;

[4923c] therefore, if we retain only the two terms $-4+4e.\cos.cv$ of the factor [4923a], the final product will be correct, in the *sixth* order. We may even neglect the term $4e.\cos.v$; because, when it is multiplied by the two greatest terms of [4879] lines 1, 2, it produces terms depending on $e^2.\cos.(2v-2mv)$, which mutually destroy each other; also,

[4923d] terms of the order e^2, connected with the angles $2v-2mv\pm 2cv$, which do not increase by integration, and are neglected in [4911, &c.]. Hence, the first term of [4924], is represented as in [4923a, b], by the following function;

[4923e]
$$\frac{6m'.u'^3}{h^2.u^4}\cdot\frac{du}{dv}\cdot\frac{\delta u}{u}.\sin.(2v-2v')=-4.a\delta u\times\text{function [4879]}.$$

It is only necessary to notice the terms $A_2^{(0)}$, $A_1^{(1)}$, $A_1^{(13)}$, in the value of $a\delta u$ [4904]; because, the function [4879] is of the *third* order, and the other terms $A_2^{(2)}e$, $A_2^{(3)}e$, &c. are of the *third*, or higher orders; so that their products are of the sixth, or higher orders,

[4923f] which are neglected. The reason for retaining the term $A_1^{(13)}$ is, because it is connected with the angle $2gv-cv$, and is much increased by integration [4828d]. Now, the part of

[4923g] $-4.a\delta u$ [4904], depending on $A_2^{(0)}$, is $-4.A_2^{(0)}.\cos.(2v-2mv)$. If we multiply this by the first line of [4879], between the braces, neglecting e^2, we shall get the term

[4923h]
$$-2ce.A_2^{(0)}.(1-\tfrac{5}{2}e'^2).\cos.(cv-\varpi);$$

and the second line of [4879], retaining the factor [4879k], produces the same term, with a different sign; so that these terms mutually destroy each other. The other terms produced by $A_2^{(0)}$, are too small to be noticed, or depend on angles which may be neglected. The

[4923i] product of the term $-4A_1^{(1)}e.\cos.(2v-2mv-cv)$, in $-4.a\delta u$ [4904], by the tenth line of [4879], between the braces, produces $gA_1^{(1)}e\gamma^2.\cos.(2gv-cv)$. Finally, the product of

$$\frac{3\bar{m}^2}{4a_{,}}\cdot\left(\begin{array}{l}
2.(1-m)A_2^{(0)}.(1-\tfrac{5}{2}e'^2) \\
+\left\{\begin{array}{l}(2-2m-c).A_1^{(1)}+(2-2m+c).A_2^{(2)}-8(1-m)A_2^{(0)} \\ +\tfrac{7}{2}(2-3m-c).A_1^{(7)}.e'^2-\tfrac{1}{2}(2-m-c).A_1^{(6)}.c'^2\end{array}\right\}.e.(1-\tfrac{5}{2}e'^2).\cos.(cv-\varpi) \\
+\{6.(1-m).A_2^{(0)}+(2-m).A_2^{(3)}+(2-3m).A_2^{(4)}\}.e'.\cos.(c'mv-\varpi') \\
+\{(2-3m-c).A_1^{(7)}-\tfrac{1}{2}(2-2m-c).A_1^{(1)}\}.ce'.\cos.(cv+c'mv-\varpi-\varpi') \\
+\{(2-m-c).A_1^{(6)}+\tfrac{7}{2}(2-2m-c).A_1^{(1)}\}.ce'.\cos.(cv-c'mv-\varpi+\varpi') \\
+(c-m).A_1^{(9)}.ee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') \\
+(c+m).A_1^{(8)}.ee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') \\
+\left\{\begin{array}{r}\tfrac{1}{4}(4g+4+m-2e)A_1^{(1)}-2(1-2m).A_1^{(13)} \\ +(2-2m-2g+c).A_1^{(16)}\end{array}\right\}.e\gamma^2.\cos.(2gv-cv-2\theta+\varpi) \\
+A_0^{(15)}.e\gamma^2.\cos.(2v-2mv-2gv+ev+2\theta-\varpi) \\
+\{(1-m).A_1^{(17)}-\tfrac{1}{2}A_0^{(18)}.e'^2+3(1-m).\lambda_2\}.\tfrac{a}{a'}.\cos.(v-mv) \\
+\{(1-2m).A_1^{(19)}-\tfrac{1}{2}(1-m).A_1^{(17)}\}.\tfrac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi') \\
+\{A_0^{(18)}+\tfrac{7}{2}(1-m).A_1^{(17)}\}.\tfrac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi')
\end{array}\right).
\quad \begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\end{array}
\qquad [4925]$$

$-4A_1^{(13)}\gamma^2.\cos.(2gv-2v+2mv)$, in $-4.a\delta u$ [4904], by the first term of [4879], between the braces, produces $-2A_1^{(13)}e\gamma^2.\cos.(2gv-cv)$. Substituting these two terms in the second member of [4923e], we get,

$$\frac{6m'.u'^3}{h^2.u^4}\cdot\frac{du}{dv}\cdot\frac{\delta u}{u}\cdot\sin.(2v-2v')=\frac{3\bar{m}^2}{4a_{,}}\cdot\{(gA_1^{(1)}-2A_1^{(13)}).e\gamma^2.\cos.(2gv-cv)\}. \qquad [4923k]$$

The third term of [4924], $\frac{3m'.u'^3.\delta v'}{h^2.u^4}\cdot\frac{du}{dv}\cdot\cos.(2v-2v')$, produces only a very small quantity, depending on the same angle as in the preceding expression [4923k]. Now, without taking the trouble to compute the whole development of this third term, we may form a satisfactory idea of its value, by taking the product of the two functions [4878, 4918]; which gives the expression of [4923l]

$$\frac{3m'.u'^3.\delta v'}{h^2.u^4}\cdot\frac{du}{dv}\cdot\sin.(2v-2v'); \qquad [4923m]$$

and, as this differs from [4923l] only by the change of *cos.* into *sin.* in its last factor, it is evident, that the two functions will produce terms of the same forms and orders; so that, what may be neglected in the one, may also be neglected in the other. Now, the greatest term of [4878], independent of its sign, is $ce.\sin.cv$; and, if we multiply it by the terms

[4926] The expression of $\left(\frac{dQ}{dv}\right)\cdot\frac{du}{h^2u^2dv}$ [4754], contains also the following

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of [4918], we obtain only quantities of the sixth order, depending on angles which may be neglected. The remaining terms of [4878] are of the second or higher orders, producing terms of the seventh or higher orders; therefore, they may all be neglected, excepting one, depending on the angle $2gv-cv$, which is retained for the reasons stated in [4828d]. A term of this form is produced in the function [4923m], by multiplying the term in line 4 [4878],

[4923n] which is nearly equal to $\frac{1}{2}\gamma^2.\sin.2gv$, by the term depending on $A_1^{(1)}e$, in the expression of

$$\frac{3m'.u'^3}{h^2.u^3}.\sin.(2v-2v').\delta v' \quad [4918] \text{ line } 1.$$

Hence, it is evident, by a similar process, that the terms of the function [4923l], depending

[4923o] on the angle $2gv-cv$, may be found, by multiplying $\frac{1}{2}\gamma^2.\sin.2gv$, by the terms depending

[4923p] on $A_1^{(1)}e$, in the function $\frac{3m'.u'^3}{h^2.u^3}.\cos.(2v-2v').\delta v'$.

Now, the term depending on $A_1^{(1)}e$, in the expression of $a\delta u$ [4904], is

$$a\delta u = A_1^{(1)}.e.\cos.(2v-2mv-cv);$$

the corresponding term of $\delta v'$ [4916, 4917], is

[4923p'] $$\delta v' = -2A_1^{(1)}.me.\sin.(2v-2mv-cv).$$

Multiplying this by the chief term of

$$\frac{3m'.u'^3}{h^2.u^3}.\cos.(2v-2v') \ [4870], \text{ which is, } \frac{3\overline{m}^2}{a_,}.\cos.(2v-2mv),$$

we get, in the function [4923p], the term

$$\frac{3\overline{m}^2}{a_,}.A_1^{(1)}.me.\sin.cv.$$

Finally, multiplying this by the factor $\frac{1}{2}\gamma^2.\sin.2gv$ [4923o], we get, for the third term of [4924], the following expression;

[4923q] $$\frac{3m'.u'^3.\delta v'}{h^2.u^4}.\frac{du}{dv}.\cos.(2v-2v') = \frac{3\overline{m}^2}{4a_,}.\{m.A_1^{(1)}.e\gamma^2.\cos.(2gv-cv)\}.$$

We shall now develop the second term of [4924], which is the most important. It may be put under the following form;

[4923r] $$-\frac{3m'.u'^3}{2h^2.u^4}.\frac{d\delta u}{dv}.\sin.(2v-2v') = -\left\{\frac{3m'.u'^3}{2h^2.u^4.a}.\sin.(2v-2v')\right\}.\frac{d.(a\delta u)}{dv}.$$

The factor between the braces, in the second member of this expression, connected with the negative sign, is evidently equal to the differential of the first member of [4885], divided by $2.adv$; and if we perform this process on the second member of [4885], we shall find, that

[4923s] the division by $2a$, makes the factor, without the braces, become $\frac{3\overline{m}^2}{2a_,}$. Moreover, by taking

variation ; [4926']

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the differential of the terms between the braces, the divisors $2-2m$, $2-2m-c$, &c., which were introduced by the integration, are effaced, and *cos.* is changed into *—sin.*; [4923t]
so that, if we represent any term, between the braces in [1885], after effacing the divisors, by $k'.\cos.v'$ the corresponding term of the first factor of the second member of [4923r], will be represented by a series of terms, of the form [4923u]

$$-\frac{3\,\overline{m}^2}{2\,a_{\prime}}\cdot k'.\sin.v' \quad [4923s,u].$$ [4923v]

Now, putting $a\delta u$ equal to a series of terms of the form $k.\cos.(iv+\varepsilon)$ [4916], or, for brevity, $k.\cos.iv$ [4918b], the corresponding term of $\frac{d.(a\,\delta u)}{dv}$ will be $-ik.\sin.iv$. [4923w]
Multiplying this by the first factor, which is given [4923v], we get the following expression of the function [4923r], or, of the second term of [1921];

$$-\frac{3m'.u'^3}{2h^2.u^4}\cdot\frac{d\,\delta u}{dv}.\sin.(2v-2v') = \frac{3\,\overline{m}^2}{4\,a_{\prime}}.\{ikk'.\cos.(iv\mp v')-ikk'.\cos.(iv+v')\}.$$ [4923x]

The factor without the braces $\frac{3\,\overline{m}^2}{4\,a_{\prime}}$, is the same in all three terms of the functions [4923k, q, x]; and is equal to that in [4925]; we shall, therefore, neglect wholly the consideration of this factor; and, in speaking of these functions, shall limit ourselves exclusively to the terms within the braces. These terms, of the function [4923x], are represented by, [4923y]

$$ik.\{k'.\cos.(iv\mp v')-k'.\cos.(iv+v')\};$$ [4923z]

in which $k.\cos.iv$ *represents the terms of* [4904], *and* $k'.\cos.v'$ *the terms between the braces in* [1885], *rejecting the divisors* $2-2m$, $2-2m-c$, &c. *which were introduced by the integration.* [4924a]

We shall now take, for $k.\cos.iv$, the terms of the function [4904]; so as to combine successively *each* of the symbols $A_2^{(0)}$, $A_1^{(1)}$, &c. with *all* the terms of [1885]. We shall neglect the terms which appear to be insensible, and shall compare those which are retained with the function [4925]; taking the terms, depending on $A_2^{(0)}$, $A_2^{(1)}$, $A_2^{(2)}$, &c. in the order in which they occur in [4904]; and, noticing also the terms [4923k, q], depending on the angle $2gv-cv$. [4924b]

First. The first line of [4904] gives $k=A_2^{(0)}$, $i=2-2m$; substituting this in [4923z], it becomes, $(2-2m).A_2^{(0)}.\{k'.\cos.([2-2m]v\mp v')-k'.\cos.(2v-2mv+v')\}$. [4924c]
The first line of [1885], neglecting e^2, gives $k'=1-\frac{5}{2}e'^2$, $v'=2v-2mv$; substituting these in the first term of [4924c], we get the first line of [4925]; the other term of [4924c] depends on the angle $(4v-4mv)$, which is neglected. In like manner, the second line of [4885], gives $k'=-2(1+m).(1-\frac{5}{2}e'^2).e$; $v'=2v-2mv-cv$; hence, the first term of [4924c] becomes, [4924d] [4924e]

$$-(2-2m).A_2^{(0)}.2(1+m).(1-\tfrac{5}{2}e'^2).e.\cos.cv=-4(1+m).\{(1-m).A_2^{(0)}.(1-\tfrac{5}{2}e'^2).e.\cos.cv\};$$ [4924f]

and, by the same process, we get, from the third line of [4885], by using the factor $1-\frac{5}{2}e'^2$

[4927] $$-\frac{\overset{2}{m}.a}{8a_{,}a'}.\{3.\sin.(v-mv)+15.\sin.(3v-3mv)\}.\frac{a.d\,\delta u}{dv};$$

[4879k], the term $-4(1-m).\{(1-m).A_2^{(0)}.(1-\frac{5}{2}e'^2).e.\cos.cv\}$. The sum of these two
[4924g] terms is $-8\{(1-m).A_2^{(0)}.(1-\frac{5}{2}e'^2).e.\cos.cv\}$, as in the second line of [4925]. It is unnecessary, in this case, to notice the second term of [4924c], because the coefficient of v is so large, that the term becomes insensible. Proceeding in the same manner with the
[4924h] fourth line of [4885], which gives $k'=\frac{7}{2}e'$, $\text{v}'=2v-2mv-c'mv$; also, with the fifth line of [4885], which gives $k'=-\frac{1}{2}e'$, $\text{v}'=2v-2mv+c'mv$, we find, that the terms corresponding to the first of the functions [4924c], are, respectively,

$$+(2-2m).A_2^{(0)}.\tfrac{7}{2}e'.\cos.c'mv, \qquad -(2-2m).A_2^{(0)}.\tfrac{1}{2}e'.\cos.c'mv;$$

[4924i] whose sum is $6.(1-m).A_2^{(0)}.e'.\cos.c'mv$, as in [4925] line 4.

The remaining terms of the function [4885], being of the *second* or higher orders in e,
[4924k] e', γ, multiplied by $\overset{2}{m}$ of the *second* order, and $A_2^{(0)}$ of the *second* order, produce only terms of the *sixth* and higher orders, which may be neglected.

Second. The second line of [4904] gives $k=A_1^{(1)}.e$, $i=2-2m-c$, hence [4923z] becomes,

[4924l] $$(2-2m-c).A_1^{(1)}.e.\{k'.\cos.([2-2m-c]\,v\mp\text{v}')-k'.\cos.(2v-2mv-cv+\text{v}')\}.$$

Substituting, in the first term of this function, the values [4924d], corresponding to the first line of [4885], we get the term $(2-2m-c)\,A_1^{(1)}.e.(1-\frac{5}{2}e'^2).\cos.cv$, as in the second line
[4924m] of [4925]. The second and third lines of [4885], produce terms having the factor $A_1^{(1)}.\overset{2}{m}.e^2$, of the fifth order; but they do not increase by integration, and are therefore
[4924n] neglected. The fourth and fifth lines of [4885] correspond to the values [4924h], and by substituting them in the first term of [4924l], we get the two terms,

[4924o] $$\tfrac{7}{2}e'.(2-2m-c).A_1^{(1)}.e.\cos.(cv-c'mv), \quad -\tfrac{1}{2}e'.(2-2m-c)\,A_1^{(1)}.e.\cos.(cv+c'mv),$$

as in [4925] lines 6, 5. All the remaining terms of [4885], excepting that in line 12, may
[4924p] be neglected as in [4924k]. This line corresponds to $k'=-\frac{1}{4}(2+m).\gamma^2$, $\text{v}'=2gv-2v+2mv$, and produces, by means of the second term [4924l], the expression,

[4924q] $$+\tfrac{1}{4}(2+m).(2-2m-c).A_1^{(1)}.e\gamma^2.\cos.(2gv-cv).$$

Connecting this with the terms, between the braces in [4923k, q], depending on $A_1^{(1)}$,
[4924r] they become $\{g+m+\frac{1}{4}(2+m).(2-2m-c)\}.A_1^{(1)}.e\gamma^2.\cos.(2gv-cv)$; and, as c is nearly equal to 1, we may, by neglecting m^2, put $\frac{1}{4}m.(2-2m-c)=\frac{1}{4}m$; consequently, the first
[4924s] factor of the expression becomes, $g+m+\frac{2}{4}(2-2m-c)+\frac{m}{4}=\frac{1}{4}(4g+4+m-2c)$, which is the same as the coefficient of $A_1^{(1)}$, in [4925] line 9.

Third. The term $A_2^{(2)}.e.\cos.(2v-2mv+cv)$ [4904], combined with [4885] line 1, gives
[4924t] the term depending on $A_2^{(2)}$ [4925] line 2. In like manner, we may combine the terms of [4904], depending on $A_2^{(3)}$, $A_2^{(4)}$, with the same terms of [4885], to obtain the terms depending on $A_2^{(3)}$, $A_2^{(4)}$ [4925] line 4; observing, that, as c' is nearly equal to 1, we have
[4924t'] very nearly $2-2m+c'm=2-m$, $2-2m-c'm=2-3m$. The term depending on $A_2^{(5)}$ produces nothing of importance.

*hence, we obtain the quantity, [4927']

Fourth. The term depending on $A_1^{(6)}$ [4901] gives $k=A_1^{(6)}.ee'$, $i=2-2m-c+c'm$, or nearly $i=2-m-c$. Substituting this in [4923z] it becomes, [4924u]

$(2-m-c).A_1^{(6)}.ee'.\{k'.\cos.([2-2m-c+c'm]v\mp v')-k'.\cos.(2v-2mv-cv+c'mv+v')\}$. [4924v]

The first line of [4885] produces, in the first term of [4924v], the quantity depending on $A_1^{(6)}$ [4925] line 6; and the fifth line of [4885], produces the terms depending on $A_1^{(6)}$, in line 3 [4925]. In like manner, the term depending on $A_1^{(7)}$ [4904], combined with [4885] lines 1, 4, produce those in [4925] lines 5, 2, depending on $A_1^{(7)}$. Also, the terms depending on $A_1^{(8)}$, $A_1^{(9)}$ [4901], being combined with the first term of [4885], produce the corresponding terms in [4925], lines 8, 7. [4924w] [4925a]

Fifth. The terms of [4904] depending on $A_2^{(10)}$, $A_1^{(11)}$, $A_2^{(12)}$, produce nothing of importance. The term in line 14 [4901], gives $k=A_1^{(13)}.\gamma^2$; $i=2g-2+2m=2m$ nearly; and the first term of line 2 [4885], gives $k'=-2e$, $v'=2v-2mv-cv$. Substituting these in the second term of [4923z], it produces $4m.e\gamma^2.A_1^{(13)}.\cos.(2gv-ev)$. Connecting this with the second term of [4923k], we obtain $-2(1-2m).A_1^{(13)}.e\gamma^2.\cos.(2gv-cv)$, as in [4925] line 9. The term depending on $A_2^{(14)}.e'^2$ [4904] produces nothing of importance. [4925b] [4925c] [4925d]

Sixth. The term in [4904] line 16, gives $k=A_0^{(15)}.e\gamma^2$, $i=2g-c=1$ nearly; and the first term of [4885] line 1, makes $k'=1$, $v'=2v-2mv$; hence, the first term of [4923z] produces $A_0^{(15)}.e\gamma^2.\cos.(2v-2mv-2gv+cv)$, as in [4925] line 11. The same values of k', v', being combined with the term in [4904] line 17, produce [4925e] [4925f]

$(2-2m-2g+c).A_1^{(16)}.e\gamma^2.\cos.(2gv-cv)$, as in [4925] line 10. [4925g]

Seventh. From [4904] line 18, we have $k=A_1^{(17)}.\frac{a}{a'}$, $i=1-m$. Combining these with k', v' [4925f], we get the term $(1-m).A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$ [4925] line 12. If we combine the same values of k, i, with the term in line 4 [4885], we get the term depending on $A_1^{(17)}$ [4925] line 14; and if we combine them with that in line 5 [4885], we obtain the term depending on $A_1^{(17)}$, in [4925] line 13. [4925h] [4925i]

Eighth. From [4904] line 19, we have $k=A_0^{(18)}.\frac{a}{a'}.e'$, $i=1-m+c'm=1$ nearly. Combining this with k', v' [4925f], we get the term depending on $A_0^{(18)}$ [4925] line 14. If we combine these values of k, i, with the term in [4885] line 5, we get the term depending on $A_0^{(18)}$ [4925] line 12. [4925k]

Ninth. From [4904] line 20, we have $k=A_0^{(19)}.\frac{a}{a'}.e'$, $i=1-m-c'm=1-2m$ nearly. Combining this with the values k', v' [4925f], we get the terms depending on $A_1^{(19)}$ [4925] line 13. [4925l]

Tenth. The term of $a\delta u$ [4912], gives $k=\lambda_2.\frac{a}{a'}$, $i=3-3m$. Combining this with the values [4925f], we obtain the term depending on λ_2, in [4925] line 12. [4925m]

Thus, we have obtained all the terms of the function [4925], as they are given by the author; and, it is evident, from the details of the calculation in this note, that, in general, the neglected terms are such as have been usually rejected. [4925n]

* (2825) Having found, in the preceding note, the variation of the first term of

[4928] $$^{*}\frac{9\overline{m}^2}{4a_{,}}\cdot(1-m)\,.\,A_2^{(0)}\cdot\frac{a}{a'}\cdot\cos.(v-m\,v).$$

$\left(\frac{dQ}{dv}\right)\cdot\frac{du}{h^2.u^2dv'}$ contained in [4876], we shall now proceed to the calculation of the next term, which is given in [4880]; and, if we put, for brevity,

[4927a] $$A=-\frac{m'.u'^4}{8h^2.u^5}\cdot\{3.\sin.(v-v')+15.\sin.(3v-3v')\};$$

this part becomes $A.\frac{du}{dv}$. Its variation, considering u, du, v', as variable, and neglecting $\delta u'$,

[4927b] as in [4909, &c.], is $\left(\frac{dA}{du}\right)\cdot\delta u\cdot\frac{du}{dv}+\left(\frac{dA}{dv'}\right)\cdot\delta v'\cdot\frac{du}{dv}+A\cdot\frac{d\,\delta u}{dv}.$

The factor $\frac{m'.u'^4}{h^2.u^5}$, in the value of A [4927a], is of the order $\overline{m}^2\cdot\frac{a}{a_{,}}\cdot\frac{a}{a'}$ [4921b],

[4927c] which is of the *fourth* order; therefore, $\left(\frac{dA}{du}\right)$, $\left(\frac{dA}{dv'}\right)$ are of the same order. Moreover, δu [4904] is of the *second* order; $\frac{du}{dv}$ [4878] is of the *first* order; $\delta v'$ is of the *third* order [4916, 4917]; consequently, $\left(\frac{dA}{du}\right)\cdot\delta u\cdot\frac{du}{dv}$ is of the *seventh* order; and $\left(\frac{dA}{dv}\right)\cdot\delta v'\cdot\frac{du}{dv}$

[4927d] of the *eighth* order; so that, by rejecting these terms, the function [4927b] is reduced to $A.\frac{d\,\delta u}{dv}$ of the *sixth* order. Then, by neglecting terms of the seventh order, we may use in A [4927a], the values [4921a—e], and the preceding expression becomes as in [4927].

* (2826) The differential of [4904], divided by dv, gives,

[4928a] $$\frac{a.d\,\delta u}{dv}=-(2-2m).A_2^{(0)}.\sin.(2v-2mv)$$
$$-(2-2m-c).A_1^{(1)}.c.\sin.(2v-2mv-cv)-\&c.;$$

which is to be substituted in [4927]. In the first place, the terms depending on $A_2^{(0)}$ [4928a], produce, in [4927], the following expression;

[4928b] $$\frac{\overline{m}^2.a}{8a_{,}a'}.(2-2m).A_2^{(0)}.\{3.\sin.(v-mv)+15.\sin.(3v-3mv)\}.\sin.(2v-2mv).$$

As this is of the sixth order, we need only notice the resulting terms which depend on the angle $(v-mv)$. Now,

$$3.\sin.(v-mv).\sin.(2v-2mv)=\tfrac{3}{2}.\cos.(v-mv)-\&c.;$$
$$15.\sin.(3v-3mv).\sin.(2v-2mv)=\tfrac{15}{2}.\cos.(v-mv)-\&c.;$$

whose sum is $\quad 9.\cos.(v-mv)-\&c.;$

hence, it is evident, that the term [4928b] is equal to

[4928c] $$\frac{\overline{m}^2.a}{8a_{,}.a'}.(2-2m).A_2^{(0)}.9.\cos.(v-mv)\,;$$

The function [4891],

$$\left(\frac{ddu}{dv^2}+u\right)\cdot\frac{2}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2},$$ [4929]

contains, in the first place, the term,

$$-\left(\frac{ddu}{dv^2}+u\right)\cdot\int\frac{3m'.u'^3.dv}{h^2.u^4}\cdot\sin.(2v-2v')\quad [4882];$$ [4930]

its variation is,*

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which is easily reduced to the form [4928]. We may proceed in the same manner with the terms of $a\delta u$ [4901], depending on $A_1^{(1)}.e$, $A_1^{(2)}.e$, &c.; but, as these terms produce only quantities of the sixth, seventh, &c. orders, they may be neglected.

* (2827) We shall put, for brevity,

$$V=\frac{ddu}{dv^2}+u,\qquad W=\frac{3m'.u'^3}{h^2.u^4}.\sin.(2v-2v');$$ [4929a]

then, we shall have the development of V, in the second member of [4890]; and the expression [4930] will become $-V.\int W.dv$. Now, as V, W, contain the variable quantities u, u', v', the variation of the function $-V.\int W.dv$, will be denoted by

$$-V.\int\left\{\left(\frac{dW}{du}\right).\delta u+\left(\frac{dW}{dv'}\right).\delta v'\right\}.dv-\delta V.\int W.dv-V.\int\left(\frac{dW}{du'}\right).\delta u'.dv.$$ [4929b]

The *three* different integrals, of which this expression is composed, correspond respectively to the three integrals in [4931], as we shall find by the following investigation; in which we shall use the abridged notation [4821f].

If we substitute the values of $\left(\frac{dW}{du}\right)$, $\left(\frac{dW}{dv'}\right)$, deduced from that of W [4929a], in the first of the integrals [4929'], it becomes,

$$-V.\int\left\{\left(\frac{dW}{du}\right).\delta u+\left(\frac{dW}{dv'}\right).\delta v'\right\}.dv=\frac{12Vm'}{h^2}.\int\frac{u'^3dv}{u^4}.\left\{\frac{\delta u}{u}.\sin.(2v-2v')+\tfrac{1}{2}\delta v'.\cos.(2v-2v')\right\};$$ [4929c]

in which the terms under the sign $\int$, are the same as in the first term of [4931]. If we substitute the values of c, g [4828e], in V [4890], and neglect terms of the order m^2e, $m^2\gamma$, e^2, $\frac{1}{4}\gamma^2$, we obtain, [4929d]

$$V=\frac{1}{a}.\{1+\tfrac{3}{4}\gamma^2.\cos.2gv\}.$$ [4929e]

Substituting this in the factor, without the sign $\int$ [4929c], it becomes as in the first term of [4931]. As the terms of $a\delta u$ [4904], are of the *second* or higher orders, it follows, from [4908g], that the terms depending on δu, under the sign $\int$ [4929c], are of the *fourth* or higher orders; and when these are multiplied by the terms of V, which we have neglected in [4929d], they will produce only terms of the *sixth* or *seventh* orders. Those of the sixth [4929f]

$$\frac{12m'}{h^2.a}.\left\{1+\tfrac{3}{4}\gamma^2.\cos.(2gv-2\theta)\right\}.\int\frac{u'^3.dv}{u^4}.\left\{\frac{\delta u}{u}.\sin.(2v-2v')+\tfrac{1}{2}\delta v'.\cos.(2v-2v')\right\} \qquad 1$$

[4931]
$$-\left(\frac{dd\delta u}{dv^2}+\delta u\right).\int\frac{3m'.u'^3.dv}{h^2.u^4}.\sin.(2v-2v') \qquad 2$$

$$-\frac{9m'}{h^2.a}.\int\frac{u'^2\,\delta u'}{u^4}.dv.\sin.(2v-2v'). \;* \qquad 3$$

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order are produced by e^2, $\frac{1}{4}\gamma^2$ [4929d], and do not depend on the angles $v-mv$, and $2gv-cv$, whose coefficients are required to a great degree of accuracy; hence, we see the propriety of neglecting the above-mentioned terms of V [4929d].

[4929g] In making this estimate, we have omitted the consideration of $\delta v'$ [4929c], because it is of the order $m.a\delta u$ [1916, 1917], and must, therefore, produce terms of still less importance than those of $a\delta u$, which we have neglected.

[4929h] Again, the value of V [1929a] gives $\delta V=\frac{dd\delta u}{dv^2}+\delta u$; substituting this in $-\delta V.\int W.dv$ [4929b], it becomes as in [4931] line 2.

Lastly, taking the partial differential of W [4929a], relative to u', and substituting it in the third integral [4929b], it becomes,

[4929i]
$$-V.\int\left(\frac{dW}{du'}\right).\delta u'.dv=-V.\int\frac{9m'.u'^2}{h^2.u^4}.\delta u'.dv\,\sin.(2v-2v').$$

Now, from [4833], we have nearly, $a'u'=e'.\cos c'v'+$, whose variation is,

$$a'\delta u'=-c'e'.\delta v'.\sin.c'v';$$

and, as $\delta v'$ is of the order $m.a\delta u$ [4929g], this quantity will be of the order $me'.a\delta u$, or of

[4929k] the *fourth* order [1901]. If we retain only the chief term of [1929c], we get $V=\frac{1}{a}$ and, by using the value [4921b, &c.], we find, that $\frac{m'.u'^2}{h^2.u^4}$ is of the order

[4929l]
$$\frac{m'.a^3}{a_{,}a^3}.aa'=\bar{m}^2.aa' \quad [4865];$$

consequently, the function [4929i] is of the *sixth* order; and, by neglecting terms of the seventh order, we may substitute the value of V [4929k], in [1929i]; by which means it becomes as in third line of [4931].

* (2828) In computing the value of the function [4931], we shall retain terms of the fifth

[4931a] order in c, e', γ, $\left(\frac{a}{a'}\right)^{\frac{1}{2}}$; also, in the coefficient of $\cos.cv$, we shall retain the factor $1-\frac{5}{2}e'^2$.

[4931b] In the terms depending on the angles $2gv-cv$, $v-mv$, $v-mv\pm c'mv$, we shall retain terms

The development of these terms, observing, that c is nearly equal [4932]

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of the *sixth* order; observing, that the divisors, arising from the integration, $2g-2+2m$, $2c-2+2m$, which occur in the terms depending on $A_2^{(11)}$, $A_2^{(12)}$ [4934], are of the order m; so that, independent of these divisors, these terms must be taken to include quantities of the *sixth* order. [4931c] [4931d]

We shall first compute the term

$$\frac{12\,m'}{h^2.a}\cdot\int\frac{u'^3.dv}{u^4}\cdot\frac{\delta u}{u}.\sin.(2v-2v') \quad [4931]. \qquad [4931e]$$

To obtain this, we shall take the differential of the equation [4885], and then multiply it by $-\frac{2}{a^2}$, neglecting such terms as we have usually done, and using the abridged notation [4821f]; hence we get, [4931f]

$$\frac{6m'}{h^2a^2}\cdot\frac{u'^3\,dv}{u^4}.\sin.(2v-2v')=\frac{6\,\overline{m}^2}{a_{,}a}.dv.\left\{\begin{array}{ll}
(1-\frac{5}{2}e'^2).\sin.(2v-2mv) & 1\\
-2(1+m).(1-\frac{5}{2}e'^2).e.\sin.(2v-2mv-cv) & 2\\
-2(1-m).(1-\frac{5}{2}e'^2).e.\sin(2v-2mv+cv) & 3\\
+\frac{7}{2}e'.\sin.(2v-2mv-c'mv) & 4\\
-\frac{1}{2}e'.\sin(2v-2mv+c'mv) & 5\\
-\frac{7}{2}(2+3m).ce'.\sin.(2v-2mv-cv-c'mv) & 6\\
-\frac{7}{2}(2-3m).ce'.\sin.(2v-2mv+cv-c'mv) & 7\\
+\frac{1}{2}(2+m).ce'.\sin.(2v-2mv-cv+c'mv) & 8\\
+\frac{1}{2}(2-m).ce'.\sin.(2v-2mv+cv+c'mv) & 9\\
-\frac{1}{4}(10+19m).e^2.\sin.(2cv-2v+2mv) & 10\\
+\frac{1}{4}(10-19m).e^2.\sin.(2cv+2v-2mv) & 11\\
-\frac{1}{4}(2+m).\gamma^2.\sin.(2gv-2v+2mv) & 12\\
+\frac{1}{4}(2-m).\gamma^2.\sin.(2gv+2v-2mv) & 13\\
+\frac{17}{2}.e'^2.\sin.(2v-2mv-2c'mv) & 14\\
-\frac{1}{4}(5+m).c\gamma^2.\sin.(2v-2mv-2gv+cv) & 15
\end{array}\right\} \qquad [4931g]$$

This is to be multiplied by the expression of $\frac{2}{u}$ [4881], to obtain the value of the function in the first member of [4931k]. By this means, the product of the factors, without the braces, becomes,

$$\frac{12\,\overline{m}^2}{a_{,}}.dv, \quad \text{as in [4931k]}; \qquad [4931h]$$

and the products of the terms, between the braces, are found as in the following table; in which, the first column contains the terms of [4884]; the second, those of [4931g]; and the third, those of [4931k], respectively;

[4933] to $1-\frac{3}{2}m^2$, and, that g is very nearly equal to $1+\frac{3}{4}m^2$ [4828e], is,

[4931i]

(Col. 1.) Terms of [4884].	(Col. 2.) Terms of [4931g].	(Col. 3.) Products, or terms of [4931k].
1	whole of [4931g]	whole function [4931g] between the braces
$-\frac{1}{2}e^2-\frac{1}{4}\gamma^2$	same	. . . .neglected
$-e.\cos.cv$	$(1-\frac{5}{2}e'^2).\sin.(2v-2mv)$	$-\frac{1}{2}e.(1-\frac{5}{2}e'^2).\{\sin.(2v-2mv+cv)+\sin.(2v-2mv-cv)\}$
	$-2(1+m)e.\sin(2v-2mv-cv)$	$-(1+m).e^2.\sin.(2cv-2v+2mv)+\&c.$
	$-2(1-m)e.\sin(2v-2mv+cv)$	$+(1-m).e^2.\sin(2cv+2v-2mv)-\&c.$
	$+\frac{7}{2}e'.\sin(2v-2mv-c'mv)$	$-\frac{7}{4}ee'.\{\sin.(2v-2mv+cv-c'mv)+\sin.(2v-2mv-cv-c'mv)\}$
	$-\frac{1}{2}e'.\sin.(2v-2mv+c'mv)$	$+\frac{1}{4}ee'.\{\sin.(2v-2mv+cv+c'mv)+\sin.(2v-2mv-cv+c'mv)\}$
$-e(-\frac{1}{4}e^2-\frac{1}{2}\gamma^2)\cos.cv$	whole of [4931g]	. . . .neglected
$+\frac{1}{2}e^2.\cos 2cv$	$+\sin(2v-2mv)$	$+\frac{1}{4}e^2.\{\sin.(2cv+2v-2mv)-\sin.(2cv-2v+2mv)\}$
$+\frac{1}{4}\gamma^2.\cos.2gv$	$+\sin.(2v-2mv)$	$+\frac{1}{8}\gamma^2.\{\sin.(2gv+2v-2mv)-\sin.(2gv-2v+2mv)\}$
	$-2(1+m)e.\sin(2v-2mv-cv)$	$-\frac{1}{4}e\gamma^2.(1+m)\sin.(2v-2mv+2gv-cv)-\&c.$
	$-2(1-m)e.\sin(2v-2mv+cv)$	$-\frac{1}{4}e\gamma^2.(1-m).\sin.(2v-2mv-2gv+cv)-\&c.$

Substituting, in the third column of this table, the value of its first line, which is equal to the terms between the braces in [4931g]; and then connecting together the terms of the same forms, it becomes equal to the terms between the braces in the second member of [4931k]; and the external factor is as in [4931h]; hence we get, by retaining terms of the usual forms and orders,

[4931k]
$$\frac{12m'}{h^2a^2}\cdot\frac{u'^3 dv}{u^4}\cdot\frac{1}{u}.\sin.(2v-2v')=\frac{12\overline{m}^2}{a_{\prime}}.dv.\left(\begin{array}{l}
(1-\frac{5}{2}e'^2)\sin.(2v-2mv) \\
-(\frac{5}{2}+2m)(1-\frac{5}{2}e'^2).e\sin.(2v-2mv-cv) \\
-(\frac{5}{2}-2m).(1-\frac{5}{2}e'^2).e\sin.(2v-2mv+cv) \\
+\frac{7}{2}e'.\sin.(2v-2mv-c'mv) \\
-\frac{1}{2}e'.\sin.(2v-2mv+c'mv) \\
-\frac{7}{2}(\frac{5}{2}+3m).ee'.\sin.(2v-2mv-cv-c'mv) \\
-\frac{7}{2}(\frac{5}{2}-3m)ee'.\sin.(2v-2mv+cv-c'mv) \\
+\frac{1}{2}(\frac{5}{2}+m).ee'.\sin.(2v-2mv-cv+c'mv) \\
+\frac{1}{2}(\frac{5}{2}-m).ee'.\sin.(2v-2mv+cv+c'mv) \\
-\frac{1}{4}(15+23m).e^2\sin.(2cv-2v+2mv) \\
+\frac{1}{4}(15-23m).e^2.\sin.(2cv+2v-2mv) \\
-\frac{1}{4}(\frac{5}{2}+m).\gamma^2.\sin(2gv-2v+2mv) \\
+\frac{1}{4}(\frac{5}{2}-m)\gamma^2.\sin(2gv+2v-2mv) \\
+\frac{17}{2}e'^2.\sin.(2v-2mv-2c'mv)
\end{array}\right).
\qquad
\begin{array}{r}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14
\end{array}$$

This is to be multiplied by $a\delta u$ [4904], and then integrated, to obtain the value of the term [4931e]. Now, if we suppose any term of $a\delta u$ to be represented, as in [4918b], by

[4931l] $a\delta u=k.\cos.iv$; and any term of the second member of [4931k], by $\frac{12\overline{m}^2}{a_{\prime}}.dv.k'\sin.i'v$;

$$-\frac{3\overline{m}^2}{4a_{\prime}.(1-m)}.\{4.(1-m)^2-1\}.A_2^{(0)}.(1-\tfrac{5}{2}e'^2) \qquad 1$$

$$-\frac{3\overline{m}^2}{a_{\prime}}.\left\{\begin{array}{l}\frac{7+(2-2m-c)^2}{4.(1-m)}.A_1^{(1)}-\{4.(1-m)^2-1\}.\left\{\frac{1+m}{2-2m-c}+\frac{1-m}{2-2m+c}\right\}.A_2^{(0)} \\ -A_1^{(6)}.e'^2+7A_1^{(7)}.e'^2\end{array}\right\}.c(1-\tfrac{5}{2}e'^2).\cos(cv-\varpi) \qquad \begin{array}{l}2\\3\end{array}$$

$$+\left(\begin{array}{l}\frac{6m}{a_{\prime}}.\{4A_2^{(0)}+A_2^{(3)}-A_2^{(1)}-10A_1^{(1)}e^2+\tfrac{5}{2}(A_1^{(7)}-A_1^{(6)}.).e^2 \\ -\frac{3\overline{m}^2}{4a_{\prime}}.\left(4.\overline{1-m}^2-1\right).A_2^{(0)}.\left\{\frac{7}{2-3m}-\frac{1}{2-m}\right\} \\ -\frac{3\overline{m}^2}{4a_{\prime}.(1-m)}.\left\{\left(\overline{2-m}^2-1\right).A_2^{(3)}+\left(\overline{2-3m}^2-1\right).A_2^{(4)}\right\} \\ -\frac{3\overline{m}^2}{a_{\prime}}.\{\tfrac{11}{2}.C_2^{(6)}+C_2^{(9)}-C_2^{(10)}\}\end{array}\right).c'.\cos.(c'mv-\varpi') \qquad \begin{array}{l}4\\5\\6\\7\end{array}$$

[4934]

Development of the variation [4931].

$$-\frac{6\overline{m}^2.A_1^{(8)}}{a_{\prime}.(2-3m-c)}.ce'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') \qquad 8$$

$$-\frac{6\overline{m}^2.A_1^{(9)}}{a_{\prime}.(2-m-c)}.ce'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') \qquad 9$$

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the product of these two terms will be represented by

$$\frac{6\overline{m}^2}{a_{\prime}}.dv.kk'.\{\sin.(i'v-iv)+\sin.(i'v+iv)\}. \qquad [4931m]$$

Its integral gives the corresponding term of [4931e]; namely,

$$\frac{12m'}{h^2a}.\int\frac{u'^3.dv}{u^4}.\frac{\delta u}{u}.\sin.(2v-2v')=\frac{6\overline{m}^2}{a_{\prime}}.\left\{-\frac{kk'}{i'-i}.\cos.(i'v-iv)-\frac{kk'}{i'+i}.\cos.(i'v+iv)\right\}; \qquad [4931n]$$

all of which have the common factor $\frac{6\overline{m}^2}{a_{\prime}}$, and the terms between the braces; namely,

$-\frac{kk'}{i'-i}.\cos.(i'v-iv)-\frac{kk'}{i'+i}.\cos.(i'v+iv)$, are computed in the following table; in which, [4931o] the first column represents the terms of $a\,\delta u$; the second, the terms of [4931k]; and the third, the terms of the function [4931o]: the operation being performed for each term separately, putting c and g equal to unity, in several of the small coefficients. When $i'=i$, the first term of [4931m] vanishes, and the function [4931o] is reduced to its second term $-\frac{kk'}{2i}.\cos.2iv$. This case occurs in the first line of [4931p], which is reduced to a term, [4931o'] depending on the angle $4v-4mv$, that may be neglected.

[4934] Continued.

$$-\frac{6\bar{m}^2}{a_{\prime}.(c-m)}.\{A_1^{(6)}+\tfrac{7}{2}A_1^{(1)}\}.ee'.\cos.(cv-c'mv-\varpi+\varpi') \qquad 10$$

[4931ρ] Development of the first term of the function [4931].

(Col. 1.) Terms of $a\delta u$ [4904].	(Col. 2.) Terms of [4931k].	(Col. 3.) Factors of $\frac{6\bar{m}^2}{a_{\prime}}$ [4931n].	
$A_2^{(0)}.\cos.(2v-2mv)$	$\left(1-\frac{5}{2}.e'^2\right).\sin.(2v-2mv)$	. . . .neglected	1
	second term	$\left(-\frac{5}{2}-2m\right).\left(1-\frac{5}{2}.e'^2\right).A_2^{(0)}.e.\cos.cv$	2
	third term	$\left(+\frac{5}{2}-2m\right).\left(1-\frac{5}{2}.e'^2\right).A_2^{(0)}.e.\cos.cv$	3
	$+\frac{7}{2}.e'.\sin.(2v-2mv-c'mv)$	$+\frac{7}{2}.A_2^{(0)}.\frac{e'}{m}.\cos.c'mv$	4
	$-\frac{1}{2}.e'.\sin.(2v-2mv+c'mv)$	$+\frac{1}{2}.A_2^{(0)}.\frac{e'}{m}.\cos.c'mv$	5
$A_1^{(1)}.e.\cos.(2v-2mv-cv)$	$\left(1-\frac{5}{2}.e'^2\right).\sin.(2v-2mv)$	$-A_1^{(1)}.e.\left(1-\frac{5}{2}.e'^2\right).\cos.cv$	6
	$+\frac{7}{2}.e'.\sin.(2v-2mv-c'mv)$	$-\frac{7}{2}.A_1^{(1)}.\frac{ee'}{c-m}.\cos.(cv-c'mv)$	7
	$-\frac{1}{2}.e'.\sin.(2v-2mv+c'mv)$	$+\frac{1}{2}.A_1^{(1)}.\frac{ee'}{c+m}.\cos.(cv+c'mv)$	8
	$-\frac{35}{4}.ee'.\sin.(2v-2mv-cv-c'mv)$	$-\frac{35}{4}.A_1^{(1)}.\frac{e^2e'}{m}.\cos.c'mv$	9
	$+\frac{5}{4}.ee'.\sin.(2v-2mv-cv+c'mv)$	$-\frac{5}{4}.A_1^{(1)}.\frac{e^2e'}{m}.\cos.c'mv$	10
	$-\frac{1}{4}\left(\frac{5}{2}+m\right).\gamma^2.\sin.(2gv-2v+2mv)$	$\left(+\frac{5}{8}+\frac{1}{4}.m\right).A_1^{(1)}.e\gamma^2.\cos.(2gv-cv)$	11
$A_2^{(2)}.e.\cos.(2v-2mv+cv)$	$\left(1-\frac{5}{2}.e'^2\right).\sin.(2v-2mv)$	$+A_2^{(2)}.e.\left(1-\frac{5}{2}.e'^2\right).\cos.cv$	12
$A_2^{(3)}.e'.\cos.(2v-2mv+c'mv)$	$\sin.(2v-2mv)$	$+A_2^{(3)}.\frac{e'}{m}.\cos.c'mv$	13
$A_2^{(4)}.e'.\cos.(2v-2mv-c'mv)$	$\sin.(2v-2mv)$	$-A_2^{(1)}.\frac{e'}{m}.\cos.c'mv$	14
$A_2^{(5)}.e'.\cos.c'mv$	$\sin.(2v-2mv)$	. . . .neglected	15
$A_1^{(6)}.ee'.\cos.(2v-2mv-cv+c'mv)$	$\sin.(2v-2mv)$	$-A_1^{(6)}.\frac{ee'}{c-m}.\cos.(cv-c'mv)$	16
	$-\frac{5}{2}.e.\sin.(2v-2mv-cv)$	$-\frac{5}{2}.A_1^{(6)}.\frac{e^2e'}{m}.\cos.c'mv$	17
	$-\frac{1}{2}.e'.\sin.(2v-2mv+c'mv)$	$+\frac{1}{2}.A_1^{(6)}.ee'^2.\cos.cv$	18
$A_1^{(7)}.ee'.\cos.(2v-2mv-cv-c'mv)$	$\sin.(2v-2mv)$	$-A_1^{(7)}.\frac{ee'}{c+m}.\cos.(cv+c'mv)$	19
	$-\frac{5}{2}.e.\sin.(2v-2mv-cv)$	$+\frac{5}{2}.A_1^{(7)}.\frac{e^2e'}{m}.\cos.c'mv$	20
	$+\frac{7}{2}.e'.\sin.(2v-2mv-c'mv)$	$-\frac{7}{2}.A_1^{(7)}.ee'^2.\cos.cv$	21
$A_1^{(8)}.ee'.\cos.(cv+c'mv)$	$\sin.(2v-2mv)$	$-A_1^{(8)}.\frac{ee'}{2-3m-c}.\cos.(2v-2mv-cv-c'mv)$	22
$A_1^{(9)}.ee'.\cos.(cv-c'mv)$	$\sin.(2v-2mv)$	$-A_1^{(9)}.\frac{ee'}{2-m-c}.\cos.(2v-2mv-cv+c'mv)$	23
$A_2^{(10)}.e^2.\cos.2cv$	$\sin.(2v-2mv)$	$A_2^{(10)}.\frac{e^2}{2c-2+2m}.\cos.(2cv-2v+2mv)$	24
$A_1^{(11)}.e^2.\cos.(2cv-2v+2mv)$	$\sin.(2v-2mv)$	. . . .neglected	25
$A_2^{(12)}.\gamma^2.\cos.2gv$	$\sin.(2v-2mv)$	$+A_2^{(12)}.\frac{\gamma^2}{2g-2+2m}.\cos.(2gv-2v+2mv)$	26
$A_1^{(13)}.\gamma^2.\cos.(2gv-2v+2mv)$	$-\frac{5}{2}.e.\sin.(2v-2mv-cv)$	$+\frac{5}{2}.A_1^{(13)}e\gamma^2.\cos.(2gv-cv)$	27
$A_2^{(14)}.e'^2.\cos.2c'mv$	[terms of 4931k]	. . . .neglected	28
$A_0^{(15)}e\gamma^2.\cos.(2gv-cv)$	$\sin.(2v-2mv)$	$-A_0^{(15)}.\frac{e\gamma^2}{2-2m-2g+c}.\cos.(2v-2mv-2gv+cv)$	29
$A_1^{(16)}.e\gamma^2.\cos.(2v-2mv-2gv+cv)$	$\sin.(2v-2mv)$	$-A_1^{(16)}.e\gamma^2.\cos.(2gv-cv)$	30
$A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$	$\sin.(2v-2mv)$	$-A_1^{(17)}.\frac{a}{a'}.\frac{1}{1-m}.\cos.(v-mv)$	31
	$+\frac{7}{2}.e'.\sin.(2v-2mv-c'mv)$	$-\frac{7}{2}.A_1^{(17)}.\frac{e'}{1-2m}.\frac{a}{a'}.\cos.(v-mv-c'mv)$	32
	$-\frac{1}{2}.e'.\sin.(2v-2mv+c'mv)$	$+\frac{1}{2}.A_1^{(17)}.e'.\frac{a}{a'}.\cos.(v-mv+c'mv)$	33
$A_0^{(18)}.\frac{a}{a'}.e'.\cos.(v-mv+c'mv)$	$\sin.(2v-2mv)$	$-A_0^{(18)}.\frac{a}{a'}.\frac{e'}{1-2m}.\cos.(v-mv-c'mv)$	34
	$-\frac{1}{2}.e.\sin.(2v-2mv+c'mv)$	$+A_0^{(18)}.\frac{a}{a'}.\frac{e'^2}{2(1-m)}.\cos.(v-mv)$	35
$A_1^{(19)}.\frac{a}{a'}.e'.\cos.(v-mv-c'mv)$	$\sin.(2v-2mv)$	$-A_1^{(19)}.\frac{a}{a'}.e'.\cos.(v-mv+c'mv)$	36
$\lambda_2.\frac{a}{a'}.\cos.(3v-3mv)$	$\sin.(2v-2mv)$	$+\lambda_2.\frac{a}{a'}.\frac{1}{1-m}.\cos.(v-mv)$.	37

All these terms have the common factor $\frac{6\bar{m}^2}{a_{\prime}}$.

$$-\frac{6\overline{m}^2}{a_{\prime}.(c+m)}.\{A_1^{(7)}-\tfrac{1}{2}A_1^{(1)}\}.ee'.\cos.(cv+c'mv-\varpi-\varpi') \qquad 11$$

$$+\frac{6\overline{m}^2.A_2^{(10)}}{a_{\prime}.(2c-2+2m)}.e^2.\cos.(2cv-2v+2mv-2\varpi) \qquad 12$$

[4934] Continued.

$$+\frac{6\overline{m}^2.A_2^{(12)}}{a_{\prime}.(2g-2+2m)}.\gamma^2.\cos.(2gv-2v+2mv-2\theta) \qquad 13$$

$$+\frac{6\overline{m}^2}{a_{\prime}}.\{2A_1^{(13)}-A_1^{(16)}+\tfrac{7m}{8}.A_1^{(1)}\}.e\gamma^2.\cos.(2gv-cv-2\theta+\varpi) \qquad 14$$

Continuation of the development of the function [4931]

$$-\frac{6\overline{m}^2.A_0^{(15)}}{a_{\prime}.(2-2m-2g+c)}.e\gamma^2.\cos.(2v-2mv-2gv+cv+2\theta-\varpi) \qquad 15$$

$$-\frac{3\overline{m}^2}{2a_{\prime}.(1-m)}.\{(4+3m).A_1^{(17)}-2A_0^{(18)}.e'^2-\tfrac{9}{2}.[1-(1-m)^2].\lambda_2\}.\frac{a}{a'}\ \cos.(v-mv) \qquad 16$$

$$+\frac{3\overline{m}^2}{a_{\prime}}.\{A_1^{(17)}-2A_1^{(19)}\}.\frac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi') \qquad 17$$

$$-\frac{6\overline{m}^2}{a_{\prime}(1-2m)}.\{A_0^{(18)}+\tfrac{7}{2}A_1^{(17)}\}.\frac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi'). \qquad 18$$

We may remark, that the sum of the terms in lines 2, 3, is reduced to

[4931p] Continued.

$$-4m.(1-\tfrac{5}{2}e'^2).A_2^{(0)}e.\cos.cv\,; \qquad (38)$$

the sum of those, in lines 4, 5, to $4.A_2^{(0)}.\frac{e'}{m}.\cos.c'mv$; and the sum of those, in lines 9, 10, (39) to $-10.A_1^{(1)}.\frac{e^2e'}{m}.\cos.c'mv$. Moreover, the term neglected in line 25, of the form $-A_1^{(11)}.\frac{e^2}{2c}.\cos.(2cv-2\varpi)$, will be used hereafter in a different calculation; also, the term [4931q] $\frac{6\overline{m}^2}{a_{\prime}}.\frac{5}{4}.A_1^{(1)}.e^2.\cos.2cv$, arising from the combination of [4904] line 2, with the first term in [4931r] line 3 [4931k].

The function [4931p] is also multiplied by $\frac{3}{4}\gamma^2.\cos.(2gv-2\theta)$, in [4931]; but the only term of [4931p], which requires any notice, is $-A_1^{(1)}.e.\cos.cv$, in line 6; because the [4931s] product of these two terms produces a quantity, depending on the angle $2gv-cv$, of the [4931t] following form;

Second term of the function [4931]

$$\frac{12m'}{h^2a}.\tfrac{3}{4}\gamma^2.\cos.(2gv-2\theta).\int\frac{u'^3.dv}{u^4}.\frac{\delta u}{u}.\sin.(2v-2v')=-\frac{6\overline{m}^2}{a_{\prime}}.\left\{\frac{3}{8}.A_1^{(1)}.e\gamma^2.\cos.(2gv-cv)\right\}. \qquad [4931u]$$

[4935] We must observe, that $C_2^{(6)}.\sin.(2v-2vm)$ is the inequality depending on

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[4931v] The next term of [4931] is $\frac{12m'}{h^2a}.\int\frac{u'^3.dv}{u^4}.\frac{1}{2}\delta v'.\cos.(2v-2v')$; which is of the order $\frac{\delta v'}{a\,\delta u}$, or m [4922l,e], in comparison with the terms produced by $a\,\delta u$ in [4931p]; and, as this last function may be considered as of the *fourth* order, that in [4931v]

[4931w] may be supposed of the *fifth* or a higher order, in all the angles which require any notice; so that it will only be necessary to retain the terms depending on the angles, whose coefficients increase considerably by integration; as cv, $2gv-cv$, $v-mv$. These are produced by the terms of $a\,\delta u$ [4901], depending on $A_1^{(1)}$, $A_1^{(17)}$; which give, by the process in [4916, 4917], the following terms of $\delta v'$; namely,

[4931x] $$\delta v'=-2m.A_1^{(1)}e.\sin.(2v-2mv-cv)-2m.A_1^{(17)}.\frac{a}{a'}.\sin.(v-mv).$$

Now, if we multiply $-\frac{8}{3}.\delta v'.dv$ by the first member of [4910k], and prefix the sign $\int$, it produces the term [4931v]. Performing the same operation on the second member of [4910k], we find, that it becomes,

[4931y] $$\frac{6\overline{m}^2}{a_{\prime}}.\int\{\delta v'.dv\times\text{terms between the braces in [4910}k]\}.$$

The first term of $\delta v'$ [4931x], being combined with the first line of [4910k], neglecting e^2,

[4931z] produces the term [4932a] line 1; the same term, combined with $\frac{1}{2}\gamma^2.\cos\,(2gv-2v+2mv)$ [4910k] line 12, gives [4932a] line 2. The second term of [4931x], being combined with the first of [4910k], produces [4932a] line 3; hence we have,

Third term of the function [4931].

[4932a] $$\frac{12m'}{h^2a}.\int\frac{u'^3dv}{u^4}.\tfrac{1}{2}\delta v'.\cos.(2v-2v')=\frac{6\overline{m}^2}{a_{\prime}}.\left\{\begin{array}{ll}-m.A_1^{(1)}.e.(1-\frac{3}{2}e'^2).\cos.cv & \quad 1\\ +\frac{1}{2}m.A_1^{(1)}.c\gamma^2.\cos.(2gv-cv) & \quad 2\\ -\frac{m}{1-m}.A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv) & \quad 3\end{array}\right\}.$$

These terms are the most important ones of those depending on $\delta v'$, and they are only of

[4932b] the fifth or sixth order; therefore, it will not be necessary to notice the terms arising from the multiplication of these by the factor $\frac{3}{4}\gamma^2.\cos.2gv$ [4931].

[4932b'] The next terms of [4931] are $-\left(\frac{dd\,\delta u}{dv^2}+\delta u\right).\int\frac{3m'.u'^3.dv}{h^2.u^4}.\sin.(2v-2v')$; which will evidently be obtained, by multiplying the function [4885], by the factor $\left(\frac{dd\,\delta u}{du^2}+\delta u\right)$.

[4932c] Now, any term of $a\delta u$ [4901, 4912], being represented by $a\,\delta u=k.\cos.(it+\varepsilon)$, the

[4932c'] corresponding term of this factor will be $-\frac{k}{a}.(i^2-1).\cos.(it+\varepsilon)$; and the product of the terms of this kind, by the corresponding ones in [4885], are computed in the following table; putting $c=1$, $g=1$, in some of the small terms; but, in the term depending on

$\sin.(2v-2mv)$, in the expression of the moon's mean longitude in terms of [4936]

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the angle $2gv-cv$ [4932f line 7], we must use $c=1-\frac{3}{2}m^2$, $g=1+\frac{3}{4}m^2$ [4932, 4933], [4932d] which give, very nearly,

$$-\frac{(2+m).\gamma^2}{4.(2g-2+2m)}=\frac{-(2+m).\gamma^2}{4.(2m+\frac{3}{2}m^2)}=-\frac{\gamma^2}{4m}.\left(\frac{1+\frac{1}{2}m}{1+\frac{3}{4}m}\right)=-\frac{(1-\frac{1}{4}m).\gamma^2}{4m}; \qquad [4932d']$$

by which means the coefficient of the term, in col. 2, line 7 [4932f], becomes $-(1-\frac{1}{4}m).\frac{\gamma^2}{4m}$. [4932$e$]

Moreover, the factor $-(i^2-1).k$ [4932c'] becomes, in this case, by neglecting m^3,

$$-\{(2-2m-c)^2-1\}.A_1^{(1)}e=-\{(1-2m+\tfrac{3}{2}m^2)^2-1\}.A_1^{(1)}e=(4m-7m^2).A_1^{(1)}e=4m(1-\tfrac{7}{4}m)A_1^{(1)}e. \qquad [4932e']$$

Multiplying this by the factor $-\frac{(1-\frac{1}{4}m)\gamma^2}{4m}$ [4932e], we get $-(1-2m).A_1^{(1)}.e\gamma^2$ for the factor of $\cos.(2gv-cv)$, in line 7 [4932f].

(Col. 1.) Terms of $a\delta u$ [4904].	(Col. 2.) Terms of [4885].	(Col. 3.) Corresponding terms of the function [4932b']. These terms have the factor $\frac{3\bar{n}^2}{2a_{\prime}}$.	
$A_2^{(0)}.\cos.(2v-2mv)$	$\frac{(1-\frac{5}{2}e'^2)}{2-2m}.\cos.(2v-2mv)$	$-\frac{\{4(1-m)^2-1\}}{2(1-m)}.A_2^{(0)}.(1-\frac{5}{2}e'^2)$	1
	$\frac{-2(1+m)}{2-2m-c}.e.\cos(2v-2mv-cv)$	$+\{4(1-m)^2-1\}.\frac{2(1+m)}{2-2m-c}.A_2^{(0)}e.\cos.cv$	2
	$\frac{-2(1-m)}{2-2m+c}.e.\cos(2v-2mv+cv)$	$+\{4(1-m)^2-1\}.\frac{2(1-m)}{2-2m+c}.A_2^{(0)}e.\cos.cv$	3
	$\frac{7e'}{2(2-3m)}.\cos(2v-2mv-c'mv)$	$-\{4(1-m)^2-1\}.\frac{7}{2(2-3m)}.A_2^{(0)}e'.\cos.c'mv$	4
	$-\frac{e'}{2(2-m)}.\cos.(2v-2mv+c'mv)$	$+\{4(1-m)^2-1\}.\frac{1}{2(2-m)}.A_2^{(0)}e'.\cos.c'mv$	5
$A_1^{(1)}e.\cos.(2v-2mv-cv)$	$\frac{(1-\frac{5}{2}e'^2)}{2-2m}.\cos.(2v-2mv)$	$-\left\{\frac{(2-2m-c)^2-1}{2.(1-m)}\right\}.A_1^{(1)}e.(1-\frac{5}{2}e'^2).\cos.cv$	6
	$-\frac{(2+m)\gamma^2}{4(2g-2+2m)}.\cos(2gv-2v+2mv)$	$-(1-2m).A_1^{(1)}e\gamma^2.\cos.(2gv-cv)$	7
$A_2^{(2)}e.\cos.(2v-2mv+cv)$	$\frac{(1-\frac{5}{2}e'^2)}{2-2m}.\cos.(2v-2mv)$	$-4(1-\frac{5}{2}e'^2).A_2^{(2)}e.\cos.cv$ nearly	8
$A_2^{(3)}e'.\cos(2v-2mv+c'mv)$	$\frac{1}{2-2m}.\cos.(2v-2mv)$	$-\left\{\frac{(2-m)^2-1}{2(1-m)}\right\}.A_2^{(3)}e'.\cos.c'mv$	9
$A_2^{(4)}e'.\cos(2v-2mv-c'mv)$	$\frac{1}{2-2m}.\cos.(2v-2mv)$	$-\left\{\frac{(2-3m)^2-1}{2(1-m)}\right\}.A_2^{(1)}e'.\cos.c'mv$	10
$A_1^{(13)}\gamma^2.\cos(2gv-2v+2mv)$	$-2e.\cos.(2v-2mv-cv)$	$-2A_1^{(13)}e\gamma^2.\cos.(2gv-cv)$	11
$A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$	$+\frac{1}{2-2m}.\cos.(2v-2mv)$	$+\left\{\frac{1-(1-m)^2}{2(1-m)}\right\}.A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$	12
$\lambda_2.\frac{a}{a'}.\cos.(3v-3mv)$	$+\frac{1}{2-2m}\cos.(2v-2mv)$	$+\left\{\frac{1-9.(1-m)^2}{2-2m}\right\}.\lambda_2.\frac{a}{a'}.\cos.(v-mv)$.	13

Development of the fourth term of the function [4931]

[4932f]

[4930f] its true longitude [5095].

The last term of the function [4931] is,

[4932g]
$$-\frac{9m'}{h^2.a}\cdot\int\frac{u'^2.\delta u'}{u^4}.dv.\sin.(2v-2v').$$

[4932h] To develop it, we have, by retaining only the first power of e', $a'u'=1+e'.\cos.c'v'$ [4833], whose variation is $a'\delta u'=-c'e'.\delta v'.\sin.c'v'=-e'\delta v'.\sin.c'mv$, nearly; and, by substituting the value of $\delta v'$ [4931 r], we find, that $\delta u'$ is of the *fourth* order; consequently,
[4932i] the expression [4932g] is composed of terms of the sixth and higher orders; and, as the integration, in [4932g], does not have the effect to increase essentially these terms of the sixth order, the whole expression may be neglected.

We have thus computed all the terms of the function [4931]. Nothing now remains, but to connect together the terms which depend on the same angles, as they are found in the
[4932k] functions [4931p, u, 4932 i, f]. The sum of these four functions ought to be equal to the development of the expression given in [4934], neglecting, for a moment, the consideration of the terms depending on C [4935, &c.], which will be noticed in [4937a, &c.]. In
[4932l] finding the sums of these coefficients, it will be necessary to make some slight alterations, to reduce them to the forms adopted by the author in [4934]. This will be done in the remainder of this note.

[4932m] *First.* The term in [4932f line 1], which is independent of any angle, corresponds to [4934 line 1], without any reduction.

Second. The second term of [4934] has the factor $-\frac{3\overline{m}^2}{a_{\prime}}e.(1-\frac{5}{2}e'^2).\cos.(cv-\varpi)$ common to all its terms; and the terms by which this factor is multiplied, in the functions which we have mentioned in [4932k], are collected in the following table, in the order in which they occur, without any reduction, except, that the two terms [4931p lines 2, 3], are reduced to one in line 38.

[4932n]

[4931p] lines 38, 6, 12, 18, 21	$+8m.A_2^{(0)}+2A_1^{(1)}-2A_2^{(2)}-A_1^{(6)}.e'^2+7.A_1^{(7)}.e'^2$	1
[4932a] line 1	$+2m.A_1^{(1)}$	2
[4932f] lines 2, 3	$-\{4.(1-m)^2-1\}.\left\{\frac{1+m}{2-2m-c}+\frac{1-m}{2-2m+c}\right\}.A_2^{(0)}$	3
[4932f] lines 6, 8	$+\left\{\frac{(2-2m-c)^2-1}{4.(1-m)}\right\}.A_1^{(1)}+2A_2^{(2)}.$	4

The coefficient of $A_1^{(1)}$, in this table, is

[4932o]
$$2+2m+\frac{(2-2m-c)^2-1}{4.(1-m)}=\frac{7-8m^2+(2-2m-c)^2}{4.(1-m)};$$

and, by neglecting the term m^2, in the numerator, which produces only terms of the sixth

$C_2^{(9)}.e'.\sin.(2v-2mv+c'mv-\varpi')$ and $C_2^{(10)}.e'.\sin.(2v-2mv-c'mv+\varpi')$ [4937]

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order in [4931], which are usually rejected, it becomes equal to the coefficient of $A_1^{(1)}$, in [4934 line 2]. We may also omit the term $8m.A_2^{(0)}$ [4932n line 1], which is of the same order; and then, the remaining terms, connected with $A_2^{(0)}$, in line 3, are the same as in [4934 line 2]. The terms depending on $A_2^{(2)}$ [4932n lines 1, 4], mutually destroy each other. The remaining terms, depending on $A_1^{(6)}$, $A_1^{(7)}$, are as in [4934 line 3]. [4932p]

Third. The third term of [4931] has the factor $e'.\cos.(c'mv-\varpi')$ common to all the terms. The coefficients of this factor, in the functions mentioned in [4932k], are given in [4932s], in the order in which they occur; observing that the two terms in [4931p lines 4, 5], as well as those in lines 9, 10, are reduced to one in [4931p line 39]. Moreover, the terms of [4931p], depending on the angle $c'mv-\varpi'$, have the divisor m, which is introduced by the integration; and they have also the common factor $\frac{6\bar{m}^2}{a_{\prime}}$; so that they are all multiplied by [4932q]

$$\frac{6\bar{m}^2}{a_{\prime}.m}=\frac{6.(m^2-\frac{1}{2}m^4)}{a_{\prime}.m}=\frac{6m}{a_{\prime}}-\frac{3m^3}{a_{\prime}}\ \text{[5094]; or}\ \frac{6m}{a_{\prime}}\ \text{nearly;}$$ [4932r]

neglecting the term $-\frac{3m^3}{a_{\prime}}$, which produces only terms of the sixth order in [4934]. Hence the factor of $e'.\cos.(c'mv-\varpi')$ becomes, without any other reduction, as in the following table;

[4931p] lines 39, 13, 14, 17, 20	$\frac{6m}{a_{\prime}}.\{4.A_2^{(0)}-10A_1^{(1)}.e^2+A_2^{(3)}-A_2^{(1)}-\frac{5}{2}A_1^{(6)}.e^2+\frac{5}{2}A_1^{(7)}.e^2\}$
[4932f] lines 4, 5	$-\frac{3\bar{m}^2}{4a_{\prime}}.\{4.(1-m)^2-1\}.A_2^{(0)}.\left\{\frac{7}{2-3m}-\frac{1}{2-m}\right\}$ [4932s]
[4932f] lines 9, 10	$-\frac{3\bar{m}^2}{4a_{\prime}.(1-m)}.\left\{\left(\overline{2-m}^2-1\right).A_2^{(3)}+(\overline{2-3m}^2-1)^2.A_2^{(4)}\right\}.$

By altering a little the arrangement of the terms in the first line of this table, it becomes as in [4934 line 4]; the second and third lines of the table, correspond respectively to [4934] lines 5, 6. The terms relative to C, in [4934 line 7], are discussed in the next note. [4932t]

Fourth. The eighth and ninth lines of [4934], correspond to [4931p lines 22, 23], respectively. The tenth line of [4934], depends on [4931p lines 7, 16]. The eleventh line of [4934] depends on [4931p lines 8, 19]. The twelfth and thirteenth lines of [4934], correspond, respectively, to [4931p lines 21, 26]. [4932u]

Fifth. The factors of $\frac{6\bar{m}^2}{a_{\prime}}.e\gamma^2.\cos.(2gv-cv)$, in the functions mentioned in [4932k], are contained in the following table;

[4937'] *are the inequalities depending on the angles $2v-2mv+c'mv-\varpi'$ and

[4932u']

[1931p] lines 11, 27, 30	$+\frac{5}{8}A_1^{(1)}+\frac{1}{4}m.A_1^{(1)}+\frac{5}{2}A_1^{(13)}-A_1^{(16)}$	1
[4931u]	$-\frac{3}{8}A_1^{(1)}$	2
[4932a] line 2	$+\frac{1}{2}m.A_1^{(1)}$	3
[1932f] lines 7, 11	$-\frac{1}{4}A_1^{(1)}+\frac{1}{2}m.A_1^{(1)}-\frac{1}{2}A_1^{(13)}$	4
Sum =	$+\frac{5}{4}m.A_1^{(1)}+2A_1^{(13)}-A_1^{(16)}$.	

[4932v]

This sum agrees with the coefficient in [4931 line 14], except in the term depending on $A_1^{(1)}$, which is $\frac{5}{4}m.A_1^{(1)}$ instead of $\frac{7}{8}m.A_1^{(1)}$. The difference is of the seventh order only, and is but of little importance, producing only terms of the fifth order, after integration, in [4847]. This discrepancy appears to have arisen from putting $g=1$, $c=1$, in the

[4932v'] calculation [4932e, e'], instead of the values [4932, 4933]. For, by using $g=1$, the factor [4932 l', e] becomes $\frac{-(1+\frac{1}{2}m)}{4m}$, and the factor [4932e'] is

$$-\{(2-2m-c)^2-1\}=-\{(1-2m)^2-1\}=4m-4m^2=4m.(1-m).$$

The product of these two factors is nearly equal to $-(1-\frac{1}{2}m)$, instead of $-(1-2m)$

[4932w] [4932f line 7]. Hence, the coefficient of m is decreased to one quarter part of its former value, and the term $\frac{1}{2}m.A_1^{(1)}$ [4932v], will be decreased in the same ratio, so as to become $\frac{1}{8}m.A_1^{(1)}$; by which means, the sum of all these terms $\frac{5}{4}m.A_1^{(1)}$ [4932v], is reduced to $\frac{7}{8}m.A_1^{(1)}$, as in [4934 line 14].

Sixth. The term in [4934 line 15], corresponds to that in [4931p line 29]. The factors of $-\frac{3\overline{m}^2}{2a_{,}.(1-m)}\cdot\frac{a}{a'}\cdot\cos.(v-mv)$, in the functions mentioned in [4932k], are contained in the following table. The sum of these factors corresponds to that in [4934] line 16, neglecting terms of the order $m^2.A_1^{(17)}$.

[4932x]

[4931p] lines 31, 35, 37	$+\ 4A_1^{(17)}-2A_0^{(18)}.e'^2-4\lambda_2$	
[4932a]	$+4m.A_1^{(17)}$	
[4932f] lines 12, 13	$-\frac{1}{2}\{1-(1-m)^2\}.A_1^{(17)}$	$-\lambda_2.\{\frac{1}{2}-\frac{9}{2}(1-m)^2\}$
Sum =	$(4+3m).A_1^{(17)}-2A_0^{(18)}.e'^2-\frac{9}{2}\lambda_2.\{1-(1-m)^2\}$.	

Seventh. The terms in [4934 line 17], correspond to those in [4931p lines 33, 36]; and

[4932y] the terms of [4934 line 18], correspond to [4931p lines 32, 34]. Hence it appears, that all the terms we have computed, agree with those in [4934].

* (2829) If we compare the value of $nt+\varepsilon$ [4828] heretofore used, with the form

[4937a] finally adopted in [5095], we shall find, that the terms depending on $C_2^{(6)}$, $C_1^{(7)}$, &c... $C_2^{(20)}$,

[4937b] have been neglected; and, if we put C for the sum of these terms, we must add C to the value of $nt+\varepsilon$ [4828], which will introduce in the second member of [4836] the term

$2v-2mv-c'mv+\varpi'$, in the same expression. We may also observe, that [4937i]
the term,

Cm; and the same quantity in the second member of [1837]; and we shall represent this
increment of r', by the expression $\delta v'=Cm$. Substituting this in $a'\delta u'$ [1932h], we [4937c]
get $a'\delta u'=-Cme'.\sin.cmv$. Now, if we select the chief terms of [4910, 4931], depending [4937d]
on $\delta r'$, $\delta u'$, they will become

$$\frac{3m'.u^3}{h^2.u^3}.\delta r'.\sin.(2v-2v')+\frac{12m'}{h^2.a}.\int\frac{u'^3.dv}{u^4}.\tfrac{1}{2}\delta r'.\cos.(2v-2v')-\frac{9m}{h^2a}.\int\frac{u^2.\delta u'}{u^4}.dv.\sin.(2v-2v'). \quad [4937e]$$

We have neglected the last term of [4921], depending on $\delta r'$, because it is multiplied by
$\frac{du}{dv}$, which is of the order e [4875]; so that this will be of the same order as the product
of the first term of [4937e] by e, which, as we shall soon see, may be neglected [4937k].
Now, substituting the values of $\delta r'$, $\delta u'$ [4937c, d], in [4937e], it becomes, by merely [4937f]
altering the arrangement of each of the terms, so as to bring them under the forms we have
already computed,

$$\left\{\frac{3m'.u'^3}{2h^2.u^3}.\sin.(2v-2v')\right\}.2Cm+\frac{9m'}{4h^2.a}.\int\frac{u'^3.dv}{u^4}.\cos.(2v-2v').\tfrac{?}{?}Cm$$
$$+\frac{9m'}{h^2.a}.\int\frac{u^2}{u^4}.dv.\sin.(2v-2v').\frac{me}{a'}.C.\sin.c'mv. \quad [4937g]$$

The value of C, to be substituted in this expression, is easily deduced from [5095, 4937h],
and is represented by

$$C=C_2^{?}.\sin.(2v-2mv)+C_1^{?}.e.\sin.(2v-2mv-cv)+C_2^{?}.e.\sin.(2v-2mv+cv)$$
$$+C_2.e'.\sin.(2v-2mv+c'mv)+C_2^{?}.e'.\sin.(2v-2mv-c'mv) \quad [4937h]$$
$$+C_2^{(1)}.e'.\sin.c'mv+\&c.\;...\;+C_1^{?}.\frac{a}{a'}.\cos.(v-mv)+\&c.$$

If we multiply together the two functions [4876e, 4937h], and the product by $2m$, we
shall get the first term of the function [4937g]. These terms of this product are of the
fifth and higher orders; so that it will only be necessary to retain those which depend on the [4937i]
angles cv, $v-mv$. These terms are found by multiplying the first term of [4876e],
namely, $\frac{3\overline{m}^2}{2a_{,}}.\sin.(2v-2mv)$, by the terms of $2mC$ [4937h] depending on $C_1^{(7)}$, $C_1^{(?)}$:
from which we get,

$$\left\{\frac{3m'.u'^3}{2h^2.u^3}.\sin.(2v-2v')\right\}.2Cm=\frac{3\overline{m}^2}{a_{,}}.\left\{\tfrac{1}{2}me.C_1^{?}.\cos.cv+\tfrac{1}{2}m.\frac{a}{a'}.C^{(9)}.\cos.(v-mv)\right\}; \quad [4937k]$$

in which we have neglected some terms of the sixth order, depending on $C_2^{(6)}$, and on the
angle cv.

[4938] $$\frac{6m}{a_{\prime}}.\{4A_2^{(0)}+A_2^{(3)}-A_2^{(4)}\}.e'.\cos.(c'mv-\varpi'),$$

[4937l] The next term of [4937g] is found by multiplying together the functions [4910k,4937h], and the product by $-\frac{3}{2}m.dv$; and then integrating the result; as in the following table;

[4937m]

Terms of [4937h].	Terms of [4910k].	Terms of [4934]. These terms have the factor $\frac{3\overline{m}^2}{a_{\prime}}$.	
$+C_2^{(6)}.\sin.(2v-2mv)$	$+\cos.(2v-2mv)$	. . . .neglected	1
	$-2e.\cos.(2v-2mv-cv)$	$+2C_2^{(6)}.me.\cos.cv$	2
	$-2e.\cos.(2v-2mv+cv)$	$-2C_2^{(6)}.me.\cos.cv$	3
	$+\frac{7}{2}e'.\cos.(2v-2mv-c'mv)$	$-\frac{7}{2}C_2^{(6)}.e'.\cos.c'mv$	4
	$-\frac{1}{2}e'.\cos.(2v-2mv+c'mv)$	$-\frac{1}{2}C_2^{(6)}.e'.\cos.c'mv$	5
$+C_1^{(7)}.e.\sin.(2v-2mv-cv)$	$+\cos.(2v-2mv)$	$+C_1^{(7)}.me.\cos.cv$	6
$+C_2^{(9)}e'.\sin.(2v-2mv+c'mv)$	$+\cos.(2v-2mv)$	$-C_2^{(9)}.e'.\cos.c'mv$	7
$+C_2^{(10)}e'.\sin(2v-2mv-c'mv)$	$+\cos.(2v-2mv)$	$+C_2^{(10)}.e'.\cos.c'mv$	8
$+C_1^{(19)}.\frac{a}{a'}.\sin.(v-mv)$	$+\cos.(2v-2mv)$	$+C_1^{(19)}.m.\frac{a}{a'}.\cos.(v-mv).$	9

The last term of [4937g] being very small, we may substitute in it the values

[4937n] $$u=\frac{1}{a};\quad u'=\frac{1}{a'};\quad v'=mv;\quad h^2=a_{\prime}\quad [4921a-c];$$

by which means it becomes,

$$\frac{9m'.a^3}{a_{\prime}.a'^3}.me'.\int dv.\sin.(2v-2mv).\sin.c'mv\times C;$$

and, by using [4865], it may be reduced to the form,

[4937o] $$\frac{9\overline{m}^2}{2a_{\prime}}.me'.\int Cdv.\{\cos.(2v-2mv-c'mv)-\cos.(2v-2mv+c'mv)\}.$$

Now, substituting the value of C [4937h], it produces terms of the sixth order, before integration; and some of them may be reduced to the fifth, after integration, if they be connected with the angle $c'mv$; we shall, therefore, retain this angle only. These terms are found, by substituting, in [4937o], the part of C [4937h] represented by

[4937p] $C_2^{(6)}.\sin.(2v-2mv)$. Combining this with each of the terms of [4937o], it produces a term, $\int\frac{1}{2}dv.\sin.c'mv=-\frac{1}{2m}.\cos.c'mv$; so that both terms, taken together, produce the following expression;

[4937q] $$\frac{9m'}{h^2.a}.\int\frac{u'^3}{u^4}.dv.\sin.(2v-2v').\frac{me'}{a'}.C.\sin.c'mv=\frac{3\overline{m}^2}{a_{\prime}}.\{-\frac{3}{2}C_2^{(6)}.e'.\cos c'mv\}.$$

appears to be of the order m^4, which would produce a quantity of the order m^3, in the expression of the moon's mean longitude; but this term is, in fact, only of the order m^5. For, we shall see, by means of the values of $A_2^{(0)}$, $A_2^{(3)}$, $A_2^{(1)}$* [5157,5160,5161], that the function $4A_2^{(0)}+A_2^{(3)}-A_2^{(1)}$ is of the order m^3; which produces, in the expression of the mean longitude, a term of the order m^4 only. We shall, however, retain it here, because we have imposed on ourselves the condition of including terms of that order, in the calculation of the terms of the third order. [4938'] [4939] [4939']

For this reason, it is indispensable, in the development of

$$-\frac{3m'.u}{h^2}.\int\frac{u'^3.dv}{u^4}.\sin.(2v-2v') \quad [4930], \qquad [4940]$$

to carry on the approximation to terms of the order δu^2; hence we obtain the term,†

Connecting together the quantities contained in [4937k,m,q], we get the terms of the function [4937e] depending on C. The coefficients of $\frac{3\overline{m}^2}{a_{\prime}}.C_2^{(6)}.c'.\cos.c'mv$, in [4937$m$ lines 4,5], and in [4937q], being connected together, become,

$$-\tfrac{7}{2}-\tfrac{1}{2}-\tfrac{3}{2}=-\tfrac{11}{2}, \quad \text{as in [4934 line 7]}; \qquad [4937r]$$

and the terms in the same line, corresponding to $C_2^{(9)}$, $C_2^{(10)}$, agree with those in [4937m] lines 7, 8. The term depending on $C_2^{(6)}$ [4937m lines 2, 3], mutually destroy each other. The quantities we have mentioned include all the terms retained by the author; who has not noticed those in [4937k], and in lines 6, 9 of [4937m], whose sum is

$$\frac{9\overline{m}^2}{2a_{\prime}}.m.\left\{C_1^{(7)}.c.\cos.cv+C_1^{(19)}.\frac{a}{a'}.\cos.(v-mv)\right\}. \qquad [4937s]$$

These neglected terms are of the fifth or sixth order, increasing also by the integration in [4847]; and are of the same orders as the terms which are usually retained with these angles; but, as we did not wish to alter the numerical calculations of the author, we have not introduced them into [4934]. [4937t]

* (2830) These values are nearly represented by $A_2^{(0)}=0{,}0071$, $A_2^{(3)}=-0{,}0030$, $A_2^{(1)}=0{,}0285$; whence, $4A_2^{(0)}+A_2^{(3)}-A_2^{(1)}=-0{,}003$, nearly. This is less than m^2 [5117], but can hardly be called of the order m^3, as in [4939']; however, as it is multiplied by e', which is much smaller than e, γ, m, we may consider the whole term [4938] as of the order m^5. [4938a] [4938b]

† (2831) The factor δu^2 is of the fourth order [4904], and, as all the terms we have

[4941] $$-\frac{30m'.u}{h^2}.\int\frac{u'^3.\delta u^2}{u^6}.dv.\sin.(2v-2v').$$

This term produces the following ;*

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computed [4910, 4924, &c.] have the factor m', or $\overline{m}^2$, except where the sign of integration
[4941a] [4932q—r] has introduced the divisor m; it follows, that these terms depending on δu^2, are generally
[4941b] of the *sixth* order; but some of them may be reduced to the *fifth* order, by the integration we have just mentioned. Therefore, we need only notice those terms where the variations are connected with the signs of integration; so that we may neglect the second powers or products of the variations in the terms [4909″, 4921, 4924, 4927, 4931, &c.], and, in fact, only
[4941c] retain the chief term of [4930 or 4931], which depends on δu^2. For, we need not notice the terms depending $\delta u.\delta v'$, $\delta u.\delta u'$, $\delta v'^2$, $\delta u'^2$, &c.; because δu is of the *second*
[4941d] order [4904], $\delta v'$ is of the *third* order [4929g], $\delta u'$ is of the *fourth* order [4929i—k]; hence, the terms depending on $\delta u.dv'$, $\delta u.\delta u'$, &c. must generally be much less than those depending on δu^2; therefore, we shall only notice this last quantity. We have already
[4941e] found, by Taylor's theorem [610, &c.], in [4929b], the increment of the function $-V.\int Wdv$, arising from the increments δu, $\delta v'$, $\delta u'$, in the values of u, v, u', respectively; and, by the same theorem, the term depending on δu^2, will evidently be represented by

$$-\tfrac{1}{2}V.\int\left(\frac{ddW}{du^2}\right).\delta u^2.dv \quad [610, 4929b].$$

Substituting the value of W [4929a], it becomes,

[4941f] $$\frac{-30m'.V}{h^2}.\int\frac{u'^3.\delta u^2}{u^6}.dv.\sin.(2v-2v');$$

and, by using the value of $V=\frac{1}{a}=u$, nearly [4929k, 4937n], it becomes as in [4941]; neglecting in V terms of the order em^2, e^2, γ^2.

* (2832) As the function [4941] is of the *sixth* order, before integration [4941b]; we may, by neglecting terms of the seventh order, substitute in it the values [4937n]; by this means, it becomes,

[4942a] $$-\frac{30m'.a^3}{a_{\prime}a'^3}.\int dv.(a\delta u)^2.\sin.(2v-2mv)=-\frac{30\,\overline{m}^2}{a_{\prime}}.\int dv.(a\delta u)^2.\sin.(2v-2mv) \quad [4865].$$

If we retain only the term of $(a\delta u)^2$, of the fourth order, we may neglect all the expression [4904], except the two first lines, and we shall have,

[4942b] $$a\delta u=A_2^{(0)}.\cos.(2v-2mv)+A_1^{(1)}.e.\cos.(2v-2mv-cv).$$

Squaring this, and reducing, by means of [20] Int. we get,

[4942c] $$(a\delta u)^2=(A_2^{(0)})^2.\{\tfrac{1}{2}+\tfrac{1}{2}.\cos.(4v-4mv)\}+A_2^{(0)}.A_1^{(1)}.e.\{\cos.cv+\cos.(4v-4mv-cv)\}$$
$$+(A_1^{(1)})^2.e^2.\{\tfrac{1}{2}+\tfrac{1}{2}.\cos.(4v-4mv-2cv)\}.$$

This must be multiplied by $\sin.(2v-2mv)$, and the product substituted in [4942a], after

$$\frac{15\overset{2}{m}}{2a_{,}} \cdot \frac{(A_1^{(1)})^2.e^2.\cos.(2cv-2v+2mv-2\varpi)}{2c-2+2m}\,;$$ [4942]

although it is only of the fifth order, yet, as it acquires by integration, in the expression of the mean longitude, the divisor* $2v-2+2m$, it is necessary to notice it. [4942']

The function

$$\left(\frac{ddu}{dv^2}+u\right)\cdot\frac{2}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}\quad [4754],$$ [4943]

gives the following ;†

$$-\left(\frac{ddu}{dv^2}+u\right)\cdot\frac{1}{h^2}\cdot\int\frac{m'.u'^4.\,dv}{4u^5}.\{3.\sin.(v-v')+15.\sin.(3v-3v')\}.$$ [4944]

Its variation produces the terms,‡

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making the reductions by [18] Int. The only term of this product, in which the coefficient of v is small, is that produced by multiplying the last term of [4942c],

$$\tfrac{1}{2}(A_1^{(1)})^2.e^2.\cos.(4v-4mv-2cv),\quad\text{by}\quad\sin.(2v-2mv)\quad[4942a],$$ [4942d]

which produces the term $\frac{1}{4}(A_1^{(1)})^2.e^2.\sin.(2cv-2v+2mv)$; and, by substituting this in [4942a], it becomes equal to the following expression ;

$$-\frac{15\overset{2}{m}}{2a_{,}}.(A_1^{(1)})^2.e^2.\int dv\,.\sin.(2cv-2v+2mv)=\frac{15\overset{2}{m}}{2a_{,}}\cdot\frac{(A_1^{(1)})^2.e^2.\cos.(2cv-2v+2mv)}{2c-2+2m}.$$ [4942e]

as in [4942]. The terms we have neglected are of the sixth or higher orders; the term [4942] is reduced to the fifth order, by means of the small divisor $2c-2+2m$, which is nearly equal to $2m$ [4828e].

* (2833) The term of u, resulting from the substitution of [4942] in [4961], is to be added to u [4812 or 4819]; and this produces in dt [4753] a term depending on the same angle. The integration gives, in t, and in $nt+\varepsilon$ [4828], a term of the same form with the new divisor $2c-2+2m$; and, by this means, it is reduced to the fourth order. [4943a]

† (2834) The terms [4809], depending on the angles $v-v'$, $3v-3v'$, are retained in [4944]; because they produce, in [4946], some terms depending on the angle $v-mv$, which require a greater degree of accuracy than the others [4906, &c.]. [4944a]

‡ (2835) Since $\delta v'$, $\delta u'$, are much smaller than δu [4941d], we may neglect them in finding the variation of the function [4944], and consider u as the only variable quantity; by this means, the variation of [4944] becomes, [4944b]

[4945]
$$-\frac{1}{a_{\prime}}\cdot\left(\frac{dd\delta u}{dv^2}+\delta u\right)\cdot\int\frac{m'.u'^4.dv}{4u^5}.\{3.\sin.(v-v')+15.\sin.(3v-3v')\}$$
$$+\frac{5\bar{m}^2}{4a_{\prime}}\cdot\frac{a}{a'}\cdot\int a\,\delta u.dv.\{3.\sin.(v-v')+15.\sin.(3v-3v')\}:$$

hence results the term,*

[4945a]
$$-\left(\frac{dd\delta u}{dv^2}+\delta u\right)\cdot\frac{1}{h^2}\cdot\int\frac{m'.u'^4.dv}{4u^5}\cdot\{3.\sin.(v-v')+15.\sin.(3v-3v')\}$$
$$+\left(\frac{ddu}{dv^2}+u\right)\cdot\frac{1}{h^2}\cdot\int\frac{5m'.u'^4.dv}{4u^5}\cdot\delta u.\{3.\sin.(v-v')+15.\sin.(3v-3v')\}.$$

Substituting, in the first line of this expression, the value $h^2=a_{\prime}$ [4937n], it becomes like the first line of [4945]. Again, by substituting, in the second line of
[4945b] [4945a], the values of u, u', h^2 [4937n], and for $\frac{ddu}{dv^2}+u$, the chief term $\frac{1}{a}$ [4890], it becomes

[4945c]
$$\frac{5m'.a^3}{4a_{\prime}.a'^3}\cdot\frac{a}{a'}\cdot\int a\,\delta u.dv.\{3.\sin.(v-v')+15.\sin.(3v-3v')\}.$$

This is easily reduced to the form in the second line of [4945], by the substitution of $\bar{m}^2$ [4865].

* (2836) The terms [4945], being of the sixth order, independent of the integrations, it is only necessary to notice the terms depending on the angle $v-mv$; and, we may, therefore, substitute the values [4937n], in [4945], and they will become, by using [4865],

[4946a]
$$-\frac{1}{a_{\prime}}\cdot\left(\frac{dd\delta u}{dv^2}+\delta u\right)\cdot\frac{\bar{m}^2}{4}\cdot\frac{a^2}{a'}\cdot\int\{3.\sin.(v-mv)+15.\sin.(3v-3mv)\}.dv$$
$$+\frac{5\bar{m}^2}{4a_{\prime}}\cdot\frac{a}{a'}\cdot\int a\,\delta u.\{3.\sin.(v-mv)+15.\sin.(3v-3mv)\}.dv.$$

In this we may substitute, for $a\,\delta u$, its two chief terms [4942b]; and a little consideration will show, that we may even neglect the part depending on $A_1^{(1)}$, because it does not
[4946b] produce, in [4946], any term connected with the angle $v-mv$; so that we shall finally have $a\,\delta u=A_2^{(0)}.\cos.(2v-2mv)$. Substituting this in [4946a], it becomes,

[4946c]
$$-\frac{\bar{m}^2}{4a_{\prime}}\cdot\frac{a}{a'}\cdot A_2^{(0)}.\{1-4.(1-m)^2\}.\cos.(2v-2mv).\int\{3.\sin.(v-mv)+15.\sin.(3v-3mv)\}.dv$$
$$+\frac{5\bar{m}^2}{4a_{\prime}}\cdot\frac{a}{a'}\cdot A_2^{(0)}.\int\cos.(2v-2mv).\{3.\sin.(v-mv)+15.\sin.(3v-3mv)\}.dv.$$

Now we have,

$$-\frac{\overline{m}^2}{2a_{\prime}.(1-m)}.\{13+8.(1-m)^2\}.A_2^{(0)}.\frac{a}{a'}.\cos.(v-mv).$$ [4946]

We must here make an important observation relative to the terms depending on $\cos.(v-mv)$, which we propose to determine with accuracy. The expressions of the radius of the sun's orbit, and its longitude, contain terms depending on the angle $v-mv$ [4324], resulting from the moon's [4947] action upon the earth. These terms produce others, in the expression of u, and in the moon's mean longitude; and it is essential that we should notice these terms. For this purpose, we shall observe, that, in consequence of the moon's action, the sun's radius vector contains the term

Indirect action of the moon.

$$\delta r'=\frac{\mu}{u}.\cos.(v-v') \quad [4315, 4316b];*$$ [4948]

$$f\{3.\sin.(v-mv)+15.\sin.(3v-3mv)\}.dv=-\frac{3}{1-m}.\cos.(v-mv)-\frac{5}{1-m}.\cos.(3v-3mv).$$ [4946d]

Multiplying this by $\cos.(2v-2mv)$, and retaining only the terms depending on $\cos.(v-mv)$, we find, that the product becomes,

$$\left(-\frac{3}{1-m}-\frac{5}{1-m}\right).\tfrac{1}{2}\cos.(v-mv)=\frac{-4}{1-m}.\cos.(v-mv);$$

hence the first line of [4946c] becomes,

$$-\frac{\overline{m}^2}{2a_{\prime}.(1-m)}.\{-2+8.(1-m)^2\}.A_2^{(0)}.\frac{a}{a'}.\cos.(v-mv).$$ [4946e]

Again

$$\cos.(2v-2mv).3.\sin.(v-mv)=-\tfrac{3}{2}.\sin.(v-mv)+\&c.$$
$$\cos.(2v-2mv).15.\sin.(3v-3mv)=\tfrac{15}{2}.\sin.(v-mv)+\&c.$$

whose sum is $6.\sin.(v-mv)+\&c.$ Substituting this under the integral sign of the second line of [4946c], that line becomes,

$$-\frac{5\overline{m}^2}{4a_{\prime}}.\frac{a}{a'}.A_2^{(0)}.\frac{6.\cos.(v-mv)}{1-m}=-\frac{\overline{m}^2}{2a_{\prime}.(1-m)}.15.A_2^{(0)}.\frac{a}{a'}.\cos.(v-mv).$$ [4946f]

Adding this to the part [4946e], it becomes as in [4946].

* (2837) The inequality of the earth's radius vector, arising from the action of the moon, is

$$\delta r''=-\frac{m}{M+m}:R.\cos.(U-v'') \quad [4315, 4316b].$$ [4948a]

To conform to the present notation, we must change U into v [4313, 4760'], R into r,

[4948′] μ *being the ratio of the moon's mass to the sum of the masses of the moon and earth.* This gives, in u', the term,*

[4949] $$\delta u' = -\frac{\mu . u'^2}{u} . \cos.(v - v').$$

The longitude of the sun v' contains also the term [4314],†

[4950] $$\delta v' = \frac{\mu . u'}{u} . \sin.(v - v').$$

This being premised, the term $\frac{m'.u'^3}{2h^2.u^3}$ [4865′] contains the following;‡

[4951] $$-\frac{3m'.\mu.u'^4}{2h^2.u^4} . \cos.(v - v').$$

[4951′] The term $\frac{3m'.u'^3}{2h^2.u^3}.\cos.(2v-2v')$ [4866′], contains the two following §

[4952] $$-\frac{9m'.\mu.u'^4}{2h^2.u^4}.\cos.(v-v').\cos.(2v-2v') + \frac{6m'.\mu.u'^4}{2h^2.u^4}.\sin.(v-v').\sin.(2v-2v');$$

[4948b] [4313, 4759], r'' into r' [4313, 4759′]; moreover, the longitude v'' of the *earth*, seen
[4948c] from the sun [4313], is equal to $180^d + v'$ of the present notation [4777d]; lastly
[4948d] $\mu = \frac{m}{M+m}$ [4757, 4757′, 4948′]. Substituting in [4948a], we get $\delta r' = \mu\, r.\cos.(v-v')$; and if we neglect the square of the inclination of the moon's orbit to the ecliptic, we may
[4948e] put $r = \frac{1}{u}$ [4776], and then the preceding value of $\delta r'$ becomes as in [4948].

[4949a] * (2838) From [4777c] we have, very nearly, $r' = \frac{1}{u'}$; whence $\delta r' = -\frac{\delta u'}{u'^2}$. Substituting the value of $\delta r'$ [4948], we get $\delta u'$ [4949].

† (2839) This term is given in [4314, 4316b], under the form

$$\delta v'' = -\frac{m}{M+m} . \frac{R}{r''}.\sin.(U - v'');$$

and, by making the changes in the symbols, as in [4948b, &c.], it becomes,

[4950a] $$\delta v' = +\mu . \frac{r}{r'}.\sin.(v-v'), \quad \text{or nearly} \quad \delta v' = \mu . \frac{u'}{u}.\sin.(v-v'), \quad \text{as in [4950].}$$

[4951a] ‡ (2840) The variation of the term $\frac{m'.u'^3}{2h^2.u^3}$ [4865′], taken relatively to u', is $\frac{3m'.u'^2}{2h^2.u^3}.\delta u'$; and, by substituting $\delta u'$ [4949], it becomes as in [4951].

[4952a] § (2841) Taking the variation of the term [4951′], relatively to u', v'; and then substituting the values of $\delta u'$, $\delta v'$ [4949, 4950], we get [4952].

which produces the term,*

$$-\frac{3m'.\mu.u'^4}{4h^2.u^4}.\cos.(v-v'). \qquad [4953]$$

Connecting it with that in [4951], we obtain,

$$-\frac{9m'.\mu.u'^4}{4h^2.u^4}.\cos.(v-v'); \qquad [4954]$$

whence results the following terms ;†

$$-\frac{9\overset{2}{\bar{m}}.\mu}{4a_{\prime}}.\frac{a}{a'}\cos.(v-mv)-\frac{9\overset{2}{\bar{m}}.\mu}{4a_{\prime}}.\frac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi') \qquad 1$$

[4955]

$$-\frac{27\overset{2}{\bar{m}}.\mu}{4a_{\prime}}.\frac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi'). \qquad 2$$

The term $-\frac{3m'}{h^2}.\int\frac{u'^3.dv}{u^4}.\sin.(2v-2v')$ [4882] gives, in like manner, the following ;‡ [4956]

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* (2842) If we retain only the angle $\cos.(v-v')$, and reduce the products by [17,20] Int., we may substitute, in [4952], the values

$$\cos.(v-v').\cos.(2v-2v')=\tfrac{1}{2}.\cos.(v-v')+\&c.\,;$$
$$\sin.(v-v').\sin.(2v-2v')=\tfrac{1}{2}.\cos.(v-v')-\&c.\,; \qquad [4953a]$$

and, since $-\frac{9}{2}.\frac{1}{2}+\frac{6}{2}.\frac{1}{2}=-\frac{3}{4}$, the expression [4952] becomes as in [4953].

† (2843) Multiplying [4872] by -2μ, and neglecting e^2, e'^2, we get [4955]. [4955a]

‡ (2844) The variation of the term [4956], is as in [4956b]; substituting the values of $\delta u'$, $\delta v'$ [4919, 4950], it becomes as in [4956c]; reducing the products of the sines and cosines, by [18, 19] Int., retaining only the angle $v-v'$, it becomes as in [4956d]. [4956a]

$$\frac{3m'}{h^2}.\int\left\{-\frac{3u'^2.\delta u'.dv}{u^4}.\sin.(2v-2v')+\frac{2u'^3.dv}{u^4}\,\delta v'.\cos.(2v-2v')\right\} \qquad [4956b]$$

$$=\frac{3m'.\mu}{h^2}.\int\left\{\frac{3u'^4.dv}{u^5}.\sin.(2v-2v').\cos.(v-v')+\frac{2u'^4.dv}{u^5}.\cos.(2v-2v').\sin.(v-v')\right\} \qquad [4956c]$$

$$=\frac{3m'.\mu}{h^2}.\int\left\{\frac{3u'^4.dv}{2u^5}.\sin.(v-v')-\frac{u'^4\,dv}{u^5}.\sin\,(v-v')\right\}=\frac{3m'.\mu}{2h^2}.\int\frac{u'^4.dv}{u^5}.\sin.(v-v'). \qquad [4956d]$$

This last expression is evidently equal to the first member of [4889] multiplied by -2μ; and, if we multiply its second member by the same factor -2μ, we shall get the development [4957]; neglecting the small terms e^2, γ^2, e'^2. [4956e]

[4957] $$-\frac{3\overset{2}{\overline{m}}.\mu}{2.(1-m)}\cdot\frac{a}{a_{\prime}}\cdot\frac{a}{a'}\cdot\cos.(v-mv)-\frac{3\overset{2}{\overline{m}}.\mu}{2}\cdot\frac{a}{a_{\prime}}\cdot\frac{a}{a'}\cdot e'.\cos.(v-mv+c'mv-\varpi')\ 1$$
$$-\frac{9\overset{2}{\overline{m}}.\mu}{2.(1-2m)}\cdot\frac{a}{a_{\prime}}\cdot\frac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi').\ 2$$

There remains yet to be considered the part of the development of
[4958] $-\frac{1}{h^2.(1+ss)^{\frac{3}{2}}}$ [4893], depending on the square of the disturbing force.
[4959] This development contains the function* $\frac{3}{2a_{\prime}}.(\delta s)^2$, which produces the following terms ;†

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* (2845) We have, by Taylor's theorem,

[4956f] $$\varphi(s+\delta s)=\varphi s+\delta s.\frac{d.\varphi(s)}{ds}+\tfrac{1}{2}(\delta s)^2.\frac{d^2.\varphi(s)}{ds^2}+\&c.\quad[617];$$

where the terms of the second order are represented by $\frac{1}{2}(\delta s)^2.\frac{d^2.\varphi(s)}{ds^2}$. Now, putting the function [4958] equal to $\varphi(s)$, and developing it, we get,

[4956g] $$\varphi(s)=-h^{-2}.(1+ss)^{-\frac{3}{2}}=-h^{-2}.(1-\tfrac{3}{2}s^2+\tfrac{15}{8}s^4-\&c.).$$

Its second differential gives,

[4956h] $$\frac{d^2.\ (s)}{ds^2}=3h^{-2}-\tfrac{45}{2}h^{-2}s^2+\&c.=\frac{3}{h^2}=\frac{3}{a_{\prime}}\quad\text{nearly}\quad[4937n];$$

neglecting s^2, &c. Substituting this in the terms depending on $(\delta s)^2$ [4956f], it becomes $\frac{3}{2a_{\prime}}.(\delta s)^2$, as in [4959]. The terms of the order $s^2.(\delta s)^2$, which we have
[4956i] here neglected, are of the order γ^2 [1811], in comparison with those which are retained and developed in [4960]; they must, therefore, be of the sixth or seventh order, and are not usually noticed.

† (2846) If we separate the terms of δs [4897] into classes, of the second, third and fourth orders, by putting

[4960a] $S_2=B_1^{(0)}.\gamma.\sin.(2v-2mv-gv)$;

[4960b] $$S_3=B_2^{(1)}.\gamma.\sin.(2v-2mv+gv)+B_1^{(7)}.e'\gamma.\sin.(gv+c'mv)+B_1^{(8)}.e'\gamma.\sin.(gv-c'mv)$$
$$+B_1^{(9)}.e'\gamma.\sin.(2v-2mv-gv+c'mv)$$
$$+B_1^{(10)}.e'\gamma.\sin.(2v-2mv-gv-c'mv)+B_0^{(11)}.e^2\gamma.\sin.(2cv-gv);$$

$S_4=B_2^{(5)}.e\gamma.\sin.(2v-2mv+gv-cv)+$ the remaining terms of δs [4897];

the index of S denoting the order of the terms; we shall have $\delta s=S_2+S_3+S_4$. Its
[4960c] square is $(\delta s)^2=S_2.S_2+2S_2.S_3+2S_2.S_4+S_3.S_3$; neglecting terms of the seventh

$$\frac{3}{4a_{\prime}} \cdot (B_1^{(0)})^2 . \gamma^2 \qquad 1$$

$$+\frac{3}{2a_{\prime}} \cdot \{B_1^{(9)}+B_1^{(10)}\} . B_1^{(0)} . e'\gamma^2 . \cos.(c'mv-\varpi') \qquad 2 \qquad [4960]$$

$$+\frac{3}{2a_{\prime}} \cdot B_1^{(0)} . B_2^{(5)} . e\gamma^2 . \cos.(2gv-cv-2\theta+\varpi). \qquad 3$$

9. We shall now collect together and reduce the different terms which we have calculated; and, by these means, we shall obtain the following development of the equation [4754];*

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order. Substituting the values of S_2, S_3, S_4, and then reducing, by means of [17—20] Int., retaining only the usual angles and terms, we get, by observing, that the terms depending on $B_2^{(1)}$ may be neglected, on account of its smallness [5177],

$$S_2 . S_2 = \tfrac{1}{2}(B_1^{(0)})^2 . \gamma^2 ;$$
$$2S_2 . S_3 = \{B_1^{(0)} . B_1^{(9)}+B_1^{(0)} . B_1^{(10)}\} . c'\gamma^2 . \cos.c'mv ;$$
$$2S_2 . S_4 = B_1^{(0)} . B_2^{(5)} . c\gamma^2 . \cos.(2gv-cv) ; \qquad [4960d]$$
$$S_3 . S_3 = \text{terms which may be neglected.}$$

The sum of these terms gives the value of $(\delta s)^2$ [4960c], which being multiplied by $\frac{3}{2a_{\prime}}$ gives $\frac{3}{2a_{\prime}} . (\delta s)^2$, as in [4960].

* (2847) We have thus finished this elaborate development of the terms composing the equation [4754]; and we must now connect together the different terms; namely, *those which are contained in the twenty functions* [4866, 4870, 4872, 4879, 4892, 4895, 4901, 4908, 4911, 4913, 4918, 4922, 4925, 4928, 4934, 4942, 4946, 4955, 4957, 4960], *and add to the sum the two first terms of* [4754]; namely, $\frac{ddu}{dv^2}+u$, as in the two first terms of [4961]. In performing this part of the operation, we shall take the terms depending on each angle separately, in the order in which they occur in [4961]. [4960e]

Functions which form the differential equation in u.

First. The constant terms of [4961 line 1], are found in [4895, 4866 line 1], without any reduction. The terms having the common factor $-\frac{3\overline{m}^2}{4a_{\prime}} . A_2^{(0)} . (1-\frac{5}{2}e'^2)$ are found by adding together the terms in the first lines of [4911, 4925, 4934]; namely, 3, $-2+2m$, $\frac{4.(1-m)^2-1}{1-m}$. Their sum is $1+2m+4.(1-m)-\frac{1}{1-m}=4-3m-m^2$, neglecting m^3 and the higher powers of m; this agrees with [4961 line 2]. Lastly, the term depending on $B_1^{(0)}$ [4960 line 1], is as in [4961 line 2]. [4961a]

Differential equation in u.

[4961]
$$0=\frac{ddu}{dv^2}+u-\frac{1}{a_{\prime}}.\{1+e^2+\tfrac{1}{4}\gamma^2+\beta''\}+\frac{\overline{m}^2}{2a_{\prime}}.\{1+e^2+\tfrac{1}{4}\gamma^2+\tfrac{3}{2}e'^2\} \qquad 1$$

$$-\frac{3\overline{m}^2}{4a_{\prime}}.(4-3m-m^2).A_2^{(0)}.(1-\tfrac{5}{2}e'^2)+\frac{3}{4a_{\prime}}.(B_1^{(0)})^2.\gamma^2 \qquad 2$$

$$-\frac{3\overline{m}^2}{4a_{\prime}}.\left\{\begin{array}{ll}
+2e+e^2+3e'^2-2.(B_2^{(2)}+B_2^{(3)}).\dfrac{\gamma^2}{\overline{m}^2}+(1+2m-c).A_2^{(2)}(1-\tfrac{5}{2}e'^2) & 3\\
-4.\left\{1+2m+\left(4.\overline{1-m}^2-1\right).\left(\dfrac{1+m}{2-2m-c}+\dfrac{1-m}{2-2m+c}\right)\right\}A_2^{(0)}(1-\tfrac{5}{2}e'^2) & 4\\
+\dfrac{\{(1+6m+c).(1-m)+7+(2-2m-c)^2\}}{1-m}.A_1^{(1)}.(1-\tfrac{5}{2}e'^2) & 5\\
-\tfrac{1}{2}.(9+m+c).A_1^{(6)}.e'^2+\tfrac{7}{2}.(9+3m+c).A_1^{(7)}.e'^2 & 6\\
+3.(A_1^{(8)}+A_1^{(9)}).e'^2 & 7
\end{array}\right\}.e.\cos(cv-\varpi)$$

(L')

$$+\frac{3\overline{m}^2}{2a_{\prime}}.\left\{\begin{array}{ll}
1+(1+2m).e^2+\tfrac{1}{4}\gamma^2-\tfrac{5}{2}e'^2 & 8\\
+\dfrac{(1+3e^2+\frac{1}{4}\gamma^2-\frac{5}{2}e'^2)}{1-m} & 9\\
-A_2^{(0)}-(B_1^{(0)}-B_2^{(1)}).\dfrac{\gamma^2}{\overline{m}^2} & 10
\end{array}\right\}.\cos.(2v-2mv)$$

Second. We shall now collect together all the terms which are connected with $\cos.cv$. For brevity, we shall divide all the terms of the twenty functions [4960c] containing this quantity, by the common factor $-\frac{3\overline{m}^2}{4a_{\prime}}.e.\cos.cv$, retaining only the quotients which ought to correspond to the terms, between the braces, in [4961 lines 3—7]. *The same*
[4961b] *process will be used with the other angles in the rest of this note.* Then we have, in [4866 line 2], the terms $2+e^2+3e'^2$, and, in [4901 line 3], the terms $-2(B_2^{(2)}+B_2^{(3)}).\frac{\gamma^2}{\overline{m}^2}$; these agree with [4961 line 3]. The rest of the quantities depend on the different terms of A, which we shall examine according to the order of the indices. The coefficients of $-4A_2^{(0)}.(1-\frac{5}{2}e'^2)$, in [4911 line 2, 4925 line 2], are, respectively, $+3$, and $-2+2m$, whose sum $1+2m$ is the same as in the two first terms of line 4 [4961]; the last terms of the same line being found, without any reduction, in [4934 line 2]. The coefficients of $A_1^{(1)}.e.(1-\frac{5}{2}e'^2)$, in [4911 line 2, 4918 line 1, 4925 line 2], are respectively 3, $4m$, $-(2-2m-c)$, whose sum is $(1+6m+c)$; multiplying and dividing this by $1-m$, it produces the three first terms in [4961 line 5], connected with the factor $(1-m)$; the remaining terms

$$
+\frac{3\bar{m}^2}{a_{\prime}}\cdot\left\{\begin{array}{ll}
\tfrac{1}{4}e.\{1+\tfrac{1}{4}(2-19m).e^2-\tfrac{5}{2}e'^2\} & 11\\
-\tfrac{1}{4}(3+4m).(1+\tfrac{1}{2}e^2-\tfrac{5}{2}e'^2) & 12\\
+\dfrac{1-c^2}{4(1-m)}-\dfrac{2(1+m)}{2-2m-c}.(1+\tfrac{7}{4}e^2-\tfrac{5}{2}e'^2) & 13\\
-\tfrac{1}{2}.(A_1^{(1)}-2A_2^{(0)})+\tfrac{1}{2}.(B_2^{(5)}-B_2^{(6)}).\dfrac{\gamma^2}{\bar{m}^2} & 14
\end{array}\right\}.e.\cos.(2v-2mv-cv+\varpi)
$$

$$
-\frac{3\bar{m}^2}{4a_{\prime}}.\left\{3+c-4m+\frac{8(1-m)}{2-2m+c}+2A_2^{(2)}\right\}.e.\cos.(2v-2mv+cv-\varpi) \qquad 15
$$

[4961]

Differential equation in u continued.

$$
-\frac{3\bar{m}^2}{4a_{\prime}}.\left\{\frac{4-m}{2-m}+2B_1^{(9)}.\frac{\gamma^2}{\bar{m}^2}+2A_2^{(3)}\right\}.e'.\cos.(2v-2mv+c'mv-\varpi') \qquad 16
$$

$$
+\frac{3\bar{m}^2}{4a_{\prime}}.\left\{\frac{7(4-3m)}{2-3m}-2B_1^{(10)}.\frac{\gamma^2}{\bar{m}^2}-2A_2^{(4)}\right\}.e'.\cos.(2v-2mv-c'mv+\varpi') \qquad 17
$$

$$
+\left\{\begin{array}{ll}
\dfrac{3\bar{m}^2}{2a_{\prime}}.\left\{\begin{array}{l}
1+e^2+\tfrac{1}{4}\gamma^2+\tfrac{9}{8}e'^2+(B_1^{(7)}+B_1^{(8)}).\dfrac{\gamma^2}{\bar{m}^2}-\tfrac{3}{2}(1+2m).A_2^{(0)}\\
-\dfrac{2(1-2m).(3-2m).(3-m)}{(2-3m).(2-m)}.A_2^{(0)}-2.A_2^{(3)}-(2-3m).A_2^{(4)}\\
+(B_1^{(9)}+B_1^{(10)}).B_1^{(0)}.\dfrac{\gamma^2}{\bar{m}^2}-A_2^{(5)}-11C_2^{(6)}-2C_2^{(9)}+2C_2^{(10)}
\end{array}\right\} & \begin{array}{l}18\\19\\20\end{array}\\
+\dfrac{6m}{a_{\prime}}.\{4A_2^{(0)}+A_2^{(3)}-A_2^{(4)}-10.A_1^{(1)}e^2+\tfrac{5}{2}(A_1^{(7)}-A_1^{(6)}).e^2\} & 21
\end{array}\right\}.e'.\cos.(c'mv-\varpi')
$$

---

of that line are found in [4934 line 2], without any reduction. The coefficients of $A_2^{(2)}\,e.(1-\tfrac{5}{2}e'^2)$, in [4911 line 2, 4925 line 2], are $3-(2-2m+c)=1+2m-c$, as in [4961 line 3]. The coefficients of $-\tfrac{1}{2}.A_1^{(6)}.e'^2$, in [4911 line 2, 4925 line 3, 4934 line 3], neglecting the factor $1-\tfrac{5}{2}e'^2$, are 3, $-(2-m-c)$, 8; whose sum is $9+m+c$ [4961 line 6]. The coefficients of $\tfrac{7}{2}.A_1^{(7)}.e'^2$, in [4911 line 2, 4925 line 3, 4934 line 3], give $3-(2-3m-c)+8=9+3m+c$ [4961 line 7]. Lastly, the terms in [4908 line 6] give, without reduction, $3.(A_1^{(8)}+A_1^{(9)}).e'^2$, as in [4961 line 7]. [4961b']

Third. The terms in [4961 lines 8—10] have the common factor $\frac{3\bar{m}^2}{2a_{\prime}}.\cos.(2v-2mv)$; and, if we divide the corresponding terms of the functions [4960c] by this factor, we shall obtain, in [4870 line 1], the terms $1+e^2+\tfrac{1}{4}\gamma^2-\tfrac{5}{2}e'^2$, and, in [4879 line 9], the term $+2me^2$ nearly; the sum of these gives [4961 line 8]. The terms [4892 line 1] are the same as [4961 line 9]; those in [4901 line 1] are the same as those depending on $B_1^{(0)}$, $B_2^{(1)}$ [4961c]

[4961] Differential equation in u continued.

$$+\frac{3\overline{m}^2}{2a_{\prime}}\cdot\left\{\begin{array}{l}\frac{3+2m-c}{4}+\frac{(2+m)}{2-m-c}-\frac{3}{2}A_1^{(7)}-A_1^{(6)} \\ -\left\{\frac{3+m-c}{2}+\frac{4}{2-m-c}\right\}\cdot A_1^{(9)}\end{array}\right\}.ee'.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') \quad \begin{array}{r}22\\23\end{array}$$

[4961 line 10]. Lastly, the first term of $a\delta u$ [4908 line 1, 4904] gives the term depending on $A_2^{(0)}$ [4961 line 10].

Fourth. The terms in [4961 lines 11–14] have the common factor $\frac{3\overline{m}^2}{a_{\prime}}.e.\cos(2v-2mv-cv)$. Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4870 line 2], the terms in [4961 line 12]; in [4879 line 1], the same terms as in [4961 line 11]; in
[4961d] [4892 line 2], the same terms as [4961 line 13]; in [4901 line 6], the terms depending on $B_2^{(5)}$, $B_2^{(6)}$ [4961 line 14]; lastly, we find, in [4908 lines 1, 2], the terms depending on $A_2^{(0)}$, $A_1^{(1)}$ [4961 line 14].

Fifth. The terms in [4961 line 15] have the factor $-\frac{3\overline{m}^2}{4a_{\prime}}.e.\cos.(2v-2mv+cv)$. Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4870 line 3], the terms $3-4m$; and, in [4879 line 2], the term $+c$; the sum of these is equal to the three
[4961e] first terms of [4961 line 15]. Again, [4892 line 3] gives $\frac{8.(1-m)}{2-2m+c}$; and [4908 line 1] gives $2A_2^{(2)}$; which are the remaining terms of [4961 line 15].

Sixth. The terms in [4961 line 16] have the factor $-\frac{3\overline{m}^2}{4a_{\prime}}.c'.\cos.(2v-2mv+c'mv)$. Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4870] line 5, the term 1; and, in [4892 line 5], the term $+\frac{2}{2-m}$; the sum of these is
[4961f] $\frac{4-m}{2-m}$, as in the first term of line 16 [4961]; the term depending on $B_1^{(9)}$ is deduced from [4901 line 8], and, that depending on $A_2^{(3)}$, from [4908 line 1].

Seventh. The terms in [4961 line 17] have the common factor $\frac{3\overline{m}^2}{4a_{\prime}}.c'.\cos.(2v-2mv-c'mv)$. Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4870] line 4, the term $+7$; and, in [4892 line 4], the term $\frac{14}{2-3m}$; the sum of these is
[4961g] $\frac{7(4-3m)}{2-3m}$, as in [4961 line 17]; then we have, in [4901 line 9], the term $-2B_1^{(10)}.\frac{\gamma^2}{\overline{m}^2}$; and, in [4908 line 1], the term $-2A_2^{(4)}$; all of which agree with [4961 line 17].

Eighth. The terms in [4961 lines 18—20] have the common factor $\frac{3\overline{m}^2}{2a_{\prime}}.e'.\cos.c'mv$.

$$-\frac{3\overline{m}^2}{2a_{\prime}}\cdot\left\{\begin{array}{l}\frac{7(3+6m-c)}{4}+\frac{7(2+3m)}{2-3m-c}+\frac{3}{2}A_1^{(1)} \\ +A_1^{(7)}+\left\{\frac{3-m-c}{2}+\frac{4}{2-3m-c}\right\}A_1^{(8)}\end{array}\right\}.ee'.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi')$$ 24 [4961] 25

Differential equation in u continued.

Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4866] line 3, the terms $1+e^2+\frac{1}{4}\gamma^2+\frac{3}{8}e'^2$; in [4901 line 7], the terms $+(B_1^{(7)}+B_1^{(8)})\cdot\frac{\gamma^2}{\overline{m}^2}$; these include the terms of [4961 line 18], except those depending on $A_2^{(0)}$. The terms depending on $A_2^{(0)}$, in [4911 line 3, 4925 line 4], are $-\frac{3}{2}A_2^{(0)}.\{3+(-2+2m)\}$, or, $-\frac{3}{2}.(1+2m).A_2^{(0)}$, as in [4961 line 18]. The other terms depending on $A_2^{(0)}$, in [4961] line 19, are the same as in [4934 line 5]; observing, that $4.(1-m)^2-1=(1-2m).(3-2m)$, [4961h] and $\frac{7}{2-3m}-\frac{1}{2-m}=\frac{4.(3-m)}{(2-3m).(2-m)}$. The factors of $A_2^{(3)}$, in [4911 line 3, 4925 line 4], are, respectively, $-\frac{3}{2}$, $1-\frac{1}{2}m$; that in [4934 line 6] is $\frac{-(2-m)^2+1}{2-2m}=-\frac{3}{2}+\frac{1}{2}m$, neglecting terms of the order m^2; the sum of these three terms gives, $-2A_2^{(3)}$, as in [4961 line 19]. The factors of $A_2^{(1)}$, in the same three functions [4911, 4925, 4934], and reduced in the same manner, are $-\frac{3}{2}$, $1-\frac{3}{2}m$, $-\frac{3}{2}+\frac{9}{2}m$; whose sum is $-2+3m$, as in [4961 line 19]. The term depending on $A_2^{(5)}$ [4908 line 1] is as in [4961 line 20]. The remaining terms of [4961 line 20] correspond, without any reduction, to those in [4960 line 2, 4934 line 7]. Lastly, the terms in [4934 line 4], are the same as in [4961 line 21].

Ninth. The terms of [4961 lines 22, 23] have the common factor

$$\frac{3\overline{m}^2}{2a_{\prime}}.ee'.\cos.(2v-2mv-cv+c'mv).$$

Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4870 line 8], the terms $\frac{1}{4}(3+2m)$; in [4879 line 5], the term $-\frac{1}{4}c$; in [4892 line 8], the term $\frac{2+m}{2-m-c}$; and, in [4908 lines 4, 1], the terms $-\frac{3}{2}A_1^{(1)}-A_1^{(6)}$; these terms, connected together in the same order, form the part in [4961 line 22]. In computing the terms which [4961i] are multiplied by $-A_1^{(1)}$, we have, in [4911 line 7], the term $\frac{3}{2}$; in [4925 line 7], the term $\frac{1}{2}(m-c)$; and, in [4934 line 9], the term $\frac{4}{2-m-c}$; the sum of these three parts is as in [4961 line 23].

Tenth. The terms of [4961 lines 24, 25] have the common factor

$$-\frac{3\overline{m}^2}{2a_{\prime}}.ee'.\cos.(2v-2mv-cv-c'mv).$$

[4961]

Differential equation in u continued.

$$-\frac{3\overline{m}^2}{2a_{\prime}} \cdot \left\{ \begin{array}{l} \frac{3+2m}{2} - \left\{ \frac{1+2m+c}{4} + \frac{2}{c+m} \right\} . A_1^{(1)} \\ + A_1^{(8)} + \left\{ \frac{1+3m+c}{2} + \frac{4}{c+m} \right\} . A_1^{(7)} \end{array} \right\} . e e'.\cos.(cv+c'mv-\varpi-\varpi') \quad \begin{array}{r} 26 \\ 27 \end{array}$$

$$-\frac{3\overline{m}^2}{2a_{\prime}} \cdot \left\{ \begin{array}{l} \frac{3-2m}{2} + A_1^{(9)} + 7 \left\{ \frac{1+2m+c}{4} + \frac{2}{c-m} \right\} . A_1^{(1)} \\ + \left\{ \frac{1+m+c}{2} + \frac{4}{c-m} \right\} . A_1^{(6)} \end{array} \right\} . e e'.\cos.(cv-c'mv-\varpi+\varpi') \quad \begin{array}{r} 28 \\ 29 \end{array}$$

Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4870] line 6, the terms $\frac{7}{4}(3+6m)$; in [4879 line 3] the term $-\frac{7}{4}c$; in [4892 line 6], the term $+\frac{7(2+3m)}{2-3m-c}$; in [4908 line 5], the term $\frac{3}{2}.A_1^{(1)}$; the sum of these terms is as in [4961 line 21]. There is also, in [4908 line 1], the term $A_1^{(7)}$, as in the first term of

[4961k] [4961 line 25]. The coefficients of $A_1^{(8)}$ are as follows; in [4911 line 6], $+\frac{3}{2}$; in [4925 line 8], $-\frac{1}{2}(m+c)$; in [4934 line 8], $\frac{4}{2-3m-c}$; the sum of these is the same as the coefficient of $A_1^{(8)}$, in [4961 line 25].

Eleventh. The terms of [4961 lines 26, 27] have the common factor

$$-\frac{3\overline{m}^2}{2a_{\prime}}.ce'.\cos.(cv+c'mv).$$

Dividing the corresponding terms of the functions [4960e] by this, we obtain in [4866] line 4, the terms $\frac{1}{2}(3+2m)$, as in the first part of line 26 [4961]. The coefficients of $-A_1^{(1)}$ are as follows; in [4911 line 5], $+\frac{3}{4}$; in [4925 line 5], $\frac{1}{4}(-2+2m+c)$; and in [4934 line 11] $+\frac{2}{c+m}$; the sum of these three parts is the same as the coefficient of

[4961l] $-A_1^{(1)}$ [4961 line 26]. The coefficients of $A_1^{(7)}$, in the same three functions [4911 line 5, 4925 line 5, 4934 line 11], are $\frac{3}{2}$, $\frac{1}{2}(-2+3m+c)$, $+\frac{4}{c+m}$; whose sum is equal to the coefficient of $A_1^{(7)}$, in [4961 line 27]. Lastly, the term depending on $A_1^{(8)}$ [4908 line 1], is the same as in [4961 line 27].

Twelfth. The terms of [4961 lines 28, 29] have the common factor

$$-\frac{3\overline{m}^2}{2a_{\prime}}.ce'.\cos.(cv-c'mv).$$

Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4866] line 5, the terms $\frac{1}{2}(3-2m)$, as in the first part of line 28 [4961]. The coefficient of $7A_1^{(1)}$, in [4911 line 4], is $+\frac{3}{4}$; in [4925 line 6], is $\frac{1}{4}(-2+2m+c)$; in [4934 line 10], is

$$+\frac{3\bar{m}^2}{2a_{,}}\cdot\left\{1-B_0^{(11)}.\frac{\gamma^2}{\bar{m}^2}-A_2^{(10)}\right\}.\,e^2.\cos.(2cv-2\varpi) \qquad 30$$

[4961]

$$+\frac{3\bar{m}^2}{4a_{,}}\cdot\left\{\begin{array}{l}\dfrac{(2+11m+8m^2)}{2}-\dfrac{(10+19m+8m^2)}{2c-2+2m} \qquad 31 \\ +4A_1^{(1)}+\dfrac{\{8A_2^{(10)}+10(A_1^{(1)})^2\}}{2c-2+2m}-2A_1^{(11)} \qquad 32\end{array}\right\}.\,e^2.\cos.(2cv-2v+2mv-2\varpi)$$

Differential equation in u continued.

$\frac{2}{c-m}$; the sum of these three parts is the same as the coefficient of $A_1^{(1)}$, in [4961] line 28. In like manner, the coefficients of $A_1^{(6)}$, in the same lines of these three functions, are $\frac{3}{2}$, $+\frac{1}{2}(-2+m+c)$, $+\frac{4}{c-m}$; whose sum is the same as the coefficient of $A_1^{(6)}$ [4961 line 29]. Lastly, the term of [4908 line 1], depending on $A_1^{(9)}$, is the same as in [4961 line 28]. [4961m]

Thirteenth. The coefficients of $\frac{3\bar{m}^2}{2a_{,}}.e^2.\cos.2cv$, in [4866 line 6, 4901 line 10, 4908 line 1], are, respectively, 1, $-B_0^{(11)}.\frac{\gamma^2}{\bar{m}^2}$, $-A_2^{(10)}$; whose sum is as in [4961 line 30]. [4961n]

Fourteenth The terms of [4961 lines 31, 32] have the common factor

$$\frac{3\bar{m}^2}{4a_{,}}\cdot e^2.\cos.(2cv-2v+2mv).$$

Dividing the corresponding terms of the functions [4960e] by this quantity, we obtain, in [4870 line 11], the terms $\frac{1}{2}(6+15m+8m^2)$; and, in [4879 line 7], the terms $-2c.(1+m)$, or, $\frac{1}{2}(-4-4m)$ nearly; the sum of these two expressions is $\frac{1}{2}(2+11m+8m^2)$, as in the first term of [4961 line 31]. The term in [4892 line 10] is the same as the second term of [4961 line 31]. The term in [4908 line 3], neglecting e'^2, is $4A_1^{(1)}$, as in the first term of [4961 line 32]; and the term of [4908 line 1], depending on $A_1^{(11)}$, is the same as in the last term of [4961 line 32]. The term [4934 line 12], is the same as that depending on $A_2^{(10)}$ in [4961 line 32]. Lastly, [4942] is the same as the term depending on $(A_1^{(1)})^2$ [4961 line 32]. [4961o]

Fifteenth. The coefficients of $-\frac{3}{4a_{,}}.\gamma^2.\cos.2gv$, in [4866 line 7, 4895, 4908 line 1], are, respectively, $-\frac{1}{2}\bar{m}^2$, $1+e^2-\frac{1}{4}\gamma^2$, $+2\bar{m}^2.A_2^{(12)}$; whose sum is as in [4961 line 33]. [4961p]

Sixteenth. The terms of [4961 lines 34, 35] have the common factor

$$\frac{3\bar{m}^2}{4a_{,}}\cdot\gamma^2.\cos.(2gv-2v+2mv).$$

$$-\frac{3}{4a_{\prime}}\cdot\left\{1+e^2-\tfrac{1}{4}\gamma^2-\tfrac{1}{2}.\bar{m}^2+2\bar{m}^2.A_2^{(12)}\right\}.\gamma^2.\cos.(2gv-2\theta) \qquad 33$$

[4961]

Differential equation in v continued.

$$+\frac{3\bar{m}^2}{4a_{\prime}}.\left\{\begin{array}{l}\dfrac{3+2m-2g}{4}+\dfrac{(4g^2-1)}{4(1-m)}-\dfrac{(2+m)}{2g-2+2m} \qquad 34\\ +\dfrac{2B_1^{(0)}}{\bar{m}^2}-2A_1^{(13)}+\dfrac{8A_2^{(12)}}{2g-2+2m} \qquad 35\end{array}\right\}.\gamma^2.\cos.(2gv-2v+2mv-2\theta)$$

$$+\frac{3\bar{m}^2}{2a_{\prime}}.\left\{\tfrac{3}{2}-A_2^{(14)}\right\}.e'^2.\cos.(2c'mv-2\varpi') \qquad 36$$

$$-\frac{3\bar{m}^2}{2a_{\prime}}.\left\{\begin{array}{l}\tfrac{1}{2}+\dfrac{B_2^{(3)}}{\bar{m}^2}+\dfrac{(1+c-2g-10m)}{4}.A_1^{(1)}-(10+5m).A_1^{(13)} \qquad 37\\ +(5+m).A_1^{(16)}-\dfrac{B_1^{(0)}.B_2^{(5)}}{\bar{m}^2}+A_0^{(15)} \qquad 38\end{array}\right\}.c\gamma^2.\cos.(2gv-cv-2\theta+\varpi)$$

[4961q] Dividing the corresponding terms of the functions [4960e] by this quantity, we obtain, in [4870 line 13], the terms $\frac{1}{4}(3+2m)$; in [4879 line 10], the term $-\frac{2}{4}g$; the sum of these two expressions is $\frac{1}{4}(3+2m-2g)$, as in the first part of [4961 line 34]; the remaining terms of this line are given in [4892 line 12]. The term depending on $B_1^{(0)}$ [4901 line 2], that depending on $A_1^{(13)}$ [4908 line 1], and that depending on $A_2^{(12)}$ [4934 line 13], correspond, respectively, to those in [4961 line 35].

[4961r] *Seventeenth.* The coefficients of $\frac{3\bar{m}^2}{2a_{\prime}}.e'^2.\cos.2c'mv$, in [4866 line 8, 4908 line 1], are $\frac{3}{2}$, $-A_2^{(14)}$, as in [4961 line 36].

Eighteenth. The terms of [4961 lines 37, 38] have the common factor

$$-\frac{3\bar{m}^2}{2a_{\prime}}.c\gamma^2.\cos.(2gv-cv).$$

Dividing the corresponding terms of the functions [4960e] by this quantity, we obtain, in [4866 line 9], the term $\frac{1}{2}$; in [4901 line 4], the term $\frac{B_2^{(3)}}{\bar{m}^2}$; these agree with the two first terms of [4961 line 37]. The coefficient of $\frac{1}{4}A_1^{(1)}$, in [4911 line 8], is $3+\frac{3}{2}m$; in [4918 line 2], is $+3m$; in [4925 line 9], is $-2g-2-\frac{1}{2}m+c$; and, in [4934 line 14],

[4961s] is $-14m$; the sum of these terms is $1+c-2g-10m$, as in [4961 line 37]. The coefficient of $-A_1^{(13)}$, in [4911 line 8], is $3+3m$; in [4925 line 9], is $-1+2m$; in [4934 line 14], is $+8$; the sum of these is $10+5m$. as in [4961 line 37]. The coefficient of $A_1^{(16)}$, in [4911 line 8], is $+\frac{3}{2}$; in [4925 line 10], is nearly $-\frac{1}{2}+m$; and, in [4934 line 14], is $+4$; the sum of these is $5+m$, as in [4961 line 38]. The

$$-\frac{3\bar{m}^2}{4a_{,}}\cdot\left\{\begin{array}{l} 1+2m+\dfrac{5+m}{1-2m}+\dfrac{3(1-m)}{3-2m}+2A_1^{(16)} \qquad 39 \\ -\dfrac{2B_2^{(4)}}{\bar{m}^2}+\dfrac{10A_0^{(15)}}{1-2m} \qquad 40 \end{array}\right\}\cdot e\gamma^2.\cos.(2v-2mv-2gv+cv+2\theta-\varpi)$$

[4961]

Differential equation in u continued.

$$+\frac{\bar{m}^2}{a_{,}}\cdot\left\{\begin{array}{l} \tfrac{9}{8}.(1-2\mu).(1+2e^2+2e'^2)+\dfrac{3(1-2\mu).(1+\frac{9}{2}e^2+2e'^2)}{4(1-m)} \qquad 41 \\ -\dfrac{(36+21m-15m^2)}{4(1-m)}.A_1^{(17)}+\dfrac{3(1+m)}{2(1-m)}.A_0^{(18)}.e'^2 \qquad 42 \\ -\dfrac{(57-38m)}{4(1-m)}.A_2^{(0)}+\tfrac{3}{2}.(B_2^{(14)}+B_2^{(15)}).\dfrac{\gamma^2}{\bar{m}^2} \qquad 43 \end{array}\right\}\cdot\frac{a}{a'}.\cos.(v-mv)$$

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term $+A_0^{(15)}$ occurs in [4908 line 1]. Lastly, the terms in [4960 line 3], are the same as in [4961 line 38].

Nineteenth. The terms of [4961 lines 39, 40] have the common factor

$$-\frac{3\bar{m}^2}{4a_{,}}.e\gamma^2.\cos.(2v-2mv-2gv+cv).$$

Dividing the corresponding terms of the functions [4960e] by this quantity, we obtain, in [4870 line 15], the terms $\frac{3}{2}+\frac{3}{4}m$; in [4879 line 12], the terms $-\frac{1}{2}+\frac{5}{4}m$; the sum of these is $1+2m$, as in the two first terms of [4961 line 39]. The terms in [4892 line 15], by putting $c=1$, $g=1$, become $\frac{5+m}{1-2m}+\frac{3(1-m)}{3-2m}$, as in [4961 line 39]. The function [4908 line 1] gives $2A_1^{(16)}$; and [4901 line 5] gives $-\frac{2B_2^{(4)}}{\bar{m}^2}$, as in [4961 lines 39,40]. [4961t]
The coefficient of $A_0^{(15)}$, in [4911 line 9], is $+3$; in [4918 line 3], is $+4m$; in [4925 line 11], is -1; the sum of these is $2+4m=2(1+2m)$; and, by neglecting m^2, it may be put under the form $\frac{2}{1-2m}$; adding this to the term [4934 line 15], which is nearly equal to $\frac{8}{1-2m}$, the sum becomes $\frac{10}{1-2m}$, as in [4961 line 40].

Twentieth. The terms of [4961 lines 41—43] have the common factor

$$\frac{\bar{m}^2}{a_{,}}.\frac{a}{a'}.\cos.(v-mv). \qquad [4961t']$$

Dividing the corresponding terms of the functions [4960e] by this quantity, we obtain, in [4872 line 1], the term $\frac{9}{8}(1+2e^2+2e'^2)$; and, in [4892 line 16], the term $\frac{3(1+\frac{9}{2}e^2+2e'^2)}{4(1-m)}$; these are the same as the terms of [4961 line 41], independent of μ. The term depending

[4961] Differential equation in u concluded.

$$+\frac{3\overline{m}^2}{2a_{\prime}}\cdot\left\{\begin{array}{l}\tfrac{5}{4}(1-2\mu)-A_0^{(18)}+\tfrac{1}{4}(4+m).A_1^{(17)}\\ -(5+m).A_1^{(19)}\end{array}\right\}\cdot\frac{a}{a'}.e'.\cos.(v-mv+c'mv-\varpi') \qquad \begin{array}{r}44\\45\end{array}$$

$$+\frac{3\overline{m}^2}{2a_{\prime}.(1-2m)}\cdot\left\{\begin{array}{l}\tfrac{1}{4}(15-8m).(1-2\mu)-\tfrac{1}{4}(76-33m).A_1^{(17)}\\ -5A_0^{(18)}-(1-2m).A_1^{(19)}\end{array}\right\}\cdot\frac{a}{a'}.e'.\cos.(v-mv-c'mv+\varpi'). \qquad \begin{array}{r}46\\47\end{array}$$

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on μ, in [4955 line 1], is $-\frac{9}{4}\mu$; and, if we neglect terms of the seventh order, we may connect it with the same factor as the other part of this term, putting it equal to $-\frac{9}{4}\mu.(1+2e^2+2e'^2)$, as in the first part of [4961 line 41]; and, we may incidentally
[4961u] remark, that this factor might be changed into $1+2e^2+2e'^2-\frac{2}{3}\gamma^2$ [4870z']. In like manner, the term depending on μ, in [4957 line 1], is $-\frac{3\mu}{2(1-m)}$; and may be connected with the corresponding factor $1+\frac{3}{2}e^2+2e'^2$, and then it becomes as in the last term of [4961 line 41]. The coefficient of $-\frac{1}{4}A_1^{(17)}$, in [4908 line 1], is $+6$; in [4911 line 10], is $+9$; in [4918 line 4], is $12m$; and, in [4925 line 12], is $-3+3m$; the sum of these is $12+15m=\frac{12+3m-15m^2}{1-m}$; adding this to the term in [4934 line 16] $\frac{6(4+3m)}{1-m}=\frac{24+18m}{1-m}$, it becomes $\frac{36+21m-15m^2}{1-m}$, as in [4961 line 42]. The coefficient
[4961u'] of $\frac{1}{2}A_0^{(18)}.e'^2$, in [4908 line 9], is $-\frac{9}{2}$; in [4911 line 10], is $+\frac{9}{4}$; in [4925 line 12], is $-\frac{3}{4}$; the sum of these is $-3=\frac{-3+3m}{1-m}$; adding this to the term in [4934 line 16], namely $\frac{6}{1-m}$, the sum becomes $\frac{3+3m}{1-m}=\frac{3(1+m)}{1-m}$, as in the last term of [4961] line 42. The coefficient of $-\frac{1}{4}A_2^{(0)}$, in [4922], is $+24$; in [4928], is $-9+9m$; the sum of these is $15+9m=\frac{15-6m}{1-m}$, neglecting m^2; adding this to the term [4946] $\frac{42-32m}{1-m}$, nearly; the sum is $\frac{57-38m}{1-m}$, as in the first part of [4961 line 43]. The terms in [4901 line 11], are the same as those depending on $B_2^{(14)}$, $B_2^{(15)}$ [4961 line 43]. Lastly, the coefficient of $\frac{1}{4}\lambda_2$, in [4913], is -9; in [4925 line 12], is $+9-9m$; and, in [4934 line 16], is $27\{1-(1-m)^2\}=54m-27m^2$; the sum of all these is $45m-27m^2$; the terms ± 9 mutually destroy each other; so that the whole term becomes of the order
[4961v] $\overline{m}^2.\frac{a}{a'}.\lambda_2.m$, or of the seventh order, as in [4962].

Twenty-first. The terms of [4961 lines 44, 45] have the common factor

$$\frac{3\overline{m}^2}{2a_{\prime}}\cdot\frac{a}{a'}.e'.\cos.(v-mv+c'mv).$$

Dividing the corresponding terms of [4960e] by this quantity, we obtain, in [4872 line 2], the

We have not noticed the terms multiplied by λ_2, because they mutually destroy each other, except in quantities of the order m^7 [4961v]. [4962]

10. To integrate this differential equation, we shall observe, that, by noticing only the parts which are not periodical, it gives,* [4963]

$$u=\frac{1}{a_{\prime}}\cdot\{1+e^2+\tfrac{1}{4}\gamma^2+\beta''\}-\frac{\overline{m}^2}{2a_{\prime}}\cdot(1+e^2+\tfrac{1}{4}\gamma^2+\tfrac{3}{2}e'^2)$$
$$+\frac{3\overline{m}^2}{4a_{\prime}}\cdot\{4-3m-m^2\}.A_2^{(0)}.(1-\tfrac{5}{2}e'^2)-\frac{3}{4a_{\prime}}\cdot(B_1^{(0)})^2.\gamma^2. \qquad [4964]$$

We have denoted this by $u=\frac{1}{a}.(1+e^2+\frac{1}{4}\gamma^2+\beta)$ [4861]. Now, if we [4965]

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term $+\frac{3}{4}$; and, in [4892 line 17], the term $\frac{2}{4}$; the sum of these two terms is $\frac{5}{4}$, as in the first term of line 41 [4961]. The term depending on μ, in [4955 line 1], is $-\frac{3}{2}\mu$; and, in [4957 line 1], is $-\mu$; the sum of these two expressions is $-\frac{5}{2}\mu$, as in the second term of [4961 line 44]. The term depending on $A_0^{(18)}$ [4908 line 1], is as in [4961 line 44]. The coefficient of $\frac{1}{4}A_1^{(17)}$, in [4908 line 7], is -6; in [4911 line 11], is $+3$; in [4925 line 13], is $-1+m$; and, in [4934 line 17], is $+8$; the sum of all these is $4+m$, as in [4961 line 44]. The coefficient of $-A_1^{(19)}$, in [4911 line 11], is $+\frac{3}{2}$. in [4925 line 13], is $-\frac{1}{2}+m$; and, in [4934 line 17], is $+4$; the sum of all these is $(5+m)$, as in [4961 line 45]. [4961v']

Twenty-second. The terms of [4961 lines 46, 47] have the common factor

$$\frac{3\overline{m}^2}{2a_{\prime}.(1-2m)}\cdot\frac{a}{a'}\cdot e'.\cos.(v-mv-c'mv).$$

Dividing the corresponding terms of the functions [4960e] by this, we obtain, in [4872 line 3], the term $\frac{9}{4}-\frac{13}{4}m$; and, in [4892 line 18], the term $\frac{6}{4}$; whose sum is $\frac{1}{4}(15-18m)$, as in the first term of [4961 line 46]. The term in [4955 line 2], is $-\frac{9}{2}(1-2m).\mu$; in [4957 line 2], is -3μ; whose sum is $-(\frac{15}{2}-9m)\,\mu=-\frac{1}{4}(15-18m).2\mu$, as in the first part of line 46 [4961]. The coefficient of $-\frac{1}{4}A_1^{(17)}$, in [4908 line 8], is $6-12m$; in [4911 line 12], is $21-42m$, in [4925 line 14], is $-7+21m$, neglecting m^2; in [4934 line 18], is $+56$; the sum of these terms is $76-33m$, as in [4961 line 46]. The coefficient of $-A_0^{(18)}$, in [4911 line 12], is $\frac{3}{2}-3m$; in [4918 line 5], is $+2m$, nearly; in [4925 line 14], is $-\frac{1}{2}+m$; in [4934 line 18], is 4; the sum of these terms is $+5$, as in [4961 line 47]. Lastly, the coefficient of $A_1^{(19)}$ [4908 line 1], is the same as in [4961 line 47]. [4961w]

* (2848) The equation [4961] being linear in u, we may compute the terms

[4966] neglect the sun's action, we shall have $\frac{1}{a}=\frac{1}{a_{\prime}}$ [4864]; so that we may

[4967] suppose* $\beta=\beta''$; therefore, we shall have,

Equation between a, $a_{\prime}$.

[4968] $$\frac{1}{a}=\frac{1}{a_{\prime}}-\frac{\overline{m}^2}{2a_{\prime}}.\{1+\tfrac{3}{2}e'^2\}+\frac{3\overline{m}^2}{4a_{\prime}}.(4-3m-m^2).A_2^{(0)}.(1-\tfrac{5}{2}e'^2)-\frac{3}{4a_{\prime}}.(B_1^{(0)})^2.\gamma^2.$$

The action of the planets produces a variation in the excentricity of the earth's orbit e', without altering its semi-major axis a', as we have seen in [1051', 1122, &c.]. Therefore, *the value of* $\frac{1}{a}$ *suffers corresponding*

[4969] *variations on account of the term*† $-\frac{3\overline{m}^2.e'^2}{4a_{\prime}}$ [4968], which it contains.

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depending on each angle separately; and, if we put A for the constant terms of that equation, we may compute the corresponding part of u by means of the equation

[4963a] $$0=\frac{ddu}{dv^2}+u+A;$$

which is evidently satisfied by putting $u=-A$. Hence it follows, that the constant part of u, is the same as the constant part of [4961], changing the signs; this agrees with

[4963b] [4964]. We may remark, that it is not necessary, in making this integration, to add an arbitrary constant quantity to $-A$; because it is implicitly included in the arbitrary quantity a, or $a_{\prime}$ [4860, 4864].

[4964a] * (2849) If we neglect the sun's disturbing force, we have $a_{\prime}=a$ [4861]; and the

[4964b] expression [4964] becomes, in this case, $u=\frac{1}{a}.\{1+e^2+\tfrac{1}{4}\gamma^2+\beta''\}$. Comparing this with the assumed value of the constant part of u, in the same hypothesis; namely, $u=\frac{1}{a}.\{1+e^2+\tfrac{1}{4}\gamma^2+\beta\}$ [4861], we get $\beta=\beta''$ [4967]; which is to be substituted

[4964c] in the second member of [4964]. We must also substitute, in the first member, the value of $u=\frac{1}{a}.(1+e^2+\tfrac{1}{4}\gamma^2+\beta)$ [4860, 4861]; hence we get,

[4964d] $$\frac{1}{a}.(1+e^2+\tfrac{1}{4}\gamma^2+\beta)=\frac{1}{a_{\prime}}.(1+e^2+\tfrac{1}{4}\gamma^2+\beta)-\frac{\overline{m}^2}{2a_{\prime}}.(1+e^2+\tfrac{1}{4}\gamma^2+\tfrac{3}{2}e'^2)$$
$$+\frac{3\overline{m}^2}{4a_{\prime}}.(4-3m-m^2).A_2^{(0)}.(1-\tfrac{5}{2}e'^2)-\frac{3}{4a_{\prime}}.(B_1^{(0)})^2.\gamma^2.$$

Dividing this by $1+e^2+\tfrac{1}{4}\gamma^2+\beta$, and neglecting terms of the sixth order, we get the value of $\frac{1}{a}$ [4968].

† (2850) The variation of the term

Moreover, as the constant term of the moon's parallax is proportional to $\frac{1}{a}$, [4970]
it is evident, that it must suffer a secular variation; *but, upon examination, it is found always to be insensible.**

The part of u, depending on $\cos.(cv-\varpi)$, is represented, in [4826], by [4971]
$\frac{e}{a}.(1+e^2).\cos.(cv-\varpi)$. If we substitute it, in the equation [4961], and then compare the sines and cosines of $cv-\varpi$, neglecting quantities of the
order $\frac{d^2.\frac{e}{a}}{dv^2}$, which can be permitted, considering the slowness of the [4972]
secular variations of the earth's orbit, we shall obtain the two following equations;†

$$-\frac{3\overline{m}^2.e'^2}{4a_{,}} \text{ [4969], is } -\tfrac{3}{2}.\frac{\overline{m}^2}{a_{,}}.e'.\delta e' = -\tfrac{3}{2}.\frac{m^2}{a}.e'.\delta e' \text{ [5094];} \qquad [4969a]$$

therefore the whole value of $\frac{1}{a}$, is to this variation, as 1 to $-\frac{3}{2}m^2.e'.\delta e'$. Substituting the values of m, e' [5117]; also $2\delta E$, or $2\delta e' = -t.0^s,187638$ [4330]; or, in parts of the radius, $\delta e' = -t.0,00000045$, nearly; we get

$$-\tfrac{3}{2}.\frac{\overline{m}^2}{a_{,}}.e'.\delta e' = t.0,00000000006, \text{ nearly;} \qquad [4969b]$$

which, in 1000 years, will not produce a single unit in the seventh decimal place of the moon's distance from the earth, taken as the unit of distance. If we multiply this expression by the constant term of the moon's horizontal parallax $3424^s,16$ [5331], we [4969c]
shall obtain the secular effect on the parallax, equal to $t.0^s,0000002$; which will not [4969d]
amount to a second in a million of years. We may remark, that the similar term of [4968], depending on $A_2^{(0)}$, is much less than that we have estimated, as is evident from the smallness of the value of $A_2^{(0)}$ [5157].

* (2851) We shall see, by the estimate made in [4987h—l], that this quantity is insensible. [4971a]

† (2852) If we put for a moment, for brevity, $E = \frac{(1+e^2).e}{a}$, and use the values of [4973a]
p, q [4975], we shall find, that the term, depending on $\cos.(cv-\varpi)$, in [4961], is
$E.(-p-q.e'^2).\cos.(cv-\varpi)$, and the corresponding part of the equation [4961], is [4973b]

$$0 = \frac{ddu}{dv^2} + u + E.(-p-q.e'^2).\cos.cv-\varpi). \qquad [4973c]$$

[4973] $$0=\frac{c.(1+e^2)}{a}\cdot\frac{dd\varpi}{dv^2}-2.\left(c-\frac{d\varpi}{dv}\right)\cdot\frac{d.\left\{c.\frac{(1+e^2)}{a}\right\}}{dv};$$

[4974] $$0=1-\left(c-\frac{d\varpi}{dv}\right)^2-p-q.e'^2;$$

the quantity $-p-q.e'^2$, *being supposed equal to the coefficient of* $\cos.(cv-\varpi)$,
[4975] *in the differential equation* [4961], *divided by* $\frac{(1+e^2).c}{a}$; where we must observe, that the values of $A_2^{(0)}$, $A_1^{(1)}$, $B_2^{(2)}$, and $B_2^{(3)}$ contain already
[4976] the factor $1-\frac{5}{2}e'^2$.* The equation [4973] gives, by integration,

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If we consider e, ϖ, as variable, and c constant [4986], we may satisfy this equation
[4973d] by assuming for u, an expression of the same form as in the purely elliptical hypothesis, which is $u=E.\cos.(cv-\varpi)$ [4826, 4973a]; substituting this in [4973c], we get,

[4973e] $$0=\left\{\frac{ddE}{dv^2}.\cos.(cv-\varpi)+2.\frac{dE}{dv}.\frac{d.\{\cos.(cv-\varpi)\}}{dv}+E.\frac{d^2.\{\cos.(cv-\varpi)\}}{dv^2}\right\}$$
$$+E.\cos.(cv-\varpi)+E.(-p-q.e'^2).\cos.(cv-\varpi).$$

Now, by neglecting quantities of the order mentioned in [4972], we may reject ddE, and we shall also have,

[4973f] $$\frac{d.\{\cos.(cv-\varpi)\}}{dv}=-\left(c-\frac{d\varpi}{dv}\right).\sin.(cv-\varpi);$$
$$\frac{d^2.\{\cos.(cv-\varpi)\}}{dv^2}=\frac{dd\varpi}{dv^2}.\sin.(cv-\varpi)-\left(c-\frac{d\varpi}{dv}\right)^2.\cos.(cv-\varpi);$$

hence, the equation [4973e] becomes,

[4973g] $$0=\left\{E.\frac{dd\varpi}{dv^2}-2\left(c-\frac{d\varpi}{dv}\right).\frac{dE}{dv}\right\}.\sin.(cv-\varpi)+\left\{E-E.\left(c-\frac{d\varpi}{dv}\right)^2+E.(-p-q.e'^2)\right\}.\cos.(cv-\varpi).$$

To satisfy this equation, for all values of $cv-\varpi$, we must put the coefficients of the
[4973h] sine and cosine of $cv-\varpi$, separately, equal to zero. The first of these conditions gives, without any reduction, the equation [4973]; the second, divided by E, gives [4974].

* (2853) The chief terms of $A_2^{(0)}$, $A_1^{(1)}$, deduced from [4998, 4999], evidently
[4976a] contain the factor $1-\frac{5}{2}e'^2$, and the expression of $B_1^{(0)}$, obtained from [5062], contains terms with the same factor; by this means it is introduced into the equations [5064, 5065], from which $B_2^{(2)}$, $B_2^{(3)}$ are derived. Hence, it appears, that the quantities $A_2^{(0)}$, $A_1^{(1)}$,
[4976b] $B_2^{(2)}$ $B_2^{(3)}$, which occur in the coefficient of $\cos.(cv-\varpi)$ [4961], contain the factor $1-\frac{5}{2}e'^2$, as in [4976]. We see, in this article, [4982, &c.], the importance of retaining

$$\frac{1}{c-\frac{d\varpi}{dv}}=\frac{k\,.\,e^2.(1+e^2)^2}{a^2}\,; \qquad [4977]$$

k being an arbitrary constant quantity*. Neglecting the square of $q.e'^2$, we obtain, from [4974],†

$$\frac{d\varpi}{dv}=c-\sqrt{1-p}+\frac{\frac{1}{2}q.e'^2}{\sqrt{1-p}}. \qquad [4978]$$

Therefore, if we consider p and q as constant, which we can do here, without any sensible error,‡ we shall have, by putting $q'=\frac{q}{\sqrt{1-p}}$, [4979]

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the term depending on e'^2, of which we have already spoken in [4910o]; since the secular inequalities of the moon's motion depend on this quantity [4984, &c.].

* (2854) We shall put for a moment, $c-\frac{d\varpi}{dv}=W$, and then, by taking its differential, [4977a]

we get $\frac{dd\varpi}{dv^2}=-\frac{dW}{dv}$. Substituting these values, and that of E [4973a], in [4973], [4977b]

we obtain

$$0=-E.\frac{dW}{dv}-2W.\frac{dE}{dv}, \quad\text{or}\quad -\frac{dW}{W}=2.\frac{dE}{E}. \qquad [4977c]$$

Its integral is

$$\log.\frac{1}{W}=\log.E^2+\log.k, \quad\text{or}\quad \frac{1}{W}=k.E^2, \text{ as in [4977]}\,; \qquad [4977c']$$

k being the arbitrary constant quantity. This satisfies the first of the equations of condition [4973]; and, if we deduce from it the value of $W=c-\frac{d\varpi}{dv}$, and substitute it in the second of these equations [4974], it becomes,

$$0=1-\frac{a^4}{k^2.e^4.(1+e^2)^4}-p-q.e'^2. \qquad [4977d]$$

This might be satisfied, if all the elements e, e', γ, &c. were invariable, by taking the arbitrary constant quantity k, so as to correspond to these elements; but e', or E [4977e] [4330], being subject to a secular inequality, it will produce secular terms in the value of c, deduced from [4977d].

† (2855) From [4974], we have

$$c-\frac{d\varpi}{dv}=\sqrt{(1-p-q.e'^2)}=\sqrt{(1-p)}-\frac{\frac{1}{2}q.e'^2}{\sqrt{(1-p)}}+\&c. \qquad [4978a]$$

If we neglect the square and higher powers of $q.e'^2$, it becomes, by reduction, as in [4978].

‡ (2856) The quantities p, q [4975], are functions of e, γ; whose secular variations are [4979a]

[4980] $$\varpi = c\,v - v\sqrt{1-p} + \tfrac{1}{2}q'.\textstyle\int c'^2.\,dv + \varepsilon;$$

ε being an arbitrary quantity*. From this equation we get,

[4981] $$\cos.(c\,v-\varpi) = \cos.\{v\sqrt{1-p} - \tfrac{1}{2}q'.\textstyle\int c'^2.dv - \varepsilon\}.$$

Hence it follows, *in conformity with observation, that the lunar perigee has a motion, which is represented by*

[4982] $$(1-\sqrt{1-p}).v + \tfrac{1}{2}q'.\textstyle\int c'^2.\,dv = \text{motion of the moon's perigee.}$$

This motion is not uniform on account of the variableness of c'; and, if we suppose, in counting from a given epoch, that c' is represented by

[4983] $$c' = E' + fv + lv^2 \quad [4330,\&c.];$$

Motion of the moon's perigee.

E' being the excentricity of the earth's orbit, at the same epoch, *the motion of the perigee will be*†

[4984] $$(1-\sqrt{1-p}+\tfrac{1}{2}q'.E'^2).v + \tfrac{1}{2}q'.E'.fv^2 + \tfrac{1}{6}.q'.(2E'l+f^2).v^3 = \text{motion of the moon's perigee.}$$

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insensible [4987, 5061]; we may, therefore, consider p and q as constant quantities, in making the integrations.

* (2857) Multiplying [4978] by dv, integrating, and substituting q' [4979], we get [4980]; whence,

[4982a] $$cv - \varpi = v\sqrt{(1-p)} - \tfrac{1}{2}q'.\textstyle\int c'^2.dv - \varepsilon;$$

whose cosine is as in [4981]. Now, we have supposed, in [4971, &c.], that $cv-\varpi$ represents the moon's anomaly, and v the moon's motion; their difference is

[4982b] $$v - v\sqrt{(1-p)} + \tfrac{1}{2}q'.\textstyle\int c'^2.dv + \varepsilon;$$

so that, while v varies from 0 to v, the corresponding motion of the perigee is represented by

[4982c] $$v - v\sqrt{(1-p)} + \tfrac{1}{2}q'.\textstyle\int c'^2.\,dv;$$

the integral $\int c'^2.dv$ being supposed to commence with $v=0$. This is easily reduced to the form [4982].

† (2858) By using the value of c' [4983], we obtain,

[4984a] $$\textstyle\int c'^2.dv = \int dv.\{E'^2+2E'f.v+(2E'l+f^2).v^2+\&c.\} = E'^2v + E'f.v^2 + \tfrac{1}{3}(2E'l+f^2).v^3 + \&c.;$$

substituting this in [4982], we obtain the expression of the motion of the perigee [4984]. The part of this expression, depending on the first power of v, represents the mean motion of the perigee, which we have put equal to $(1-c).v$ [4817]; hence we get

[4984b] $$(1-c).v = (1-\sqrt{1-p}+\tfrac{1}{2}q'.E'^2).v.$$

This expression may be used for two thousand years before or after the epoch [4984f, i]. The part of it, included in the following formula, expresses the secular equation of the motion of the perigee, which is decreasing from age to age [5232']; [4984']

$$\tfrac{1}{2}q'.E'.fv^2+\tfrac{1}{6}q'.(2E'l+f^2).v^3 = \text{secular equation of the perigee [4984}d\text{]}. \quad [4985]$$

Value of c.

The value of the constant quantity c *may be represented by*

$$c=\sqrt{1-p}-\tfrac{1}{2}q'.E'^2 \quad [4984c]; \quad [4986]$$

the angle ϖ *is then equal to the constant quantity* ε, *increased by the secular equation of the motion of the perigee* [4985].* [4986']

Secular variation of e is insensible.

The excentricity e *of the lunar orbit is subjected to a secular variation, similar to that of the parallax, and like it is insensible*† [4970]; these variations [4987]

Dividing by v, and reducing, we obtain

$$c=\sqrt{(1-p)}-\tfrac{1}{2}q'.E'^2 \quad [4986]. \quad [4984c]$$

The remaining terms of [4984], *depending on* v^2, v^3, *give the secular motion* [4985]; in which terms of the order v^4 are neglected. To make a rough estimate of the value of these neglected terms, without the labor of a direct calculation, we shall observe, that the secular motion of the perigee is about *three* times as great as that of the moon's mean motion [5235]; and this last quantity is very nearly represented by $10^s.i^2+0^s,018.i^3$ [5543]; i, being the number of centuries elapsed from the epoch of 1750. If we suppose $i=20$, corresponding to 2000 years [4984'], these two terms, of the orders v^2, v^3, respectively, will become 4000^s, 144^s; which are nearly in the ratio of 28 to 1; and, if the term of the order v^4 decrease in the same ratio, it will become $\frac{144^s}{28}$, or 5^s, nearly. Now, a term of this order, in the secular motion of the moon, or one of *three* times that value in the motion of the perigee [4984e], is wholly undeserving of notice in such distant observations; and, we may, therefore, restrict ourselves to the terms of the orders v^2, v^3, included in the formula [4984]. This is conformable to the remarks of the author in [4984']. [4984d] [4984e] [4984f] [4984g] [4984h] [4984i]

* (2859) Substituting the values of c and $fe'^2.dv$ [4986, 4984a], in [4980], we get as in [4986'],

$$\varpi=\varepsilon+\{\tfrac{1}{2}q'.E'fv^2+\tfrac{1}{6}q'.(2E'l+f^2).v^3\}=\varepsilon+\text{secular equation [4985]}. \quad [4986a]$$

† (2860) Using the value of q' [4979], we get, successively, from [4978, 4983, 4986], by neglecting terms of the order l and f^2,

being proportional to $\frac{d\varpi}{dv}$, which become sensible only in the integral
[4987'] $\int \frac{d\varpi}{dv} \cdot dv$.

[4988] *If we represent any term whatever of the equation* [4961] by $\frac{H}{a_{,}}.\cos(iv+\beta)$, *and denote the corresponding part of* u *by*

[4987a]
$$c-\frac{d\varpi}{dt}=\sqrt{(1-p)}-\tfrac{1}{2}q'.e'^2=\sqrt{(1-p)}-\tfrac{1}{2}q'.(E'^2+2E'fv)$$
$$=\sqrt{(1-p)}-\tfrac{1}{2}q'.E'^2-q'.E'fv=c-q'.E'fv.$$

Substituting this in [4977], and neglecting e^4, e^6, in its second member, we get

[4987b]
$$e^2=\frac{a^2}{k}\cdot\frac{1}{c-q'.E'fv}, \quad \text{or} \quad e=\frac{a}{\sqrt{(ck)}}\cdot\left(1+\frac{q'.E'fv}{2c}\right), \quad \text{nearly;}$$

consequently, the secular variation of e is represented by

[4987c]
$$\delta e=\frac{a}{\sqrt{(ck)}}\cdot\frac{q'.E'fv}{2c}, \quad \text{or} \quad \delta e=e.\frac{q'.E'f}{2}.v;$$

observing, that, if we neglect this secular variation, we have, very nearly,

[4987c']
$$e=\frac{a}{\sqrt{(ck)}} \quad [4987b], \quad \text{and} \quad c=1 \quad [4828e].$$

If we compare this with the chief term of the secular motion of the perigee, which we shall
[4987d] represent by $\delta\varpi=\frac{1}{2}q'.E'fv^2$ [4985], we shall get $\delta e=\frac{e}{v}.\delta\varpi$. Now, from [4984e,f], we have, by neglecting the signs,

[4987e]
$$\delta\varpi=30^s.i^2, \quad \text{and} \quad v=\frac{360^d}{m}.i=\frac{1296000^s.i}{m}, \text{ nearly; hence } \delta e=\frac{30m.e.i}{1296000};$$

and, by substituting the values of m, e [5117, 5120], it becomes

[4987f]
$$\delta e=i.0{,}0000001, \quad \text{nearly.}$$

[4987g] This is wholly insensible, since, in 20 centuries, which corresponds to $i=20$, it only amounts to 0,000002.

If we retain terms of the order v^2, in the calculation of e [4987b], its value will be
[4987h] increased by a term of the form $al_{,}.v^2$; $l_{,}$ being of the same order as f^2, or l; and
[4987i] this value of e gives $d^2.\frac{e}{a}=l_{,}$. Hence it appears, that the quantities neglected in [4972] are of the order f^2, or l. Now, we have seen, in [4987f], that the expression
[4987k] of the part of δe, depending on the first power of f, is insensible; and, by proceeding as in [4984e—i], it must be evident, that these terms of the second order f^2, or l, will
[4987l] be still less, and may, therefore, be neglected, as wholly insensible, even in the most ancient observations.

$$u = P.\cos.(iv+\beta)+Q.\sin.(iv+\beta),$$ [4989]

we shall have the two following equations to determine P and Q;*

$$0=\left\{1-\left(i+\frac{d\beta}{dv}\right)^2\right\}.P+\frac{H}{a_{\prime}};$$ [4990]

$$Q=\frac{2.\left(i+\frac{d\beta}{dv}\right).\frac{dP}{dv}+P.\frac{dd\beta}{dv^2}}{1-\left(i+\frac{d\beta}{dv}\right)^2}.$$ [4991]

The variations of β *and* P *being extremely slow, and* i *very great, relatively to* $\frac{d\beta}{dv}$, *the value of* Q *is insensible* [4990e], and we have, from [4990], [4992]

$$P=\frac{H}{a_{\prime}.\left\{\left(i+\frac{d\beta}{dv}\right)^2-1\right\}};$$ [4993]

in which we must observe, that, as $i+\frac{d\beta}{dv}$ is the coefficient of dv, in

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* (2861) If we substitute the assumed value of u [4989], in

$$0=\frac{ddu}{dv^2}+u+\frac{H}{a_{\prime}}.\cos.(iv+\beta) \quad [4961, 4988],$$ [4990a]

supposing v, P, Q, β, to be variable, it will become as in [4990b]; observing, that β is composed of terms of ϖ, ϖ', &c., similar to [4986a]; and P, Q, of terms e, e', γ, &c., whose secular variations are similar to that in [4987f];

$$0=\left\{P-P.\left(i+\frac{d\beta}{dv}\right)^2+\frac{H}{a_{\prime}}+2.\frac{dQ}{dv}.\left(i+\frac{d\beta}{dv}\right)+Q.\frac{dd\beta}{dv^2}+\frac{ddP}{dv^2}\right\}.\cos.(iv+\beta)$$
$$+\left\{Q-Q.\left(i+\frac{d\beta}{dv}\right)^2-2.\frac{dP}{dv}.\left(i+\frac{d\beta}{dv}\right)-P.\frac{dd\beta}{dv^2}+\frac{ddQ}{dv^2}\right\}.\sin.(iv+\beta).$$ [4990b]

To satisfy this equation for all values of the angle $iv+\beta$, we must put the coefficients of $\sin.(iv+\beta)$, $\cos.(iv+\beta)$, separately, equal to nothing; hence we have,

$$0=\left\{1-\left(i+\frac{d\beta}{dv}\right)^2\right\}.P+\frac{H}{a_{\prime}}+2.\frac{dQ}{dv}.\left(i+\frac{d\beta}{dv}\right)+Q.\frac{dd\beta}{dv^2}+\frac{ddP}{dv^2};$$ [4990c]

$$0=\left\{1-\left(i+\frac{d\beta}{dv}\right)^2\right\}.Q-2.\frac{dP}{dv}.\left(i+\frac{d\beta}{dv}\right)-P.\frac{dd\beta}{dv^2}+\frac{ddQ}{dv^2}.$$ [4990d]

If we neglect the term ddQ [4990d], which is very small, as we shall soon see, and

[4994] the differential of the angle $iv+\beta$, *we may suppose* β *to be constant in that angle, provided we take, for* i, *the coefficient of* v *corresponding to the epoch* for which the calculation is made. Thus, we shall determine the
[4995] coefficients $A_2^{(0)}$, $A_1^{(1)}$, &c., in the expression of $a\delta u$.

Relatively to the terms, where the coefficient of v differs from unity, by a quantity of the second order, and which depend on the angles

[4995′] $$2gv-cv-2\theta+\varpi \quad \text{and} \quad v-mv+c'mv-\varpi',$$

the consideration of the terms, depending on the cube of the disturbing force,* becomes necessary; but, by carrying on the approximation as we have
[4996] done, to quantities of the fourth order inclusively, the terms depending on the cube of the disturbing force, which might become sensible, will be found to be included in the preceding results.

This being premised, if we substitute, in the equation [4961], instead of u, the following function;†

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divide the remaining terms of that equation by the coefficient of Q, we get its value [4991]. Now, the secular variations of β, P, being of the order $\delta\varpi$, δc, &c. [4987*e*,*f*, &c.],
[4990*e*] they must be very small; and their products and differentials, which occur in the expression of Q [4991] must, therefore, be insensible. Neglecting the quantity Q, and the second differential of P, in [4990*c*], it becomes as in [4990]; which is easily reduced to the form [4993].

* (2862) Terms of this kind have been noticed in the differential equation in u. Thus, for example, the term multiplied by $\overset{2}{\overline{m}}.(A_1^{(1)})^2$, in the coefficient of

[4995*a*] $$\cos.(2cv-2v+2mv-2\varpi) \quad [4961 \text{ line } 32],$$

is of the order of the *cube* of the disturbing force; because $\overset{2}{\overline{m}}$, $A_1^{(1)}$, are each of the same order as the first power of this force.

† (2863) The function connected with δu [4997], is the same as the value of u [4826], augmented by the term β, of the fourth order [4858, &c.], and taking the
[4997*a*] coefficient of $\cos.(2gv-2\theta)$, so as to include terms of the fourth order. These neglected terms are easily computed. For, in the first place, the term s^4 [4812*a*] introduces the factor $1-\frac{1}{4}\gamma^2$, in the coefficient of $\cos.(2v-2\theta)$ [4816], by which means it is changed from $-\frac{1}{4}\gamma^2$ [4816] to $-\frac{1}{4}\gamma^2.(1-\frac{1}{4}\gamma^2)$ [4812*a*]. The same change being
[4997*b*] made in the coefficient of $\cos.(2gv-2\theta)$ [4819], it becomes, by using [4823*c*],

$$u=\frac{1}{a}\cdot\left\{\begin{array}{l}1+e^2+\frac{1}{4}\gamma^2+\beta+e.(1+ee).\cos.(cv-\varpi)\\ -\frac{1}{4}\gamma^2.(1+e^2-\frac{1}{4}\gamma^2).\cos.(2gv-2\theta)\end{array}\right\}+\delta u; \qquad [4997]$$

the comparison of the different cosines will give the following equations;*

$$-\frac{1}{h^2.(1+\gamma^2)}\cdot\tfrac{1}{4}\gamma^2.(1-\tfrac{1}{4}\gamma^2).\cos.(2gv-2\theta)=-\frac{1}{a}.(1+e^2).\tfrac{1}{4}\gamma^2.(1-\tfrac{1}{4}\gamma^2).\cos.(2gv-2\theta)$$
$$=-\frac{1}{a}.\tfrac{1}{4}\gamma^2.(1+e^2-\tfrac{1}{4}\gamma^2).\cos.(2gv-2\theta), \qquad [4997c]$$

as in [4997].

* (2864) If we take the value of i, corresponding to the epoch, as in [4994], and neglect the variations of $d\beta$, we may put the equation [4990] under the form

$$0=\{1-i^2\}.P+\frac{H}{a_{\prime}}; \qquad [4998a]$$

or, as it may be written,

$$0=\{1-i^2\}.Pa+\frac{a}{a_{\prime}}.H. \qquad [4998b]$$

Now, multiplying [4989] by a, and neglecting Q, as in [4992], we get, for au, the expression $au=Pa.\cos.(iv+\beta)$, corresponding to the term $\frac{H}{a_{\prime}}.\cos.(iv+\beta)$ [4988], in [4998c] the equation [4961]. Hence, it appears, that the coefficient Pa, corresponding to any angle $iv+\beta$, is found by multiplying the expression [4997] by a, and substituting the value $a\delta u$ [4904]. These values of Pa, together with the corresponding ones of $\frac{H}{a_{\prime}}$ [4961], being substituted, successively, in [4998b], give the equations [4998—5017]. [4998d]
For the constant part of au [4997]; namely, $1+e^2+\frac{1}{4}\gamma^2+\beta$ satisfies the equation [4998e] [4961], as has been proved in [4964—4968]. The term of au [4997], represented by $e.(1+e^2).\cos.(ev-\varpi)$, satisfies the equation [4961], as in [4973d, &c.]. The term of au [4997], depending on $2gv-2\theta$, is $\{-\frac{1}{4}(1+e^2-\frac{1}{4}\gamma^2)+A_2^{(12)}\}.\gamma^2.\cos.(2gv-2\theta)$; hence, [4998f]

$$Pa=\{-\tfrac{1}{4}(1+e^2-\tfrac{1}{4}\gamma^2)+A_2^{(12)}\}; \qquad [4998g]$$

the corresponding value of H, [4988, 4961 line 33], is

$$H=-\tfrac{3}{4}\{(1+e^2-\tfrac{1}{4}\gamma^2)-\tfrac{1}{2}\overline{m}^2+2\overline{m}^2.A_2^{(12)}\}.\gamma^2; \text{ and } i=2g; \qquad [4998h]$$

substituting these in [4998b], and dividing by γ^2, we get,

$$0=(1-4g^2).\{-\tfrac{1}{4}(1+e^2-\tfrac{1}{4}\gamma^2)+A_2^{(12)}\}-\tfrac{3}{4}.\frac{a}{a_{\prime}}.\left\{(1+e^2-\tfrac{1}{4}\gamma^2)-\tfrac{1}{2}\overline{m}^2+2\overline{m}^2.A_2^{(12)}\right\} \qquad [4998i]$$
$$=(1-4g^2).A_2^{(12)}-(1+e^2-\tfrac{1}{4}\gamma^2).\left\{\frac{1-4g^2}{4}+\tfrac{3}{4}.\frac{a}{a_{\prime}}\right\}+\tfrac{3}{4}.\frac{a}{a_{\prime}}.\left\{\tfrac{1}{2}\overline{m}^2-2\overline{m}^2.A_2^{(12)}\right\}$$
$$=(1-4g^2).A_2^{(12)}-(1+e^2-\tfrac{1}{4}\gamma^2).\left\{1-g^2+\tfrac{3}{4}.\frac{(a-a_{\prime})}{a_{\prime}}\right\}+\tfrac{3}{4}.\frac{a}{a_{\prime}}.\left\{\tfrac{1}{2}\overline{m}^2-2\overline{m}^2.A_2^{(12)}\right\}. \qquad [4998k]$$

[4998] $$0=\{1-4(1-m)^2\}.A_2^{(0)}+\tfrac{3}{2}\overline{m}^2.\frac{a}{a_{,}}.\left\{\begin{array}{l}1+(1+2m).e^2+\tfrac{1}{4}\gamma^2-\tfrac{5}{2}e'^2 \\ \dfrac{+(1+3e^2+\frac{1}{4}\gamma^2-\frac{5}{2}e'^2)}{1-m} \\ -A_2^{(0)}-(B_1^{(0)}-B_2^{(1)}).\dfrac{\gamma^2}{\overline{m}^2}\end{array}\right\};$$

Equations for the determination of A.

[4999] $$0=\{1-(2-2m-c)^2\}.A_1^{(1)}+3\overline{m}^2.\frac{a}{a_{,}}.\left\{\begin{array}{l}\tfrac{1}{4}c.\{1+\tfrac{1}{4}(2-19m).e^2-\tfrac{5}{2}e'^2\} \\ -\tfrac{1}{4}(3+4m).(1+\tfrac{1}{2}e^2-\tfrac{5}{2}e'^2)+\dfrac{1-c^2}{4(1-m)} \\ -\dfrac{2(1+m)}{2-2m-c}.(1+\tfrac{7}{4}e^2-\tfrac{5}{2}e'^2) \\ -\tfrac{1}{2}.(A_1^{(1)}-2A_2^{(0)})+\tfrac{1}{2}.(B_2^{(5)}-B_2^{(6)}).\dfrac{\gamma^2}{\overline{m}^2}\end{array}\right\};$$

[5000] $$0=\{1-(2-2m+c)^2\}.A_2^{(2)}-\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{,}}.\left\{3+c-4m+\frac{8(1-m)}{2-2m+c}+2A_2^{(2)}\right\};$$

[5001] $$0=\{1-(2-m)^2\}.A_2^{(3)}-\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{,}}.\left\{\frac{4-m}{2-m}+2B_1^{(9)}.\frac{\gamma^2}{\overline{m}^2}+2A_2^{(3)}\right\};$$

[5002] $$0=\{1-(2-3m)^2\}.A_2^{(4)}+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{,}}.\left\{\frac{7(4-3m)}{2-3m}-2B_1^{(10)}.\frac{\gamma^2}{\overline{m}^2}-2A_2^{(4)}\right\};$$

[5003] $$0=(1-m^2).A_2^{(5)}+\tfrac{3}{2}\overline{m}^2.\frac{a}{a_{,}}.\left\{\begin{array}{l}1+e^2+\tfrac{1}{4}\gamma^2+\tfrac{9}{8}e'^2+(B_1^{(7)}+B_1^{(8)}).\dfrac{\gamma^2}{\overline{m}^2}-\tfrac{3}{2}(1+2m).A_2^{(0)} \\ -\dfrac{2(1-2m).(3-2m).(3-m)}{(2-3m).(2-m)}.A_2^{(0)}-2A_2^{(3)}-(2-3m).A_2^{(4)} \\ +(B_1^{(9)}+B_1^{(10)}).B_1^{(0)}.\dfrac{\gamma^2}{\overline{m}^2}-A_2^{(5)}-11C_2^{(6)}-2C_2^{(9)}+2C_2^{(10)}\end{array}\right\}$$
$$+6m.\{4A_2^{(0)}+A_2^{(3)}-A_2^{(4)}-10.A_1^{(1)}e^2+\tfrac{5}{2}(A_1^{(7)}-A_1^{(6)}).e^2\};*$$

On account of the smallness of the terms $1-g^2$, $\frac{a-a_{,}}{a_{,}}$ [4828e, 4968], we may change the factor $1+e^2-\frac{1}{4}\gamma^2$ into 1, or $\frac{a}{a_{,}}$ [4968], and then the equation becomes as in

[4998l] [5010]. The rest of the terms of au [4997] depend wholly on $a\delta u$; therefore, the remaining terms of Pa [4998c, b], will be represented by the coefficients of $a\delta u$ [4904]; and, by taking them in the order in which they occur, we shall obtain, with but very little reduction, the equations [4998—5017].

[5003a] * (2865) This line might, for greater accuracy, be multiplied by the factor $\frac{a}{a_{,}}$, like the

$$0=\{1-(2-m-c)^2\}.A_1^{(6)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_,}.\left\{\begin{array}{l}\dfrac{3+2m-c}{4}+\dfrac{2+m}{2-m-c}-\tfrac{3}{2}A_1^{(1)}-A_1^{(6)}\\ -\left\{\dfrac{3+m-c}{2}+\dfrac{4}{2-m-c}\right\}.A_1^{(9)}\end{array}\right\}; \qquad [5004]$$

$$0=\{1-(2-3m-c)^2\}.A_1^{(7)}-\tfrac{3}{2}\bar{m}^2.\frac{a}{a_,}.\left\{\begin{array}{l}\dfrac{7(3+6m-c)}{4}+\dfrac{7(2+3m)}{2-3m-c}+\tfrac{3}{2}A_1^{(1)}\\ +A_1^{(7)}+\left\{\dfrac{3-m-c}{2}+\dfrac{4}{2-3m-c}\right\}A_1^{(8)}\end{array}\right\}; \qquad [5005]$$

$$0=\{1-(c+m)^2\}.A_1^{(8)}-\tfrac{3}{2}\bar{m}^2.\frac{a}{a_,}.\left\{\begin{array}{l}\dfrac{3+2m}{2}-\left\{\dfrac{1+2m+c}{4}+\dfrac{2}{c+m}\right\}.A_1^{(1)}\\ +A_1^{(8)}+\left\{\dfrac{1+3m+c}{2}+\dfrac{4}{c+m}\right\}.A_1^{(7)}\end{array}\right\}; \qquad [5006]$$

$$0=\{1-(c-m)^2\}.A_1^{(9)}-\tfrac{3}{2}\bar{m}^2.\frac{a}{a_,}.\left\{\begin{array}{l}\dfrac{3-2m}{2}+A_1^{(9)}+7\left\{\dfrac{1+2m+c}{4}+\dfrac{2}{c-m}\right\}.A_1^{(1)}\\ +\left\{\dfrac{1+m+c}{2}+\dfrac{4}{c-m}\right\}.A_1^{(6)}\end{array}\right\}; \qquad [5007]$$

$$0=(1-4c^2).A_2^{(10)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_,}.\left\{1-B_0^{(11)}.\frac{\gamma^2}{\bar{m}^2}-A_2^{(10)}\right\}; \qquad [5008]$$

$$0=\{1-(2c-2+2m)^2\}.A_1^{(11)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_,}.\left\{\begin{array}{l}\dfrac{(2+11m+8m^2)}{2}-\dfrac{(10+19m+8m^2)}{2c-2+2m}\\ +4A_1^{(1)}+\dfrac{\{8A_2^{(10)}+10(A_1^{(1)})^2\}}{2c-2+2m}-2A_1^{(11)}\end{array}\right\}; \qquad [5009]$$

$$0=\{1-4g^2\}A_2^{(12)}+\frac{a}{a_,}.\left\{g^2-1-\tfrac{3}{4}.\left(\frac{a-a_,}{a}\right)+\tfrac{3}{8}\bar{m}^2-\tfrac{3}{2}\bar{m}^2.A_2^{(12)}\right\} \quad [4998k,l]; \qquad [5010]$$

$$0=\{1-(2g-2+2m)^2\}.A_1^{(13)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_,}.\left\{\begin{array}{l}\dfrac{3+2m-2g}{4}+\dfrac{(4g^2-1)}{4(1-m)}-\dfrac{(2+m)}{2g-2+2m}\\ +\dfrac{2B_1^{(0)}}{\bar{m}^2}-2A_1^{(13)}+\dfrac{8A_2^{(12)}}{2g-2+2m}\end{array}\right\}; \qquad [5011]$$

$$0=(1-4m^2).A_2^{(14)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_,}.\left\{\tfrac{3}{2}-A_2^{(14)}\right\}; \qquad [5012]$$

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other terms of these equations, as is evident by comparing it with the corresponding terms of [4961 line 21].

[5013] $$0=\{1-(2g-c)^2\}.A_0^{(15)}-\tfrac{3}{2}\bar{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{l}\tfrac{1}{2}+\frac{B_2^{(3)}}{\bar{m}^2}+\frac{(1+c-2g-10m)}{4}.A_1^{(1)}-(10+5m).A_1^{(13)} \\ +(5+m).A_1^{(16)}-\frac{B_1^{(0)}.B_2^{(5)}}{\bar{m}^2}+A_0^{(15)}\end{array}\right\};$$

[5014] $$0=\{1-(2-2m-2g+c)^2\}A_1^{(16)}-\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{l}1+2m+\frac{(5+m)}{1-2m}+\frac{3(1-m)}{3-2m}+2A_1^{(16)} \\ -\frac{2B_2^{(4)}}{\bar{m}^2}+\frac{10A_0^{(15)}}{1-2m}\end{array}\right\};$$

[5015] $$0=\{1-(1-m)^2\}A_1^{(17)}+\bar{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{l}\tfrac{9}{8}.(1-2\mu).(1+2e^2+2e'^2)+\frac{3(1-2\mu).(1+\frac{9}{2}e^2+2e'^2)}{4(1-m)} \\ -\frac{(36+21m-15m^2)}{4(1-m)}.A_1^{(17)}+\frac{3(1+m)}{2(1-m)}.A_0^{(18)}.e'^2 \\ -\frac{(57-38m)}{4(1-m)}.A_2^{(0)}+\tfrac{3}{2}.(B_2^{(14)}+B_2^{(15)}).\frac{\gamma^2}{\bar{m}^2}\end{array}\right\};$$

[5016] $$0=\frac{5.(1-2\mu)}{4}-A_0^{(18)}+\frac{(4+m)}{4}.A_1^{(17)}-(5+m).A_1^{(19)};$$

[5017] $$0=\{1-(1-2m)^2\}.A_1^{(19)}+\frac{3\,\bar{m}^2}{2.(1-2m)}.\frac{a}{a_{\prime}}.\left\{\begin{array}{l}\tfrac{1}{4}(15-8m).(1-2\mu)-\tfrac{1}{4}(76-33m).A_1^{(17)} \\ -5A_0^{(18)}-(1-2m).A_1^{(19)}\end{array}\right\}.$$

11. *We shall now take into consideration the equation* [4755]. The function

[5018] $$-\frac{s}{h^2u}.\left(\frac{dQ}{du}\right)-\frac{(1+s^2)}{h^2.u^2}.\left(\frac{dQ}{ds}\right),$$

which occurs in this equation, produces the terms,*

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* (2866) Multiplying the equation [4808] by $-\frac{s}{h^2u}$; also [4810] by $-\frac{1}{h^2.u^2}$;
[5019a] and taking the sum of the products, we find that the first member of the sum is equal to the function [5018]; consequently, the second member of this sum will express the
[5019b] development of this function. The first terms of these products, with the divisor $(1+ss)^{\frac{3}{2}}$, mutually destroy each other. The remaining terms of this sum, being written down in the order in which they occur, without any reduction, become

$$\frac{3m'.u'^3.s}{2h^2.u^4}+\frac{3m'.u'^3.s}{2h^2.u^4}.\cos.(2v-2v')+\frac{3m'.u'^4.s}{8h^2.u^5}.\{11.\cos.(v-v')+5.\cos.(3v-3v')\}.$$ [5019]

These terms are successively calculated in the following manner. The quantity $\frac{3m'.u'^3.s}{2h^2.u^4}$ becomes, by development,* [5020]

$$\frac{3m'.u'^3.s}{2h^2.u^4}=\tfrac{3}{2}\overline{m}^2.\frac{a}{a_{,}}.\gamma.\left\{\begin{array}{ll}(1+2e^2-\frac{1}{2}\gamma^2+\frac{3}{2}e'^2).\sin.(gv-\theta) & 1\\ -2e.\sin.(gv+cv-\theta-\varpi) & 2\\ -2e.\sin.(gv-cv-\theta+\varpi) & 3\\ +\frac{3}{2}e'.\sin.(gv+c'mv-\theta-\varpi') & 4\\ +\frac{3}{2}e'.\sin.(gv-c'mv-\theta+\varpi') & 5\\ -\frac{5}{2}e^2.\sin.(2cv-gv-2\varpi+\theta) & 6\end{array}\right\}.$$ [5021]

$$\frac{m'.u'^3s}{2h^2.u^4}.\{1+3.\cos.(2v-2v')\}+\frac{3m'.u'^4s}{8h^2.u^5}.\{(3-4s^2).\cos.(v-v')+5.\cos.(3v-3v')\}$$ [5019c]

$$+\frac{m'.u'^3s}{h^2.u^4}+\frac{3m'.u'^4s}{h^2.u^5}.\cos.(v-v').$$ [5019d]

If we neglect the terms of the order s^2, and connect together the other terms, it becomes as in [5019].

* (2867) Using always the abridged notation [4821f], we have $\frac{3}{2}s=\frac{3}{2}\gamma.\sin.gv$, nearly [4818]. Multiplying this by the function [4884], and reducing the products by [18, 19] Int., we get the following expression, which corresponds, line for line with the four first lines of [4884], neglecting terms of the fourth order; [5020a]

$$\frac{3s}{u}=3a.\gamma.\left\{\begin{array}{ll}(1-\frac{1}{2}e^2-\frac{1}{4}\gamma^2).\sin.gv & 1\\ -\frac{1}{2}e.(1-\frac{1}{4}e^2-\frac{1}{2}\gamma^2).\{\sin.(gv+cv)+\sin.(gv-cv)\} & 2\\ +\frac{1}{4}e^2.\{\sin.(2cv+gv)-\sin.(2cv-gv)\} & 3\\ +\frac{1}{8}\gamma^2.\{-\sin.gv+\sin.3gv\} & 4\end{array}\right\}.$$ [5020b]

The coefficients of $\sin.gv$, between the braces, by connecting the terms, become $1-\frac{1}{2}e^2-\frac{3}{8}\gamma^2$. 5

Multiplying together the two expressions [4866, 5020b], we get [5021]. The detail of the calculation is in the following table [5020d—f]; in which the first column contains the terms between the braces in [5020b], the second, the terms between the braces in [4866], the six remaining columns contain the coefficients between the braces in [5021], corresponding to each of the sines, marked at the top of the columns [5020d]. The sums of the coefficients [5020f], agree with the coefficients between the braces in [5021]. [5020c]

[5022] The development of $\frac{3m'.u'^3.s}{2h^2.u^4}.\cos.(2v-2v')$, is obtained by multiplying the value of $\frac{3m'.u'^3}{2h^2.u^3}.\cos.(2v-2v')$, which we have given in [4870], by $\frac{s}{u}$, and we shall have,*

[5023]
$$\frac{3m'.u'^3.s}{2h^2.u^4}.\cos.(2v-2v')=\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{,}}.\gamma.\left\{\begin{array}{ll}
-\{1+2e^2-\frac{1}{4}(2+m).\gamma^2-\frac{5}{2}e'^2\}.\sin(2v-2mv-gv+\theta) & 1\\
+\sin.(2v-2mv+gv-\theta) & 2\\
-2.(1+m).e.\sin.(2v-2mv+gv-cv-\theta+\varpi) & 3\\
+2.(1+m).e.\sin.(2v-2mv-gv-cv+\theta+\varpi) & 4\\
+2.(1-m).e.\sin.(2v-2mv-gv+cv+\theta-\varpi) & 5\\
-2.(1-m).e.\sin.(2v-2mv+gv+cv-\theta-\varpi) & 6\\
-\frac{7}{2}.e'.\sin.(2v-2mv-gv-c'mv+\theta+\varpi') & 7\\
+\frac{7}{2}.e'.\sin.(2v-3mv+gv-c'mv-\theta+\varpi') & 8\\
+\frac{1}{2}.e'.\sin.(2v-2mv-gv+c'mv+\theta-\varpi') & 9\\
-\frac{1}{2}.e'.\sin.(2v-2mv+gv+c'mv-\theta-\varpi') & 10\\
+\frac{1}{4}(10+19m+8m^2)e^2.\left\{\begin{array}{l}\sin.(2v-2mv-2cv+gv+2\varpi-\theta)\\ +\sin(2cv+gv-2v+2mv-2\varpi-\theta)\end{array}\right\} & \begin{array}{l}11\\12\end{array}
\end{array}\right\}.$$

	(Col. 1.)	(Col. 2.)	Terms of [5021], having the common factor $\frac{3}{2}.\overline{m}^2.\frac{a}{a_{,}}.\gamma.$					
[5020d]	Terms of [5020b].	Terms of [4866].	$\sin.gv$	$\sin(gv+cv)$	$\sin(gv-cv)$	$\sin.(gv+c'mv)$	$\sin.(gv-c'mv)$	$\sin.(2cv-gv)$
	$(1-\frac{1}{2}e^2-\frac{3}{8}\gamma^2)\sin gv$	$1+e^2+\frac{1}{4}\gamma^2+\frac{3}{2}e'^2$	$1+\frac{1}{2}e^2-\frac{1}{8}\gamma^2+\frac{3}{2}e'^2$					
	$\sin.gv$	$-3e.\cos.cv$	.	$-\frac{3}{2}e$	$-\frac{3}{2}e$			
		$+3e'.\cos.c'mv$	.			$+\frac{3}{2}e'$	$+\frac{3}{2}e'$	
		$+3e^2.\cos.2cv$	.					$-\frac{3}{2}e^2$
		$+\frac{3}{4}\gamma^2.\cos.2gv$	. $-\frac{3}{8}\gamma^2$					
[5020e]	$-\frac{1}{2}e.\sin.(gv+cv)$	1	.	$-\frac{1}{2}e$				
		$-3e.\cos.cv$	$+\frac{3}{4}e^2$					
	$-\frac{1}{2}e.\sin.(gv-cv)$	1			$-\frac{1}{2}e$			
		$-3e.\cos.cv$	$+\frac{3}{4}e^2$					$-\frac{3}{4}e^2$
	$-\frac{1}{4}e^2.\sin.(2cv-gv)$	1						$-\frac{1}{4}e^2$
[5020f]		Sum	$1+2e^2-\frac{1}{2}\gamma^2+\frac{3}{2}e'^2$	$-2e$	$-2e$	$+\frac{3}{2}e'$	$+\frac{3}{2}e'$	$-\frac{5}{2}e^2$

* (2868) Multiplying *one third* part of the expression of $\frac{3s}{u}$ [5020b], by that of

The term $\frac{33\,m'.u'^4.s}{8\,h^2.u^5}.\cos.(v-v')$ [5019], produces the following; [5024]

$\frac{3m'.u'^3}{2h^2.u^3}.\cos.(2v-2v')$ [4870], we shall evidently obtain the value of $\frac{3m'.u'^3 s}{2h^2.u^4}.\cos.(2v-2v')$; [5023a]
which we shall find to agree with the expression [5023], as will appear by the following calculation. If any term of [5020b] be represented by $3a\gamma.A.\sin.V$, and any term of
[4870] by $\frac{3\overline{m}^2}{2a_{\prime}}.A'.\cos.V'$, one third part of the product of these two terms, or the [5023b]
corresponding part of $\frac{3m'.u'^3 s}{2h^2.u^4}.\cos.(2v-2v')$, will be represented by

$$\tfrac{3}{2}\overline{m}^2.\frac{a}{a_{\prime}}.\gamma.AA'.\sin.V.\cos.V'=\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\gamma.\{AA'.\sin.(V+V')+AA'.\sin.(V-V')\}; \qquad [5023c]$$

where the factor $\frac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\gamma$ is the same as that without the braces [5023]; consequently, [5023d]
the terms between the braces [5023], must be represented by the function

$$AA'.\sin.(V+V')+AA'.\sin.(V-V'); \quad \text{or} \quad AA'.\sin.(V+V')-AA'.\sin.(V'-V); \qquad [5023e]$$

$A.\sin.V$ *representing the terms between the braces in* [5020b], *and* $A'.\cos.V'$ *the terms* [5023f]
between the braces in [4870]. By means of this formula, we may compute the terms between the braces [5023] in the following manner.

First. The coefficients of $\sin.(2v-2mv-gv)$ are contained in the four lines of the annexed table. The first is obtained by combining $(1-\frac{1}{2}e^2-\frac{3}{8}\gamma^2).\sin.gv$ [5020b line 5] with $(1+e^2+\frac{1}{4}\gamma^2-\frac{5}{2}e'^2).\cos.(2v-2mv)$ [4870] line 1, and using the second term of [5023e]. The second is produced by $\sin.gv$ [5020b line 1] and $\frac{1}{8}(3+2m).\gamma^2.\cos.(2gv-2v+2mv)$ [4870] line 13. The third line is produced by $-\frac{1}{2}e.\sin.(gv+cv)$ [5020b line 2] and $-\frac{3}{2}e.\cos.(2v-2mv+cv)$ [4870 line 3]. Lastly, the fourth line is produced by $-\frac{1}{2}e.\sin.(gv-cv)$ [5020b line 2] and $-\frac{3}{2}e.\cos.(2v-2mv-cv)$ [4870 line 2]. The sum of these four terms is given in line 5, and is the same as in [5023 line 1].

[5023g]

$-1-\frac{1}{2}e^2+\frac{1}{8}\gamma^2$	$+\frac{5}{2}e'^2$	1
$\cdot\ +\frac{3}{8}\gamma^2+\frac{1}{4}m\gamma^2$		2
$-\frac{3}{4}e^2$ $\cdot$ $\cdot$		3
$-\frac{3}{4}e^2$ $\cdot$ $\cdot$		4
Sum $=-1-2e^2+\frac{1}{2}\gamma^2+\frac{1}{4}m\gamma^2$	$+\frac{5}{2}e'^2$	5

Second. The term $\sin.gv$ [5020b line 1], combined with $\cos.(2v-2mv)$ [4870 line 1], and using the first of the forms [5023e], gives [5023 line 2]. [5023h]

Third. The terms of [5023 lines 3—6] are computed in the following table; in which the first column contains the terms of $A.\sin.V$ [5020b]; the second, the terms of $A'.\cos.V'$ [4870]; the remaining columns contain the corresponding terms of [5023e], connected with [5023i] the sines of the angles marked at the top of these columns [5023k] respectively. The

[5025] $$\overset{*}{}\frac{33\,\overset{2}{m}}{16}\cdot\frac{a}{a_{\prime}}\cdot\frac{a}{a'}\cdot\gamma\cdot\{\sin.(gv-v+mv-\theta)+\sin.(gv+v-mv-\theta)\}.$$

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sums of these terms, in the bottom line of the table, agree with the coefficients in [5023 lines 3—6].

	(Col. 1.) $A.\sin.V$ [5020b].	(Col. 2.) $A'.\cos.V'$ [4870].	Corresponding terms of [5023e or 5023]. $\sin(2v-2mv+gv-cv)$	$\sin(2v-2mv-gv-cv)$	$\sin(2v-2mv-gv+cv)$	$\sin(2v-2mv+gv+cv)$
[5023k]	$\sin.gv$	$-\frac{1}{2}(3+4m)e.\cos(2v-2mv-cv)$	$(-\frac{3}{2}-2m).e$	$(\frac{3}{2}+2m).e$		
	$\sin.gv$	$-\frac{1}{2}(3-4m)e.\cos(2v-2mv+cv)$			$(\frac{3}{2}-2m).e$	$(-\frac{3}{2}+2m).e$
[5023k']	$-\frac{1}{2}e.\sin(gv+cv)$	$\cos.(2v-2mv)$		$(\frac{1}{2}\quad).e$		$(-\frac{1}{2}\quad).e$
	$-\frac{1}{2}e.\sin.(gv-cv)$	$\cos.(2v-2mv)$	$(-\frac{1}{2}\quad).e$		$(\frac{1}{2}\quad).e$	
[5023l]		Sums	$(-2-2m).e$	$(2+2m).e$	$(2-2m).e$	$(-2+2m).e$

Fourth. The term $\sin.gv$ [5020b] combined with $+\frac{7}{2}e'.\cos.(2v-2mv-c'mv)$
[5023m] [4870 line 4] gives, by [5023e], the terms in [5023 lines 7, 8]. In like manner, the same term $\sin.gv$, being combined with $-\frac{1}{2}e'.\cos.(2v-2mv+c'mv)$ [4870 line 5], gives [5023 lines 9, 10].

Fifth. The terms [5023 lines 11, 12] are computed in the following table, which is arranged in the same manner as that in [5023k'];

	(Col. 1.) $A.\sin.V$ [5020b].	(Col. 2.) $A'.\cos.V'$ [4870].	Corresponding terms of [5023e or 5023]. $\sin.(2v-2mv-2cv+gv)$	$\sin.(2cv+gv-2v+2mv)$
	$\sin.gv$	$\frac{1}{4}(6+15m+8m^2).e^2.\cos.(2cv-2v+2mv)$	$+\frac{1}{4}e^2.(6+15m+8m^2)$	$+\frac{1}{4}e^2.(6+15m+8m^2)$
	$-\frac{1}{2}e.\sin.(gv+cv)$	$-\frac{1}{2}(3+4m).e.\cos.(2v-2mv-cv)$		$+\frac{1}{4}e^2.(3+4m\quad)$
[5023n]	$-\frac{1}{2}e.\sin.(gv-cv)$	$-\frac{1}{2}(3+4m).e.\cos.(2v-2mv-cv)$	$+\frac{1}{4}e^2.(3+4m\quad)$	
	$-\frac{1}{4}e^2.\sin(2cv-gv)$	$\cos.(2v-2mv)$	$+\frac{1}{4}e^2.(1\quad)$	
	$+\frac{1}{4}e^2.\sin(2cv+gv)$	$\cos.(2v-2mv)$		$+\frac{1}{4}e^2.(1\quad)$
		Sums	$\frac{1}{4}e^2.(10+19m+8m^2)$	$+\frac{1}{4}e^2.(10+19m+8m^2)$

This sum agrees with the two last terms of [5023 lines 11, 12]. The other terms of the development of the function [5023], of the fifth and higher orders, are neglected.

* (2869) Substituting successively the values [4937n, 4865, 4818], and reducing, we get,

[5024a] $$\frac{33m'.u'^4.s}{8h^2.u^5}.\cos.(v-v')=\frac{33m'.a^5.s}{8a_{\prime}.a'^4}.\cos.(v-mv)=\frac{33\overset{2}{m}.a^2.s}{8a_{\prime}.a'}.\cos.(v-mv)$$

$$=\frac{33\overset{2}{m}.a^2.\gamma}{8a_{\prime}.a'}.\sin.(gv-\theta).\cos.(v-mv)$$

[5024b] $$=\frac{33\overset{2}{m}.a^2.\gamma}{16a_{\prime}.a'}.\{\sin.(gv-v+mv-\theta)+\sin.(gv+v-mv-\theta)\}.$$

This last expression is the same as in [5025]. It is of the *fifth* order; moreover, the

The term depending on $\cos.(3v-3v')$ is insensible.* We have noticed the two preceding terms solely on account of their having a little influence on the argument of the moon's longitude, depending on $v-mv$. [5026]

The function $\frac{1}{h^2.u^2}\cdot\frac{ds}{dv}\cdot\left(\frac{dQ}{dv}\right)$, contained in the equation [4755], gives the following term;† [5027]

$$-\frac{3m'.u'^3}{2h^2.u^4}.g\gamma.\cos.(gv-\theta).\sin.(2v-2v'). \qquad [5028]$$

We shall have the value of this term by increasing, in the development of $\frac{3m'.u'^3.s}{2h^2.u^4}.\cos.(2v-2v')$ [5023], the angles gv and $2v$, by a right angle, and then multiplying it by g, which gives,‡ [5029]

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neglected terms of the *sixth* order, do not depend on the angle $v-mv$ [4875]; therefore, it is unnecessary to notice them. [5024c]

* (2870) By means of the values of [4937n, 4865, 4818], which are used in the last note, we find, that the term of [5019], depending on $3v-3v'$, becomes

$$\frac{15\overline{m}^2}{8}\cdot\frac{a}{a_,}\cdot\frac{a}{a'}.\gamma.\sin.(gv-\theta).\cos.(3v-3mv). \qquad [5026a]$$

This term is of the fifth order, and depends on the angles $3v-3mv\pm gv=0$, which have not been noticed in these calculations; and a little consideration will show, that if we develop it so as to include terms of the sixth order, it will not produce any quantity connected with the angle $v-mv$ [4875]. With other angles, the terms of the sixth order are usually neglected. [5026b]

† (2871) The differential of [4818], using the abridged notation [4821f], gives $\frac{ds}{dv}=g\gamma.\cos.gv$; substituting this in [5027], it becomes $\frac{g\gamma}{h^2.u^2}.\cos.gv.\left(\frac{dQ}{dv}\right)$; and, by using [4809], we get the three terms in the second member of the following equation; [5028a]

$$\frac{1}{h^2.u^2}\cdot\frac{ds}{dv}\cdot\left(\frac{dQ}{dv}\right)=\frac{g\gamma.\cos.gv}{h^2.u^2}\cdot\left\{-\frac{3m'.u'^3}{2u^2}.\sin.(2v-2v')-\frac{m'.u'^4}{8u^3}.[3\sin.(v-v')+15\sin(3v-3v')]\right\}. \qquad [5028b]$$

The first of these terms is noticed in [5028, &c.], the others in [5031, 5032b].

‡ (2872) Substituting the value of s [4818], in the first member of [5023], and omitting θ for brevity [4821f], it becomes $\frac{3m'.u'^3.\gamma}{2h^2.u^4}.\sin.gv.\cos.(2v-2v')$. Now, a slight attention will show, that the process made use of in [5023a—n], in computing [5023], will be the same, if we change $2v$ into $2v+90^d$, and gv into $gv+90^d$, without altering [5029a] [5029b]

[5030]
$$-\frac{3m'u'^3}{2h^2u^4}\cdot\frac{ds}{dv}.\sin.(2v-2v') = -\tfrac34\overset{2}{m}.\frac{a}{a_,}.g\gamma.\left\{\begin{array}{lr}
\{1+2e^2-\tfrac14(2+m).\gamma^2-\tfrac52 e'^2\}.\sin(2v-2mv-gv+\theta) & 1\\
+\sin.(2v-2mv+gv-\theta) & 2\\
-2.(1+m).e.\sin.(2v-2mv+gv-cv-\theta+\varpi) & 3\\
-2.(1+m).e.\sin.(2v-2mv-gv-cv+\theta+\varpi) & 4\\
-2.(1-m).e.\sin.(2v-2mv-gv+cv+\theta-\varpi) & 5\\
-2.(1-m).e.\sin.(2v-2mv+gv+cv-\theta-\varpi) & 6\\
+\tfrac72.e'.\sin.(2v-2mv-gv-c'mv+\theta+\varpi') & 7\\
+\tfrac72.e'.\sin.(2v-2mv+gv-c'mv-\theta+\varpi') & 8\\
-\tfrac12.e'.\sin.(2v-2mv-gv+c'mv+\theta-\varpi') & 9\\
-\tfrac12.e'.\sin.(2v-2mv+gv+c'mv-\theta-\varpi') & 10\\
+\tfrac14(10+19m+8m^2)e^2.\left\{\begin{array}{l}\sin.(2v-2mv-2cv+gv+2\varpi-\theta)\\ -\sin(2cv+gv-2v+2mv-2\varpi-\theta)\end{array}\right\} & \begin{array}{r}11\\12\end{array}
\end{array}\right\}$$

[5031] The terms of the function $\frac{1}{h^2.u^2}\cdot\frac{ds}{dv}\cdot\frac{dQ}{dv}$ [4755 or 5028*b*], which depend on u'^4, produce the following;*

[5032]
$$\frac{3\overset{2}{m}}{16}.\frac{a}{a_,}.\frac{a}{a'}.\gamma.\{\sin.(gv-v+mv-\theta)-\sin.(gv+v-mv-\theta).$$

---

the angles mv, cv, $c'mv$; by a method of derivation similar to that in [4876*a*—*d*]. These changes being made in $\sin.gv$, $\cos.(2v-2v')$, they become $\cos.gv$, $-\sin.(2v-2v')$, respectively; and the function [5029*a*] becomes

[5029*c*]
$$-\frac{3m'.u'^3.\gamma}{2h^2.u^4}.\cos.gv.\sin.(2v-2v').$$

Multiplying this by g, it becomes similar to [5028]. Hence we see, that the method of derivation [5029] is correct.

* (2873) The second term of [5028*b*] is $-\frac{3m'.u'^4}{8h^2.u^5}.g\gamma.\cos.gv.\sin.(v-v')$; and, by

[5032*a*] substituting the values [4937*n*, 4865], it becomes

$$-\frac{3\overset{2}{m}}{8}.\frac{a}{a_,}.\frac{a}{a'}.g\gamma.\cos.gv.\sin.(v-mv):$$

which is easily reduced to the form [5032], by using [19] Int. This term is of the *fifth*

[5032*b*] order, and those of the *sixth* may be neglected, as in the similar term [5021*c*]. Moreover, the term of [5028*b*], depending on the angle $3v-3v'$, may be neglected, for the same reasons as in [5026*a*, *b*].

The product $\left(\frac{dds}{dv^2}+s\right).\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$ contained in the equation [4755], [5033]
is reduced to*

$$\frac{2}{h^2}.(1-g^2).\gamma.\sin.(gv-\theta).\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}. \qquad [5034]$$

$1-g^2$ being of the order m^2 [4828e], we shall retain, in this product, only the term depending on $\sin.(2v-2mv-gv+\theta)$; and it follows, from the preceding development of $\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$, that this term is equal to [5035]

$$-\frac{3\overset{2}{\overline{m}}.(1-g^2)}{4.(1-m)}.\frac{a}{a_{\prime}}.\gamma.\sin.(2v-2mv-gv+\theta). \qquad [5036]$$

Thus, the equation [4755] is reduced to the following form;*

Differential equation in s.

$$0=\frac{dds}{dv^2}+s+\Gamma; \qquad [5037]$$

Γ *being the sum of the terms we have just considered. But, for greater* [5037′]

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* (2874) Using the abridgments [4821f], we have $s=\gamma.\sin.gv$ [4818]; whence we obtain

$$\frac{dds}{dv^2}+s=(1-g^2).\gamma.\sin.gv; \qquad [5034a]$$

substituting this in [5033], it becomes as in [5034]. Now, $(1-g^2).\gamma$ is of the order $m^2\gamma$ [4828e], or of the *third* order; and $\left(\frac{dQ}{dv}\right)$ [4809] is of the *second* order; hence, the function [5034] is of the *fifth* order; therefore, we need only notice its chief term. Now, the chief part of $\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$ [4881′, 4882] has been computed in [4885], and [5034b]
its chief term is

$$3\overset{2}{\overline{m}}.\frac{a}{a_{\prime}}.\frac{1}{2-2m}.\cos.(2v-2mv).$$

Multiplying this by the factor $(1-g^2).\gamma.\sin.gv$, we get the corresponding part of [5034],

$$\frac{3\overset{2}{\overline{m}}.(1-g^2)}{2(1-m)}.\frac{a}{a_{\prime}}.\gamma.\sin.gv.\cos.(2v-2mv). \qquad [5034c]$$

Reducing this by [19] Int., we get the term [5036], and another similar term, depending on the angle $2v-2mv+gv$; but this is neglected, because it is of the *fifth* order, and is not increased by the integration of the equation [4755], as in [4897o, &c.]. [5034d]

* (2875) Substituting, in [4755], the development of the terms given in [5021, 5023, 5025, 5030, 5032, 5036], it evidently becomes of the form [5037]. [5037a]

accuracy, we must add the terms depending on the square of the disturbing force, which might have a sensible influence.

[5038] 12. The term $\frac{3m'.u'^3\,s}{2h^2.u^4}$ [5029] gives, by its variation, the following ones;

[5039] $$\frac{3\,m'.u'^3.\delta s}{2h^2.u^4}-\frac{6\,m'.u'^3.s\,\delta u}{h^2.u^5};$$

from which we obtain the function,*

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* (2876) In finding the variation of the function [5038], s, u, u', are the variable

[5040a] quantities; but we may neglect $\delta u'$, on account of its smallness, as in [4909, 4932i,&c.]; and the variation becomes as in [5039]. We shall now separately compute the two terms of which this function is composed. The first of these terms $\frac{3m'.u'^3}{2h^2.u^4}.\delta s$, is evidently equal to the first member of [4908f], multiplied by $-a.\delta s$; and, as the factor without the

[5040b] braces, in the second member of [4908f], is $-\frac{3\,\overline{m}^2}{2a_,}$, the required function will evidently be equal to the product of $\frac{3\overline{m}^2}{2}.\frac{a}{a_,}.\delta s$ by the terms between the braces in the second member of [4908f]. We shall now compute this product in the following table; in which the first column represents the terms of δs; the second, the terms between the braces in

[5040c] the function [4908f]; and, in the third column, the corresponding terms of the function [5040]; rejecting such terms as have been usually neglected.

[5040d]

(Col. 1.) Terms of δs [4897].	(Col. 2.) Terms of [4908f].	(Col. 3.) Corresponding terms of [5040]. [All these terms must be multiplied by $\frac{3}{2}.\overline{m}^2.\frac{a}{a}$].	
Whole value of δs	1	δs [4897]	1
$B_{\iota}^{(0)}\gamma.\sin.(2v-2mv-gv)$	$-4e.\cos.cv$	$+2B_{\iota}^{(0)}e\gamma.\{-\sin.(2v-2mv-gv-cv)-\sin(2v-2mv-gv+cv)\}$	2
	$+3e'.\cos c'mv$	$+\frac{3}{2}B_{\iota}^{(0)}e'\gamma\{\sin(2v-2mv-gv+c'mv)+\sin(2v-2mv-gv-c'mv)\}$	3
	$+5e^2.\cos.2cv$	$-\frac{5}{2}B_{\iota}^{(0)}e^2\gamma.\sin(2cv+gv-2v+2mv)+$&c.	4
$B_{2}^{(1)}\gamma.\sin.(2v-2mv+gv)$	all its terms	....neglected	5
$B_{\iota}^{(7)}e'\gamma.\sin.(gv+c'mv)$	$+3e'.\cos c'mv$	$+\frac{3}{2}B_{\iota}^{(7)}e'^2\gamma.\sin.gv$	6
$B_{\iota}^{(8)}e'\gamma.\sin(gv-c'mv)$	$+3e'.\cos c'mv$	$+\frac{3}{2}B_{\iota}^{(8)}e'^2\gamma.\sin gv.$	7

This table contains all the terms of the function [5040] depending on the coefficients B. Thus, [5040d line 1] is the same as [5040 line 1]; the terms in [5040d lines 6, 7] are the

[5040e] same as in [5040 line 2]; the terms in [5040d line 2] are in [5040 lines 6, 4]; the terms in [5040d line 3] are in [5040 lines 7, 8]; lastly, the term in [5040d line 4] is the same as in [5040 line 9].

$$\frac{3\overline{m}^2}{2} \cdot \frac{a}{a_{\prime}} \cdot \delta s \quad [4897]$$ 1

$$+\frac{9\overline{m}^2}{4} \cdot \frac{a}{a_{\prime}} \cdot \{ . B_1^{(7)} + B_1^{(8)} \} . c'^2 \gamma . \sin.(gv - \theta)$$ 2

$$+3\overline{m}^2 . \frac{a}{a_{\prime}} . \{ A_2^{(0)} - \tfrac{5}{2} A_1^{(1)} . e^2 \} . \gamma . \sin.(2v - 2mv - gv + \theta)$$ 3

$$-3\overline{m}^2 . \frac{a}{a_{\prime}} . B_1^{(0)} . e\gamma . \sin.(2v - 2mv - gv + cv + \theta - \varpi)$$ 4

Development of the variation [5039].

$$-3\overline{m}^2 . \frac{a}{a_{\prime}} . A_1^{(1)} . e\gamma . \sin.(2v - 2mv + gv - cv - \theta + \varpi)$$ 5

[5040]

$$-3\overline{m}^2 . \frac{a}{a_{\prime}} . \{ B_1^{(0)} - A_1^{(1)} \} . e\gamma . \sin.(2v - 2mv - gv - cv + \theta + \varpi)$$ 6

$$+\frac{9\overline{m}^2}{4} . \frac{a}{a_{\prime}} . B_1^{(0)} . e'\gamma . \left\{ \begin{array}{l} \sin.(2v - 2mv - gv + c'mv + \theta - \varpi') \quad 7 \\ + \sin.(2v - 2mv - gv - c'mv + \theta + \varpi') \quad 8 \end{array} \right\}$$

$$+\frac{3\overline{m}^2}{2} \cdot \frac{a}{a_{\prime}} . \{ 5A_1^{(1)} - 2A_1^{(11)} - \tfrac{5}{2} B_1^{(0)} \} . e^2 \gamma . \sin.(2cv + gv - 2v + 2mv - 2\varpi - \theta)$$ 9

$$+\frac{3\overline{m}^2}{2} \cdot \frac{a}{a_{\prime}} . \{ 5A_1^{(1)} - 2A_1^{(11)} \} . e^2 \gamma . \sin.(2v - 2mv - 2cv + gv + 2\varpi - \theta).$$ 10

The second term of [5039] $-\frac{6m'.u'^3.s.\delta u}{h^2.u^5}$, is evidently equal to the product of the first member of [4908] by $\frac{4}{3} \times \frac{3s}{u}$; therefore, the development of this term will be obtained by multiplying the second member of [4908] by $\frac{3s}{u}$ [5020b], and the product by $\frac{4}{3}$. This process is performed in the following table; in which the first column contains the terms of $\frac{3s}{u}$ between the braces in [5020b]; the second column contains the terms of [4908] between the braces; the third column, the corresponding terms of the function [5040f or 5040], retaining terms of the usual forms and orders; observing moreover, that the product of the above factor $\frac{4}{3}$, by the terms without the braces [4908, 5020b], is [5040f] [5040g]

$$-\frac{4}{3} \times \frac{3\overline{m}^2 . (1 + \frac{3}{2} e^2)}{2a_{\prime}} . 3a\gamma = -6\gamma . \overline{m}^2 . \frac{a}{a_{\prime}} \quad \text{nearly.}$$ [5040g]

[5041] The term $\frac{3m'.u'^3.s}{2h^2.u^4}.\cos.(2v-2v')$ [5022] gives, by its variation, the following terms ;*

[5042]
$$\frac{3m'.u'^3.\delta s}{2h^2.u^4}.\cos.(2v-2v')-\frac{6m'.u'^3.s.\delta u}{h^2.u^5}.\cos.(2v-2v')$$
$$+\frac{3m'.u'^3.s.\delta v'}{h^2.u^4}.\sin.(2v-2v')\ ;$$

[5040h]

(Col. 1.) Terms of [5020b].	(Col. 2.) Terms of [4908].	(Col. 3.) Corresponding terms of [5040]. [All these terms must be multiplied by $-6\,\overline{m}^2.\frac{a}{a_{,}}$].	
$\sin.gv$	$A_2^{(0)}.\cos.(2v-2mv)$	$-\frac{1}{2}.A_2^{(0)}.\gamma.\sin.(2v-2mv-gv)$	1
	$A_1^{(1)}.e.\cos.(2v-2mv-cv)$	$\frac{1}{2}.A_1^{(1)}e\gamma\,\{\sin.(2v-2mv+gv-cv)-\sin.(2v-2mv-gv-cv)\}$	2
	$A_1^{(11)}e^2.\cos.(2cv-2v+2mv)$	$\frac{1}{2}.A_1^{(11)}.e^2\gamma.\sin.(2cv+gv-2v+2mv)$	3
	$A_1^{(11)}e^2.\cos.(2cv-2v+2mv)$	$\frac{1}{2}.A_1^{(11)}.e^2\gamma.\sin.(2v-2mv-2cv+gv)$	4
	$-2A_1^{(1)}e^2.\cos.(2v-2mv-2cv)$	$-A_1^{(1)}e^2\gamma.\{\sin.(2cv+gv-2v+2mv)+\sin(2v-2mv-2cv+gv)\}$	5
$-\frac{1}{2}e.\sin.(gv+cv)$	$A_1^{(1)}.e.\cos.(2v-2mv-cv)$	$-\frac{1}{4}A_1^{(1)}.e^2\gamma.\sin.(2cv+gv-2v+2mv)$	6
$-\frac{1}{2}e.\sin.(gv-cv)$	$A_1^{(1)}.e.\cos.(2v-2mv-cv)$	$-\frac{1}{4}A_1^{(1)}e^2\gamma.\{\sin(2v-2mv-2cv+gv)-\sin(2v-2mv-gv)\}$	7
$\sin.gv$	$-2A_1^{(1)}.e^2.\cos.(2v-2mv)$	$A_1^{(1)}.e^2\gamma.\sin.(2v-2mv-gv)$.	8

[5040i] The last term of the function [4908], included in this table, is $-2A_1^{(1)}.e^2.\cos(2v-2mv)$, which is not expressly given in [4908], though it is produced by the term $-4e.\cos.cv.a\delta u$ [4908g line 2], neglecting, for brevity, the consideration of the factor $\frac{3\overline{m}^2}{2a_{,}}.(1+\frac{3}{2}e'^2)$, without the braces. For, by substituting the term $a\delta u=A_1^{(1)}.e.\cos.(2v-2mv-cv)$ [4904 line 2], and reducing by [20] Int., we get,

$$-4e.\cos.cv.a\delta u=-2A_1^{(1)}.e^2.\cos.(2v-2mv-2cv)-2A_1^{(1)}.e^2.\cos.(2v-2mv).$$

The first term of the second member, is given in [4908 line 3], but the second is not given;
[5040k] we have, however, introduced it, because it is necessary to make the development [5040] agree with [5039]. This table contains the remaining terms of [5040] depending on A. Thus, the term depending on $A_2^{(0)}$ [5040 line 3] is the same as in [5040h line 1]. The coefficient of $A_1^{(1)}.\sin.(2v-2mv-gv)$, in [5040 line 3], is equal to the sum of the two terms in [5040h lines 7,8]. The coefficients of $A_1^{(1)}$ [5040 lines 5,6] are as in [5040h line 2].
[5040l] The coefficient of $A_1^{(11)}$ [5040 line 9] is the same as [5040h line 3]. The coefficient of $A_1^{(11)}$ [5040 line 10] is the same as [5040h line 4]. The coefficient of $A_1^{(1)}$ [5040 line 9] is the same as the sum of the two corresponding terms in [5040h lines 5, 6]. Lastly, the coefficient of $A_1^{(1)}$ [5040 line 10] is the same as the sum of the two corresponding terms in [5040h lines 5, 7].

[5041a] * (2877) In finding this variation, we neglect the terms depending on $\delta u'$, as in [5040a]

hence results the function,*

$$-\frac{3\overline{m}^2}{4}\cdot\frac{a}{a_{\prime}}\cdot\{B_1^{(0)}+4A_2^{(0)}+\tfrac{7}{2}B_1^{(10)}.e'^2-\tfrac{1}{2}B_1^{(9)}.e'^2\}.(1-\tfrac{5}{2}e'^2).\gamma.\sin.(gv-\theta) \qquad 1$$

$$+\frac{3\overline{m}^2}{2}\cdot\frac{a}{a_{\prime}}\cdot e\gamma\cdot\left\{\begin{array}{l}\{(1+m).B_1^{(0)}-A_1^{(1)}\}.\sin.(gv-cv-\theta+\varpi) \qquad 2\\ +\{(1-m).B_1^{(0)}-A_1^{(1)}\}.\sin.(gv+cv-\theta-\varpi) \qquad 3\end{array}\right\}$$

Development of the variation [5042]. [5043]

$$-\frac{3\overline{m}^2}{4}\cdot\frac{a}{a_{\prime}}\cdot e'\gamma.\left\{\begin{array}{l}+\{B_1^{(9)}+\tfrac{7}{2}B_1^{(0)}\}.\sin.(gv-c'mv-\theta+\varpi') \qquad 4\\ +\{B_1^{(10)}-\tfrac{1}{2}B_1^{(0)}\}.\sin.(gv+c'mv-\theta-\varpi') \qquad 5\\ +B_1^{(8)}.\sin.(2v-2mv-gv+c'mv+\theta-\varpi') \qquad 6\\ +B_1^{(7)}.\sin.(2v-2mv-gv-c'mv+\theta+\varpi') \qquad 7\end{array}\right\}$$

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* (2878) If we multiply the first member of [4910k] by $-\frac{2a}{3}.\delta s$, it produces the first term of the expression [5042] $\frac{3m'.u'^3.\delta s}{2h^2.u^4}.\cos.(2v-2v')$. Performing the same process on the second member of [4910k], we find, that the preceding term will be represented by the product of $\frac{3\overline{m}^2}{2}\cdot\frac{a}{a_{\prime}}.\delta s$ by the terms between the braces in [4910k]; or, in other words, it will be found, by multiplying the expression of δs [4897] by the terms between the braces in [4910k], *and then annexing the common factor* $\frac{3\overline{m}^2}{2}\cdot\frac{a}{a_{\prime}}$ *to all the terms.* Taking now, successively, the different terms of δs [4897], multiplying, reducing and retaining terms of the usual forms and orders, we shall find, that this first term of [5042] produces all the terms of [5043], which contain the symbol B; as will appear by the following calculation.

[5043a] [5043b] [5043c]

First. The product of $B_1^{(0)}.\gamma.\sin.(2v-2mv-gv)$ [4897 line 1], by the first line of [4910k], gives

$$\frac{3\overline{m}^2}{2}\cdot\frac{a}{a_{\prime}}\cdot(1+2e^2-\tfrac{5}{2}e'^2).\{\tfrac{1}{2}.\sin.(4v-4mv-gv)-\tfrac{1}{2}.\sin.gv\}. \qquad [5043d]$$

The first of these terms, which depends on the *sum* of the two angles is neglected, as usual, because it produces nothing of importance; and the same happens with the sums of all the other angles, which deserve notice, in this first term of [5042]; provided we change the signs of the angles in [4910k lines 10, 12]; which does not alter their cosines; so that the term between the braces, in [4910k line 10], may be put under the form $\frac{1}{4}(10+19m+8m^2).e^2.\cos.(2v-2mv-2cv)$, &c. Taking, therefore, the second term of [5043e]

[5043] continued. $$-\frac{3\overline{m}^2}{4}\cdot\frac{a}{a_{\prime}}\cdot e^2\gamma\cdot\left\{\begin{array}{l}+5A_1^{(1)}-2A_1^{(11)}+B_1^{(12)} \\ -\tfrac{1}{4}(10+19m+8m^2).B_1^{(0)}\end{array}\right\}.\sin.(2cv-gv-2\varpi+\theta)$$ 8, 9

[5043d], depending on the *difference of the angles*, and neglecting the quantities depending on e^2, e'^2, it becomes

$$-\frac{3\overline{m}^2}{4}\cdot\frac{a}{a_{\prime}}.B_1^{(0)}.\gamma.\sin.gv, \quad \text{as in [5043 line 1].}$$

[5043e′] We may remark, that the circumstance of only using the *difference* of the angles in this function, enables us to apply the principle of derivation with much facility, in finding the development of the function treated of in [5046a, &c.]. The same term of [4897 line 1] being combined with [4910k line 2], produces the term depending on $B_1^{(0)}$ [5043 line 2]; [5043e″] neglecting e^2, γ^2, e'^2; moreover, the term in [4910k line 3], gives, in like manner, the term depending on $B_1^{(1)}$ [5043 line 3]; the term in [4910k line 4] produces that in [5043 line 4]; lastly, the term in [4910k line 5] gives that in [5043 line 5]. The remaining terms of [4910k] produce quantities of the sixth and higher orders, which are neglected, with [5043f] the exception of that in [4910k line 10 or 5043e], which produces the term in [5043 line 9], depending on the angle $2cv-gv$.

Second. The terms in [4897 line 2—7] produce only terms of the fifth and higher orders [5043f′] which may be neglected. The term $\cos.(2v-2mv)$ [4910k line 1], being combined with [4897 line 8], produces the term depending on $B_1^{(7)}$ [5043 line 7]; with [4897 line 9], it produces the term depending on $B_1^{(8)}$ [5043 line 6]; with [4897 line 10], it produces the term depending on $B_1^{(9)}$ [5043 line4]; with [4897 line 11], it produces the term depending [5043g] on $B_1^{(10)}$ [5043 line 5]; with [4897 line 12], it produces the term depending on $B_0^{(11)}$ [5043 line 10]; lastly, with [4897 line 13], it produces the term depending on $B_0^{(12)}$ [5043 line 8]. The terms [4910k line 4] and [4897 line 11] produce the term depending on $B_1^{(10)}$ [5043 line 1]. The terms [4910k line 5] and [4897 line 10] produce the term depending on $B_1^{(9)}$ [5043 line 1]. This includes all the terms connected with the symbol B in [5043].

[5043h] The second term of [5042] is $-\frac{6m'.u'^3.s.\delta u}{h^2.u^5}.\cos.(2v-2v')$; this is evidently equal to the continued product of the first member of [4911] by the function $\frac{3s}{u}$ [5020b], and by the factor $\frac{4}{9}$. Now, this factor $\frac{4}{9}$, being multiplied by the factor $-\frac{9\overline{m}^2}{4a_{\prime}}$, without the [5043i] braces [4911], and by the factor $3a\gamma$ [5020b], produces $-3\overline{m}^2.\frac{a}{a_{\prime}}.\gamma$. Hence, it is evident, that the second term of [5042] will be obtained, by multiplying together the functions between the braces in [4911, 5020b]; then reducing, and annexing the common factor $-3\overline{m}^2.\frac{a}{a_{\prime}}.\gamma$. This calculation is made in the following table, which requires no particular explanation.

$$-\frac{3\overline{m}^2}{4}\cdot\frac{a}{a_{\prime}}\cdot B_0^{(11)}.e^2\gamma.\sin.(2v-2mv-2cv+gv+2\varpi-\theta).$$ 10 [5043] concluded

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(Col. 1.) Terms of [5020b], between the braces.	(Col. 2.) Terms of [4911], between the braces.	(Col. 3.) Corresponding terms of the function [5043]. [All these terms must be multiplied by $-3\overline{m}^2.\frac{a}{a_{\prime}}.\gamma$].		
$\sin.gv$	$A_2^{(0)}.(1-\frac{5}{2}e'^2)$	$A_2^{(0)}.(1-\frac{5}{2}e'^2).\sin.gv$	1	
$\sin.gv$	$A_1^{(1)}e.\cos.cv$	$\frac{1}{2}A_1^{(1)}e.\{\sin.(gv-cv)+\sin.(gv+cv)\}$	2	
$-\frac{1}{2}e.\sin.(gv+cv)$	$A_1^{(1)}e.\cos.cv$	....neglected	3	[5043k]
$-\frac{1}{2}e.\sin.(gv-cv)$	$A_1^{(1)}e.\cos.cv$	$+\frac{1}{4}A_1^{(1)}e^2.\sin.(2cv-gv)$	4	
$\sin.gv$	$-2A_1^{(1)}e^2.\cos.2cv$	$+\ A_1^{(1)}e^2.\sin.(2cv-gv)$	5	
$\sin.gv$	$A_1^{(11)}e^2.\cos.2cv$	$-\frac{1}{2}A_1^{(11)}e^2.\sin.(2cv-gv).$	6	

The two lower terms of column 2, lines 5, 6, correspond in [4911] to

$$-\frac{9\overline{m}^2}{4a_{\prime}}.\{-2A_1^{(1)}+A_1^{(11)}\}.e^2.\cos.2cv;$$ [5043l]

which are not expressly mentioned in [4911]; but are easily computed, as in [4910k, &c.]. For, the function [4911] is found, in [4910l], by multiplying the function [4910k] by the expression of $2a\delta u$, deduced from [4904]. Now, the term

$$\frac{9\overline{m}^2}{4a_{\prime}}.2e.\cos.(2v-2mv+cv)\quad [4910k\text{ line }3],$$ [5043m]

being multiplied by the term $2A_1^{(1)}.e.\cos.(2v-2mv-cv)$ of $2a\delta u$ [4904], produces the term

$$-\frac{9\overline{m}^2}{4a_{\prime}}.\{-2A_1^{(1)}.e^2.\cos.2cv\},$$ [5043n]

which is used in [5043k line 5]. In like manner, the term

$$-\frac{9\overline{m}^2}{4a_{\prime}}.\cos.(2v-2mv)\quad [4910k\text{ line }1],$$

being multiplied by the term $2A_1^{(11)}e^2.\cos.(2cv-2v+2mv)$ of $2a\delta u$ [4904], produces, in [4911], the term [5043o]

$$-\frac{9\overline{m}^2}{4a_{\prime}}.\{A_1^{(11)}e^2.\cos.2cv\},\quad\text{as in [5043k line 6].}$$

If we now compare the terms of [5043k] with those in [5043], depending on A, we shall find that they agree. For, the term in [5043k line 1], depending on $A_2^{(0)}$, is the same as in [5043 line 1]; those in [5043k line 2], depending on $A_1^{(1)}$, are the same as in [5043 lines 2, 3]; [5043p] the sum of those in [5043 lines 4, 5] is $-3\overline{m}^2.\frac{a}{a_{\prime}}.\gamma.\{\frac{5}{4}A_1^{(1)}.e^2.\cos.(2cv-gv)\}$, as in [5043 line 8]; lastly, the term in [5043k line 6], depending on $A_1^{(11)}$, is the same as in [5043 line 8].

[5044] The term $-\frac{3m'.u'^3}{2h^2.u^4}\cdot\frac{ds}{dv}.\sin.(2v-2v')$* gives, by its variation,

[5045]
$$-\frac{3m'.u'^3}{2h^2.u^4}\cdot\frac{d.\delta s}{dv}.\sin.(2v-2v')+\frac{6m'.u'^3}{h^2.u^5}.\delta u.\frac{ds}{dv}.\sin.(2v-2v')$$
$$+\frac{3m'.u'^3}{h^2.u^4}\cdot\frac{ds}{dv}\delta v'.\cos.(2v-2v').$$

Hence results the following function ;†

The third or last term of [5042] $\frac{3m'.u'^3.s.\delta v'}{h^2.u^4}.\sin.(2v-2v')$, is evidently equal to the continued product of the functions in the first members of [4918, 5020b] by the factor $\frac{1}{3}$;
[5043q] and, as this product gives terms of the sixth and higher orders, it may be neglected; consequently, the value of the terms in [5042] is accurately given, within the prescribed limits, by the function [5043].

* (2879) This term is the same as [5027], substituting [4809 line 1]; its chief part is
[5045a] computed in [5028, &c.]. Taking the variation of [5044], and neglecting $\delta u'$, as in [5040a, &c.], we get [5045].

[5046a] † (2880) The first term of [5045] $-\frac{3m'.u'^3}{2h^2.u^4}\cdot\frac{d.\delta s}{dv}.\sin.(2v-2v')$, may be computed in the same manner as the first term of [5042], in [5043a—g]; and, by this means, we shall obtain all the terms depending on the symbol B, in [5046]. But, we may obtain the
[5046b] same result in a more simple manner, by the principle of derivation used in [4876a—d]; deducing the terms of [5046], depending on any symbol $B^{(m)}$, from those in [5043], depending on the same symbol, in the following manner. If we denote any term of δs
[5046c] [4897], by $\delta s=B^{(m)}.\sin.iv$, we shall have, by taking its differential,

[5046d]
$$\frac{d.\delta s}{dv}=B^{(m)}.i.\cos.iv=B^{(m)}.i.\sin.(iv+90^d);$$

so that $\frac{d.\delta s}{dv}$ may be derived from δs, by increasing the angle iv by 90^d, and
[5046e] multiplying the coefficient by i. Moreover, if we increase $2v$ by 90^d, in the same manner as in [4876a—d], the term $\cos.(2v-2v')$, which occurs in [5043a], will change into $-\sin.(2v-2v')$, as in [5046a]. This increase of the angles iv and $2v$ by
[5046f] 90^d, does not alter the *differences* of these angles in the terms [5043d—g], which depend solely on these differences [5043c']. Hence it follows, that the terms of the function [5046a], may be deduced from the corresponding ones of [5043a], by merely multiplying by the coefficient i, corresponding to each term respectively; or, in other words, we must multiply each of the terms of [5043], depending on $B^{(m)}$, by
[5046g] the coefficient i, which corresponds to this term of $\delta s=B^{(m)}.\sin.iv$ [4897]. Thus, in

Development of the variation [5045].

$$-\frac{3\bar{m}^2}{4}\cdot\frac{a}{a_{\prime}}\cdot\left\{\begin{array}{l}(2-2m-g).B_1^{(0)}+\tfrac{7}{2}(2-3m-g).B_1^{(10)}.e'^2 \\ -\tfrac{1}{2}.(2-m-g).B_1^{(9)}.e'^2\end{array}\right\}.(1-\tfrac{5}{2}e'^2).\gamma.\sin.(gv-\theta) \qquad \begin{matrix}1\\2\end{matrix}$$

$$+\frac{3\bar{m}^2}{2}\cdot\frac{a}{a_{\prime}}\cdot e\gamma\cdot\left\{\begin{array}{l}\{(1+m).(2-2m-g).B_1^{(0)}-A_1^{(1)}\}.\sin.(gv-cv-\theta+\varpi) \\ +\{(1-m).(2-2m-g).B_1^{(0)}+A_1^{(1)}\}.\sin.(gv+cv-\theta-\varpi)\end{array}\right\} \qquad \begin{matrix}3\\4\end{matrix}$$

$$-\frac{3\bar{m}^2}{4}\cdot\frac{a}{a_{\prime}}\cdot e'\gamma.\left\langle\begin{array}{l}\left\{\begin{array}{l}(2-m-g).B_1^{(9)} \\ +\tfrac{7}{2}.(2-2m-g).B_1^{(0)}\end{array}\right\}.\sin.(gv-c'mv-\theta+\varpi') \\ +\left\{\begin{array}{l}(2-3m-g).B_1^{(10)} \\ -\tfrac{1}{2}.(2-2m-g)\,B_1^{(0)}\end{array}\right\}.\sin.(gv+c'mv-\theta-\varpi') \\ +(g-m).B_1^{(8)}.\sin.(2v-2mv-gv+c'mv+\theta-\varpi') \\ +(g+m).B_1^{(7)}.\sin.(2v-2mv-gv-c'mv+\theta+\varpi')\end{array}\right\rangle \qquad \begin{matrix}5\\6\\7\\8\\9\\10\end{matrix} \qquad [5046]$$

$$-\frac{3\bar{m}^2}{4}\cdot\frac{a}{a_{\prime}}\cdot e^2\gamma\cdot\left\{\begin{array}{l}5A_1^{(1)}-2A_1^{(11)}+(2-2m-2c+g).B_1^{(12)} \\ -\tfrac{1}{4}(10+19m+8m^2).(2-2m-g).B_1^{(0)}\end{array}\right\}.\sin.(2cv-gv-2\varpi+\theta) \qquad \begin{matrix}11\\12\end{matrix}$$

$$-\frac{3\bar{m}^2}{4}\cdot\frac{a}{a_{\prime}}\cdot B_0^{(11)}.e^2\gamma.\sin.(2v-2mv-2cv+gv+2\varpi-\theta). \qquad 13$$

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the first term of [4897 line 1], we have $B^{(m)}=B_1^{(0)}\gamma$, $i=2-2m-g$; therefore, the terms depending on $B_1^{(0)}$ [5043] must be multiplied by $2-2m-g$, to produce the corresponding terms of [5046]; by this means, the term in [5043 line 1] produces that in [5046 line 1]; and those in [5043 lines 2, 3, 4, 5] give those in [5046 lines 3, 4, 6, 8], respectively. Again, in the third term of [5043 line 1], corresponding to $B_1^{(10)}$, we have $i=2-2m-g-c'm=2-3m-g$, nearly; and this produces the term depending on $B_1^{(10)}$ [5046 line 1]. In like manner, the terms depending on $B_1^{(7)}$, $B_1^{(8)}$, $B_1^{(9)}$, $B_1^{(10)}$, $B_0^{(11)}$, $B_1^{(12)}$, produce the corresponding ones in [5046]; observing always, that c, g and c', are very nearly equal to unity. [5046h] [5046i]

Substituting the value of $\frac{ds}{dv}$ [5028a], in the second term of [5045], it becomes,

$$\frac{6m'.u'^3}{h^2.u^5}\cdot\delta u\cdot\sin.(2v-2v').g\gamma.\cos.gv; \qquad [5046k]$$

which, by putting $g=1$, is evidently equal to the differential of the first member of the

[5047] Lastly, the function $\left(\frac{dds}{dv^2}+s\right)\cdot\frac{2}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}$ [5033] gives, by its variation, the terms*

[5046l] function [4931n or 4931p], multiplied by $\frac{a\gamma}{2dv}.\cos.gv$; and, as the resulting function is composed of terms of the *fifth* and higher orders, we need only notice the chief terms of the
[5046l′] differential of the function [4931p]. These, after reduction, are contained in [4931p line 6],
[5046m] and in [4931q,r], whose differential, divided by $2dv$, produces the following terms, nearly;

[5046n] $$\frac{6\overline{m}^2}{2a_{\prime}}\cdot\{A_1^{(1)}e\sin.cv+A_1^{(11)}e^2.\sin.2cv\}-\frac{15\overline{m}^2}{2a_{\prime}}\cdot A_1^{(1)}e^2.\sin.2cv;$$

which must be multiplied by the factor $a\gamma.\cos.gv$. The first term of [5046n] evidently
[5046o] produces the two terms, depending on $A_1^{(1)}$ [5046 lines 3, 4]; the second term produces that depending on $A_1^{(11)}$ [5046 line 11]; the third term produces that depending on $A_1^{(1)}$, in [5046 line 11].

As the last term of [5045] is very small, we may substitute in it the values [4937n], and [4865, 5028a]; by which means it becomes

[5046p] $$\frac{3\overline{m}^2.a}{a_{\prime}}.\delta v'.\cos.(2v-2mv).\{g\gamma.\cos.gv\};$$

and, as $\delta v'$ is of the *third* order [4931x], the whole expression must be of the *sixth* or
[5046q] higher orders. Now, as it does not contain any quantities of the sixth order, depending on the angle $v-mv$ [4875], it may be neglected; therefore, the function [5045] will be represented by the quantities depending on its two first terms, which are given in [5046].

[5048a] * (2881) The chief term $\frac{2}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}$ [4809] is represented by $-\int W.dv$ [4929a], and if we put, in this case, $V_{\prime}=\frac{dds}{dv^2}+s$, the function [5047] will become of the
[5048b] form $-V_{\prime}.\int W.dv$; *whose variation is given in* [4929b], *changing V into $V_{\prime}$*. Now, we see, in [5049], that $V_{\prime}=\frac{dds}{dv^2}+s$ is of the order $\overline{m}^2.\gamma$, or of the *third* order; also W [4929a], or its differential coefficient, as well as $a\delta u$, are of the *second* or a higher order; hence it appears, that all the terms of the variation [5048b, 4929b], excepting
[5048c] $-\delta V_{\prime}.\int W.dv$, may be neglected, as of the seventh or a higher order. Now, the function $-\int W.dv$ [4929a] is evidently equal to the first member of [4885], and
[5048d] $\delta V_{\prime}=\frac{dd.\delta s}{dv^2}+\delta s$; hence, the function $-\delta V_{\prime}.\int W.dv$ [5048c], or the chief part of the development of the function [5047], will be represented by

$$\frac{3\overline{m}^2}{4} \cdot \frac{a}{a_{\prime}} \cdot \{(2-2m-g)^2-1\} \cdot B_1^{(0)} \cdot (1-\tfrac{5}{2}e'^2) \cdot \left\{ \begin{array}{l} \frac{1}{1-m} \cdot \gamma \cdot \sin.(gv-\theta) \\ +\frac{(10+19m+8m^2)}{2.(2c-2+2m)} \cdot e^2\gamma \cdot \sin.(2cv-gv-2\varpi+\theta) \end{array} \right\}. \quad \begin{array}{l} 1 \\ 2 \end{array}$$ Development of [5047]. [5048]

The terms depending on the cube of the disturbing force are insensible.*

13. Connecting together all the terms of this development, we find, that the equation [4755] becomes,

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$$\left(\frac{dd.\delta s}{dv^2}+\delta s\right) \times \text{ by the second member of the function [4885];}$$ [5048e]

in which we must substitute the value of δs [4897]. Now, the first term of this value gives, in $\frac{dd.\delta s}{dv^2}+\delta s$, the term $-\{(2-2m-g)^2-1\} . B_1^{(0)} . \gamma . \sin.(2v-2mv-gv)$; [5048f] multiplying this by the terms in [4885 lines 1, 10], it produces the terms in [5048 lines 1,2], respectively; neglecting terms of the order e^2, e'^2, in the factor $(1+2e^2-\frac{5}{2}e'^2)$ [5048g] [4885 line 1]. The factor $(2-2m-g)^2-1$, being of the order m, renders the term in [5048 line 1] of the fifth order; which is retained, though small, because the term connected with the angle gv gives the motion of the nodes in [5050, &c.]; and the term in [5048 line 2], depending on $2cv-gv$, is retained for reasons similar to those in [4828d]. The term depending on $B_1^{(0)}$ [5048f], being multiplied by the remaining terms of [4885], [5048h] produces terms of the sixth and higher orders, connected with angles which have been usually neglected. The next term of δs [4897 line 2] has the coefficient $B_2^{(1)}.\gamma$, which is marked of the *third* order; but, if we examine the value of $B_2^{(1)}$ [5177], we shall find it to be so very small, that it may be neglected. The terms in [4897 lines 3—7] are of the *fourth* order, producing in [5048e] terms of the *sixth* or higher orders, which may be neglected. The terms [4897 lines 8—11] are of the form $B_1^{(m)}.e'\gamma.\sin.iv$; [5048i] in which i differs from unity, by a quantity of the order m; so that $1-i^2$ is of the order m. This gives, in $\frac{dd.\delta s}{dv^2}+\delta s$, a term of the form $B_1^{(m)}.e'\gamma.(1-i^2).\sin.iv$, which is of the *fourth* order; producing only terms of the *sixth* order, in [5048e]. In like manner, we find, that the remaining terms of [4897] may be neglected, and the whole [5048k] function [5017] is reduced to the two small terms [5048].

* (2882) If we compare the value of Π [4902, 4961], with that of Γ [5037, 5049], we shall easily perceive, that the terms of Γ are of the order $\Pi.\gamma$; and, as the terms of [5048l] Π, depending on the cube of the disturbing force, are of the fifth or a higher order [4995a, 4941, 4942, &c.], the corresponding ones of Γ must be of the sixth or of a higher order, which may be neglected.

$$0=\frac{dds}{dv^2}+s+\tfrac{3}{2}\overline{m}^2\cdot\frac{a}{a_{\prime}}\cdot\left\{\begin{array}{l}1+2e^2-\tfrac{1}{2}\gamma^2+\tfrac{3}{2}e'^2 \\ -\tfrac{1}{2}\cdot\left\{\frac{(3-2m-g)\cdot(g+m)}{1-m}\cdot B_1^{(0)}+4A_2^{(0)}\right\}\cdot(1-\tfrac{5}{2}e'^2) \\ -\tfrac{7}{4}\cdot\{3-3m-g\}\cdot B_1^{(10)}\cdot e'^2+\tfrac{1}{4}\cdot(3-m-g)\cdot B_1^{(9)}\cdot e'^2 \\ +\tfrac{3}{2}\cdot\{B_1^{(7)}+B_1^{(8)}\}\cdot e'^2\end{array}\right\}\cdot\gamma\cdot\sin(gv-\theta) \quad \begin{array}{r}1\\2\\3\\4\end{array}$$ *

[5049] (L'')

$$-\tfrac{3}{4}\overline{m}^2\cdot\frac{a}{a_{\prime}}\cdot\left\{\begin{array}{l}(1+g)\cdot\{1+2e^2-\tfrac{1}{4}\cdot(2+m)\cdot\gamma^2-\tfrac{5}{2}e'^2\} \\ +\frac{(1-g^2)}{1-m}-4A_2^{(0)}+10A_1^{(1)}\cdot e^2-2B_1^{(0)}\end{array}\right\}\cdot\gamma\cdot\sin.(2v-2mv-gv+\theta) \quad \begin{array}{r}5\\6\end{array}$$

Differential equation in s.

$$+\tfrac{3}{2}\overline{m}^2\cdot\frac{a}{a_{\prime}}\cdot\{\tfrac{1}{2}\cdot(1-g)+B_2^{(1)}\}\cdot\gamma\cdot\sin.(2v-2mv+gv-\theta) \quad 7$$

$$+\tfrac{3}{2}\overline{m}^2\cdot\frac{a}{a_{\prime}}\cdot\{B_2^{(2)}-2+(1-m)\cdot(3-2m-g)\cdot B_1^{(0)}\}\cdot e\gamma\cdot\sin.(gv+cv-\theta-\varpi) \quad 8$$

$$+\tfrac{3}{2}\overline{m}^2\cdot\frac{a}{a_{\prime}}\cdot\{B_2^{(3)}-2-2A_1^{(1)}+(1+m)\cdot(3-2m-g)\cdot B_1^{(0)}\}\cdot e\gamma\cdot\sin.(gv-cv-\theta+\varpi) \quad 9$$

$$+\tfrac{3}{2}\overline{m}^2\cdot\frac{a}{a_{\prime}}\cdot\{(1+g)\cdot(1-m)-2B_1^{(0)}+B_2^{(4)}\}\cdot e\gamma\cdot\sin.(2v-2mv-gv+cv+\theta-\varpi) \quad 10$$

$$+\tfrac{3}{2}\overline{m}^2\cdot\frac{a}{a_{\prime}}\cdot\{(g-1)\cdot(1+m)+B_2^{(5)}-2A_1^{(1)}\}\cdot e\gamma\cdot\sin.(2v-2mv+gv-cv-\theta+\varpi) \quad 11$$

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* (2883) The equation [5049] is the same as [5037], taking for Γ all the terms we

[5049a] have computed in the *ten* functions [5021, 5023, 5025, 5030, 5032, 5036, 5040, 5043, 5046, 5048]. In finding the sum of these terms, we shall proceed as in note 2847 [4960e, &c.], taking the quantities depending on each angle separately, in the order in which they occur in [5049]; after dividing them by the factor which is common to all the terms as in [4961b].

First. The terms depending on $\sin.gv$ [5049 lines 1—4] have the common factor $\tfrac{3}{2}\overline{m}^2\cdot\frac{a}{a_{\prime}}\cdot\gamma\cdot\sin.gv$; and if we divide all the terms of the functions [5049a], depending on this angle, by this factor, we shall obtain in [5021 line 1] the terms $1+2e^2-\tfrac{1}{2}\gamma^2+\tfrac{3}{2}e'^2$, as in [5049 line 1]. The terms in [5040 line 2] are the same as in [5049 line 4]. The coefficient of $B_1^{(0)}\cdot(1-\tfrac{5}{2}e'^2)$, in [5043 line 1], is $-\tfrac{1}{2}$; in [5046 line 1], is $-\tfrac{1}{2}(2-2m-g)$; their sum is

$$-\tfrac{1}{2}\cdot(3-2m-g)=-\tfrac{1}{2}\cdot\left(\frac{3-2m-g}{1-m}\right)\cdot(1-m).$$

[5049b] Lastly, the term in [5048 line 1] is

$$\tfrac{1}{2}\cdot\left\{\frac{(2-2m-g)^2-1}{1-m}\right\}=-\tfrac{1}{2}\cdot\left(\frac{3-2m-g}{1-m}\right)\cdot(g+2m-1);$$

$$+\tfrac{3}{2}\overline{m}^2.\frac{a}{a_{\prime}}.\{(1+g).(1+m)+B_2^{(6)}+2A_1^{(1)}-2B_1^{(0)}\}.e\gamma.\sin.(2v-2mv-gv-cv+\theta+\varpi) \qquad 12$$

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\{3+2B_1^{(7)}+\tfrac{1}{2}(3-2m-g).B_1^{(0)}-(3-3m-g).B_1^{(10)}\}.e'\gamma.\sin.(gv+c'mv-\theta-\varpi') \qquad 13$$

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\{3+2B_1^{(8)}-\tfrac{7}{2}(3-2m-g).B_1^{(0)}-(3-m-g).B_1^{(9)}\}.e'\gamma.\sin.(gv-c'mv-\theta+\varpi') \qquad 14$$

Differential equation in s, concluded.

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\{\tfrac{1}{2}(1+g)+2B_1^{(9)}+3B_1^{(0)}-(1+g-m).B_1^{(8)}\}.e'\gamma.\sin.(2v-2mv-gv+c'mv+\theta-\varpi') \qquad 15$$

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\{2B_1^{(10)}-\tfrac{7}{2}(1+g)+3B_1^{(0)}-(1+g+m)B_1^{(7)}\}.e'\gamma.\sin.(2v-2mv-gv-c'mv+\theta+\varpi') \qquad 16$$

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{ll}2B_0^{(11)}-5-10A_1^{(1)}+4A_1^{(11)}-(3-2m-2c+g).B_1^{(12)} & 17\\ +\left\{\frac{3-2m-g}{4}+\frac{\{(2-2m-g)^2-1\}}{2(2c-2+2m)}\right\}.(10+19m+8m^2).B_1^{(0)} & 18\end{array}\right\}.e^2\gamma.\sin(2cv-gv-2\theta+\varpi) \qquad [5049]$$

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{ll}2B_1^{(12)}+\tfrac{1}{4}(1-g).(10+19m+8m^2) & 19\\ +10A_1^{(1)}-4A_1^{(11)}-2B_0^{(11)} & 20\end{array}\right\}.e^2\gamma.\sin.(2v-2mv-2cv+gv+2\varpi-\theta)$$

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{ll}\tfrac{1}{2}.(10+19m+8m^2)+2B_1^{(13)} & 21\\ +10A_1^{(1)}-4A_1^{(11)}-5B_1^{(0)} & 22\end{array}\right\}.e^2\gamma.\sin.(2cv+gv-2v+2mv-2\varpi-\theta)$$

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\left\{3+2B_2^{(14)}\right\}.\frac{a}{a'}.\gamma.\sin.(gv-v+mv-\theta) \qquad 23$$

$$+\tfrac{3}{4}\overline{m}^2.\frac{a}{a_{\prime}}.\left\{\tfrac{5}{2}+2B_2^{(15)}\right\}.\frac{a}{a'}.\gamma.\sin.(gv+v-mv-\theta). \qquad 24$$

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adding this to the sum of the two preceding terms, it becomes

$$-\tfrac{1}{2}.\left(\frac{3-2m-g}{1-m}\right).(g+m)\,; \qquad [5049b]$$

which is the same as the coefficient of $B_1^{(0)}$ [5049 line 2]. The term depending on $A_2^{(0)}$ [5043 line 1], is the same as in [5049 line 2]. The coefficient of $B_1^{(9)}.e'^2.(1-\tfrac{5}{2}e'^2)$, in [5043 line 1], is $\tfrac{1}{4}$; in [5046 line 2], is $\tfrac{1}{4}(2-m-g)$; whose sum is $\tfrac{1}{4}(3-m-g)$, as in [5049 line 3]. The coefficient of $B_1^{(10)}.e'^2.(1-\tfrac{5}{2}e'^2)$, in [5043 line 1], is $-\tfrac{7}{4}$; and, in [5046 line 1], is $-\tfrac{7}{4}(2-3m-g)$; their sum is $-\tfrac{7}{4}(3-3m-g)$, as in [5049 line 3]. The term depending on $A_2^{(0)}$ [5043 line 1], is the same as in [5049 line 2].

Second. The terms in [5049 lines 5, 6] have the common factor

$$-\tfrac{3}{4}\,\overline{m}^2.\frac{a}{a_{\prime}}.\gamma.\sin.(2v-2mv-gv)\,; \qquad [5049c]$$

[5049'] 14. In finding the integral of the equation [5049], we must proceed in

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and, if we divide the corresponding terms of the functions [5049a] by this factor, we shall obtain, in [5023 line 1], the terms $1+2e^2-\frac{1}{4}(2+m).\gamma^2-\frac{5}{2}e'^2$; and, in [5030 line 1], the same terms, multiplied by g; their sum is the same as in [5049 line 5]. The expression [5036] is the same as the first term in [5049 line 6]. The terms depending on $A_2^{(0)}$, $A_1^{(1)}$ [5040 line 3], are the same as in [5049 line 6]. The terms in [5040 line 1, 4897 line 1] give the term depending on $B_1^{(0)}$ [5049 line 6].

[5049c'] *Third.* Of the three terms in [5049 line 7], the first is found in [5023 line 2]; the second, in [5030 line 2]; and the third, in [5040 line 1, 4897 line 2].

Fourth. The terms in [5049 line 8] have the common factor

$$\tfrac{3}{2}\,\overline{m}^2\,.\,\frac{a}{a_{\prime}}\,.\,c\gamma\,.\,\sin.(gv+cv)\;;$$

[5049d] and, if we divide the corresponding terms of the functions [5049a] by this factor, we shall get, in [5021 line 2], the term -2; in [5040 line 1], the term $B_2^{(2)}$; as in the second and first terms of [5049 line 8]. The coefficient of $(1-m).B_1^{(0)}$, in [5043 line 3], is 1; and in [5046 line 4], is $2-2m-g$; whose sum is $3-2m-g$, as in [5049 line 8]. Lastly, the terms depending on $A_1^{(1)}$ [5043 line 3, 5046 line 4] mutually destroy each other.

Fifth. The terms in [5049 line 9] have the common factor

$$\tfrac{3}{2}\,\overline{m}^2\,.\,\frac{a}{a_{\prime}}\,.\,c\gamma\,.\,\sin.(gv-cv)\;;$$

[5049e] and, if we divide the corresponding terms of the functions [5049a] by it, we shall obtain, in [5021 line 3], the term -2; in [5040 line 1], the term $B_2^{(3)}$; as in the two first terms of [5049 line 9]. The coefficient of $(1+m).B_1^{(0)}$, in [5043 line 2], is 1; and, in [5046 line 3], is $(2-2m-g)$; whose sum is $(3-2m-g)$, as in [5049 line 9]. Lastly, the terms depending on $A_1^{(1)}$ [5043 line 2, 5046 line 3], being added, give $-2A_1^{(1)}$ [5049 line 9].

Sixth. The common factor of the terms in [5049 line 10] is

$$\tfrac{3}{2}\,\overline{m}^2\,.\,\frac{a}{a_{\prime}}\,.\,c\gamma\,.\,\sin.(2v-2mv-gv+cv).$$

The term connected with it, in [5023 line 5], is $1-m$; in [5030 line 5], is $g(1-m)$; whose sum is $(1+g).(1-m)$, as in the first part of [5049 line 10]; [5040 line 4] gives $-2B_1^{(0)}$; and [5040 line 1] gives $B_2^{(4)}$; as in [5049 line 10]. In the same manner we

[5049f] obtain the terms connected with $\frac{3}{2}\,\overline{m}^2\,.\,\frac{a}{a_{\prime}}\,.\,c\gamma.\sin.(2v-2mv+gv-cv)$; namely, in [5023 line 3], $-(1+m)$; in [5030 line 3], $g(1+m)$; whose sum is $(g-1).(1+m)$; in [5040 line 5], the term $-2A_1^{(1)}$; and, in [5040 line 1], the term $B_2^{(5)}$; all these agree with [5049 line 11].

a similar manner to that in [4971, &c.]. We shall, therefore, suppose [5049′

Seventh. The common factor of the terms in [5049 line 12] is

$$\tfrac{3}{2}\overline{m}^2 \cdot \frac{a}{a_{\prime}} \cdot c\gamma \cdot \sin.(2v-2mv-gv-cv).$$

The term connected with it, in [5023 line 4], is $(1+m)$; in [5030 line 4], is $g(1+m)$; whose sum is $(1+g).(1+m)$; in [5040 line 6], is $-2(B_1^{(0)}-A_1^{(1)})$; and, in [5040 line 1], is $B_2^{(6)}$. These agree with [5049 line 12]. [5049g]

Eighth. The terms connected with the common factor

$$\tfrac{3}{2}\overline{m}^2 \cdot \frac{a}{a_{\prime}} \cdot c\gamma \cdot \sin.(2v-2mv+gv+cv),$$

are as follows. In [5023 line 6], $-(1-m)$; in [5030 line 6], $g(1-m)$; whose sum $(g-1).(1-m)$ is of the second order [4828c]; or, of the sixth order in [5049]; and, as this is not increased by the integration [4897o, &c.], it is neglected. [5049h]

Ninth. The common factor of the terms in [5049 line 13] is

$$\tfrac{3}{4}\overline{m}^2 \cdot \frac{a}{a_{\prime}} \cdot c'\gamma \cdot \sin.(gv+c'mv).$$

The term connected with it, in [5021 line 4], is 3; in [5040 line 1], is $2B_1^{(7)}$; as in the two first terms of [5049 line 13]. The coefficient of $B_1^{(0)}$, in [5043 line 5], is $\tfrac{1}{2}$; in [5046 line 8], is $\tfrac{1}{2}(2-2m-g)$; whose sum is $\tfrac{1}{2}(3-2m-g)$, as in [5049 line 13]. The coefficient of $B_1^{(10)}$, in [5043 line 5], is -1; in [5046 line 7] is $-(2-3m-g)$; whose sum is $-(3-3m-g)$, as in [5049 line 13]. [5049i]

Tenth. The common factor of the terms in [5049 line 14] is

$$\tfrac{3}{4}\overline{m}^2 \cdot \frac{a}{a_{\prime}} \cdot c'\gamma \cdot \sin.(gv-c'mv).$$

The term connected with it, in [5021 line 5], is 3; in [5040 line 1], is $2B_1^{(8)}$, as in the two first terms of [5049 line 13]. The coefficient of $B_1^{(0)}$, in [5043 line 4], is $-\tfrac{7}{2}$; in [5046 line 6], is $-\tfrac{7}{2}(2-2m-g)$; whose sum is $-\tfrac{7}{2}(3-2m-g)$, as in [5049 line 14]. The coefficient of $B_1^{(9)}$, in [5043 line 4], is -1; in [5046 line 5], is $-(2-m-g)$; whose sum is $-(3-m-g)$, as in [5049 line 14]. [5049k]

Eleventh. The common factor of the terms in [5049 line 15] is

$$\tfrac{3}{4}\overline{m}^2 \cdot \frac{a}{a_{\prime}} \cdot c'\gamma \cdot \sin.(2v-2mv-gv+c'mv).$$

The term connected with it, in [5023 line 9], is $\tfrac{1}{2}$; in [5030 line 9], is $\tfrac{1}{2}g$; in [5040 line 1], is $2B_1^{(9)}$; in [5040 line 7], is $3B_1^{(0)}$; in [5043 line 6], is $-B_1^{(8)}$; and, in [5046 line 9], is $-(g-m).B_1^{(8)}$. These terms, taken in the same order, are as in [5049 line 15]. [5049l]

[5049'''] γ and θ to be variable; in consequence of the variation of the excentricity

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Twelfth. The common factor of the terms in [5049 line 16] is

$$\tfrac{3}{4}\,\overline{m}^2 \,.\, \frac{a}{a_,} \,.\, e'\gamma \,.\, \sin.(2v-2mv-gv-c'mv).$$

[5049m] The term connected with it, in [5023 line 7], is $-\frac{7}{2}$; in [5030 line 7], is $-\frac{7}{2}g$; whose sum is $-\frac{7}{2}(1+g)$, as in the second term of [5049 line 16]. The term, in [5040 line 1], is $2B_1^{(10)}$; in [5040 line 8], is $3B_1^{(9)}$; as in the first and third terms of [5049 line 16]. Lastly, the coefficient of $B_1^{(7)}$, in [5043 line 7], is -1; and, in [5046 line 10], is $-(g+m)$; whose sum is $-(1+g+m)$; as in the last term of [5049 line 16].

Thirteenth. The term connected with the common factor

$$\tfrac{3}{4}\,\overline{m}^2 \,.\, \frac{a}{a_,} \,.\, e'\gamma \,.\, \sin.(2v-2mv+gv-c'mv),$$

in [5023 line 8], $\frac{7}{2}$; and, in [5030 line 8], is $-\frac{7}{2}g$; whose sum $\frac{7}{2}(1-g)$ is of the order m^2 [4828c], producing terms of the *sixth* order, which may be neglected. In like
[5049n] manner, the terms connected with the factor $\frac{3}{4}\,\overline{m}^2 \,.\, \frac{a}{a_,} \,.\, e'\gamma \,.\, \sin.(2v-2mv+gv+c'mv)$, in [5023 line 10], is $-\frac{1}{2}$; and, in [5030 line 10], is $+\frac{1}{2}g$; whose sum $-\frac{1}{2}(1-g)$, is of the second order, producing terms of the sixth order, in [5049], which may be neglected.

Fourteenth. The common factor of the terms in [5049 lines 17, 18] is

$$\tfrac{3}{4}\,\overline{m}^2 \,.\, \frac{a}{a_,} \,.\, e^2\gamma \,.\, \sin.(2cv-gv).$$

The term connected with it, in [5021 line 6], is -5; in [5040 line 1], is $2B_0^{(11)}$; as in the two first terms of [5049 line 17]. The coefficient of $A_1^{(1)}$, in [5043 line 8], is -5; in [5046 line 11], is -5; whose sum is -10, as in [5049 line 17]. The coefficient
[5049o] of $A_1^{(11)}$, in [5043 line 8], is $+2$, in [5046 line 11], is $+2$; whose sum is $+4$, as [5049 line 17]. The coefficient of $B_1^{(12)}$, in [5043 line 8], is -1; in [5046 line 11], is $-(2-2m-2c+g)$; whose sum is $-(3-2m-2c+g)$, as in [5049 line 17]. The coefficient of $(10+19m+8m^2).B_1^{(0)}$, in [5043 line 9], is $\frac{1}{4}$; in [5046 line 12], is $\frac{1}{4}(2-2m-g)$; whose sum is $\frac{1}{4}(3-2m-g)$, as in the first term of [5049 line 18]; the remaining term is as in [5048 line 2], neglecting the factor $(1-\frac{5}{2}e'^2)$.

Fifteenth. The common factor of the terms in [5049 lines 19, 20] is

$$\tfrac{3}{4}\,\overline{m}^2 \,.\, \frac{a}{a_,} \,.\, e^2\gamma.\sin.(2v-2mv-2cv+gv).$$

[5049p] The term connected with it, in [5040 line 1], is $2B_1^{(12)}$, as in the first term of [5049 line 19]. The coefficient of $\frac{1}{4}(10+19m+8m^2)$, in [5023 line 11], is 1; and, in [5030 line 11], is $-g$; whose sum is $(1-g)$, as in the second term of [5049 line 19]. The terms depending on $A_1^{(1)}$, $A_1^{(11)}$ [5040 line 10], are as in [5049 line 20]. The coefficient of

of the earth's orbit. Then, by substituting, for s, the expression [4897*i*]

$$s=\gamma.\sin.(gv-\theta)+\delta s,$$ [5050]

and comparing at first, the sines and cosines of $gv-\theta$, we shall obtain the two following equations;*

$B_0^{(11)}$ [5043 line 10], is -1; and that in [5046 line 13], is also -1; whose sum -2, is as in [5049 line 20].

Sixteenth. The common factor of the terms in [5049 lines 21, 22] is

$$\tfrac{3}{4}\,\overline{m}^2\,.\frac{a}{a_{\prime}}\,.\,e^2\gamma.\sin.(2cv+gv-2v+2mv).$$

This is multiplied by the factor $\frac{1}{4}(10+19m+8m^2)$, in [5023 line 12]; and also, in [5030 line 12]; their sum is as in the first term of [5049 line 21]. In [5040 line 1], we have $2B_1^{(13)}$, as in the second term of [5049 line 21]. The terms, in [5040 line 9], give those in [5049 line 22]. [5049*q*]

Seventeenth. The common factor of the terms in [5049 line 23] is

$$\tfrac{3}{4}\,\overline{m}^2\,.\frac{a}{a_{\prime}}\,.\,\frac{a}{a'}\,.\gamma.\sin.(gv-v+mv).$$

The terms connected with this, in [5025], is $\frac{11}{4}$; and, in [5032], is $\frac{1}{4}$; whose sum is 3, as in [5049 line 23]. Lastly, the term [5040 line 1], depending on $B_2^{(14)}$, is as in [5049 line 23]. [5049*r*]

Eighteenth. The common factor of the terms in [5049 line 24] is

$$\tfrac{3}{4}\,\overline{m}^2\,.\frac{a}{a_{\prime}}\,.\,\frac{a}{a'}\,.\gamma.\sin.(gv+v-mv).$$

The term connected with this, in [5025], is $\frac{11}{4}$; and, in [5032], is $-\frac{1}{4}$; whose sum is $\frac{5}{2}$, as in [5049 line 24]. The term [5040 line 1], depending on $B_2^{(15)}$, is as in the last term of [5049 line 24]. Hence it appears, that all the terms of the equation [5049] agree with the preceding developments. [5049*s*]

* (2884) The quantities $B_1^{(0)}$, $A_2^{(0)}$, in the factor of $\gamma.\sin.(gv-\theta)$ [5049 line 1-4], are multiplied by $1-\frac{5}{2}e'^2$, and some of the other terms of that factor are multiplied by e'^2; so that we may put the whole factor under the form $p''+q''.e'^2$ [5053]. Moreover, as the equation [5049] is linear, we may notice the terms depending on $\sin.(gv-\theta)$ separately; and, by restricting the value of s to this term, we shall have, from [5049, 5053], [5051*a*]

$$0=\frac{dds}{dv^2}+s+(p''+q''.e'^2).\gamma.\sin.(gv-\theta).$$ [5051*b*]

Substituting in this, the value $s=\gamma.\sin.(gv-\theta)$, and its second differential,

$$\frac{dds}{dv^2}=\frac{dd\gamma}{dv^2}.\sin.(gv-\theta)+2.\frac{d\gamma}{dv}.\left(g-\frac{d\theta}{dv}\right).\cos.(gv-\theta)-\gamma.\frac{dd\theta}{dv^2}.\cos.(gv-\theta)-\gamma.\left(g-\frac{d\theta}{dv}\right)^2.\sin.(gv-\theta);$$ [5051*c*]

[5051] $$0=\gamma\cdot\frac{dd\theta}{dv^2}-2\cdot\frac{d\gamma}{dv}\cdot\left(g-\frac{d\theta}{dv}\right);$$

[5052] $$0=\frac{dd\gamma}{dv^2}-\gamma\cdot\left\{\left(g-\frac{d\theta}{dv}\right)^2-1\right\}+(p''+q''.e'^2).\gamma;$$

[5053] $p''+q''.e'^2$ *denoting the coefficient of* $\gamma.\sin.(gv-\theta)$ *in the differential*
p'', q''. equation [5049]; in which we must observe, that $B_1^{(0)}$ and $A_2^{(0)}$ contain
[5054] inplicitly the factor $(1-\frac{5}{2}e'^2)$ [4976a,b]. The first of these equations gives, by integration,*

[5055] $$\frac{1}{g-\frac{d\theta}{dv}}=H.\gamma^2;$$

H being an arbitrary constant quantity. The equation [5052] gives, by
[5055′] neglecting $\frac{dd\gamma}{dv^2}$, and the square of $q''.e'^2$,†

[5056] $$\frac{d\theta}{dv}=g-\sqrt{1+p''}-\frac{\frac{1}{2}q''.e'^2}{\sqrt{1+p''}};$$

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considering v, γ, θ, as variable quantities, we get,

[5051d] $$0=-\left\{\gamma\cdot\frac{dd\theta}{dv^2}-2.\frac{d\gamma}{dv}.\left(g-\frac{d\theta}{dv}\right)\right\}.\cos.(gv-\theta)+\left\{\frac{dd\gamma}{dv^2}-\gamma.\left[\left(g-\frac{d\theta}{dv}\right)^2-1\right]+(p''+q''e'^2)\gamma\right\}\sin(gv-\theta).$$

This equation is satisfied by putting the coefficients of $\cos.(gv-\theta)$, $\sin.(gv-\theta)$ separately
[5051e] equal to nothing; by which means we obtain the equations [5051, 5052], respectively. The whole calculation being similar to that for the motion of the perigee, in note 2852 [4973a—h].

* (2885) The equations [5051, 5052] are similar to [4973, 4974], and are solved in the same manner as in [4977a, &c.]. Putting, in this case, $g-\frac{d\theta}{dv}=W_{,}$, we get, for
[5055a] its differential, $-\frac{dd\theta}{dv^2}=\frac{dW_{,}}{dv}$. Substituting these in [5051], we obtain,

[5055b] $$0=-\gamma\frac{dW_{,}}{dv}-2W_{,}.\frac{d\gamma}{dv};\quad\text{or,}\quad-\frac{dW_{,}}{W_{,}}=2.\frac{d\gamma}{\gamma};$$

whose integral is $\frac{1}{W_{,}}=H.\gamma^2$; H being the arbitrary constant quantity. This is the same as [5055], and is similar to [4977 or 4977c].

† (2886) In like manner as we have neglected ddE, or dde, in [4973e—h], we may neglect $dd\gamma$, in [5052]; and then, dividing by γ, we get,

therefore, if we consider p'', q'', as constant, which may here be done without any sensible error [4979a, &c.], we shall have, [5056']

$$\theta = gv - \sqrt{1+p''}.v - \frac{\frac{1}{2}q''}{\sqrt{1+p''}}.\int e'^2.dv + \lambda; \qquad [5057]$$

λ being an arbitrary quantity. This gives, [5057']

$$\sin.(gv-\theta) = \sin.\left\{\sqrt{1+p''}.v + \frac{\frac{1}{2}q''}{\sqrt{1+p''}}.\int e'^2.dv - \lambda\right\}. \qquad [5058]$$

Hence it follows, in conformity with observation, that the nodes of the moon's orbit have a retrograde motion upon the apparent ecliptic, which is represented by,*

$$\text{Retrograde motion of the nodes} = \{\sqrt{1+p''}-1\}.v + \frac{\frac{1}{2}q''}{\sqrt{1+p''}}.\int e'^2.dv. \qquad [5059]$$

This motion is not uniform by reason of the variableness of e'; and *the secular equation of the longitude of the node is to the secular equation of the perigee* as $\frac{q''}{\sqrt{1+p''}}$ is to $-\frac{q}{\sqrt{1-p}}$.† [5060] Motion of the nodes.

$$0 = -\left(g - \frac{d\theta}{dv}\right)^2 + 1 + p'' + q''.e'^2;$$

or, by reduction,

$$g - \frac{d\theta}{dv} = \sqrt{(1+p''+q''.e'^2)} = \sqrt{(1+p'')} + \frac{\frac{1}{2}q''.e'^2}{\sqrt{(1+p'')}} + \&c. \qquad [5056a]$$

Neglecting the square of $q''.e'^2$ [5055'], and reducing, we obtain [5056]. Multiplying this by dv, and integrating, we get [5057]; or, as it may be written,

$$gv - \theta = \left\{\sqrt{(1+p'')}.v + \frac{\frac{1}{2}q''}{\sqrt{(1+p'')}}.\int e'^2.dv - \lambda\right\}; \qquad [5056b]$$

and, by taking the sine of both members, it becomes as in [5058].

* (2887) In [4818 or 5051b] the quantity $gv-\theta$ represents the moon's distance from the node, which is equal to

$$\left\{\sqrt{(1+p'')}.v + \frac{\frac{1}{2}q''}{\sqrt{(1+p'')}}.\int e'^2.dv - \lambda\right\} \text{ [5056b].} \qquad [5059a]$$

Subtracting from this the moon's longitude v, we get the expression of the retrograde motion of the nodes [5059]; observing, that by taking the integral $\int e'^2.dv$ from $v=0$, where the motion of the node is commenced, we may neglect the quantity λ. [5059b]

† (2888) The term of the expression of the motion of the moon's perigee, upon which its secular motion depends, is represented in [4982] by $\frac{1}{2}q'.\int e'^2.dv = \frac{\frac{1}{2}q}{\sqrt{(1-p)}}.\int e'^2.dv$ [4979]. [5060a]

The tangent γ *of the inclination of the moon's longitude to the apparent ecliptic* [4813], *is also variable, since it is represented by*,*

[5061] $$\gamma = \left\{ H.\left(g - \frac{d\theta}{dv}\right) \right\}^{-\frac{1}{2}} \quad [5055].$$

The secular variation of γ is insensible. *But it is evident, that its variation is insensible;* and this is the reason why the most ancient observations do not indicate any change in the inclination, although the position of the ecliptic has varied sensibly, during that interval.

We shall then have the following equations;†

The similar term in the motion of the node is $-\frac{\frac{1}{2}q''}{\sqrt{(1+p'')}}.fe'^2.dv$ [5059]; the negative sign being prefixed, because the motion is retrograde. This last expression is to the former in the ratio mentioned in [5060].

* (2889) We may observe, that the equations [5051—5059] are similar to those in [4973—1982], and may be derived from them. Thus, by changing c, ϖ, $-p$, $-q$,
[5061a] $\frac{e.(1+e^2)}{a}$, into g, θ, p'', q'', γ, respectively, we find, that the equations [4973] and [4974] change into [5051, 5052], neglecting $dd\gamma$; [4977] becomes like [5055];
[5061b] [4978] like [5056]; [4980] like [5057]; [4981] like [5058], changing *cos.* into *sin.*; lastly, [4982] like [5059]. Hence it is evident, that we may apply the same method, to prove, that the secular inequality of γ is insensible, that we have used for e, or
[5061c] $\frac{e.(1+e^2)}{a}$, in [4987, &c.]; observing, that both these inequalities depend on terms of a similar form and order.

† (2890) If we suppose any term of δs [4897] to be represented by $B.\sin.(iv+\varepsilon)$,
[5062a] it will produce, in $\frac{dds}{dv^2}+s$, the term $\{1-i^2\}.B.\sin.(iv+\varepsilon)$. Substituting this in [5049], and putting the coefficient of each sine equal to nothing, we shall obtain the
[5062b] equations [5062—5077]; taking them in the same order as they occur in [5049]; and reducing them, by dividing the equations by the factors depending on e, e', γ, without the braces. No other reduction is necessary in any of the terms, except in that in [5049 line 18]; in which we must substitute

$$(2-2m-g)^2-1 = (3-2m-g).(1-2m-g);$$

by which means we have,

[5062c] $$\frac{3-2m-g}{4} + \frac{\{(2-2m-g)^2-1\}}{2.(2c-2+2m)} = (3-2m-g).\left\{\frac{1}{4} + \frac{1-2m-g}{2.(2c-2+2m)}\right\}$$
$$= -(3-2m-g).\frac{(g+m-c)}{2.(2c-2+2m)}.$$

$$0=\{1-(2-2m-g)^2\}.B_1^{(0)}-\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{l}(1+g).\{1+2e^2-\frac{1}{4}.(2+m).\gamma^2-\frac{5}{2}e'^2\}\\ +\dfrac{(1-g^2)}{1-m}-4A_2^{(0)}+10A_1^{(1)}.e^2-2B_1^{(0)}\end{array}\right\};$$ [5062]

$$0=\{1-(2-2m+g)^2\}.B_2^{(1)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_{\prime}}.\{\tfrac{1}{2}.(1-g)+B_2^{(1)}\};$$ [5063]

$$0=\{1-(g+c)^2\}.B_2^{(2)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_{\prime}}.\{B_2^{(2)}-2+(1-m).(3-2m-g).B_1^{(0)}\};$$ [5064]

$$0=\{1-(g-c)^2\}.B_2^{(3)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_{\prime}}.\{B_2^{(3)}-2-2A_1^{(1)}+(1+m).(3-2m-g).B_1^{(0)}\};$$ [5065]

$$0=\{1-(2-2m-g+c)^2\}.B_2^{(4)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_{\prime}}.\{(1+g).(1-m)-2B_1^{(0)}+B_2^{(4)}\};$$ [5066]

$$0=\{1-(2-2m+g-c)^2\}.B_2^{(5)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_{\prime}}.\{(g-1).(1+m)+B_2^{(5)}-2A_1^{(1)}\};$$ [5067]

$$0=\{1-(2-2m-g-c)^2\}.B_2^{(6)}+\tfrac{3}{2}\bar{m}^2.\frac{a}{a_{\prime}}.\{(1+g).(1+m)+B_2^{(6)}+2A_1^{(1)}-2B_1^{(0)}\};$$ [5068]

$$0=\{1-(g+m)^2\}.B_1^{(7)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\{3+2B_1^{(7)}+\tfrac{1}{2}(3-2m-g).B_1^{(0)}-(3-3m-g).B_1^{(10)}\};$$ [5069]

$$0=\{1-(g-m)^2\}.B_1^{(8)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\{3+2B_1^{(8)}-\tfrac{7}{2}(3-2m-g).B_1^{(0)}-(3-m-g).B_1^{(9)}\};$$ [5070]

$$0=\{1-(2-m-g)^2\}.B_1^{(9)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\{\tfrac{1}{2}(1+g)+2B_1^{(9)}+3B_1^{(0)}-(1+g-m).B_1^{(8)}\};$$ [5071]

$$0=\{1-(2-3m-g)^2\}.B_1^{(10)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\{2B_1^{(10)}-\tfrac{7}{2}(1+g)+3B_1^{(0)}-(1+g+m).B_1^{(7)}\};$$ [5072]

$$0=\{1-(2c-g)^2\}.B_0^{(11)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{l}2B_0^{(11)}-5-10A_1^{(1)}+4A_1^{(11)}-(3-2m-2c+g).B_1^{(12)}\\ -(3-2m-g).(g+m-c).\dfrac{(10+19m+8m^2)}{2.(2c-2+2m)}.B_1^{(0)}\end{array}\right\};$$ [5073]

$$0=\{1-(2-2m-2c+g)^2\}.B_1^{(12)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{l}2B_1^{(12)}+\frac{1}{4}(1-g).(10+19m+8m^2)\\ +10A_1^{(1)}-4A_1^{(11)}-2B_0^{(11)}\end{array}\right\};$$ [5074]

$$0=\{1-(2c+g-2+2m)^2\}.B_1^{(13)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\left\{\begin{array}{l}\frac{1}{2}.(10+19m+8m^2)+2B_1^{(13)}\\ +10A_1^{(1)}-4A_1^{(11)}-5B_1^{(0)}\end{array}\right\};$$ [5075]

$$0=\{1-(g+m-1)^2\}.B_2^{(14)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\left\{3+2B_2^{(14)}\right\};$$ [5076]

$$0=\{1-(g+1-m)^2\}.B_2^{(15)}+\tfrac{3}{4}\bar{m}^2.\frac{a}{a_{\prime}}.\left\{\tfrac{5}{2}+2B_2^{(15)}\right\}.$$ [5077]

15. *It now remains to determine the value of* t, *in terms of* v. For this purpose, we shall resume the equation [4753],

[5078] $$dt=\frac{dv}{hu^2.\sqrt{1+\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}}}.$$

We must substitute in it the value of u [4997]; namely,

[5079] $$u=\frac{1}{a}.\left\{\begin{array}{l}1+e^2+\tfrac{1}{4}\gamma^2+\beta+e.(1+e^2).\cos.(cv-\varpi)\\ -\tfrac{1}{4}\gamma^2.(1+e^2-\tfrac{1}{4}\gamma^2).\cos.(2gv-2\theta)\end{array}\right\}+\delta u.$$

We shall have, in the first place, by developing the factor $\frac{dv}{hu^2}$, a term independent of the cosines, which, by the nature of the elliptical motion,

[5080] must be equal to* $\frac{a^2.dv}{\sqrt{a_{,}}}$ [5081o,p].

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* (2891) If we put, for brevity,

[5081a] $$Q'=\frac{1}{\left\{1+\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}\right\}^{\frac{1}{2}}},$$

[5081a'] the expression [5078] will become, $dt=\frac{dv}{hu^2}.Q'$. The development of Q', in a

[5081b] series, gives, $$Q'=1-\frac{1}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}+\frac{3}{2h^4}.\left(\int\frac{dQ}{dv}.\frac{dv}{u^2}\right)^2-\&c.;$$

[5081c] which is of the same form as the factor of [5081], depending on Q. The terms of u [5079], independent of δu, have been heretofore denoted by u [4826, 4861, &c.; 4997]; and, by retaining this value, the second member of [5079] will be $u+\delta u$. Substituting this complete value of u in dt [5081a'] it becomes,

[5081d] $$dt=\frac{dv.Q'}{h.(u+\delta u)^2}=\frac{dv.Q'}{hu^2}.\left(1+\frac{\delta u}{u}\right)^{-2}=\left\{\frac{1}{hu^2}-\frac{2\delta u}{hu^3}+\frac{3\delta u^2}{hu^4}-\&c.\right\}.dv.Q'$$

[5081e] $$=\left\{\frac{1}{hu^2}-2(a\delta u).\frac{1}{hu^3.a}+3(a\delta u)^2.\frac{1}{hu^4.a^2}-\&c.\right\}.dv.Q';$$

observing, that we must substitute in [5081e], for u, all the terms of the second member of [5079], excepting δu. Now, by neglecting terms of the fourth order, we have,

[5081f] $$\frac{1}{h}=\frac{1}{a_{,}^{\frac{1}{2}}}.(1+\tfrac{1}{2}e^2+\tfrac{1}{2}\gamma^2)\quad[4866l];\quad\text{whence,}\quad\frac{1}{ha}=\frac{1}{aa_{,}^{\frac{1}{2}}}.(1+\tfrac{1}{2}e^2+\tfrac{1}{2}\gamma^2).$$

Multiplying this by u^{-3} [4866k], we get,

[5081g] $$\frac{1}{hu^3.a}=\frac{a^2}{a_{,}^{\frac{1}{2}}}.\{(1+\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2)-3e.(1-\tfrac{1}{2}\gamma^2).\cos.cv+3e^2.\cos.2cv+\tfrac{3}{4}\gamma^2.\cos.2gv-\tfrac{3}{2}e\gamma^2.\cos(2gv-cv)\}.$$

Then we shall have,

Differential equation in t.

$$dt=\frac{a^2dv}{\sqrt{a_{\prime}}}\cdot\left\{\begin{array}{ll}
\left\{\begin{array}{l}
1-2e.(1-\tfrac{1}{4}\gamma^2).\cos.(cv-\varpi) \\
+\tfrac{3}{2}e^2.(1+\tfrac{1}{6}e^2-\tfrac{1}{2}\gamma^2).\cos.(2cv-2\varpi) \\
+\tfrac{1}{2}\gamma^2(1+\tfrac{3}{2}e^2-\tfrac{1}{2}\gamma^2).\cos(2gv-2\theta)-e^3.\cos(3cv-3\varpi) \\
-\tfrac{3}{4}e\gamma^2\{\cos(2gv-cv-2\theta+\varpi)+\cos(2gv+cv-2\theta-\varpi)\}
\end{array}\right\}\cdot\left\{\begin{array}{l}
1-\frac{1}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2} \\
+\frac{3}{2h^4}\cdot\left[\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}\right]^2 \\
-\ \&c.
\end{array}\right\} & \begin{array}{l}1\\2\\3\\4\end{array} \\
-2a\delta u.\left\{\begin{array}{l}
1+\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2-3e.\cos.(cv-\varpi)+3e^2.\cos.(2cv-2\varpi) \\
+\tfrac{3}{4}\gamma^2.\cos.(2gv-2\theta)-\tfrac{3}{2}e\gamma^2.\cos.(2gv-cv-2\theta+\varpi)
\end{array}\right\}\cdot\left\{\begin{array}{l}
1-\frac{1}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2} \\
+\ \&c.
\end{array}\right\} & \begin{array}{l}5\\6\end{array} \\
+3.(a\delta u)^2.\{1-4e.\cos.(cv-\varpi)\}.\{1-\&c.\} & 7
\end{array}\right. \qquad [5081]$$

Substituting this and Q' [5081b], in the term of [5081e] depending on the first power of $a\delta u$, we get the corresponding terms of [5081 lines 5,6]; neglecting the very small term of the fifth order, depending on $e\gamma^2.\cos.cv$. Again, we have in [4870b],

$$u^{-4}=a^4.\{1-4e.\cos.cv+\&c.\}. \qquad [5081h]$$

Multiplying this by $\frac{1}{ha^2}=\frac{1}{a^2a_{\prime}^{\frac{1}{2}}}$ nearly [5081f], we get,

$$\frac{1}{hu^4.a^2}=\frac{a^2}{a_{\prime}^{\frac{1}{2}}}.\{1-4e.\cos.cv+\&c.\}. \qquad [5081i]$$

Substituting this and the value of Q' [5081b], in the term [5081e] depending on $(a\delta u)^2$, we get the corresponding terms of [5081 line 7]. We may observe, that $a\delta u$ [4904] is of the second order; so that, in these terms of [5081 lines 5—7], we have explicitly retained terms as far as the *fourth* order inclusively. The only remaining term of dt [5081e] is the first, $\frac{Q'.dv}{hu^2}$; and the quantity Q' is represented by the terms depending on Q, in [5081 lines 1—4]. The factor connected with Q', is of the same form as the value of dt, in the first of the equations [531]; namely, $dt=\frac{dv}{hu^2}$; from which the elliptical value [531c, 535] is deduced. This has the constant factor $a^{\frac{3}{2}}$. If we compare this factor, or $\frac{a^2}{\sqrt{a}}$, with the calculation in [534b, &c.], we shall easily perceive, that the numerator a^2, is introduced by the term u^2 [5081m], which is not altered in the disturbed orbit [4861]; but the denominator $\sqrt{a}$, which is deduced from h [534b,&c.], is changed into $\sqrt{a_{\prime}}$, in the disturbed orbit [4863]; and, by this means, it becomes $\frac{a^2}{\sqrt{a_{\prime}}}$.

[5081k] [5081l] [5081m] [5081n] [5081o]

If we take the differential of [4828], and divide it by $n=a^{-\frac{3}{2}}$ [4827], it becomes, by using the abridged notation [4821f],

[5081'] That part of the second member of this equation, which is not periodical, is

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[5081p] $$dt=\frac{a^2}{\sqrt{a}}.dv.\left\{\begin{array}{l}1-2e.(1-\tfrac{1}{4}\gamma^2).\cos.cv+\tfrac{3}{2}e^2.\cos.2cv-e^3\cos.3cv+\tfrac{1}{2}\gamma^2.\cos.2gv \\ -\tfrac{3}{4}e\gamma^2.\cos.(2gv-cv)-\tfrac{3}{4}e\gamma^2.\cos.(2gv+cv)\end{array}\right\}.$$

Now, changing the term $\frac{a^2}{\sqrt{a}}$ into $\frac{a^2}{\sqrt{a_{\prime}}}$ [5081o], we ought to get the factor which is independent of Q, in [5081 lines 1—4]; and, upon examination, we shall find they
[5081q] agree; except in some terms of the fourth order, connected with $\cos.2cv$, $\cos.2gv$, which were neglected in computing the function [5081p or 4828]. To prove this, we shall repeat the calculation [4821h—m]; *retaining only the terms which produce quantities of the fourth*
[5081r] *order* in e, γ, and are connected with the angles $2cv$, $2gv$. By this means, [4821i] becomes as in [5081u]; observing, that the last term arises from $+5(f+e.\cos.cv)^4$, which
[5081s] is omitted in [4821i]. Now, from $f=\frac{1}{4}\gamma^2-\frac{1}{4}\gamma^2.\cos.2gv$ [4821e], we obtain, by noticing only the angle $2gv$,

$$f=-\tfrac{1}{4}\gamma^2.\cos.2gv\,; \qquad\qquad f^2=-\tfrac{1}{8}\gamma^4.\cos.2gv\,.$$

[5081t] The first of these expressions ought to be changed into $f=-(\frac{1}{4}\gamma^2-\frac{1}{16}\gamma^4).\cos.2gv$, in order to notice the term of the fourth order, which was neglected in [1812a, b]. Finally, the term $-4(3fe^2.\cos.^2cv)$ gives, by noticing only the terms depending on $\cos.2gv$, $\cos.2cv$,

$$-6fe^2-6fe^2.\cos.2cv=\tfrac{3}{2}e^2\gamma^2.\cos.2gv-\tfrac{3}{2}e^2\gamma^2.\cos.2cv \quad [5081s].$$

Hence [5081u] becomes as in [5081v]; and, by substituting $\cos.^2cv=\frac{1}{2}\cos.2cv+\&c.$; $\cos.^4cv=\frac{1}{2}\cos.2cv+\&c.$ [6, 8] Int., we obtain [5081w];

[5081u] $$dt=h^3.(1+2\gamma^2).dv.\{-2f+3(e^2.\cos.^2cv+f^2)-4(3fe^2.\cos.^2cv)+5(e^4.\cos.^4cv)\}$$

[5081v] $$=h^3\,(1+2\gamma^2).dv.\left\{\begin{array}{l}2(\tfrac{1}{4}\gamma^2-\tfrac{1}{16}\gamma^4).\cos.2gv+3(e^2.\cos.^2cv-\tfrac{1}{8}\gamma^4.\cos.2gv \\ \qquad +\tfrac{3}{2}e^2\gamma^2.\cos.2gv-\tfrac{3}{2}e^2\gamma^2.\cos.2cv+5e^4.\cos.^4cv\end{array}\right\}$$

[5081w] $$=h^3.(1+2\gamma^2).dv.\{(\tfrac{3}{2}e^2-\tfrac{3}{2}e^2\gamma^2+\tfrac{5}{2}e^4).\cos.2cv+(\tfrac{1}{2}\gamma^2+\tfrac{3}{2}e^2\gamma^2-\tfrac{1}{2}\gamma^4).\cos.2gv\}.$$

The terms between the braces are of the *second* and higher orders; therefore, in finding the terms of this function, of the fourth order, we must obtain the factor $h^3.(1+2\gamma^2)$ correctly, in terms of the *second* order. This value is easily found from [4823]; which gives,

[5081x] $$h^3.(1+2\gamma^2)=a^{\frac{3}{2}}.(1-\tfrac{3}{2}e^2+\tfrac{1}{2}\gamma^2).$$

If the factor $1-\frac{3}{2}e^2+\frac{1}{2}\gamma^2$ be connected with the two terms of the second order in [5081w], it will produce some terms of the fourth order; and, by retaining terms of this order only, we obtain the expression [5081y], which is easily reduced to the form [5081z];

represented by,* [5081$''$]

$$dt = a^{\frac{3}{2}}.dv.\left\{ \begin{array}{c} \{\tfrac{3}{2}e^2(-\tfrac{3}{2}e^2+\tfrac{1}{2}\gamma^2)-\tfrac{3}{2}e^2\gamma^2+\tfrac{5}{2}e^4\}.\cos.2cv \\ +\{\tfrac{1}{2}\gamma^2.(-\tfrac{3}{2}e^2+\tfrac{1}{2}\gamma^2)+\tfrac{3}{2}e^2\gamma^2-\tfrac{1}{2}\gamma^4\}.\cos.2gv \end{array} \right\} \qquad [5081y]$$

$$= a^{\frac{3}{2}}.dv.\left\{ (\tfrac{1}{4}e^4-\tfrac{3}{4}e^2\gamma^2).\cos.2cv+(\tfrac{3}{4}e^2\gamma^2-\tfrac{1}{4}\gamma^4).\cos.2gv \right\}. \qquad [5081z]$$

The terms between the braces in this expression are the same as the terms of the fourth order in [5081 lines 2, 3]. Hence it is evident, that the development [5081] is correctly made.

* (2892) The function $\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$, whose powers and multiples occur in [5082a] [5081], has already been developed in [4881$'$, 4885, 4889, &c.], and in the variations of these quantities [4930, &c.]. If we put the function [4885] equal to M_2; and the [5082b] function [4889] equal to M_4, the indices denoting the order of the functions; we shall [5082b$'$] have $\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2} = M_2 + M_4$; whose square is $\frac{4}{h^4}.\left[\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}\right]^2 = M_2^2$; neglecting terms of the sixth order; hence, Q' [5081b] becomes,

$$Q' = 1-\frac{1}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}+\frac{3}{2h^4}.\left[\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}\right]^2 - \&c. = 1-\tfrac{1}{2}M_2-\tfrac{1}{2}M_4+\tfrac{3}{8}M_2^2. \qquad [5082c]$$

We must add to this value of Q' the terms arising from the variations of the function $-\frac{1}{2}M_2$; the variations of the other terms being so small, that they may be neglected. The chief term of the value of $-\frac{1}{2}M_2$ is that which is noticed in [4929, 4930]; namely,

$$\int\frac{3\,m'.\,u'^3.\,dv}{2\,h^2.\,u^4}.\sin.(2\,v-2\,v'); \qquad [5082d]$$

whose variation, relative to the characteristic δ, is evidently represented by,

$$-\frac{6m'}{h^2}.\int\frac{u'^3.\,dv}{u^4}.\left\{\frac{\delta u}{u}.\sin.(2v-2v')+\tfrac{1}{2}\delta v'.\cos.(2v-2v')\right\}; \qquad [5082e]$$

$\delta u'$ being neglected [5040a]. The function in the first member of [4931n], is developed in [4931p], and we shall put this last expression equal to N_4, and that in [4932a] equal to N_5; then we shall evidently have, for the two terms of the variation [5082e], the following expression; [5082f]

$$-\tfrac{1}{2}\,a.N_4-\tfrac{1}{2}\,a.N_5. \qquad [5082g]$$

The second variation of the same function $-\frac{1}{2}M_2$, is easily deduced from that of M_2u [4942], by dividing it by $-2u = -2a^{-1}$, nearly [4826]; using also [5082h]

[5082] $$\frac{a^2.dv}{\sqrt{a_{,}}}.\left\{1+\frac{27m^4}{64.(1-m)^2}+\frac{3m^2.A_2^{(0)}}{4(1-m)}+\tfrac{3}{2}.[(A_2^{(0)})^2+(A_1^{(1)}.e)^2]\right\} \quad [5092a].$$

[5082h'] $\frac{\overset{2}{\overline{m}}.a}{a_{,}}=m^2$ [5094]; whence we get,

[5082i] $$-\frac{15m^2}{4}.\frac{(A_1^{(1)})^2e^2.\cos.(2cv-2v+2mv)}{2c-2+2m}.$$

The other terms of $-\frac{1}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$, which are noticed in [4944, 4945], produce the following terms, which may be deduced from [4945 line 2], by dividing by $-2a^{-1}$, as in [5082h];

[5082k] $$-\frac{5\overset{2}{\overline{m}}.a}{8a_{,}}.\frac{a}{a'}.\int a\delta u.dv.\{3.\sin.(v-v')+15.\sin.(3v-3v')\}.$$

The terms resulting from this expression may be obtained in the same manner as [4946f'] is deduced from [4945 line 2]; or, more simply, by dividing [4946f] by $-2a^{-1}$ [5082h]. By this means it becomes, by using the value of m^2 [5082h'],

[5082l] $$\frac{15m^2}{4}.\frac{a}{a'}.A_2^{(0)}.\frac{\cos.(v-mv)}{1-m}.$$

Now, adding together the functions [5082c, g, i, l], we obtain,

[5082m] $$\begin{aligned}Q'&=1-\frac{1}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}+\frac{3}{2h^4}.\left(\int\frac{dQ}{dv}.\frac{dv}{u^2}\right)^2-\&c.\\&=1-\tfrac{1}{2}M_2-\tfrac{1}{2}M_4+\tfrac{3}{8}M_2^2-\tfrac{1}{2}a.N_4-\tfrac{1}{2}a.N_5\\&\quad-\frac{15m^2}{4}.\frac{(A_1^{(1)})^2e^2.\cos.(2cv-2v+2mv)}{2c-2+2m}\\&\quad+\frac{15m^2}{4}.\frac{a}{a'}.A_2^{(0)}.\frac{\cos.(v-mv)}{1-m}.\end{aligned}$$

Substituting m^2 [5082h'], in the value of M_2 [4885, 5082b], and neglecting terms of
[5082m'] the *second* order, between the braces, which produce only terms of the *sixth* order in M_2^2, it becomes of the form,

[5082n] $$M_2=3m^2.\left\{\frac{1}{2-2m}.\cos.(2v-2mv)+\Sigma P_{,}.\cos.(2v-2mv+T)\right\};$$

as is evident, by mere inspection; the symbol $P_{,}$ being the coefficient of the first order of any term between the braces in [4885], and $2v-2mv+T$ the corresponding

The coefficient of dv, in this function, is not rigorously constant. [5082′]

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angle. The square of this gives, by neglecting $P_{\prime}^2$, and the angles $4v-4mv$, $4v-4mv+V$,

$$\tfrac{3}{8}M_2^2=\frac{27m^4}{8}\cdot\left\{\frac{1}{2(2-2m)^2}+\frac{1}{2-2m}\cdot\Sigma P_{\prime}\cdot\cos.V\right\}$$
$$=\frac{27m^4}{16(1-m)}\cdot\left\{\frac{1}{4(1-m)}+\Sigma P_{\prime}\cdot\cos.V\right\}.$$ [5082o]

Now, it is evident, by inspection, that the terms between the braces in this last expression, are easily derived from those between the braces in [4885], by rejecting $2v-2mv$ from all the angles, and taking half of the first term in [4885 line 1]; hence we get,

$$\tfrac{3}{8}M_2^2=\frac{27m^4}{16(1-m)}\cdot\left\{\begin{array}{l}\frac{1}{4(1-m)}-\left(\frac{2(1+m)}{2-2m-c}+\frac{2(1-m)}{2-2m+c}\right)\cdot c\cdot\cos.cv\\ \qquad+\left(\frac{7}{2-3m}-\frac{1}{2-m}\right)\cdot\tfrac{1}{2}c'\cdot\cos.c'mv\end{array}\right\}.$$ [5082p]

Substituting this in [5082m], and for M_2, M_4, N_4, N_5, writing the functions to which they correspond [5082b, b′, f], we obtain,

$$Q'=1-\frac{1}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}+\frac{3}{2h^4}\cdot\left[\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}\right]^2-\&c.$$ 1

$$=1-\tfrac{1}{2}.\text{function }[4885]-\tfrac{1}{2}.\text{function }[4889]-\tfrac{1}{2}a.\text{function }[4931p]-\tfrac{1}{2}a.\text{function }[4932a]$$ 2

$$+\frac{27m^4}{16(1-m)}\cdot\left\{\begin{array}{l}\frac{1}{4(1-m)}-\left(\frac{2(1+m)}{2-2m-c}+\frac{2(1-m)}{2-2m+c}\right)\cdot c\cdot\cos.cv\\ \qquad+\left(\frac{7}{2-3m}-\frac{1}{2-m}\right)\cdot\tfrac{1}{2}c'\cdot\cos.c'mv\end{array}\right\}$$ 3

[5082q]

$$-\frac{15m^2}{4}\cdot\frac{(A_1^{(1)})^2e^2\cdot\cos.(2cv-2v+2mv)}{2c-2+2m}$$ 4

$$+\frac{15m^2}{4}\cdot\frac{a}{a'}\cdot A_2^{(0)}\cdot\frac{\cos.(v-mv)}{1-m}.$$ 5

The expression is now reduced to so simple a form, that we can, by the mere addition of the terms, obtain the complete value of Q', as in the following table; rejecting such terms and angles as have been usually omitted; and putting [5082r]

$$\overset{0}{m}\cdot\frac{a}{a_{\prime}}=m^2,\quad\text{as in [5082h′]};$$

[5082r] We have seen, in [4968], that the expression of $\frac{1}{a}$ contains the term

	Terms of [5082q].	Corresponding terms of $Q'=1-\frac{1}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}+\frac{3}{2h^4}\left(\int\frac{dQ}{dv}.\frac{dv}{u^2}\right)^2-\&c.$	
	[5082q lines 2, 3]	$1+\frac{27}{64}\cdot\frac{m^4}{(1-m)^2}$ [This line has no factor.]	1
	[4885 line 1]	$-\frac{(1+2e^2-\frac{5}{2}e'^2)}{2-2m}.\cos.(2v-2mv)$	2
	[4885 line 2]	$+\frac{2(1+m)}{2-2m-c}.(1+\frac{3}{4}e^2-\frac{1}{4}\gamma^2-\frac{5}{2}e'^2).e.\cos.(2v-2mv-cv)$	3
	[4885 line 3]	$+\frac{2(1-m)}{2-2m+c}.e.\cos.(2v-2mv+cv)$	4
	[4885 line 4]	$-\frac{7e'}{2(2-3m)}.\cos.(2v-2mv-c'mv)$ All the terms except the first line have the common factor $\frac{3}{2}m^2$.	5
	[4885 line 5]	$+\frac{e'}{2(2-m)}.\cos.(2v-2mv+c'mv)$	6
	[4885 line 6]	$+\frac{7(2+3m).ee'}{2(2-3m-c)}.\cos.(2v-2mv-cv-c'mv)$	7
Expression of Q'.	[4885 line 7]	$+\frac{7(2-3m).ee'}{2(2-3m+c)}.\cos.(2v-2mv+cv-c'mv)$	8
	[4885 line 8]	$-\frac{(2+m).ee'}{2(2-m-c)}.\cos.(2v-2mv-cv+c'mv)$	9
	[4885 line 10]	$+\left\{\begin{array}{l}\frac{1}{4}(10+19m+8m^2)\\-2A_2^{(10)}\\-\frac{5}{2}(A_1^{(1)})^2\end{array}\right\}.\frac{e^2}{2c-2+2m}.\cos.(2cv-2v+2mv)$	10
[5082s]	[4931p line 24]		11
	[5082q line 4]		12
	[4885 line 12]	$+\left\{\begin{array}{l}\frac{1}{4}(2+m)\\-2A_2^{(12)}\end{array}\right\}.\frac{\gamma^2}{2g-2+2m}.\cos.(2gv-2v+2mv)$	13
	[4931p line 26]		14
	[4885 line 13]	$-\frac{(2-m).\gamma^2}{4(2g+2-2m)}.\cos.(2gv+2v-2mv)$	15
	[4885 line 14]	$-\frac{17e'^2}{2(2-4m)}.\cos.(2v-2mv-2c'mv)$	16
	[4885 line 15]	$+\left\{\begin{array}{l}\frac{1}{4}(5+m)\\+2A_0^{(15)}\end{array}\right\}.\frac{e\gamma^2}{2-2m-2g+c}.\cos.(2v-2mv-2gv+cv)$	17
	[4931p line 29]		18
	[4889 line 1]	$+\left\{\begin{array}{l}-\frac{1}{4}(1+\frac{7}{2}e^2-\frac{1}{4}\gamma^2+2e'^2)\\+2A_1^{(7)}\\+2m.A_1^{(17)}\\\text{....neglected}\end{array}\right\}.\frac{a}{a'}.\frac{\cos.(v-mv)}{1-m}$	19
	[4931p line 31]		20
	[4932a line 3]		21
	[5082q line 5]		22
	[4831p lines 39, 13]	$+\left\{\begin{array}{l}-8A_2^{(0)}+20A_1^{(1)}e^2-2A_2^{(3)}\\+2A_2^{(1)}+5A_1^{(6)}e^2-5A_1^{(7)}e^2\\+\frac{9}{16}.\frac{m^3}{1-m}.\left\{\frac{7}{2-3m}-\frac{1}{2-m}\right\}\end{array}\right\}.\frac{e'}{m}.\cos.c'mv$	23
	[4831p lines 14,17,20]		24
	[5082q line 3]		25
	[4931p lines 7, 16]	$+\{7A_1^{(1)}+2A_1^{(6)}\}.\frac{ee'}{1-m}.\cos.(cv-c'mv)$	26
	[4931p line 6]	$+2A_1^{(1)}e.\cos.cv.$	27

$-\frac{3\overline{m}^2.c'^2}{4a_{\prime}}$; which gives, in a^2, the term* $\frac{3}{2}\overline{m}^2 a_{\prime}^2.e'^2$: thus, the quantity [5083]
$\frac{a^2.dv}{\sqrt{a_{\prime}}}$ contains the term $\frac{3}{2}a_{\prime}^{\frac{3}{2}}.dv.\overline{m}^2.e'^2$ [5083d]; now we have nearly, [5084]

$$a_{\prime}^{\frac{3}{2}}=\frac{1}{n},\qquad \overline{m}^2=m^2\qquad [5092', 5093];$$ [5085]

therefore, the expression of the time t contains the term $\frac{3m^2}{2n}.\int e'^2.dv$; [5086]
consequently, the value of the moon's true longitude, in terms of the mean

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We have omitted, in the preceding table, several terms on account of their smallness. Thus, we have neglected, in line 7, the terms depending on $A_1^{(5)}$ [4931p line 22]; in line 9, the terms depending on $A_1^{(9)}$ [4931p line 23]; in line 20, the terms depending on $A_0^{(18)}$, λ_2 [4931p lines 35, 37]; in line 27, several terms of [4931p, 5082q], of the fifth and sixth orders. Besides these, there are others depending on the angles

$$2v-2mv+cv+c'mv,\quad 2cv+2v-2mv,\quad v-mv\pm c'mv,\quad 2gv-cv,\quad cv+c'mv,\quad 2cv.$$ [5082t]

These are neglected, because the terms are of the fifth or sixth order, or are connected with angles which do not increase the coefficients by integration in finding t, from dt [5081]. In the terms depending on $\cos.(v-mv)$, we have retained the terms depending on $A_1^{(17)}$ [5082s line 21], and neglected a term of the same order, depending on $A_2^{(0)}$ [5082s line 22]. [5082u]
This is done, because $A_1^{(17)}$ is required to a great degree of accuracy in [4871, 4901 line 18].
The function Q' [5082s] is to be substituted in [5081], and then we may obtain the [5082v]
constant terms [5082], as we shall see in note 2898 [5090a].

* (2893) If we put, for a moment, $\frac{m_{\prime}}{a_{\prime}}$ to represent the terms of the second member of [4968], exclusive of the first and third, we shall have,

$$\frac{1}{a}=\frac{1}{a_{\prime}}.\left(1-\frac{3\overline{m}^2}{4}.c'^2+m_{\prime}\right).$$ [5083a]

The quantity $m_{\prime}$ contains another term, depending on c'^2, of the order $\overline{m}^2.A_2^{(0)}c'^2$,
which may be neglected, in comparison with the retained term $-\frac{3\overline{m}^2}{4}e'^2$ [5083a]. Involving [5083b]
[5083a] to the power -2, we get,

$$a^2=a_{\prime}^2.\left(1+\tfrac{3}{2}\overline{m}^2.c'^2+\&c.\right);\quad \text{hence,}\quad \frac{a^2.dv}{\sqrt{a_{\prime}}}=a_{\prime}^{\frac{3}{2}}.dv.\left(1+\tfrac{3}{2}\overline{m}^2.c'^2+\&c.\right).$$ [5083c]

[5087] longitude, contains the term $-\frac{3}{2}m^2.\int e'^2.dv$, or $-\frac{3}{2}m^2.\int e'^2.n\,dt$. *Hence*
Ratio of the secular motions of the longitude, perigee and nodes.
[5088] *it follows, that the three secular equations of the mean longitude of the moon, its perigee and its nodes, are to each other as the three quantities**

[5089] $$3\,\overline{m}^2, \qquad \frac{-q}{\sqrt{1-p}}, \qquad \frac{q''}{\sqrt{1+p''}}.$$

It is true,† that the terms, depending on the square of the disturbing force,

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This expression contains the term $\frac{3}{2}a_{\prime}^{\frac{3}{2}}.dv.\overline{m}^2.e'^2$, as in [5084], and by using the values
[5083d] [5085], it is reduced to the form $\frac{3m^2}{2n}.e'^2.dv$, which evidently represents the chief term, depending on e'^2, in the value of dt [5081]; and, by integration, we get, in t, the
[5083e] term $\frac{3m^2}{2n}.\int e'^2.dv$ [5086]. Changing its sign, and multiplying by n, we evidently obtain
[5083f] the corresponding expression in the moon's apparent longitude v [5095], $-\frac{3m^2}{2}.\int e'^2.dv$; which becomes $-\frac{3m^2}{2}\int e'^2.ndt$ [5087], by substituting the mean value of $dv=n\,dt$ [4828].

* (2894) The secular equation of the moon's longitude is

$$-\tfrac{3}{2}m^2.\int e'^2.dv \quad [5087];$$

[5089a] that of the perigee is

$$\tfrac{1}{2}.\frac{-q}{\sqrt{(1-p)}}.\int e'^2.dv \quad [4982, 4979];$$

and, that of the nodes is

[5089b] $$-\tfrac{1}{2}.\frac{q''}{\sqrt{(1+p'')}}.\int e'^2.dv \quad [5060a—5061a].$$

Dividing these three expressions by the common factor $-\frac{1}{2}.\int e'^2.dv$, we find, that these three secular motions are to each other as the quantities

$$3\,m^2, \qquad -\frac{q}{\sqrt{(1-p)}}, \qquad \frac{q''}{\sqrt{(1+p'')}}, \qquad \text{as in [5089]}.$$

† (2895) We shall, in this note, make some developments of the functions which occur in [5081], preparatory to the calculation of the values of $C_1^{(0)}$, $C_0^{(1)}$, &c. [5096—5116].

[5090a] We shall commence with the computation of the terms of the *first part* of dt, or that which is independent of $a\delta u$, and arises from the product of the two factors included in [5081 lines 1—4]. These are found in the following table, which does not require any particular explanation;

produce a little alteration in the secular equation of the mean longitude ; [5089′]

Terms of the first factor in [5081], between the braces.	Factor Q' [5081 or 5082s].	Corresponding terms of [5081].		
1	whole of [5082s]	whole function [5082s] *multiplied by* $\frac{a^2.dv}{\sqrt{a_{,}}}$	1	
$-2e.(1-\frac{1}{4}\gamma^2).\cos.cv$	1	$-2e(1-\frac{1}{4}\gamma^2).\cos.cv$	2	
	[5082s line 2]	$\frac{3}{4}\frac{m^2}{1-m}.e(1+2e^2-\frac{1}{4}\gamma^2-\frac{5}{2}e'^2).\left\{\begin{array}{l}+\cos(2v-2mv-cv)\\+\cos(2v-2mv+cv)\end{array}\right\}$	3 4	
	[5082s line 3]	$\frac{3m^2(1+m)}{2-2m-c}(1+\frac{3}{4}e^2-\frac{1}{2}\gamma^2-\frac{5}{2}e'^2)e^2.\left\{\begin{array}{l}-\cos(2v-2mv)\\-\cos(2v-2mv-2cv)\end{array}\right\}$	5 6	
	[5082s line 4]	$-\frac{3m^2(1-m)}{2-2m+c}.e^2.\cos.(2v-2mv)$	7	
	[5082s line 5]	$+\frac{3}{4}m^2.\frac{7}{2-3m}.ce'.\cos.(2v-2mv-cv-c'mv)$	8	First part of the expression of dt.
	[5082s line 6]	$-\frac{3}{4}\frac{m^2}{2-m}.ce'.\cos.(2v-2mv-cv+c'mv)$	9	
	[5082s line 10]	$-\frac{3m^2.(10+19m+8m^2).e^3}{8.(2c-2+2m)}.\cos.(2v-2mv-cv)$	10	[5090b]
$e^2.(\frac{3}{2}+\frac{1}{4}e^2-\frac{3}{4}\gamma^2).\cos.2cv$	1	$+(\frac{3}{2}+\frac{1}{4}e^2-\frac{3}{4}\gamma^2).e^2.\cos.2cv$ All these terms have the common factor $\frac{a^2.dv}{\sqrt{a_{,}}}$.	11	
$\frac{3}{2}e^2.\cos.2cv$	$-\frac{3m^2}{4(1-m)}\cos(2v-2mv)$	$-\frac{9m^2}{16(1-m)}.e^2.\cos.(2cv-2v+2mv)$	12	
$\frac{1}{2}\gamma^2(1+\frac{3}{2}e^2-\frac{1}{2}\gamma^2)\cos.2gv$	1	$+(1+\frac{3}{2}e^2-\frac{1}{2}\gamma^2).\frac{1}{2}\gamma^2.\cos.2gv$	13	
	[5082s line 2]	$-\frac{3m^2\gamma^2}{16(1-m)}.(1+\frac{7}{2}e^2-\frac{1}{2}\gamma^2-\frac{5}{2}e'^2).\cos.(2gv-2v+2mv)$	14	
$-e^3.\cos.3cv$	1	$-e^3.\cos.3cv$	15	
$-\frac{3}{4}e\gamma^2.\cos.(2gv-cv)$	1	$-\frac{3}{4}e\gamma^2.\cos.(2gv-cv)$	16	
$-\frac{3}{4}e\gamma^2.\cos.(2gv+cv)$	1	$-\frac{3}{4}e\gamma^2.\cos.(2gv+cv)$.	17	

In the next place, we shall compute the *second part* of the value of dt, depending on $a\,\delta u$, which is contained in [5081 lines 5, 6]. Now, $a\delta u$ is of the *second* order; therefore, in calculating the product of the two factors by which $a\delta u$ is multiplied, we [5090c] shall not want any terms beyond the *fourth* order, and, in general, it will suffice to compute them to the second or third order. We shall find, in the following table, the product of the

[5089''] but, it is evident, that the terms which have a very sensible

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two factors of $-2a\delta u$ [5081 lines 5, 6]; or, in other words, the product of the expression Q' [5082s], by the following function, contained in [5081 lines 5, 6]; namely,

[5090d] $$1+\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2-3e.\cos.cv+3e^2.\cos.2cv+\tfrac{3}{4}\gamma^2.\cos.2gv-\tfrac{3}{2}e\gamma^2.\cos.(2gv-cv).$$

[5090e]

Terms of [5090d].	Factor Q' [5082s].	Corresponding terms of Q', multiplied by the factor [5090d].	
1	Q'	whole function [5082s]	1
$\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2$	1	$\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2$	2
$-3e.\cos.cv$	1	$-3e.\cos.cv$	3
	$-\frac{3m^2}{2}.\frac{(1+2e^2-\frac{5}{2}e'^2)}{2(1-m)}\cos(2v-2mv)$	$+\frac{9m^2}{8(1-m)}.(1+2e^2-\tfrac{5}{2}e'^2)e.\left\{\begin{array}{l}+\cos(2v-2mv-cv)\\+\cos(2v-2mv+cv)\end{array}\right\}$	4 5
	$\frac{3m^2}{2}.\frac{2(1+m)e}{2-2m-c}\cos(2v-2mv-cv)$	$+\frac{9m^2.(1+m)}{2(2-2m-c)}.e^2.\left\{\begin{array}{l}-\cos.(2v-2mv)\\-\cos.(2v-2mv-2cv)\end{array}\right\}$	6 7
	$\frac{3m^2}{2}.\frac{2(1-m)e}{2-2m+c}.\cos(2v-2mv+cv)$	$+\frac{9m^2.(1-m)}{2(2-2m+c)}.e^2.\left\{\begin{array}{l}-\cos.(2v-2mv)\\-\cos.(2v-2mv+2cv)\end{array}\right\}$	8 9
$3e^2.\cos.2cv$	1	$+3e^2.\cos.2cv$	
	$-\frac{3m^2}{2}.\frac{1}{2(1-m)}.\cos.(2v-2mv)$	$+\frac{9m^2}{8(1-m)}.e^2.\left\{\begin{array}{l}-\cos.(2v-2mv+2cv)\\-\cos.(2v-2mv-2cv)\end{array}\right\}$	10 11
$\tfrac{3}{4}\gamma^2.\cos.2gv$	1	$+\tfrac{3}{4}\gamma^2.\cos.2gv$	
	$-\frac{3m^2}{2}.\frac{1}{2(1-m)}.\cos(2v-2mv)$	$+\frac{9m^2}{32(1-m)}.\gamma^2.\left\{\begin{array}{l}-\cos.(2gv-2v+2mv)\\-\cos.(2gv+2v-2mv)\end{array}\right\}$	12 13

This function [5090e] is to be multiplied by $-2a\delta u.\frac{a^2.dv}{\sqrt{a_i}}$, to obtain the second part of dt, contained in [5081 lines 5, 6]. This process is performed in the following table

[5090f] [5090g]. In the first column are given the terms of $-2a\delta u$ [1904]; in the second, the terms of [5090e], which includes, in its first line, the function [5082s]; these terms are taken in the same order in which they first occur in [5082s], and then in [5090e lines 2—13], omitting those terms and angles which are usually rejected;

effect on the equation of the perigee, have but a very small and [5089‴]

Terms of $-2a\delta u$ [4904].	Terms of [5090e].	Terms of [5081 lines 5, 6].		
whole of $-2a\delta u$	1	$-2a\delta u$ [4904] All these terms have the common factor $\frac{a^2.dv}{\sqrt{a_{\prime}}}$.	1	
$-2A_2^{(0)}.\cos.(2v-2mv)$	$-\frac{3m^2}{4(1-m)}.\cos.(2v-2mv)$	$\frac{3m^2}{4(1-m)}.A_2^{(0)}$	2	
	$-\frac{21m^2e'}{4(2-3m)}\cos(2v-2mv-c'mv)$	$\frac{3m^2.A_2^{(0)}}{4}.\left\{+\frac{7}{2-3m}\right.$	3	
	$\frac{3m^2e'}{4(2-m)}.\cos(2v-2mv+c'mv)$	$\left.-\frac{1}{2-m}\right\}.e'\cos.c'mv$	4	
	$\frac{1}{2}e^2-\frac{1}{4}\gamma^2$	$-2A_2^{(0)}.\{\frac{1}{2}e^2-\frac{1}{4}\gamma^2\}.\cos(2v-2mv)$	5	
	$-3e.\cos.cv$	$+3A_2^{(0)}e.\{+\cos(2v-2mv-cv)$	6	
		$+\cos(2v-2mv+cv)\}$	7	
	$+3e^2.\cos.2cv$	$-3A_2^{(0)}e^2.\{+\cos(2cv-2v+2mv)$	8	
		$+\cos(2cv+2v-2mv)\}$	9	
	$+\frac{3}{4}\gamma^2.\cos.2gv$	$-\frac{3}{4}A_2^{(0)}\gamma^2.\cos.(2gv-2v+2mv)$	10	
$-2A_1^{(1)}e.\cos.(2v-2mv-cv)$	$-\frac{3m^2}{4(1-m)}.\cos(2v-2mv)$	$\frac{3m^2}{4(1-m)}.A_1^{(1)}e.\cos.cv$	11	Second part of the expression of dt.
	$\frac{1}{2}e^2-\frac{1}{4}\gamma^2$	$-2A_1^{(1)}.(\frac{1}{2}e^2-\frac{1}{4}\gamma^2).e.\cos(2v-2mv-cv)$	12	
	$-3e.\cos.cv$	$+3A_1^{(1)}e^2.\{+\cos(2v-2mv)$	13	
		$+\cos(2cv-2v+2mv)\}$	14	
	$+3e^2.\cos.2cv$	$-3A_1^{(1)}e^3.\cos.(2v-2mv+cv)$	15	
$-2A_2^{(2)}e.\cos.(2v-2mv+cv)$	$-3e.\cos.cv$	$+3A_2^{(2)}e^2.\cos.(2v-2mv)$	16	
$-2A_2^{(3)}e'\cos(2v-2mv+c'mv)$	$-\frac{3m^2}{4(1-m)}.\cos.(2v-2mv)$	$+\frac{3m^2}{4(1-m)}.A_2^{(3)}e'.\cos.c'mv$	17	
	$-3e.\cos.cv$	$+3A_2^{(3)}ee'\cos(2v-2mv-cv+c'mv)$	18	[5090g]
$-2A_2^{(4)}e'\cos(2v-2mv-c'mv)$	$-\frac{3m^2}{4(1-m)}.\cos.(2v-2mv)$	$+\frac{3m^2}{4(1-m)}.A_2^{(4)}e'.\cos.c'mv$	19	
	$-3e.\cos.cv$	$+3A_2^{(4)}ee'.\cos(2v-2mv-cv-c'mv)$	20	
$-2A_2^{(5)}e'.\cos.c'mv$	$-3e.\cos.cv$	$+3A_2^{(5)}ee'.\{+\cos.(cv+c'mv)$	21	
		$+\cos.(cv-c'mv)\}$	22	
	$\frac{1}{2}e^2-\frac{1}{4}\gamma^2$	$-2A_2^{(5)}.(\frac{1}{2}e^2-\frac{1}{4}\gamma^2).e'.\cos.c'mv$	23	
$-2A_1^{(6)}ee'\cos(2v-2mv-cv+c'mv)$	$-3e.\cos.cv$	$+3A_1^{(6)}e^2e'.\cos(2v-2mv+c'mv)$	24	
$-2A_1^{(7)}ee'\cos(2v-2mv-cv-c'mv)$	$-3e.\cos.cv$	$+3A_1^{(7)}e^2e'.\cos(2v-2mv-c'mv)$	25	
$-2A_1^{(8)}ee'.\cos.(cv+c'mv)$	$-3e.\cos.cv$	$+3A_1^{(8)}e^2e'.\cos.c'mv$	26	
$-2A_1^{(9)}ee'.\cos.(cv-c'mv)$	$-3e.\cos.cv$	$+3A_1^{(9)}e^2e'.\cos.c'mv$	27	
$-2A_1^{(11)}e^2\cos(2cv-2v+2mv)$	$-3e.\cos.cv$	$+3A_1^{(11)}e^3.\cos.(2v-2mv-cv)$	28	
$-2A_0^{(15)}e\gamma^2.\cos.(2gv-cv)$	$-3e.\cos.cv$	$+3A_0^{(15)}e^2\gamma^2.\cos.2gv$	29	
$-2A_1^{(16)}e\gamma^2\cos(2v-2mv-2gv+cv)$	$-3e.\cos.cv$	$+3A_1^{(16)}e^2\gamma^2.\cos(2gv-2v+2mv)$	30	
$-2A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$	$-\frac{3m^2}{4(1-m)}.\cos.(2v-2mv)$	$+\frac{3m^2}{4(1-m)}.A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$	31	
	$\frac{1}{2}e^2-\frac{1}{4}\gamma^2$	$-2A_1^{(17)}.(\frac{1}{2}e^2-\frac{1}{4}\gamma^2).\frac{a}{a'}.\cos.(v-mv)$	32	

[5089iv] insensible effect on that of the mean motion [5090q, &c.]*.

The *third part* of dt, which depends on the second power of $a\delta u$, is contained in [5081 line 7], and, by neglecting terms of the sixth order, it may be put under the form,

[5090h] $$\frac{a^2.dv}{\sqrt{a_{\prime}}}\cdot\left\{3.(a\delta u)^2-3.(a\delta u)^2.4\,e.\cos.cv\right\}.$$

We shall, in the first place, compute the first of these terms, by means of [4904], as in the following table;

Third part of the expression of dt.

[5090i]

Terms of $a\delta u$ [4904].	Terms of $3.a\delta u$ [4904].	Corresponding terms of the function [5090h or 5081 line 7].	
$A_2^{(0)}.\cos.(2v-2mv)$	$3A_2^{(0)}.\cos.(2v-2mv)$	$\frac{3}{2}(A_2^{(0)})^2$	1
	$3A_1^{(1)}e.\cos.(2v-2mv-cv)$	$\frac{3}{2}A_2^{(0)}.A_1^{(1)}.e.\cos.cv$	2
	$3A_2^{(3)}e'.\cos.(2v-2mv+c'mv)$	$\frac{3}{2}A_2^{(0)}.A_2^{(3)}e'.\cos.c'mv$	3
	$3A_2^{(4)}e'.\cos.(2v-2mv-c'mv)$	$\frac{3}{2}A_2^{(0)}.A_2^{(4)}e'.\cos.c'mv$	4
	$3A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$	$\frac{3}{2}A_2^{(0)}.A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$	5
$A_1^{(1)}e.\cos.(2v-2mv-cv)$	$3A_2^{(0)}.\cos.(2v-2mv)$	$\frac{3}{2}A_2^{(0)}.A_1^{(1)}e.\cos.cv$	6
	$3A_1^{(1)}e.\cos.(2v-2mv-cv)$	$\frac{3}{2}(A_1^{(1)})^2e^2$	7
	$3A_1^{(6)}ee'\cos(2v-2mv-cv+c'mv)$	$\frac{3}{2}A_1^{(1)}.A_1^{(6)}e^2e'\cos.c'mv$	8
	$3A_1^{(7)}ee'.\cos.(2v-2mv-cv-c'mv)$	$\frac{3}{2}A_1^{(1)}.A_1^{(7)}e^2e'.\cos.c'mv$	9
$A_2^{(3)}e'.\cos.(2v-2mv+c'mv)$	$3A_2^{(0)}.\cos.(2v-2mv)$	$\frac{3}{2}A_2^{(0)}.A_2^{(3)}e'.\cos.c'mv$	10
$A_2^{(4)}e'.\cos.(2v-2mv-c'mv)$	$3A_2^{(0)}.\cos.(2v-2mv)$	$\frac{3}{2}A_2^{(0)}.A_2^{(1)}c'.\cos.c'mv$	11
$A_1^{(6)}ee'\cos(2v-2mv-cv+c'mv)$	$3A_1^{(1)}e.\cos.(2v-2mv-cv)$	$\frac{3}{2}A_1^{(1)}.A_1^{(6)}e^2e'.\cos.c'mv$	12
$A_1^{(7)}ee'.\cos.(2v-2mv-cv-c'mv)$	$3A_1^{(1)}e.\cos.(2v-2mv-cv)$	$\frac{3}{2}A_1^{(1)}.A_1^{(7)}e^2e'.\cos.c'mv$	13
$A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$	$3A_2^{(0)}.\cos.(2v-2mv)$	$\frac{3}{2}A_2^{(0)}.A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv).$	14

All these terms have the factor $\frac{a^2.dv}{\sqrt{a_{\prime}}}$.

This table contains the development of the *first* term of [5090h], $\frac{a^2.dv}{\sqrt{a_{\prime}}}.3.(a\delta u)^2$. The

[5090k] second term of [5090h] is deduced from the preceding, by multiplying it by $-4e.\cos.cv$; but, we may neglect this part, because it produces only terms of the fifth and higher orders, and of the forms which have been usually neglected.

In computing the part of Q', [5082e—g], we have neglected the term depending on

The part of $\frac{dt}{dv}$, which is not periodical, is equal to $\frac{1}{n}$ [4828,&c.5095]; [5090]

$\delta u'$ or C [5082e,&c.,4937d]; also, that part of $\delta v'$, which depends on the same quantity C [4937c, h]. We shall now compute the effect of these terms, noticing only those arising from the variation of the quantity $\int \frac{3m'.u'^3\,dv}{2h^2.u^4}.\sin.(2v-2v')$ [5082d], which is the most important part. From this we get, by taking the variation relative to $\delta u'$, $\delta v'$, [5090l]

$$-\frac{6m'}{2h^2}.\int\frac{u'^3.dv}{u^4}.\delta v'.\cos.(2v-2v')+\frac{9m'}{2h^2}.\int\frac{u'^2.\delta u'}{u^4}.dv.\sin.(2v-2v'). \qquad [5090m]$$

These two terms of Q' are equal to the product of the two integrals in [4937e] by $-\frac{1}{2}a$. [5090m'] Now, the terms of [4937e] are developed in [4937m, q]; and their sum, reduced as in [4937r, &c.], becomes, by retaining only the most important terms, which increase by integration,

$$-\frac{3\overline{m}^2}{a_{\prime}}.\{\tfrac{11}{2}C_2^{(6)}+C_2^{(9)}-C_2^{(10)}\}.e'.\cos.c'mv. \qquad [5090n]$$

Multiplying this by the factor $-\frac{1}{2}a$ [5090m'], and substituting, for $\overline{m}^2.\frac{a}{a_{\prime}}$, its value m^2 [5082h'], we get the following expression of these terms of Q' [5090m]; namely,

$$\tfrac{3}{2}m^2.\{\tfrac{11}{2}C_2^{(6)}+C_2^{(9)}-C_2^{(10)}\}.e'.\cos.c'mv. \qquad [5090o]$$

This is to be multiplied by the common factor $\frac{a^2.dv}{\sqrt{a_{\prime}}}$ [5081], and the product added to the other terms of the second member of this value of dt. Hence, the complete value of dt is found, by connecting together the terms of [5090b, g, i, o]. This may be reduced to the following form;

$$dt = \text{function [5082s]}\times\frac{a^2.dv}{\sqrt{a_{\prime}}}+\text{function [5090b omitting line 1]} \qquad 1$$

$$-\text{function [4904]}\times 2.\frac{a^2.dv}{\sqrt{a_{\prime}}}+\text{function [5090g omitting line 1]} \qquad 2 \quad [5090p]$$

$$+\text{function [5090i]}+\frac{a^2.dv}{\sqrt{a_{\prime}}}.\tfrac{3}{2}m^2.\{\tfrac{11}{2}C_2^{(6)}+C_2^{(9)}-C_2^{(10)}\}.e'.\cos.c'mv. \qquad 3$$

We shall use this expression in the rest of this article, always taking the functions in the same order in which they occur in [5090p].

* (2896) In finding the chief part of the secular equation of the mean motion in [5083,&c.] we have only noticed the first term $\frac{a^2.dv}{\sqrt{a_{\prime}}}$ of the non-periodical part of dt [5082], and have neglected the remaining terms of the fourth order, which are evidently much less than the retained part. [5090q] But this is not the case with the terms, on which the secular motion of the perigee depends [4982, 4979], since the term of q [4974, &c.], of the fourth order, [5090r] depending on the square of the disturbing force, is as great as any other of the retained quantities. This is evident, by the inspection of the coefficient of $c.\cos.cv$ [4961], upon which $-p-qe'^2$ depends [4975]. For, the term depending on $A_1^{(1)}$ [5090s]

[5090'] and, if we neglect quantities of the order m^4, this coefficient will be * $\frac{a^2}{\sqrt{a_{,}}}$.

[5091] We then have $\frac{1}{a}=\frac{1}{a_{,}}.(1-\frac{1}{2}m^2)$ [4968]; which gives $\frac{a}{a_{,}}=1+\frac{1}{2}m^2$, and

[5092] $\frac{a^2}{\sqrt{a_{,}}}=\frac{1}{n}=a^{\frac{3}{2}}.(1+\frac{1}{4}m^2)$†. Moreover, we have, by [605'], $n'=a'^{-\frac{3}{2}}.\sqrt{m'}$; therefore,‡

[4961 line 5, 4999, 5158], which arises from the square of the disturbing force, is as large as the other terms of q.

* (2897) If we collect together the terms of dt [5090p], which are independent of the cosines, we shall find, that they all have the common factor $\frac{a^2.dv}{\sqrt{a_{,}}}$; and the quantities connected with this factor are,

[5092a]
$$1+\frac{27m^4}{64.(1-m)^2}\ \ [5082s \text{ line } 1];\quad \frac{3m^2}{4(1-m)}.A_2^{(0)}\ \ [5090g \text{ line } 2];$$
$$\tfrac{3}{2}(A_2^{(0)})^2+\tfrac{3}{2}(A_1^{(1)})^2.e^2\ \ [5090i \text{ lines } 1, 7];$$

whose sum is as in [5082]. At the epoch, the constant term of $\frac{dt}{dv}$, assumed in [4828]

[5092b] or [5095], is $\frac{1}{n}$; putting this equal to the factor of dv, in [5082], and neglecting terms of

[5092c] the fourth order, we get $\frac{1}{n}=\frac{a^2}{\sqrt{a_{,}}}$ [5090, 5090'].

† (2898). Neglecting terms of the fourth order, as in the last note, we have, from

[5093a] [4968], $\frac{1}{a}=\frac{1}{a_{,}}.(1-\frac{1}{2}m^2)$, as in [5091]. This gives $\frac{a}{a_{,}}=1+\frac{1}{2}m^2$; whose square root, multiplied by $a^{\frac{3}{2}}$, is

[5093b]
$$\frac{a^2}{\sqrt{a_{,}}}=a^{\frac{3}{2}}.(1+\tfrac{1}{4}m^2)=\frac{1}{n}\ \ [5090'], \text{ as in } [5092].$$

Now, by neglecting, in [605'], the mass of the earth, in comparison with that of the sun, we get $n=a^{-\frac{3}{2}}.\sqrt{M}$; and, by changing n, a, M, into n', a', m', respectively,

[5093c] to conform to the present notation, we get $n'=a'^{-\frac{3}{2}}.\sqrt{m'}$, as in [5092].

‡ (2899) Multiplying together the values of $\frac{1}{n}=a^{\frac{3}{2}}.(1+\frac{1}{4}m^2)$, $n'=a'^{-\frac{3}{2}}.\sqrt{m'}$

[5093d] [5092], we obtain the expression of m [4835], or $\frac{n'}{n}=m=\frac{a^{\frac{3}{2}}.\sqrt{m'}}{a'^{\frac{3}{2}}}.(1+\frac{1}{4}m^2)$. Squaring

$$\frac{n'^2}{n^2} = m^2 = \frac{a^3 . m'}{a'^3}.(1+\tfrac{1}{2}m^2) = \overset{2}{\bar{m}}.(1+\tfrac{1}{2}m^2). \qquad [5093]$$

Hence we deduce,*

$$\overset{2}{\bar{m}} = m^2.(1-\tfrac{1}{2}m^2)\ ; \qquad \overset{2}{\bar{m}}.\frac{a}{a_{\prime}} = m^2. \qquad [5094]$$

We shall now suppose the value of $nt+\varepsilon$ to be of the following form ;†

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this, and neglecting terms of the order m^4, we get the first part of [5093] ; and, by using the value of $\overset{2}{\bar{m}}$ [4865], we get the last expression [5093]. [5093e]

* (2900) From [5093], we have $m^2 = \overset{2}{\bar{m}}.(1+\frac{1}{2}m^2)$; dividing this by $(1+\frac{1}{2}m^2)$, [5094a]
and neglecting terms of the order m^4, we get,

$$\overset{2}{\bar{m}} = m^2.(1-\tfrac{1}{2}m^2) \quad [5094]. \qquad [5094b]$$

Moreover, by substituting the value of $1+\frac{1}{2}m^2$ [5091], in m^2 [5094a], we obtain the second equation [5094].

† (2901) If we examine the functions which form the expression of dt [5090p], we shall find, that it is composed of terms depending on the cosines of the angles included in the function [5095], with a few others, which will be noticed hereafter [5239, 5244, &c.]. [5095a]
This expression, being multiplied by n, and integrated, gives the terms depending on the same angles in [5095]. Moreover, the expression of $\frac{dt}{dv}$, has the constant term $\frac{1}{n}$ [5090] ; therefore, $n\,dt$ contains the term dv ; and, its integral $nt+\varepsilon$, [5095b]
the term v ; as in [5095 line 1]. Again, the expression of t has the secular term,

$$\frac{3m^2}{2n}.\textstyle\int e'^2.dv \quad [5086] ; \qquad [5095c]$$

and, by multiplying it by n, we find, that the quantity $nt+\varepsilon$ contains the term,

$$\tfrac{3}{2}m^2.\textstyle\int e'^2.dv. \qquad [5095c']$$

At the epoch, when $e'=E'$, the secular term is supposed to vanish ; and this is effected by putting it under the form,

$$\tfrac{3}{2}m^2.\textstyle\int(e'^2-E'^2).dv, \qquad [5095d]$$

and making the integral commence with the epoch.

$nt+\varepsilon=v+\frac{3}{2}m^2.\int(e'^2-E'^2).dv+C_0^{(0)}.e.\sin.(cv-\varpi)$ 1

$+C_0^{(1)}.e^2.\sin.(2cv-2\varpi)$ 2

$+C_0^{(2)}.e^3.\sin.(3cv-3\varpi)$ 3

$+C_0^{(3)}.\gamma^2.\sin.(2gv-2\theta)$ 4

$+C_0^{(4)}.e\gamma^2.\sin.(2gv-cv-2\theta+\varpi)$ 5

$+C_0^{(5)}.e\gamma^2.\sin.(2gv+cv-2\theta-\varpi)$ 6

$+C_2^{(6)}.\sin.(2v-2mv)$ 7

$+C_1^{(7)}.e.\sin(2v-2mv-cv+\varpi)$ 8

Expression of t. $+C_2^{(8)}.e.\sin.(2v-2mv+cv-\varpi)$ 9

$+C_2^{(9)}.e'.\sin.(2v-2mv+c'mv-\varpi')$ 10

[5095] $+C_2^{(10)}.e'.\sin.(2v-2mv-c'mv+\varpi')$ 11

$+C_1^{(11)}.e'.\sin.(c'mv-\varpi')$ 12

$+C_1^{(12)}.ee'.\sin.(2v-2mv-cv+c'mv+\varpi-\varpi')$ 13

$+C_1^{(13)}.ee'.\sin.(2v-2mv-cv-c'mv+\varpi+\varpi')$ 14

$+C_1^{(14)}.ee'.\sin.(cv+c'mv-\varpi-\varpi')$ 15

$+C_1^{(15)}.ee'.\sin.(cv-c'mv-\varpi+\varpi')$ 16

$+C_1^{(16)}.e^2.\sin.(2cv-2v+2mv-2\varpi)$ 17

$+C_1^{(17)}.\gamma^2.\sin.(2gv-2v+2mv-2\theta)$ 18

$+C_1^{(18)}.e'^2.\sin.(2c'mv-2\varpi')$ 19

$+C_1^{(19)}.\frac{a}{a'}.\sin.(v-mv)$ 20

$+C_1^{(20)}.\frac{a}{a'}.e'.\sin.(v-mv+c'mv-\varpi').$ 21

Then we shall have,*

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* (2902) Using the sign Σ, of finite integrals, and putting any periodical term of [5096a] [5095] under the form $C.\sin.(iv+\beta)$, the expression of $nt+\varepsilon$ becomes of the form [5096b]. Its differential, multiplied by $\frac{1}{n}=\frac{a^2}{\sqrt{a_{\prime}}}$ [5092c], becomes as in [5096c].

[5096b] $$nt+\varepsilon=v+\tfrac{3}{2}m^2.\int(e'^2-E'^2)\,dv+\Sigma C.\sin.(iv+\beta)\,;$$

[5096c] $$dt=\frac{a^2.dv}{\sqrt{a_{\prime}}}.\{1+\tfrac{3}{2}m^2.(e'^2-E'^2)+\Sigma\, i\,C.\cos.(iv+\beta)\}.$$

$$C_0^{(0)}=\frac{-2(1-\frac{1}{4}\gamma^2)+\frac{15m^2.{}'A_1^{(1)}}{4\,(1-m)}+3A_2^{(0)}.A_1^{(1)}}{c}\,; \qquad [5096]$$

$$C_0^{(1)}=\frac{\frac{3}{2}+\frac{1}{4}e^2-\frac{3}{4}\gamma^2-2A_2^{(10)}}{2c}\,; \qquad [5097]$$

$$C_0^{(2)}=-\frac{1}{3c}\,; \qquad [5098]$$

$$C_0^{(3)}=\frac{\frac{1}{2}\,(1+\frac{3}{2}c^2-\frac{1}{2}\gamma^2)-2A_2^{(12)}+3A_0^{(15)}e^2}{2g}\,; \qquad [5099]$$

$$C_0^{(4)}=\frac{-\frac{3}{4}-2A_0^{(15)}}{2g-c}\,; \qquad [5100]$$

$$C_0^{(5)}=-\frac{\frac{3}{4}}{2g+c}\,; \qquad [5101]$$

Values of C.

$$C_2^{(6)}=\frac{\left\{\begin{array}{ll}\frac{-3m^2.(1+2e^2-\frac{5}{2}e'^2)}{4(1-m)}-3m^2c^2.\left\{\frac{1+m}{2-2m-c}+\frac{1-m}{2-2m+c}\right\} & \qquad 1\\ -2A_2^{(0)}.(1+\frac{1}{2}e^2-\frac{1}{4}\gamma^2)+3A_1^{(1)}.e^2+3A_2^{(2)}.e^2 & \qquad 2\end{array}\right\}}{2-2m}\,; \qquad [5102]$$

$$C_1^{(7)}=\frac{\left\{\begin{array}{ll}\frac{3m^2.(1+2e^2-\frac{1}{4}\gamma^2-\frac{5}{2}e'^2)}{4(1-m)}+\frac{3m^2.(1+m).\{1+\frac{3}{4}e^2-\frac{1}{4}\gamma^2-\frac{5}{2}e'^2\}}{2-2m-c} & \qquad 1\\ -\frac{3m^2e^2.(10+19m+8m^2)}{8\,(2c-2+2m)}-2A_1^{(1)}.(1+\frac{1}{2}e^2-\frac{1}{4}\gamma^2)+3A_2^{(0)}+3A_1^{(11)}.e^2 & \qquad 2\end{array}\right\}}{2-2m-c}\,; \qquad [5103]$$

Comparing this with the expression of dt [5090p], we evidently see, that the coefficient C, of any term $C.\sin.(iv+\beta)$ of the second member of [5095], may be deduced from the term depending on the cosine of the same angle in the second member of [5090p], by rejecting the common factor $\frac{a^2.dv}{\sqrt{a_{\prime}}}$, and dividing by the coefficient i, corresponding to the proposed angle $iv+\beta$. By this means, we obtain the values of $C_0^{(0)}$, $C_0^{(1)}$, &c. [5096—5116], as will appear, by collecting together the terms of the six functions [5090p], relative to each of the angles separately, taking the terms in the same order as they occur in [5090p]. [5096d] [5096e]

First. Comparing the general form $C.\sin.(iv+\beta)$ [5096b], with that depending on $C_0^{(0)}$ [5095 line 1], we get $C=C_0^{(0)}c$, $i=c$. The terms of C, taken in the order in

[5104] $$C_2^{(8)}=\frac{\frac{3m^2}{4.(1-m)}+\frac{3m^2.(1-m)}{2-2m+c}-2A_2^{(2)}+3A_2^{(0)}-3A_1^{(1)}.e^2}{2-2m+c};$$

[5105] $$C_2^{(9)}=\frac{\frac{3m^2}{4.(2-m)}-2A_2^{(3)}+3A_1^{(6)}.e^2}{2-m};$$

[5106] $$C_2^{(10)}=\frac{-\frac{21m^2}{4.(2-3m)}-2A_2^{(4)}+3A_1^{(7)}.e^2}{2-3m};$$

Values of C.

[5107] $$C_1^{(11)}=\frac{\left\{\begin{array}{ll}-3m.\{4A_2^{(0)}+A_2^{(3)}-A_2^{(4)}-10.A_1^{(1)}e^2+\frac{5}{2}(A_1^{(7)}-A_1^{(6)}).e^2\} & 1\\ +\left\{\frac{3m^2.A_2^{(0)}}{4}+\frac{27m^4}{32.(1-m)}\right\}.\left\{\frac{7}{2-3m}-\frac{1}{2-m}\right\} & 2\\ +\left\{\frac{3m^2}{4.(1-m)}+3A_2^{(0)}\right\}.\{A_2^{(3)}+A_2^{(4)}\}-2A_2^{(5)}.(1+\frac{1}{2}e^2-\frac{1}{4}\gamma^2) & 3\\ +3.(A_1^{(8)}+A_1^{(9)}).e^2+3.A_1^{(1)}.e^2.(A_1^{(6)}+A_1^{(7)}) & 4\\ +\frac{3}{4}m^2.(11.C_2^{(6)}+2C_2^{(9)}-2C_2^{(10)}) & 5\end{array}\right\}}{m};$$

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which they occur in [5082s line 27, 5090b line 2, 5090g line 11, 5090i lines 2, 6], give, without any reduction,

[5096f] $$c.C=3m^2.A_1^{(1)}c-2c.(1-\tfrac{1}{4}\gamma^2)+\frac{3m^2}{4(1-m)}.A_1^{(1)}c+\tfrac{3}{2}A_2^{(0)}.A_1^{(1)}c+\tfrac{3}{2}A_2^{(0)}.A_1^{(1)}e.$$

Connecting together the first and third terms, also the two last terms of the second member; substituting also $C=C_0^{(0)}e$, and dividing by ce, we get $C_0^{(0)}$ [5096], neglecting terms of the order $m^3.A_1^{(1)}e$.

Second. In the term [5095 line 2], we have $C=C_0^{(1)}e^2$, $i=2c$, and then we get, by connecting the terms depending on the angle $2cv$, in [5090b line 11, 4904 line 11],

[5097a] $$2c.C_0^{(1)}e^2=(\tfrac{3}{2}+\tfrac{1}{4}e^2-\tfrac{3}{4}\gamma^2).e^2-2.A_2^{(10)}e^2.$$

Hence we obtain $C_0^{(1)}$ [5097]. In like manner, $C_0^{(2)}$ [5098] is obtained from [5090b line 15].

Third. In the term [5095 line 4], we have $C=C_0^{(3)}\gamma^2$, $i=2g$; and the terms in [5090b line 13, 4904 line 13, 5090g line 29], being connected together, give,

[5099a] $$2g.C_0^{(3)}\gamma^2=(1+\tfrac{3}{2}e^2-\tfrac{1}{2}\gamma^2).\tfrac{1}{2}\gamma^2-2A_2^{(12)}\gamma^2+3A_0^{(15)}e^2\gamma^2;$$

whence we get [5099]. In like manner, from [5090b line 16, 4904 line 16], we obtain,

$$(2g-c).C_0^{(4)}e\gamma^2=-\tfrac{3}{4}e\gamma^2-2A_0^{(15)}e\gamma^2;$$

$$C_1^{(12)}=\frac{-\frac{3m^2.(2+m)}{4.(2-m-c)}-\frac{3m^2}{4.(2-m)}-2A_1^{(6)}+3A_2^{(3)}}{2-m-c};$$ [5108]

$$C_1^{(13)}=\frac{\frac{21m^2.(2+3m)}{4.(2-3m-c)}+\frac{21m^2}{4.(2-3m)}-2A_1^{(7)}+3A_2^{(4)}}{2-3m-c};$$ [5109]

Values of C.

$$C_1^{(14)}=\frac{-2A_1^{(8)}+3A_2^{(5)}}{c+m};$$ [5110]

$$C_1^{(15)}=\frac{-2A_1^{(9)}+3A_2^{(5)}}{c-m};$$ [5111]

$$C_1^{(16)}=\frac{\left\{\begin{array}{l}\frac{3m^2.(10+19m+8m^2)}{8.(2c-2+2m)}-\frac{3m^2.(1+m)}{2-2m-c}-\frac{9m^2}{16.(1-m)} \qquad 1\\ -3A_2^{(0)}+3A_1^{(1)}-2A_1^{(11)}-\frac{\{3m^2.A_2^{(10)}+\frac{15}{4}m^2.(A_1^{(1)})^2\}}{2c-2+2m} \qquad 2\end{array}\right\}}{2c-2+2m}.$$ [5112]

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whence we get [5100]. Also, from [5090b line 17],

$$(2g+c).C_0^{(5)}e\gamma^2=-\tfrac{3}{4}e\gamma^2, \quad \text{as in [5101]}.$$

Fourth. In the term [5095 line 7], we have $C=C_2^{(6)}$, $i=2-2m$; and, by connecting together the terms depending on the angle $2v-2mv$, we shall obtain, for the expression of $(2-2m).C_2^{(6)}$, the same expression as in the numerator of the value of $C_2^{(6)}$ [5102]. For, the first term of this numerator, with the factor $-3m^2$, is the same as in [5082s line 2]; the second term, with the factor $-3m^2e^2$, is as in [5090b line 5], neglecting terms of the order m^2e^4; the third term, with the same factor, is as in [5090b line 7]. The terms depending on $A_2^{(0)}$, are as in [4904 line 1, 5090g line 5]; that connected with $A_1^{(1)}$, is as in [5090g line 13]; lastly, that depending on $A_2^{(2)}$, is as in [5090g line 16]. [5102a]

Fifth. In the term [5095 line 8], we have $C=C_1^{(7)}e$, $i=2-2m-c$; hence we get, for $(2-2m-c).C_1^{(7)}e$, the same expression as is given by [5103]. For, of the two terms of the first line of the numerator of [5103], the *first* is found in [5090b line 3]; the *second*, in [5082s line 3]. The first term of the second line is found in [5090b line 10]; the terms depending on $A_1^{(1)}$, are in [4904 line 2, 5090g line 12]; that on $A_2^{(0)}$, in [5090g line 6]; lastly, that on $A_1^{(11)}$, in [5090g line 28]. [5103a]

Sixth. In the term [5095 line 9], we have $C=C_2^{(8)}e$, $i=2-2m+c$; hence we get, for $(2-2m+c).C_2^{(8)}e$, the same expression as is given by [5104]. For, the first term [5104a]

[5112'] It would seem as if this value of $C_1^{(16)}$ ought to be of the order zero; for,

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of [5104] is obtained from [5090*b* line 4], neglecting quantities of the order m^2e^3; the second term from [5082*s* line 4]; the third term from [4904 line 3]; the fourth from [5090*g* line 7]; the fifth from [5090*g* line 15].

Seventh. In [5095 line 10] we have $C=C_2^{(9)}e'$, $i=2-2m+c'm=2-m$, nearly;
[5105*a*] hence we get $(2-m).C_2^{(9)}e'$, corresponding to [5105]. The terms being found in [5082*s* line 6, 4904 line 4, 5090*g* line 24], respectively. In like manner, [5095 line 11] gives $C=C_2^{(10)}e'$, $i=2-2m-c'm=2-3m$, nearly; and the terms of $(2-3m).C_2^{(10)}e'$ are found in [5082*s* line 5, 4904 line 5, 5090*g* line 25].

Eighth. In [5095 line 12] we have $C=C_1^{(11)}e'$, $i=c'm=m$, nearly; hence we
[5107*a*] get $m.C_1^{(11)}e'$, corresponding to [5107]. For, by comparing the terms of the five lines of the numerator of [5107], with those in the preceding functions, we shall find that they agree, as will appear by the following examination. The terms in [5082*s* lines 23, 24] give those in [5107 line 1]. Those in [5082*s* line 25, 5090*g* lines 3, 4] give [5107 line 2]. The terms in [5090*g* lines 17, 19] are $\frac{3m^2}{4(1-m)}.(A_2^{(3)}+A_2^{(4)})$, as in the first term of [5107 line 3]. The two terms in [5090*i* lines 3, 10] make $3A_2^{(0)}.A_2^{(3)}$; and those in [5090*i* lines 4, 11], $3A_2^{(0)}.A_2^{(4)}$ the sum of these two expressions is $3A_2^{(0)}.(A_2^{(3)}+A_2^{(4)})$, as in the second term of [5107 line 3]. In [4904 line 6] we have $-2A_2^{(5)}$; and, in [5090*g* line 23], $-2A_2^{(5)}.(\frac{1}{2}e^2-\frac{1}{4}\gamma^2)$; whose sum is $-2A_2^{(5)}.(1+\frac{1}{2}e^2-\frac{1}{4}\gamma^2)$, as in [5107 line 3]. The terms depending on $A_1^{(6)}$, $A_1^{(7)}$ [5090*g* lines 26, 27], give those in [5107 line 4]. The sum of the two terms [5090*i* lines 8, 12] gives $3A_1^{(1)}.A_1^{(6)}e^2$; those in [5090*i* lines 9, 13] give $3A_1^{(1)}.A_1^{(7)}e^2$; the sum of these two expressions is $3A_1^{(1)}e^2.(A_1^{(6)}+A_1^{(7)})$, as in [5107 line 4]. Lastly, the terms depending on $C_2^{(6)}$, $C_2^{(9)}$, $C_2^{(10)}$ [5090*p* line 3], give the terms in [5107 line 5].

Ninth. In the term [5095 line 13] we have $C=C_1^{(12)}ee'$, $i=2-2m-c+c'm=2-m-c$, nearly; hence we get $(2-m-c).C_1^{(12)}ee'$, corresponding to [5108]. For, the four
[5108*a*] terms of the numerator of [5108], correspond respectively to [5082*s* line 9, 5090*b* line 9] and [4904 line 7, 5090*g* line 18]. In like manner, [5095 line 14] gives $C=C_1^{(13)}ee'$, $i=2-2m-c-c'm=2-3m-c$, nearly; corresponding to [5109]; the four terms in the numerator being obtained from [5082*s* line 7, 5090*b* line 8, 4904 line 8, 5090*g* line 20].

Tenth. In the term [5095 line 15] we have $C=C_1^{(14)}ee'$, $i=c+c'm=c+m$, nearly;
[5110*a*] hence we get $(c+m).C_1^{(14)}ee'$, corresponding to [5110]; the two terms of the numerator of $C_1^{(14)}$ being deduced from [4904 line 9, 5090*g* line 21]. In like manner, we get [5095 line 16 or 5111] from [4904 line 10, 5090*g* line 22].

Eleventh. In the term [5095 line 17] we have $C=C_1^{(16)}e^2$, $i=2c-2+2m$; hence

its numerator contains several terms of the order m,* and its divisor is of [5112']

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we get $(2e-2+2m).C_1^{(16)}e^2$, corresponding to [5112]. For, the terms in [5082s lines 10, 11, 12] give the first term and two last terms of the numerator of [5112]. In [5090b line 6], we get the term of [5112 line 1], having the factor $(1+m)$; and in [5090b line 12] the last term of the same line; in [5090g lines 8, 14], the terms depending on $A_2^{(0)}$, $A_1^{(1)}$; in [4904 line 12], the term depending on $A_1^{(11)}$. [5112a]

Twelfth. In the term [5095 line 18] we have $C=C_1^{(17)}\gamma^2$, $i=2g-2+2m$; hence we get $(2g-2+2m).C_1^{(17)}\gamma^2$, corresponding to [5113]. For, the terms in [5082s] lines 13, 14, give the first and last terms of [5113]. In [5090b line 14], we get the second term of [5113], neglecting terms of the fourth order [5112''']. In [4904 line 14] we have $-2A_1^{(13)}$; and, in [5090g line 10], the term $-\frac{3}{4}A_2^{(0)}$, as in [5113]. [5113a]

Thirteenth. In the term [5095 line 19] we have $C=C_1^{(18)}e'^2$, $i=2c'm=2m$, nearly; hence we get $2m.C_1^{(18)}e'^2$, corresponding to [5114]. For, the term in [4904 line 15], gives $-2A_2^{(14)}e'^2$; whence we get $C_1^{(18)}$ [5114]. [5114a]

Fourteenth. In the term [5095 line 20] we have $C=C_1^{(19)}.\frac{a}{a'}$, $i=1-m$; hence we get $(1-m).C_1^{(19)}.\frac{a}{a'}$, corresponding to [5115]. For, the first term of [5082s line 19] gives the first term of the numerator of [5115]. The terms in [5082s lines 20, 21] give $\frac{3m^2.A_1^{(17)}}{4(1-m)}.(4+4m)$; adding this to the term deduced from [5090g line 31], namely, $\frac{3m^2.A_1^{(17)}}{4(1-m)}$, the sum becomes $\frac{3m^2.A_1^{(17)}}{4(1-m)}.(5+4m)$. This differs a little from the author, who makes the factor equal to $5+3m$, instead of $5+4m$. The term [4904 line 18] gives $-2A_1^{(17)}$; and [5090g line 32] gives $-2.A_1^{(17)}.(\frac{1}{2}e^2-\frac{1}{4}\gamma^2)$; the sum of these is $-2A_1^{(17)}.(1+\frac{1}{2}e^2-\frac{1}{4}\gamma^2)$, as in the third term of [5115]. Lastly, the sum of the terms in [5090i lines 5, 14] gives $3A_2^{(0)}.A_1^{(17)}$, as in [5115]. [5115a] [5115b]

Fifteenth. In [5095 line 21] we have $C=C_1^{(20)}.\frac{a}{a'}.e'$, $i=1-m+c'm=1$, nearly; hence we get $C_1^{(20)}.\frac{a}{a'}.e'$, corresponding to [5116]; this term being deduced from [4904 line 19], $-2A_0^{(18)}.\frac{a}{a'}.e'$. Hence, the values of $C_0^{(0)}$, $C_0^{(1)}$, &c. [5096—5116] agree with those given by the author, except in the small term of the fourth order, mentioned in [5115b]. [5116a]

* (2903) The two terms $3A_1^{(1)}$, $2A_1^{(11)}$, of the numerator of the value of $C_1^{(16)}$

the same order. But, we have seen, in [4855], that if we retain only the first power of the disturbing force, the value of $C_1^{(16)}$ cannot have, for a divisor, the square of $2c-2+2m$; it must, therefore happen, that all these terms, taken together, destroy each other, except in quantities of the order m; which is a fact confirmed *a posteriori* by calculation. Hence it follows, that, in the values of $A_1^{(1)}$ and $A_1^{(11)}$ [4999, 5009], in the expression
[5112'''] of $C_1^{(16)}$ [5112], we *ought to reject the terms* depending on the squares of e, e' and γ. Each of these terms introduces in $C_1^{(16)}$ quantities of the order e^2; while their sum produces only a quantity of the order me^2, which we may neglect.* There is, therefore, a disadvantage in retaining only a part of
[5112^iv] these terms, and it is best to reject all of them. *This is one of those singular cases of approximation, in which we deviate more from the truth, by noticing a greater number of terms.*

Remarkable case of approximation.

We then have,

[5116b] [5112], are of the order m [4999, 5009], and the denominator $2c-2+2m$, of the same expression [5112], is also of the order m, being very nearly equal to $2m-3m^2$ [4828e].

[5116c] * (2904) Several terms of the order e^2, e'^2, γ^2, have been neglected in the investigation of the analytical expression of $C_1^{(16)}$ [5112]; as, for example, the factor $1+\frac{3}{4}e^2-\frac{1}{2}\gamma^2-\frac{5}{2}e'^2$ [5090b line 6] is omitted in [5112a]; hence, it becomes necessary,
[5116d] upon the principles adopted in [5112'''], to reject terms of the order e^2, e'^2, γ^2, in computing the values of $A_2^{(0)}$, $A_1^{(1)}$, $A_2^{(10)}$, $A_1^{(11)}$, &c., which are to be used in [5112]. Therefore, if the expression of $A_1^{(11)}$ be deduced from [5009], and put under the form

[5116e] $$A_1^{(11)}=\tfrac{3}{4}\overline{m}^2\cdot\frac{a}{a_{,}}\cdot\left\{k_1+k_2e^2+k_3e'^2+k_4\gamma^2\right\};$$

k_1 being independent of e, e', γ, we must use

[5116f] $$A_1^{(11)}=\tfrac{3}{4}\overline{m}^2\cdot\frac{a}{a_{,}}\cdot k_1,$$

in finding the value of $A_1^{(11)}$ [5212, 5112]; observing, that the terms k_1, k_2, &c. have the divisor $2c-2+2m$ in [5009 lines 1, 2]; and this introduces, in $C_1^{(16)}$ [5112],
[5116g] the divisor $(2c-2+2m)^2$, by means of the term $\dfrac{-2A_1^{(11)}}{2c-2+2m}$, &c. Now, as a divisor of the order $(2c-2+2m)^2$ cannot occur in the first power of the disturbing
[5116h] forces [4855], it is necessary, that the terms of which k_1 is composed should mutually balance each other, so as to reduce it to the order m. The same is to be observed relative
[5116i] to k_2, k_3, k_4, Similar remarks may be made upon the value of $C_1^{(17)}$ [5113], and upon those of $A_2^{(0)}$, $A_2^{(10)}$, $A_1^{(13)}$, $B_1^{(0)}$, &c., which occur in [5112, 5113, &c.].

$$C_1^{(17)} = \frac{\dfrac{3m^2.(2+m)}{8(2g-2+2m)} - \dfrac{3m^2}{16.(1-m)} - 2A_1^{(13)} - \tfrac{3}{4}A_2^{(0)} - \dfrac{3m^2.A_2^{(12)}}{2g-2+2m}}{2g-2+2m}. \qquad [5113]$$

We must apply to this value of $C_1^{(17)}$ a remark analogous to that made on $C_1^{(16)}$ [5112′—5112^iv]. Lastly, we have, [5113′]

Values of C.

$$C_1^{(18)} = -\frac{A_2^{(14)}}{m}; \qquad [5114]$$

$$C_1^{(19)} = \frac{\left\{\dfrac{-3m^2}{8.(1-m)} + \dfrac{3m^2.(5+3m)}{4.(1-m)}.A_1^{(17)} - 2A_1^{(17)}.(1+\tfrac{1}{2}e^2-\tfrac{1}{4}\gamma^2) + 3A_2^{(0)}.A_1^{(17)}\right\}}{1-m}; \qquad [5115]$$

$$C_1^{(20)} = -2A_0^{(18)}. \qquad [5116]$$

16. We shall now determine the numerical values of these different coefficients. For this purpose, we shall remark, that we have by observation;*

Data from observation.

$m = 0{,}0748013$;	$\log. m = 8{,}8739091$.
$c = 0{,}99154801$;	$\log. c = 9{,}9963137$.
$g = 1{,}00402175$;	$\log. g = 0{,}0017431$. [5117]
$e' = 0{,}016814$, at the epoch of 1750;	$\log. e' = 8{,}2256710$.
$\gamma = 0{,}0900807 = \text{tang}.\ 5^d\ 8^m\ 50^s{,}4$;	$\log. \gamma = 8{,}9546318$.

According to observation, the term $C_0^{(0)}.c.\sin.(cv-\varpi)$ is nearly equal to $-22677^s{,}5.\sin.(cv-\varpi)$ [5574]. We have given the analytical value of [5118]

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* (2905) The values [5117] agree very nearly with Burg's tables; observing, that the moon's motion is represented by v; the motion from the perigee is cv, and, from the node, gv [4817]; the sun's motion, neglecting the periodical terms, is mv [4835, 4836]. The eccentricity of the solar orbit is represented by e'; it is the same as e'' [4080], taken to six places of decimals; the neglect of 5, in the eighth decimal place of e', produces a small difference in the logarithm of e'' or e', given in [4080, 5117]. Lastly, γ represents the tangent of the inclination of the lunar orbit to the apparent ecliptic [4813, 4818, &c.]. The value of m [5117] gives $m^2 = 0{,}0055952$, which is frequently used in this volume. [5117a] [5117b] [5117c] [5117d]

[5119] $C_0^{(0)}$ in [5096]; and, if we substitute in it the values of $A_2^{(0)}$, $A_1^{(1)}$, given
Assumed value of e. by a first approximation, we obtain,*

[5120] $$e = 0,05487293.$$

This value is sufficiently accurate for the determination of the coefficients $A_2^{(0)}$, $A_1^{(1)}$, $A_2^{(2)}$, &c. We have supposed, in conformity with the phenomena
[5121] of the tides, that the moon's mass is $\frac{1}{58.7}$ of that of the earth.† This being premised, the equations between these coefficients [4998—5017, 5062—5077] become,‡

[5122] $$A_2^{(0)} = 0,00723508-0,00501814.\{B_1^{(0)}-B_2^{(1)}\};$$

[5123] $$A_1^{(1)} = 0,204044-0,0660894.A_2^{(0)}-0,0480577.\{B_2^{(5)}-B_2^{(6)}\};$$

[5124] $$A_2^{(2)} = -0,00372953;$$

[5125] $$A_2^{(3)} = -0,00315160-0,00449610.B_1^{(9)};$$

[5126] $$A_2^{(4)} = 0,0289026-0,00564793.B_1^{(10)};$$

[5127] $$A_1^{(6)} = -0,193315+0,104996.A_1^{(1)}+0,372796.A_1^{(9)};$$

[5128] $$A_1^{(7)} = 0,538027+0,0334044.A_1^{(1)}+0,135144.A_1^{(8)};$$

[5129] $$A_1^{(8)} = -0,0908432+0,139071.A_1^{(1)}-0,280299.A_1^{(7)};$$

[5130] $$A_1^{(9)} = 0,0791193+1,055799.A_1^{(1)}+0,270902.A_1^{(6)};§$$

[5120a] * (2906) The assumed value of e [5120] differs but very little from that finally adopted in [5194].

[5121a] † (2907) This value agrees nearly with the result obtained in [4321]; the author afterwards decreased it to $\frac{1}{74.946}$ [4631a—b].

‡ (2908) The equations [5122—5140] are obtained from [4998—5002, 5004—5017], by taking them in the same order, and dividing by the coefficients of $A_2^{(0)}$, $A_1^{(1)}$, $A_2^{(2)}$,
[5122a] &c. respectively. The equation [5003] is afterwards used in finding $A_2^{(5)}$ [5205]; and, in like manner the equations [5141—5156] are derived from [5062–5077], using the values
[5122b] of m, c, g, c', γ, e [5117,5120]; also $\frac{\overset{2}{m}\,.\,a}{a_{,}}=m^2$ [5082h']. Upon examination it will be found, that the numerical results obtained by the author are, in general, very correct; the differences being rarely more than one or two units in the last decimal place.
[5122c] The few cases, in which a greater difference was discovered, will be mentioned in the following notes.

§ (2909) It will be found, by examination, that the coefficient of $A_1^{(6)}$, in this equation,

$A_2^{(10)} = 0,00285368-0,00415018.B_0^{(11)}$; [5131]

$A_1^{(11)} = 0,366100-0,0172338.A_1^{(1)}-0,259744.A_2^{(10)}-0,324680.(A_1^{(1)})^2$; [5132]

$A_2^{(12)} = 0,00265066$;* [5133]

$A_1^{(13)} = 0,0523335-1,555935.B_1^{(0)}-0,220276.A_2^{(12)}$; [5134]

$A_2^{(14)} = -0,0129890$; [5135]

$A_0^{(15)} = -0,1007403+0,0385084.A_1^{(1)}+2,09016.A_1^{(13)}$
$-1,022473.A_1^{(16)}-36,11032.\{B_2^{(3)}-B_1^{(0)}.B_2^{(5)}\}$; [5136]

$A_1^{(16)} = 0,114623+0,166591.A_0^{(15)}-5,07811.B_2^{(4)}$; [5137]

Fundamental equations to determine A, B.

$A_1^{(17)} = -0,121028\dagger+0,937593.A_2^{(0)}-0,000031563.A_0^{(18)}$
$-0,139767.\{B_2^{(14)}+B_2^{(15)}\}$; [5138]

$A_0^{(18)} = 1,208124+1,018700.A_1^{(17)}-5,074801.A_1^{(19)}$; [5139]

$A_1^{(19)} = -0,121295+0,675879.A_1^{(17)}+0,183834.A_0^{(18)}$; [5140]

$B_1^{(0)} = 0,0287031-0,0574772.A_2^{(0)}+0,000432665.A_1^{(1)}$; [5141]

$B_2^{(1)} = -0,00000236395$; [5142]

$B_2^{(2)} = -0,00564433+0,0048210.B_1^{(0)}$; [5143]

$B_2^{(3)} = 0,0166486+0,0166486.A_1^{(1)}-0,0165194.B_1^{(0)}$; [5144]

$B_2^{(4)} = 0,00656716-0,00708386.B_1^{(0)}$; [5145]

$B_2^{(5)} = 0,0000147361-0,00681821.A_1^{(1)}$; [5146]

$B_2^{(6)} = -0,0183098-0,0170013.\{A_1^{(1)}-B_1^{(0)}\}$; [5147]

$B_1^{(7)} = 0,0809777+0,0249192.B_1^{(0)}-0,0478194.B_1^{(10)}$; [5148]

$B_1^{(8)} = -0,0368568+0,187099.B_1^{(0)}+0,0556224.B_1^{(9)}$; [5149]

$B_1^{(9)} = -0,0263090-0,0787687.B_1^{(0)}+0,0506541.B_1^{(8)}$; [5150]

$B_1^{(10)} = 0,0712575-0,03047765.B_1^{(0)}+0,0211192.B_1^{(7)}$; [5151]

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ought to be increased about one tenth part ; but, as this difference does not materially affect the results, no notice is taken of it. [5130a]

* (2910) Upon repeating the calculation of this value of $A_2^{(12)}$, it is found to be greater by about $\frac{1}{500}$ part, or five units in the sixth decimal place. This difference is unimportant. [5133a]

† (2911) The numerical values of the coefficients [5138] agree with the equation [5015]. A very small change in the constant part $-0,121028$ would be made, by introducing the term depending on $-\frac{2}{3}\gamma^2$ [4961u] ; but the effect is insensible. [5138a]

[5152] $$B_0^{(11)} = 0{,}421270 + 0{,}842540.A_1^{(1)} - 0{,}337016.A_1^{(11)} + 0{,}586564.B_1^{(0)} + 0{,}157666.B_1^{(12)};$$

[5153] $$B_1^{(12)} = 0{,}000194141 - 0{,}168403.A_1^{(1)} + 0{,}0673614.\{A_1^{(11)} + \tfrac{1}{2}B_0^{(11)}\};$$

[5154] $$B_1^{(13)} = 0{,}0847889 + 0{,}147896.\{A_1^{(1)} - \tfrac{1}{2}B_1^{(0)}\} - 0{,}0591586.A_1^{(11)};$$

[5155] $$B_2^{(14)} = -0{,}0125619;$$

[5156] $$B_2^{(15)} = 0{,}00386625.$$

From these equations, we have obtained the following values ;*

[5157] $$A_2^{(0)} = 0{,}00709262;$$

[5158] $$A_1^{(1)} = 0{,}202619;$$

[5159] $$A_2^{(2)} = -0{,}00372953;$$

[5160] $$A_2^{(3)} = -0{,}00300427;$$

[5156a] * (2912) Substituting the value of $B_2^{(1)}$ [5142] in [5122], we obtain a linear equation in $A_2^{(0)}$, $B_1^{(0)}$. Combining this with the four *linear* equations [5123, 5141, 5146, 5147], containing the five unknown quantities $A_2^{(0)}$, $A_1^{(1)}$, $B_1^{(0)}$, $B_2^{(5)}$, $B_2^{(6)}$, we obtain *five* linear equations; from which we may deduce these five unknown quantities, by the usual rules, as in [5157, 5158, 5176, 5181, 5182]. Substituting these values in [5143, 5144, 5145], we
[5156b] get $B_2^{(2)}$, $B_2^{(3)}$, $B_2^{(4)}$ [5178—5180]. Using the value of $B_1^{(0)}$ [5176], we obtain from [5148, 5151] two *linear* equations, for the determination of $B_1^{(7)}$, $B_1^{(10)}$ [5183, 5186];
[5156c] and, from [5149, 5150], two *linear* equations, to find $B_1^{(8)}$, $B_1^{(9)}$ [5184, 5185]. Hence we easily obtain, from [5125, 5126], the values of $A_2^{(3)}$, $A_2^{(4)}$ [5160, 5161]. Substituting $A_1^{(1)}$ [5158] in [5128, 5129], we get two *linear* equations, to find $A_1^{(7)}$, $A_1^{(8)}$ [5163, 5164]; and, in like manner, [5127, 5130] give $A_1^{(6)}$, $A_1^{(9)}$ [5162, 5165]. We may remark,
[5156d] that these values of $A_1^{(6)}$, $A_1^{(9)}$, are both affected by the small correction [5130a]; but the effect of this correction is insensible. Substituting the values of $A_1^{(1)}$, $B_1^{(0)}$ [5158, 5176] in [5131, 5132, 5152, 5153], we get four *linear* equations, for the
[5156e] determination of $A_2^{(10)}$, $A_1^{(11)}$, $B_0^{(11)}$, $B_1^{(12)}$ [5166, 5167, 5187, 5188]. Substituting $A_2^{(12)}$, $B_1^{(0)}$ [5168, 5176] in [5134], we get $A_1^{(13)}$ [5169]. Substituting, in [5136, 5137], the values of $A_1^{(1)}$, $A_1^{(13)}$, &c., which we have already investigated, we obtain two *linear* equations, for the determination of $A_0^{(15)}$, $A_1^{(16)}$ [5171, 5172]. In like manner,
[5156f] the three equations [5138—5140], are *linear* in $A_1^{(17)}$, $A_0^{(18)}$, $A_1^{(19)}$, and give their
[5156g] values [5173, 5174, 5175]; which would be altered a little by the introduction of the correction [5138a]. This correction is, however, quite unimportant. Finally, with the
[5156h] values we have already computed, we easily obtain, from [5154], that of $B_1^{(13)}$ [5189]. This completes the investigation of the series of terms contained in the equations [5157—5191].

$A_2^{(4)} = 0,0284957;$ [5161]

$A_1^{(6)} = -0,0698493;$ [5162]

$A_1^{(7)} = 0,516751;$ [5163]

$A_1^{(8)} = -0,207510;$ [5164]

$A_1^{(9)} = 0,274122;$ [5165]

$A_2^{(10)} = 0,00081065;$ [5166]

$A_1^{(11)} = 0,349068;$ [5167]

$A_2^{(12)} = 0,00265066;$ [5168]

$A_1^{(13)} = 0,0075875,$ [5169]

$A_2^{(14)} = -0,0129890;$ [5170]

$A_0^{(15)} = -0,742373;$ [5171]

$A_1^{(16)} = -0,041378;$ [5172]

Values of A, B.

$A_1^{(17)} = -0,113197;$ [5173]

$A_0^{(18)} = 1,08469;$ [5174]

$A_1^{(19)} = 0,001601;$ [5175]

$B_1^{(0)} = 0,0283831;$ [5176]

$B_2^{(1)} = -0,00000236395;$ [5177]

$B_2^{(2)} = -0,00550748;$ [5178]

$B_2^{(3)} = 0,0195530;$ [5179]

$B_2^{(4)} = 0,00636608;$ [5180]

$B_2^{(5)} = -0,00136676;$ [5181]

$B_2^{(6)} = -0,0212720;$ [5182]

$B_1^{(7)} = 0,0782400;$ [5183]

$B_1^{(8)} = -0,0833684;$ [5184]

$B_1^{(9)} = -0,0327678;$ [5185]

$B_1^{(10)} = 0,0720448;$ [5186]

$B_0^{(11)} = 0,491954;$ [5187]

$B_1^{(12)} = 0,0061023;$ [5188]

[5189] $$B_1^{(13)} = 0,0920621\,;$$

[5190] $$B_2^{(14)} = -0,0125619\,;$$

[5191] $$B_2^{(15)} = 0,00386625.$$

By means of these values, we have corrected the expression of e [5120], making use of the equation,*

[5192] $$C_0^{(0)}e = -22677^s,5.$$

The expression of $C_0^{(0)}$ [5096] gives,

[5193] $$C_0^{(0)} = -2,003974\,;$$

Corrected value of e.

hence we obtain,

[5194] $$e = 0,05486281\,; \qquad \log.e = 8,7392781\,;$$

which differs but very little from the value before used [5120]. Then we find,†

[5195] $$C_0^{(1)} = 0,752886\,;$$

[5196] $$C_0^{(2)} = -0,336175\,;$$

[5197] $$C_0^{(3)} = 0,243118\,;$$

[5198] $$C_0^{(4)} = 0,722823\,;$$

[5199] $$C_0^{(5)} = -0,250034\,;$$

[5200] $$C_2^{(6)} = -0,00919876\,;$$

[5192a] * (2913) Comparing the expression $C_0^{(0)}e.\sin.(cv-\varpi)$ [5095 line 1], with its value, deduced from observation, $-22677^s,5.\sin.(cv-\varpi)$ [5574], and adopted in Burg's tables [5574a], we get the expression of $C_0^{(0)}e$ [5192]. Now, substituting in [5096],

[5192b] the values of m, c, γ, $A_2^{(0)}$, $A_1^{(1)}$ [5117, 5157, 5158], we get the value of $C_0^{(0)}$ [5193]; and then, from [5192], we obtain the corrected value of e [5194].

[5195a] † (2914) Substituting the values [5117, 5157—5175, 5194], in [5097—5106], we get [5195—5201]. Having thus obtained $C_2^{(6)}$, $C_2^{(9)}$, $C_2^{(10)}$ [5200, 5203, 5204], we may compute $A_2^{(5)}$ [5205], by means of the formula [5003]. The values $C_1^{(11)}$, $C_1^{(12)}$, $C_1^{(15)}$, are derived from [5107, 5108, 5111], which contain $A_1^{(6)}$, $A_1^{(9)}$; but the effect of the correction [5156d] is insensible. The expressions [5208, 5209], are deduced from [5109, 5110].

$$C_1^{(7)} = -0,414046; \qquad [5201]$$
$$C_2^{(8)} = 0,0129865; \qquad [5202]$$
$$C_2^{(9)} = 0,00392546; \qquad [5203]$$
$$C_2^{(10)} = -0,0387853; \qquad [5204]$$
$$A_2^{(5)} = -0,00571628; \qquad [5205]$$
$$C_1^{(11)} = 0,196755; \qquad [5206]$$
$$C_1^{(12)} = 0,127650; \qquad [5207]$$
$$C_1^{(13)} = -1,081734; \qquad [5208]$$
$$C_1^{(14)} = 0,373115; \qquad [5209]$$
$$C_1^{(15)} = -0,616738. \qquad [5210]$$

We must, by the preceding article [5112‴, 5113′], in calculating the values of $C_1^{(16)}$, $C_1^{(17)}$, use the values of $A_1^{(1)}$, $A_1^{(11)}$, $A_1^{(13)}$, determined by neglecting the squares of the excentricity and inclination of the lunar orbit. We have thus found the following values of $A_1^{(1)}$, $A_1^{(11)}$, $A_1^{(13)}$ and $B_1^{(0)}$, which must be used in this calculation;*

$$A_1^{(1)} = 0,201816; \qquad [5211]$$
$$A_1^{(11)} = 0,349187; \qquad [5212]$$
$$A_1^{(13)} = 0,0077734; \qquad [5213]$$
$$B_1^{(0)} = 0,0282636; \qquad [5214]$$

hence we deduce,

$$C_1^{(16)} = 0,272377; \qquad [5215]$$
$$C_1^{(17)} = 0,033325. \qquad [5216]$$

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* (2915) The principles upon which these quantities are neglected have been explained in [5112′, &c.; 5116c—i]. The quantities $A_2^{(0)}$, $A_2^{(10)}$ [5157, 5166], being [5211a]
very small, their corrections are unimportant; and the author seems not to have noticed [5211b]
these corrections in [5211, &c.]. The calculation of the terms [5211—5216] is made in the following order. $A_2^{(0)}$ is given by [1998]; then $A_1^{(1)}$, by [1999]; $A_2^{(10)}$, $A_2^{(12)}$, by [5211c]
[5008, 5010]; $A_1^{(11)}$, by [5009]; $B_1^{(0)}$, by [5032]; and $A_1^{(13)}$, by [5011]. The values thus found, differ but little from those in [5211—5214]; and, by substituting them in [5211d]
[5112, 5113], we get [5215, 5216], neglecting always e^2, e'^2, γ^2.

Then we have,*

[5217] $$C_1^{(18)} = 0{,}173647;$$

[5218] $$C_1^{(19)} = 0{,}236616;$$

[5219] $$C_0^{(20)} = -2{,}16938.$$

This being premised, the expression of $nt+\varepsilon$ [5095], becomes, by reducing its coefficients to seconds,†

Formula for the determination of t.

[5220]

$$nt+\varepsilon = v+\tfrac{3}{2}m^2.\textstyle\int(e'^2-E'^2).dv \qquad 1$$
$$-22677^s{,}5\ .\sin.(cv-\varpi) \qquad 2$$
$$+\ 467^s{,}42.\sin.(2cv-2\varpi) \qquad 3$$
$$-\ 11^s{,}45.\sin.(3cv-3\varpi) \qquad 4$$
$$+\ 406^s{,}92.\sin.(2gv-2\theta) \qquad 5$$
$$+\ 66^s{,}37.\sin.(2gv-cv-2\theta+\varpi) \qquad 6$$
$$-\ 22^s{,}96.\sin.(2gv+cv-2\theta-\varpi) \qquad 7$$
$$-\ 1897^s{,}38.\sin.(2v-2mv) \qquad 8$$
$$-\ 4685^s{,}45.\sin.(2v-2mv-cv+\varpi) \qquad 9$$
$$+\ 146^s{,}96.\sin.(2v-2mv+cv-\varpi) \qquad 10$$
$$+\ 13^s{,}61\ .\sin.(2v-2mv+c'mv-\varpi') \qquad 11$$
$$-\ 134^s{,}51\ .\sin.(2v-2mv-c'mv+\varpi') \qquad 12$$
$$+\ 682^s{,}37.\sin.(c'mv-\varpi') \qquad 13$$
$$+\ 24^s{,}29.\sin.(2v-2mv-cv+c'mv+\varpi-\varpi') \qquad 14$$
$$-\ 205^s{,}82.\sin.(2v-2mv-cv-c'mv+\varpi+\varpi') \qquad 15$$
$$+\ 70^s{,}99.\sin.(cv+c'mv-\varpi-\varpi') \qquad 16$$
$$-\ 117^s{,}35.\sin.(cv-c'mv-\varpi+\varpi') \qquad 17$$
$$+\ 169^s{,}10.\sin.(2cv-2v+2mv-2\varpi) \qquad 18$$
$$+\ 56^s{,}62.\sin.(2gv-2v+2mv-2\theta) \qquad 19$$
$$+\ 10^s{,}13.\sin.(2c'mv-2\varpi') \qquad 20$$
$$+\ 122^s{,}014.(1+i).\sin.(v-mv) \qquad 21$$
$$-\ 18^s{,}809.(1+i).\sin.(v-mv+c'mv-\varpi'). \qquad 22$$

[5217a] * (2916) The values of $C_1^{(18)}$, $C_1^{(19)}$, $C_0^{(20)}$, deduced from [5114—5116], agree very nearly with those given by the author in [5217—5219].

[5220a] † (2917) Substituting, in [5095], the values of e', γ [5117], e [5194], and those of $C_0^{(1)}$, $C_0^{(2)}$, &c. [5195—5219]; also $\frac{a}{a'}$ [5221], we get [5220].

The two last terms were determined by supposing $\frac{a}{a'}=\frac{1+i}{400}$. *This fraction depends on the parallaxes of the sun and moon; it differs but very little from* $\frac{1}{400}$; *but, for greater generality, we have connected it with the indeterminate coefficient* $1+i$; *and, by comparing the term depending on* $\sin.(v-mv)$, *with the result of observation, we shall hereafter determine the solar parallax* [5589]. [5221] Sun's parallax. [5222]

It is evident, by what has been said, that the perturbations of the earth's orbit, by the moon, introduce in $A_1^{(17)}$, the quantity $0{,}25044.\mu$;* and, therefore, in $C_1^{(19)}$, the quantity $-0{,}54139.\mu$; whence arises, in the expression of the moon's apparent longitude, the inequality,† [5223] [5224]

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* (2918) Using the value of m^2 [5082h′], we find, that the coefficient of $A_1^{(17)}$, in [5015], is

$$1-(1-m)^2-\frac{m^2.(36+21m-15m^2)}{4(1-m)};\qquad [5223a]$$

and, the term depending on μ, is

$$-2m^2.\mu.\left\{\tfrac{9}{8}(1+2e^2+2e'^2)+\frac{3}{4(1-m)}.(1+\tfrac{9}{2}e^2+2e'^2)\right\}.\qquad [5223b]$$

Dividing this last expression by the preceding, and changing its sign, we get the term of $A_1^{(17)}$, depending upon μ. Substituting the values of m, e', e [5117,5194], it becomes $0{,}25044.\mu$, as in [5223]; *μ being the ratio of the moon's mass, to the sum of the masses of the moon and earth* [4948′]. [5223c]

† (2919) The symbol μ is introduced into the expression of $C_1^{(19)}$ [5115], by means of the value of $A_1^{(17)}$. Now, the coefficient of $A_1^{(17)}$, in [5115], is

$$\frac{3m^2.(5+3m)}{4(1-m)^2}-\frac{2(1+\frac{1}{2}e^2-\frac{1}{4}\gamma^2)}{1-m}+\frac{3}{1-m}.A_2^{(0)};\qquad [5225a]$$

and, if we use the values of m, γ, e, $A_2^{(0)}$ [5117, 5194, 5157], it becomes $-2{,}1326$. Multiplying this by $0{,}25044.\mu$ [5223], we get $-0{,}534.\mu$, instead of $-0{,}54139.\mu$ [5224]. This part of $C_1^{(19)}$ produces, in the expression of $nt+\varepsilon$ [5095 line 20, or 5220 line 21], the term

$$-0{,}534.\mu.\frac{a}{a'}.\sin.(v-mv);\qquad [5225b]$$

and, by changing its sign, we get the corresponding term of the moon's longitude v [5225]. The inequality of the earth's motion, depending on the direct action of the moon [4314, 4316b], using the same symbols as in this article, is [5225c]

$$\mu.\frac{a}{a'}.\sin.(v-mv)\quad [5225'],\text{ nearly};\qquad [5225d]$$

as is evident by comparing the notation [4313] with that in [4757, &c.]. The ratio of the two inequalities [5225, 5225′] is as in [5226].

[5225] $$0,54139 \,.\, \mu \,.\, \frac{a}{a'} \,.\, \sin.(v - mv).$$

Indirect action of the moon.

The direct action of the moon upon the earth produces, in the motion of the earth, the inequality,

[5225'] $$\mu \,.\, \frac{a}{a'} \,.\, \sin.(v - mv);$$

this action is, therefore, reflected to the moon, by means of the sun, but decreased
[5226] *in the ratio of* $0,54139$ *to unity.*

The preceding expression of $nt+\varepsilon$, contains the coefficients c and g,
[5227] which depend on the sun's action. We have given their analytical values in [4986, 5228x], and, by reducing them to numbers, we have,*

[5228] $$c = 0,991567;$$

[5229] $$g = 1,0040105.$$

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* (2920) Dividing the coefficient of $\cos.(cv - \varpi)$ [4961 lines 3—7], by $-\frac{(1+e^2).e}{a}$, we get $p + qe'^2$ [4975], as in the following expression, using the value of m^2 [5082h'];

[5228a] $$p + qe'^2 = \frac{\frac{3}{4} m^2}{1+e^2} \,.\, \left\{ \begin{array}{l} 2 + e^2 + 3e'^2 - 2(B_2^{(2)} + B_2^{(3)}) \,.\, \frac{\gamma^2}{\overline{m}^2} + (1 + 2m - c) \,.\, A_2^{(2)} \,.\, (1 - \frac{5}{2}e'^2) \\ -4 \left\{ 1 + 2m + (4.\overline{1-m}|^2 - 1) \,.\, \left(\frac{1+m}{2-2m-c} + \frac{1-m}{2-2m+c} \right) \right\} \,.\, A_2^{(0)} \,.\, (1 - \frac{5}{2}e'^2) \\ + \frac{1}{1-m} \,.\, \{ (1 + 6m + c) \,.\, (1 - m) + 7 + (2 - 2m - c)^2 \} \,.\, A_1^{(1)} \,.\, (1 - \frac{5}{2}e'^2) \\ - \frac{1}{2}(9 + m + c) \,.\, A_1^{(6)} e'^2 + \frac{7}{2}(9 + 3m + c) \,.\, A_1^{(7)} e'^2 + 3(A_1^{(8)} + A_1^{(9)}) \,.\, e'^2 \end{array} \right\}.$$

[5228b] We have seen, in [4976a, b], that the quantities $A_2^{(0)}$, $A_1^{(1)}$, $B_1^{(0)}$, $B_2^{(2)}$, $B_2^{(3)}$ contain implicitly the factor $1 - \frac{5}{2}e'^2$; which must be particularly noticed when finding the
[5228c] values of p, q, from [5228a]. Thus, if we neglect terms of the *sixth* order in the equation [4998], we shall find, that the term [4998 line 1] may be put under the form

[5228d] $$\frac{3}{2}\overline{m}^2 \,.\, \frac{a}{a_{\prime}} \,.\, \{ 1 + (1 + 2m) \,.\, e^2 + \frac{1}{4}\gamma^2 \} \,.\, (1 - \frac{5}{2}e'^2).$$

The factor $1 - \frac{5}{2}e'^2$ is equal to $0,99929322$ [5117]; and, if we put, for brevity,
[5228e] $\frac{1}{k} = 0,99929322$, we shall have $1 = k.(1 - \frac{5}{2}e'^2)$. Hence it is evident, that, if we have found, by a previous computation, the numerical value of the first line of [4998],
[5228f] which we shall represent by A_1, we can put it under the form $A_1 k \,.\, (1 - \frac{5}{2}e'^2)$; and,

The motion $(1-c).v$ of the lunar perigee [4817] is, therefore, by the preceding theory, equal to $0{,}008433.v$ [5228]. *This motion is, by* [5230]

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by this means, it is reduced, by a very simple method, to the form $-p-qe'^2$, adopted in [4975]. In like manner, the second line of [4998], which may be represented by A_2, can be put under the form $A_2k.(1-\frac{5}{2}e'^2)$. The term $B_1^{(0)}$, which occurs in the third line of [4998], can be put under the form $B_1^{(0)}k.(1-\frac{5}{2}e'^2)$; as is evident, from the inspection of the formula [5062], neglecting the small terms, similar to those omitted in [5228c]. Lastly, the term $B_2^{(1)}$, which occurs in the third line of [4998], is nearly equal to $-0{,}000002$ [5177]; and, as this is so very small, we may put it equal to $B_2^{(1)}k.(1-\frac{5}{2}e'^2)$. Hence it appears, that, if the analytical value of $A_2^{(0)}$ be deduced from [4998], the terms depending on e'^2, will appear very nearly under the form of the factor $(1-\frac{5}{2}e'^2)$; so that we may deduce, from the numerical value of $A_2^{(0)}$ [5157], the term depending on e'^2, by changing $A_2^{(0)}$ into $A_2^{(0)}k.(1-\frac{5}{2}e'^2)$. Proceeding in the same manner with [4999], we find, that the terms depending on e'^2 may be obtained, by changing $A_1^{(1)}$ into $A_1^{(1)}k.(1-\frac{5}{2}e'^2)$, and using the numerical value of $A_1^{(1)}$ [5158]. In the equation [5000], from which $A_2^{(2)}$ is deduced, the terms depending on e'^2 are omitted, on account of their smallness. But, if we inspect the functions which are enumerated in [4961d, e], and used in the formation of the equations [4999, 5000], we shall see, by noticing the terms depending on e'^2, that the chief terms of $A_1^{(1)}$, $A_2^{(2)}$, are formed in the same manner, with the factor $1-\frac{5}{2}e'^2$, as in [4879k, 4879f line 1] and [4876e lines 2, 3, &c.]. Hence, it is evident, that we may proceed with $A_2^{(2)}$ as we have with $A_1^{(1)}$ [5228k], and put $A_2^{(2)}=A_2^{(2)}k.(1-\frac{5}{2}e'^2)$. The terms of e'^2, which occur in the values of $B_2^{(2)}$, $B_2^{(3)}$ [5064, 5065], produce not much effect in the computation of $\frac{1}{2}qe'^2$, or $\frac{1}{2}qE'^2$, in the value of c [4986]; so that we may, without any sensible error, change $B_2^{(2)}$ into $B_2^{(2)}k.(1-\frac{5}{2}e'^2)$, and $B_2^{(3)}$ into $B_2^{(3)}k.(1-\frac{5}{2}e'^2)$, as the author has done. Hence, it appears, that if we neglect terms of the order e'^4, we shall obtain very nearly the terms depending on e'^2, in the second member of [5228a], by substituting

[5228g]

[5228h]

[5228i]

[5228k]

[5228l]

[5228m]

[5228n]

$$A_2^{(0)}.(1-\tfrac{5}{2}e'^2)=A_2^{(0)}k.(1-5e'^2)\,; \qquad A_1^{(1)}.(1-\tfrac{5}{2}e'^2)=A_1^{(1)}k.(1-5e'^2)\,;$$

$$A_2^{(2)}.(1-\tfrac{5}{2}e'^2)=A_2^{(2)}k.(1-5e'^2)\,; \quad B_2^{(2)}=B_2^{(2)}k.(1-\tfrac{5}{2}e'^2)\,; \quad B_2^{(3)}=B_2^{(3)}k.(1-\tfrac{5}{2}e'^2)\,; \qquad [5228o]$$

and then putting the terms independent of e'^2 equal to p, and the rest equal to qe'^2. Having thus obtained the analytical expressions of p, q, we must substitute in them the values of $A_2^{(0)}$, $A_1^{(1)}$, &c. [5157—5179], and we shall obtain very nearly, [5228p]

$$p=0{,}016781\,; \qquad q=0{,}04973\,. \qquad [5228q]$$

Substituting these values, and $E'=e'=0{,}016814$ [5117], in the expression of c [4986], it becomes very nearly as in [5228]. From this we obtain the expression of the motion of [5228r]

Motion of the perigee. [5231] *observation, equal to* $0,008452.v$ [5117 line 2]; *which differs from the preceding but by its four hundred and forty-fifth part.*

[5231′] *The motion of the perigee is subjected to a secular equation, whose analytical expression is given in* [4982, &c.]. *Reducing it to numbers, it becomes,**

[5228r′] the perigee $(1-c).v$ [4817, 5228], as in [5230]; which agrees very nearly with that deduced from observation $0,00845199.v$ [5117 line 2].

[5228s] The coefficient of $\gamma.\sin.(gv-\theta)$, in [5049] is put equal to $p''+q''e'^2$ [5053]; hence we get, by using [5082h′],

[5228t]
$$p''+q''e'^2=\tfrac{3}{2}m^2.\left\{\begin{array}{l}1+2e^2-\tfrac{1}{2}\gamma^2+\tfrac{3}{2}e'^2\\ -\tfrac{1}{2}\left\{\frac{(3-2m-g).(g+m)}{1-m}.B_1^{(0)}+4A_2^{(0)}\right\}.(1-\tfrac{5}{2}e'^2)\\ -\tfrac{7}{4}(3-3m-g).B_1^{(10)}e'^2+\tfrac{1}{4}(3-m-g).B_1^{(9)}e'^2\\ +\tfrac{3}{2}\{B_1^{(7)}+B_1^{(8)}\}.e'^2\end{array}\right\}.$$

[5228u] Substituting the values of $B_1^{(0)}k.(1-\frac{5}{2}e'^2)$, &c. [5228g, o]; and then putting the terms which are independent of e'^2 equal to p'', and the rest equal to $q''e'^2$; we shall get the analytical expressions of p'', q''. Reducing these values to numbers, by means of [5157—5186] we get, very nearly,

[5228v]
$$p''=0,0080337; \qquad q''=0,0123967.$$

These values and that of E' [5228r], are to be used in finding the retrograde motion of the nodes [5059], which becomes, by retaining only the terms depending on the first power of v,

[5228w]
$$\left\{\sqrt{(1+p'')}-1+\frac{\frac{1}{2}q''}{\sqrt{(1+p'')}}.E'^2\right\}.v.$$

Putting this equal to the expression $(g-1).v$, which is assumed in [4817] we get,

[5228x]
$$g=\sqrt{(1+p'')}+\frac{\frac{1}{2}q''E'^2}{\sqrt{(1+p'')}};$$

and, by substituting the values of p'', q'', E' [5228v, r], we obtain g [5229].

* (2921) The secular motion of the perigee depends upon the term $\frac{1}{2}q'.\int e'^2.dv$ [5232a] [4982]; which may be put under the form $\frac{1}{2}q'.\int_0^v(e'^2-E'^2).dv$ [5095c—d]; supposing the integral to commence at the epoch where $e'=E'$. Using the value of q' [4979],

[5232b] and multiplying by $\frac{3m^2}{3m^2}$, it becomes, $\left(\frac{q}{3m^2.\sqrt{(1-p)}}\right).\frac{3}{2}m^2.\int_0^v(e'^2-E'^2).dv$ Substituting, in the factor between the braces, the values of p, q, m [5228q, 5117], we obtain very

[5232c] nearly the same expression as in [5232]. The secular motion of the moon's longitude is $-\frac{3}{2}m^2.\int_0^v(e'^2-E'^2).dv$ [5089a, 5232a], corresponding to [5232′].

$$\delta\varpi = 3{,}00052.\tfrac{3}{2}.m^2.\int(e'^2-E'^2).dv.$$ [5232]

It has a contrary sign to the secular equation of the mean motion [5232c], *and is nearly three times as great.* [5232']

The retrograde motion of the node of the moon's orbit, $(g-1).v$ [4817], *is, by the preceding theory,* 0,0040105.v [5229]. *This motion is, by observation, equal to* 0,00402175.v [5117 line 3], which does not differ from the preceding, by its three hundred and fiftieth part. [5233]

Secular motions of the moon's longitude, perigee and node.

This motion of the node is subjected to a secular equation, whose analytical expression is given in [5059]. *Reducing it to numbers it becomes,**

$$\delta\theta = 0{,}735452.\tfrac{3}{2}.m^2.\int(e'^2-E'^2).dv.$$ [5234]

It has a contrary sign to that of the moon's mean longitude [5232c]. *Hence it follows, that the motions of the nodes and perigee are retarded, whilst the moon's mean motion is accelerated; and the secular equations of these three motions are always in the ratio of the numbers* 3,00052, 0,73542 and 1 [5232, 5234, 5232c]. Therefore, in the preceding expression of $nt+\varepsilon$, we must substitute, for the angles cv, gv, the following quantities ;† [5234'] [5235]

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* (2922) The secular motion of the node depends upon the term,

$$\frac{\frac{1}{2}q''}{\sqrt{(1+p'')}}.\int e'^2.dv \quad [5056b];$$ [5233a]

which may be changed, as in the preceding note, to

$$\frac{\frac{1}{2}q''}{\sqrt{(1+p'')}}.\int_0^v(e'^2-E'^2).dv = \left(\frac{q''}{3m^2.\sqrt{(1+p'')}}\right).\tfrac{3}{2}m^2.\int_0^v(e'^2-E'^2).dv.$$ [5233b]

Substituting, in the first factor, the values of p'', q'' [5228v], it becomes very nearly as in [5234].

† (2923) The motions of the perigee and node $(1-c).v$, $(g-1).v$ [4817], are changed, by means of the secular equations, into [5236a]

$$(1-c).v+3{,}00052.\tfrac{3}{2}m^2.\int_0^v(e'^2-E'^2).dv \quad [5232],$$
$$(g-1).v+0{,}735452.\tfrac{3}{2}m^2.\int_0^v(e'^2-E'^2).dv \quad [5234],$$ [5236b]

respectively. This requires, that we should change cv into

$$cv-3{,}00052.\tfrac{3}{2}m^2.\int_0^v(e'^2-E'^2).dv, \quad \text{as in } [5236];$$

and, gv into [5236c]

$$gv+0{,}735452.\tfrac{3}{2}m^2.\int_0^v(e'^2-E'^2).dv, \quad \text{as in } [5237].$$

[5236] $$cv - 3{,}00052\,.\tfrac{3}{2}.m^2.\int(e'^2 - E'^2).dv\,;$$

[5237] $$gv + 0{,}735452.\tfrac{3}{2}.m^2.\int(e'^2 - E'^2).dv\,.$$

Hence, *the secular equation of the mean anomaly* is,*

[5238] $$-4{,}00052.\tfrac{3}{2}.m^2.\int(e'^2 - E'^2).dv\,;$$

or, nearly four times that of the mean motion.

Investigation of some terms of the fourth order.

17. *We shall now proceed to determine some of the most sensible inequalities of the fourth order.* One of these inequalities depends upon the angle $2v - 2mv - 2gv + cv + 2\theta - \varpi$, and we have determined, in [4904 line 17, 5014], the part of $a\delta u$, which depends on the cosine of this angle. Then we find, by §15, that the expression of $nt+\varepsilon$, contains the inequality,†

[5239] $$\frac{\left\{-\frac{3m^2.(2+m)}{8.(2g-2+2m)} - 2A_1^{(16)} + 3A_1^{(13)}\right\}}{2-2m-2g+c}.e\gamma^2.\sin.(2v-2mv-2gv+cv+2\theta-\varpi).$$

This inequality, reduced to numbers, is

[5240] $$8^s{,}67.\sin.(2v-2mv-2gv+cv+2\theta-\varpi).$$

We shall now consider the inequality, relative to the angle $(2cv+2v-2mv-2\varpi)$. If we connect all the terms, depending on the cosine of this angle, in

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[5238a] * (2924) Subtracting the secular equation of the perigee [5232], from that of the mean motion [5232c], we get the secular equation of the mean anomaly, as in [5238].

† (2925) The part of dt, which would correspond to the term of $nt+\varepsilon$ [5239], may be deduced from it by taking the differential, and multiplying by $\frac{1}{n} = \frac{a^2}{\sqrt{a_{\prime}}}$ [5092c], by which means it becomes

[5239a] $$\left\{-\frac{3m^2.(2+m)}{8.(2g-2+2m)} - 2A_1^{(16)} + 3\,A_1^{(13)}\right\}.\ \frac{a^2.dv}{\sqrt{a_{\prime}}}\ .\ c\gamma^2.\cos.(2v-2mv-2gv+cv).$$

[5239b] Now the *three* terms of this function are contained in the expression of dt [5090p], as we shall see, by the following examination. The *first* term, between the braces, $-\frac{3m^2.(2+m)}{8.(2g-2+2m)}$ [5239a], occurs in the table [5090b]; by multiplying the term $-2c.\cos.cv$ in its first column, by that of [5082s line 13] in its second column. The *second* term $-2A_1^{(16)}$, arises
[5239c] from [5090g line 1, 4904 line 17]. The *third* term $3A_1^{(13)}$ is deduced from the table [5090g]. It corresponds to $-2A_1^{(13)}\gamma^2.\cos.(2gv-2v+2mv)$ in its first column, or in
[5239d] [4904 line 14]; and to $-3c.\cos.cv$ in the second column. Substituting in [5239] the values of m, g, c, γ [5117], e [5194], and $A_1^{(13)}$, $A_1^{(16)}$ [5169,5172], we get [5240].

the development of the equation [4754], which we have made in § 6, this equation becomes, by noticing only these terms,*

$$0=\frac{ddu}{dv^2}+u+\frac{3\overline{m}^2}{2a_{\prime}}\cdot\frac{(10-19\,m+8\,m^2)\cdot(2-m+c)}{4.(c+1-m)}.e^2.\cos.(2cv+2v-2mv-2\varpi);$$ [5241]

therefore, by putting $A_2'^{(0)}.e^2.\cos.(2cv+2v-2mv-2\varpi)$, for the corresponding term of $a'u$ [4904], we shall have,† [5242]

$$A_2'^{(0)}=\frac{\frac{3}{2}m^2.(10-19m+8m^2).(2-m+c)}{4.(c+1-m).\{4.(c+1-m)^2-1\}}.$$ [5243]

Then, if we put $C_2'^{(0)}.e^2.\sin.(2cv+2v-2mv-2\varpi)$, for the corresponding term of the expression of $nt+\varepsilon$, we shall find, by §15,‡ [5244]

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* (2926) The terms depending on the angle $2cv+2v-2mv$, in the equation [4961], are included in the functions which are enumerated in [4960c], and if we divide these terms by the common factor $\frac{3m^2}{2a_{\prime}}.e^2.\cos.(2cv+2v-2mv)$; we shall obtain in [4870 line 12] the term $\frac{1}{4}(6-15m+8m^2)$; and in [4879 line 8] the term $\frac{1}{4}(4-4m)$ nearly. The sum of these two expressions is $\frac{1}{4}(10-19m+8m^2)$; adding this to $\frac{1}{4}(10-19m+8m^2).\frac{1}{c+1-m}$ [4892 line 11], we obtain $\frac{1}{4}(10-19m+8m^2).\frac{2-m+c}{c+1-m}$. Connecting this with the two first terms of [4754] $\frac{ddu}{dv^2}+u$, according to the directions in [4960c, &c.], we get [5241]. [5241a] [5241b] [5241c]

† (2927) Integrating the equation [5241], by the method in [1998a—c], we find, that if $\frac{H}{a_{\prime}}.\cos.(iv+\beta)$ represent any term of [5241], the corresponding term of au or $a\delta u$ [4998c, a] will become, [5242a]

$$a\delta u=\frac{H}{i^2-1}\cdot\frac{a}{a_{\prime}}.\cos.(iv+\beta).$$ [5242b]

In the present case, we have,

$$i=2(c+1-m)\,;\qquad \frac{H}{a_{\prime}}=\frac{3}{2}.\frac{\overline{m}^2.a}{a_{\prime}}\cdot\frac{(10-19m+8m^2).(2-m+c)}{4(c+1-m)}.e^2.$$ [5242c]

Substituting these in [5242b], and putting the result equal to the assumed expression [5242], we get, by using m^2 [5082h'], the value of $A_2'^{(0)}$ [5243].

‡ (2928) If $nt+\varepsilon$ contain a term of the form [5244], its differential will give, in ndt, the expression

$$n\,dt=(2c+2-2m).C_2'^{(0)}e^2.\cos.(2cv+2v-2mv).dv.$$ [5245a]

[5245]
$$C''^{(0)}_2 = \frac{\left\{\begin{array}{l} -\frac{3}{2}m^2.\frac{(10-19m+8m^2)}{8.(c+1-m)} - \frac{3m^2.(1-m)}{2-2m+c} - \frac{9m^2}{16.(1-m)} \\ -2A'^{(0)}_2+3A^{(2)}_2-3A^{(0)}_2 \end{array}\right\}}{2c+2-2m}.$$

Reducing the formulas [5243, 5245] to numbers, we get,

[5246] $$A'^{(0)}_2 = 0,00201041 ;$$

[5247] $$C''^{(0)}_2 = -0,0130618 ;$$

hence we obtain, in $nt+\varepsilon$, the following inequality,

[5248] $$-8^s,11.\sin.(2cv+2v-2mv-2\varpi) \quad [5244].$$

Multiplying this by $\frac{1}{n}=\frac{a^2}{\sqrt{a_{,}}}$ [5092c], we shall get, in dt, the term,

[5245b] $$dt = (2c+2-2m).C''^{(0)}_2 c^2.\frac{a^2.dv}{\sqrt{a_{,}}}.\cos.(2cv+2v-2mv).$$

Comparing this with the terms of the functions [5090p], depending on the angle $2cv+2v-2mv$, we shall get, for $(2c+2-2m).C''^{(0)}_2$, the terms of the numerator of [5245]; namely,

[5245c]
$$-\frac{3}{2}m^2.\frac{(10-19m+8m^2)}{8.(c+1-m)} - \frac{3m^2.(1-m)}{2-2m+c} - \frac{9m^2}{16.(1-m)} \qquad 1$$
$$-2A'^{(0)}_2+3A^{(2)}_2-3A^{(0)}_2 ; \qquad 2$$

as will appear by the following examination of the functions [5090p], divided by the common factor

[5245d] $$c^2.\frac{a^2.dv}{\sqrt{a_{,}}}.\cos.(2cv+2v-2mv).$$

The function [5082s line 10] contains a term, depending on the angle $2cv-2v+2mv$, deduced from [4885 line 10]; and, we find in [4885 line 11], a similar expression

[5245e] $-\frac{3}{2}m^2.\frac{(10-19m+8m^2)}{8(c+1-m)}$, corresponding to the first term of [5245c]. The term neglected, in [5090b line 7], produces the second term of [5245c], $-\frac{3m^2.(1-m)}{2-2m+c}$; and, that in [5090b line 12], is $-\frac{9m^2}{16(1-m)}$; as in the third term of [5245c]. The term of $a\delta u$

[5245f] [5242] produces, in [5090p line 2], the term $-2.A'^{(0)}_2$ [5245c line 2]. The term neglected in [5090g line 1], gives $3.A^{(2)}_2$ [5245c line 2]. The term $-3A^{(0)}_2$ [5090g line 9], is the same as in [5045c line 2]. Now, substituting in [5243, 5245]. the

[5245g] values [5117, 5194, 5157, 5159], we get [5246, 5247]. Lastly, we get, from [5244], the expression [5248], by using the values [5247, 5194].

The expression of dt § 15, gives in $nt+\varepsilon$, the term,*

$$\frac{3A_2^{(4)}.ce'.\sin.(2v-2mv+cv-c'mv-\varpi+\varpi')}{2-3m+c}.$$ [5249]

This term is sensible, on account of the magnitude of the factor $A_2^{(4)}$ [5161]; [5249′]
it is, therefore, useful to consider the inequality relative to the argument $2v-2mv+cv-c'mv-\varpi+\varpi'$. The equation [4754] gives, by noticing only the terms we have developed in § 6,†

$$0=\frac{ddu}{dv^2}+u-\frac{\overline{m}^2}{a_{,}}.\frac{21.(2-3m).(4-3m+c)}{4.(2-3m+c)}.ee'.\cos.(2v-2mv+cv-c'mv-\varpi+\varpi').$$ [5250]

We shall put

$$A_2'^{(1)}.ee'.\cos.(2v-2mv+cv-c'mv-\varpi+\varpi')$$ [5251]

for the part of $a\delta u$ depending on the argument in question; we shall have,

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* (2929) This term is omitted in the product of the two quantities in [5090g line 20]; but, it is introduced in this place on account of the magnitude of $A_2^{(4)}$ [5161, 5249′]. [5249a]
Having noticed this part of the expression, it becomes convenient to introduce the smaller quantities, depending on the same angle, as in [5250—5257].

† (2930) The equation [5250] is obtained in the same manner as [5241], by dividing the terms of [4960c], depending upon the angle $2v-2mv+cv-c'mv$, by the common factor,

$$-\tfrac{21}{4}.\frac{\overline{m}^2}{a_{,}}.ee'.\cos.(2v-2mv+cv-c'mv);$$ [5250a]

and connecting the resulting quotients in the following manner. The term in [4870 line 7], gives $\frac{3}{2}(1-2m)$; in [4879 line 4], $\frac{1}{2}$; their sum is $2-3m$; adding this to the term [4892 line 7], $\frac{2(2-3m)}{2-3m+c}$, we obtain,

$$(2-3m).\left\{1+\frac{2}{2-3m+c}\right\}=(2-3m).\frac{(4-3m+c)}{2-3m+c}.$$ [5250b]

Connecting this with the common factor [5250a], and adding the two terms [5241c], we get the equation [5250]; in which we have corrected a typographical mistake in the original, where m^2 is written for $\frac{\overline{m}^2}{a_{,}}$. Comparing this with [5242a], we get, [5250c]

$$H=-\overline{m}^2.\frac{21.(2-3m).(4-3m+c)}{4.(2-3m+c)}.ee'; \qquad i=2-2m+c-c'm=2-3m+c, \text{ nearly};$$

substituting these in $a\delta u$ [5242b], and putting the result equal to the assumed value of this term of $a\delta u$ [5251], we get $A_2'^{(1)}$, as in [5252], using m^2 [5082h′]. [5250d]

[5252] $$A'^{(1)}_{2}=\frac{-21m^2.(2-3m).(4-3m+c)}{4.(2-3m+c).\{(2-3m+c)^2-1\}}.$$

Then, if we put

[5253] $$C''^{(1)}_{2}ee'.\sin.(2v-2mv+cv-c'mv-\varpi+\varpi')$$

for the part of $nt+\varepsilon$, relative to the same argument, we shall find, by § 15,*

[5254] $$C''^{(1)}_{2}=\frac{\frac{21m^2.(2-3m)}{4.(2-3m+c)}+\frac{21m^2}{4.(2-3m)}-2A'^{(1)}_{2}+3A^{(4)}_{2}}{2-3m+c}.$$

Reducing these formulas to numbers, we find,*

[5255] $$A'^{(1)}_{2}=-0,0134975;$$

[5256] $$C''^{(1)}_{2}=0,0534480;$$

which gives, in $nt+\varepsilon$, the inequality,

[5257] $$10'',17.\sin.(2v-2mv+cv-c'mv-\varpi+\varpi').$$

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* (2931) Proceeding as in [5215a, &c.], we find, that if $nt+\varepsilon$ contain a term of the form [5253], it will produce, in its differential ndt, the term,

[5253a] $$(2-3m+c).C''^{(1)}_{2}ee'.\cos.(2v-2mv+cv-c'mv),\quad\text{nearly};$$

and, by multiplying by $\frac{1}{n}=\frac{a^2}{\sqrt{a_{\prime}}}$ [5092c], it will produce in dt, the term,

[5253b] $$(2-3m+c).C''^{(1)}_{2}ee'.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.(2v-2mv+cv-c'mv).$$

Comparing this with the terms of the functions [5090p], depending on the same angle, we shall get, for $(2-3m+c).C''^{(1)}_{2}$, the terms of the numerator of [5254]; namely,

[5253c] $$\frac{21m^2.(2-3m)}{4.(2-3m+c)}+\frac{21m^2}{4.(2-3m)}-2A'^{(1)}_{1}+3A^{(4)}_{2};$$

as will appear by the following examination of the functions [5090p]. The term

[5253d] [5082s line 8] is the same as the first term of [5253c], $\frac{21m^2.(2-3m)}{4.(2-3m+c)}$; the term $\frac{21m^2}{4.(2-3m)}$, omitted in [5090b line 8], is the same as the second term of [5253c]; the term

[5253e] of $a\delta u$ [5251] produces, in [5090p line 2], the term $-2A'^{(1)}_{2}$, [5253c]; lastly, the term omitted in [5090g line 20] produces $3A^{(4)}_{2}$ in [5253c].

(2932) Substituting the values of c, m [5117] in [5252], we get [5255], and then, from [5254], we obtain [5256]. Substituting this value of $C''^{(1)}_{2}$, and the values

[5255a] of e, e' [5117, 5194]; in [5253] we get [5257].

It would seem, that the inequalities depending on the angles

$$2cv-2v+2mv\pm c'mv-2\varpi\mp\varpi'$$ [5257′]

ought to be sensible, on account of the great divisors which they acquire by integration; it is therefore important to ascertain them carefully. By following [5257″] the analysis before explained, noticing only quantities of the fourth order, and representing the corresponding part of $a\delta u$, by

$$a\delta u = A'^{(2)}_{1}\, e^2c'.\cos.(2cv-2v+2mv+c'mv-2\varpi-\varpi') \\ +A'^{(3)}_{1}e^2e'.\cos.(2cv-2v+2mv-c'mv-2\varpi+\varpi');$$ [5258]

we shall find, that the differential equation will become,*

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* (2933) We shall put, for brevity,

$$S = 2cv-2v+2mv+c'mv-2\varpi-\varpi'; \qquad D = 2cv-2v+2mv-c'mv-2\varpi+\varpi';$$ [5259 a]

and the assumed value of $a\delta u$ [5258] will become,

$$a\delta u = A'^{(2)}_{1}\, e^2c'.\cos.S+A'^{(3)}_{1}\, e^2c'.\cos.D.$$ [5259 b]

The terms of the equation [4961], depending upon the angles S, D, may be found in the functions which are enumerated in [4960e]; and, to obtain all the terms, we must review the whole calculation [4835—4961], in order to notice the quantities which have [5259c] the factor e^2c'. This great degree of accuracy is however unnecessary, on account of the smallness of the coefficients in [5259], which are of the fourth and higher orders; we shall, therefore, only notice the most important terms which are given by the author in [5259]. The first of the functions [4960e], which is noticed by him, is that in [4870]. [5259d] We may deduce this selected part of the factor of e^2c', from that of e^2 [4870 line 11], upon similar principles to those which are used in developing a function of e, c', by Taylor's theorem, by which the coefficient of e^2c', may be derived from that of e^2, &c. If we use the value of $\overline{m}^2$ [4865], and put, for brevity,

$$M = \frac{3\overline{m}^2}{8a_{,}}.(6+15m+8m^2).e^2 = \frac{3}{8a_{,}}.\frac{m'a^3}{a'^3}.(6+15m+8m^2).e^2;$$ [5259e]

we shall find, that the term of e^2 [4870 line 11] is represented by

$$M.\cos.(2cv-2v+2mv).$$ [5259f]

As this quantity does not contain c', it is evident, that it can be derived from the first member of [4870] $\frac{3m'u'^3}{2h^2u^3}.\cos.(2v-2v')$, by substituting the values of h, u, u', v' [5259g] [4825,4826,4837,4838]; then, neglecting the terms depending on c', and retaining only those connected with e^2. Now, by using merely the first terms of [4837,4838], and those depending on the first power of c', we have,

$$0=\frac{ddu}{dv^2}+u+\frac{3\overline{m}^2}{a_,}\cdot\left\{\begin{array}{l}\frac{7(2+11m+8m^2)}{16}-\frac{7(10+19m+8m^2)}{8.(2c-2+3m)} \\ -\frac{5A_1^{(8)}}{2c-2+3m}-\frac{1}{2}A_1^{\prime(2)}\end{array}\right\}.c^2e'.\cos(2cv-2v+2mv+c'mv-2\varpi-\varpi')$$ 1, 2

[5259]

$$+\frac{3\overline{m}^2}{a_,}\cdot\left\{\begin{array}{l}\frac{(10+19m+8m^2)}{8.(2c-2+m)}-\frac{(2+11m+8m^2)}{16} \\ -\frac{5A_1^{(9)}}{2c-2+m}-\frac{1}{2}A_1^{\prime(3)}\end{array}\right\}.c^2e'.\cos(2cv-2v+2mv-c'mv-2\varpi+\varpi');$$ 3, 4

[5259h] $$v'=mv+2e'.\sin.e'mv;\qquad u'=\frac{1}{a'}.\{1+e'.\cos.c'mv\}.$$

If we retain these terms of e', in forming the function [5259f], it will change $\frac{1}{a'^3}$ [5259e]

[5259i] into $\frac{1}{a'^3}.(1+3\,e'.\cos.c'mv)$, and $2mv$ into $2mv+4e'.\sin.c'mv$. By this means the function [5259f], will be increased by the terms,

[5259k] $$M.(3e'.\cos.c'mv).\cos.(2cv-2v+2mv)-M.(4e'.\sin.c'mv).\sin.(2cv-2v+2mv);$$

the second of these quantities being obtained by means of [61] Int. by putting

[5259l] $$z=2cv-2v+2mv,\qquad a=4e'.\sin.\,c'mv\ [5259i].$$

Reducing the terms of [5259k], by means of [17, 20] Int. we get, by using the abridged symbols [5259a],

[5259m] $$M.(3e'.\cos.c'mv).\cos.(2cv-2v+2mv)=\tfrac{3}{2}Me'.\cos.S+\tfrac{3}{2}Me'.\cos.D;$$

[5259n] $$-M.(4e'.\sin.c'mv).\sin.(2cv-2v+2mv)=2Me'.\cos.S-2Me'.\cos.D.$$

The sum of these two expressions gives the value of the function [5259k]; and, by re-substituting the value of M [5259e], it becomes,

[5259o] $$\tfrac{7}{2}Me'.\cos.S-\tfrac{1}{2}Me'.\cos.D=\frac{3\overline{m}^2}{a_,}.(6+15m+8m^2).c^2e'.\{\tfrac{7}{16}\cos.S-\tfrac{1}{16}.\cos.D\}.$$

A similar expression is obtained from the terms in [4879 line 7], putting $e=1$, and

[5259p] $M=-\frac{3\overline{m}^2}{2a_,}.(1+m).c^2$. Substituting this value of M, in the first number of [5259o], we get the terms,

[5259q] $$-\frac{3\overline{m}^2}{2a_,}.(1+m).c^2e'.\{\tfrac{7}{2}.\cos.S-\tfrac{1}{2}.\cos.D\}=\frac{3\overline{m}^2}{a_,}.(-4-4m).c^2e'.\{\tfrac{7}{16}.\cos.S-\tfrac{1}{16}.\cos.D\}.$$

Adding together the terms in the second members of [5259o, q], we get

[5259r] $$\frac{3\overline{m}^2}{a_,}.(2+11m+8m^2).c^2e'.\{\tfrac{7}{16}\cos.S-\tfrac{1}{16}.\cos.D\};$$

which are the same as the first term, connected with S [5259 line 1]; and the second term,

[5259s] connected with D, in [5259 line 3].

therefore, we shall have, [5259r']

We may proceed in a somewhat similar manner with the term in [4892 line 10], taking in the first place its differential, so as to make it of a like form; and, after reducing the products, which introduce the angles S, D, again integrating, to correspond to the integral in the first member of [4892]. Now, if we put [5259s'] [5259t]

$$M=\frac{3\,\overline{m}^2}{4a_{\prime}}(10+19\,m+8\,m^2)\,.\,e^2, \qquad [5259t']$$

the differential of [4892 line 10] becomes,

$$Mdv.\sin.(2cv-2v+2mv) = Mdv.\cos.(2cv-2v+2mv-90^d). \qquad [5259u]$$

The second of these expressions may be derived from [5259f], by decreasing the angle $2cv-2v+2mv$ by 90^d; which requires that S, D [5259a] should be changed into $S-90^d$, $D-90^d$, respectively, and then multiplying by dv. The same changes being made in the resulting correction, in the first member of [5259o], we obtain, [5259u']

$$\tfrac{7}{2}Me'.dv.\cos.(S-90^d)-\tfrac{1}{2}Me'.dv.\cos.(D-90^d)=\tfrac{7}{2}Me'.dv.\sin.S-\tfrac{1}{2}Me'.dv.\sin.D. \qquad [5259v]$$

Now, integrating this second expression, according to the directions in [5259t], we get the additional terms of [4892], as in the first member of the following equation, and, by re-substituting the value of M [5259t'], we get its second member,

$$-\frac{7Me'}{2.(2c-2+3m)}.\cos.S+\frac{Me'}{2.(2c-2+m)}.\cos.D$$
$$=\frac{3\,\overline{m}^2}{8a_{\prime}}.(10+19m+8m^2).e^2e'.\left\{-\frac{7}{2c-2+3m}.\cos.S+\frac{1}{2c-2+m}.\cos.D\right\}. \qquad [5259w]$$

The terms of this last expression are the same as the second term connected with S [5259 line 1], and the first term connected with D, in [5259 line 3].

The next terms of [5259] arise from the part of the function [4934 or 4932k] which is included in the table [4931p]. For, if we take, in the first column of this table, the term

$$A_1^{(8)}ee'.\cos.(cv+c'mv) \quad [4931p\text{ line }22], \qquad [5259x]$$

and in the second column, the term

$$-\frac{5e}{2}.\sin.(2v-2mv-cv),$$

which occurs also in [4931p line 17], it produces, by the process used in [4931n], the term

$$-\frac{6\overline{m}^2}{a_{\prime}}\cdot\frac{5e^2e'}{2.(2c-2+3m)}.A_1^{(8)}.\cos.(2cv-2v+2mv+c'mv); \qquad [5259x']$$

which is the same as the term depending on $A_1^{(8)}$ [5259 line 2]. In like manner, by combining the term

[5260] $$A'^{(2)}_1 = \frac{-3m^2}{1-\frac{3}{2}m^2-(2c-2+3m)^2}\cdot\left\{\begin{array}{l}\frac{7.(2+11m+8m^2)}{16} \\ -\frac{7.(10+19m+8m^2)-40A^{(8)}_1}{8.(2c-2+3m)}\end{array}\right\};$$ 1 2

$$A^{(9)}_1 c c'.\cos.(cv-c'mv) \quad [4931p \text{ line } 23],$$

with the same term

$$-\frac{5e}{2}.\sin.(2v-2mv-cv), \quad \text{in } [4931p \text{ line } 17],$$

we get, by the method used in [4931n], the term,

[5259y] $$-\frac{6\overline{m}^2}{a_{,}}\cdot\frac{5e^2e'}{2.(2c-2+m)}.A^{(9)}_1.\cos.(2cv-2v+2mv-c'mv), \quad \text{as in } [5259 \text{ line } 4].$$

The function [4908 line 1] contains the term $-\frac{3\overline{m}^2}{2a_{,}}.a\delta u$; and, by substituting the value of $a\,\delta u$ [5259b], we get,

[5259z] $$-\frac{3\overline{m}^2}{2a_{,}}.e^2e'.\{A'^{(2)}_1.\cos.S+A'^{(3)}_1.\cos.D\}, \quad \text{as in } [5259 \text{ lines } 2, 4].$$

This includes all the terms noticed by the author in [5259]; there are other terms, having the factor $\overline{m}^2.m.e^2e'$, which he has neglected on account of their smallness. Connecting these terms with $\frac{ddu}{dv^2}+u$ [5241e], it becomes as in [5259].

[5260a] * (2934) Taking separately into consideration the terms in the two first lines of [5259], which depend on the angle $S=2cv-2v+2mv+c'mv$ [5259a], they become

[5260b] of the same form as in [4990a], by putting

[5260c] $$H=3\overline{m}^2.\left\{\frac{7.(2+11m+8m^2)}{16}-\frac{7.(10+19m+8m^2)}{8.(2c-2+3m)}-\frac{5A^{(8)}_1}{2c-2+3m}-\tfrac{1}{2}A'^{(2)}_1\right\}.e^2e';$$

[5260d] $$i=2c-2+2m+c'm=2c-2+3m, \quad \text{nearly}.$$

The corresponding term of au, or $a\delta u$, is represented by $Pa.\cos.(iv+\beta)$ [4998c]; and, if we compare it with the assumed form of this term of $a\delta u$, in [5258 line 1],

[5260d'] we get $Pa=A'^{(2)}_1e^2e'$; hence [4998a] becomes, by multiplying by a, and substituting this value of Pa,

[5260e] $$0=(1-i^2).A'^{(2)}_1e^2e'+\frac{Ha}{a_{,}}.$$

[5260f] Substituting in this, the value of H [5260c], rejecting the common factor e^2e', and using m^2 [5082h'], we get [5260]. Proceeding in the same way with the terms depending on the angle D [5258 line 2, and 5259 lines 3, 4], corresponding to

[5260g] $$i=2c-2+2m-c'm=2c-2+m, \quad \text{nearly},$$

we easily obtain the value of $A'^{(3)}_1$ [5261].

$$A_1^{\prime(3)} = \frac{-3m^2}{1-\frac{3}{2}m^2-(2c-2+m)^2} \cdot \left\{ \frac{(10+19m+8m^2-40A_1^{(9)})}{8.(2c-2+m)} - \frac{(2+11m+8m^2)}{16} \right\}. \qquad [5261]$$

If we denote the corresponding part of $nt+\varepsilon$ by,*

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* (2935) If we take the differential of the term of $nt+\varepsilon$ [5262 line 1], depending on the angle $S=2cv-2v+2mv+c'mv$, and multiply it by $\frac{1}{n}=\frac{a^2}{\sqrt{a_{\prime}}}$ [5092c], putting also $c'=1$, we get, in dt, the term,

$$dt=(2c-2+3m).C_1^{\prime(2)}\,e^2c'.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.S. \qquad [5261a]$$

Substituting in this, the assumed value of $C_1^{\prime(2)}$ [5263], we find, that the result is represented by the function [5261c], or the numerator of the expression [5263], multiplied by the common factor $e^2c'.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.S$; and it will appear by the examination in [5261f–x], that the corresponding terms of the value of dt [5090p], neglecting the same factor [5261b], agree exactly with this function [5261c]; [5261b]

$$\frac{21m^2.(10+19m+8m^2)+120m^2.A_1^{(8)}}{16.(2c-2+3m)} - \frac{21m^2.(2+3m)}{4.(2-3m+c)} - \frac{63m^2}{16.(2-3m)} \qquad 1$$
$$-2A_1^{\prime(2)}+3A_1^{(7)}-3A_2^{(4)}+3A_1^{(8)}..A_1^{(1)}. \qquad 2 \qquad [5261c]$$

By a similar process with the term depending on the angle $D=2cv-2v+2mv-c'mv$, and the assumed value of $C_1^{\prime(3)}$ [5264], we find, as in [5261f–x], that the corresponding terms of dt [5090p], neglecting the common factor $e^2c'.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.D$, are represented by the function [5261e], corresponding to the numerator of [5264]; [5261d]

$$\frac{-3m^2.(10+19m+8m^2)+120m^2..A_1^{(9)}}{16.(2c-2+m)} + \frac{3m^2.(2+m)}{4.(2-m-c)} + \frac{9m^2}{16.(2-m)} \qquad 1$$
$$-2A_1^{\prime(3)}+3A_1^{(6)}-3A_2^{(3)}+3A_1^{(9)}..A_1^{(1)}. \qquad 2 \qquad [5261e]$$

We shall now proceed in the examination of the functions [5261c, e] in order to prove, that they agree with those in [5090p]. The first term of [5090p line 1] depends upon the function [5082s], which, when fully developed, contains terms of the required form, with the factor e^2c'. The terms of this function, which are retained by the author, may be derived, in a very simple manner, from those depending on e^2 [5082s line 10]; namely, [5261f]

$$\tfrac{3}{2}m^2.\tfrac{1}{4}(10+19m+8m^2).\frac{e^2}{2c-2+2m}.\cos.(2cv-2v+2mv); \qquad [5261g]$$

by the process used in [5259s′—w]. For, if we substitute in the expression [5261g], the value of the common factor

$$\tfrac{3}{2}m^2=\tfrac{3}{2}\overline{m}^2.\frac{a}{a_{\prime}} \quad [5082r, \&c.], \quad \text{and} \quad M=\frac{3\overline{m}^2}{4a_{\prime}}.(10+19m+8m^2).e^2 \quad [5259t'], \qquad [5261h]$$

[5262]
$$C_1'^{(2)}.\, e^2e'.\sin.(2cv-2v+2mv+c'mv-2\varpi-\varpi') \qquad 1$$
$$+\, C_1'^{(3)}.\, e^2e'.\sin.(2cv-2v+2mv-c'mv-2\varpi+\varpi'), \qquad 2$$

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it becomes,

[5261i]
$$\frac{\frac{1}{2}Ma}{2c-2+2m}.\cos.(2cv-2v+2mv).$$

Taking the differential of this expression, according to the direction in [5259s'], it becomes,

[5261k]
$$-\tfrac{1}{2}Ma\,.\,dv\,.\sin.(2cv-2v+2mv).$$

This is of the same form as the first member of [5259u], and may be derived from it, by changing M into $-\frac{1}{2}Ma$; so that, if we make the same change in the resulting
[5261l] terms, in the second member of [5259w], we shall get the corresponding terms of [5082s], depending on e^2e'; namely,

[5261m]
$$\frac{3\overset{2}{\overline{m}}.a}{16a_{\prime}}\,.\,(10+19m+8m^2).e^2e'.\left\{\frac{7}{2c-2+3m}.\cos.S-\frac{1}{2c-2+m}.\cos.D\right\}.$$

Re-substituting the value of $\frac{3}{2}m^2$ [5261h], we find, that the coefficient of $e^2e'.\cos.S$, is the same as the first term of [5261c], which is connected with the factor $10+19m+8m^2$;
[5261n] and, the coefficient of $e^2e'.\cos.D$, is the same as the first term of [5261e], connected with the same factor.

The second of the functions enumerated in [5090p], is that contained in the table [5090b]. We shall make the following additions, so as to include those terms of e^2e'
[5261o] which were neglected in the former computation. The three columns of the table are here marked the same as in [5090b]; and all the terms in the third column have the common factor $\frac{a^2.dv}{\sqrt{a_{\prime}}}$.

	(Col. 1.) Terms of the first factor in [5081], between the braces.	(Col. 2.) Factor Q' [5081 or 5082s].	(Col. 3.) Corresponding terms of [5081], or [5261c,e].		
	$-2e.\cos.cv$	$\frac{21m^2.(2+3m).ee'}{4.(2-3m-c)}.\cos.(2v-2mv-cv-c'mv)$	$-\frac{21m^2.(2+3m).e^2e'}{4.(2-3m-c)}.\cos.S$		1
		$-\frac{3m^2.(2+m).ee'}{4.(2-m-c)}.\cos.(2v-2mv-cv+c'mv)$	$+\frac{3m^2.(2+m).e^2e'}{4.(2-m-c)}.\cos.D$	Common factor $\frac{a^2.dv}{\sqrt{a_{\prime}}}$.	2
[5261p]	$\frac{3}{2}e^2.\cos.2cv$	$-\frac{21m^2e'}{4.(2-3m)}.\cos(2v-2mv-c'mv)$	$-\frac{63m^2.e^2e'}{16.(2-3m)}.\cos.S$		3
		$+\frac{3m^2e'}{4.(2-m)}.\cos.(2v-2mv+c'mv)$	$+\frac{9m^2.e^2e'}{16.(2-m)}.\cos.D.$		4

The terms in the third column, depending on $\cos.S$, correspond to the two last terms of [5261c line 1]; and, those depending on $\cos.D$, correspond to the two last terms of [5261e line 1].

We shall have, by § 15,

$$C'^{(2)}_1 = \frac{\left\{\begin{array}{l} \frac{21m^2.(10+19m+8m^2)+120m^2.A_1^{(8)}}{16.(2c-2+3m)} - \frac{21m^2.(2+3m)}{4.(2-3m+c)} - \frac{63m^2}{16.(2-3m)} \\ -2A'^{(2)}_1+3A_1^{(7)}-3A_2^{(4)}+3A_1^{(8)}.A_1^{(1)}. \end{array}\right\}}{2c-2+3m} \quad \begin{array}{l}1\\2\end{array} \qquad [5263]$$

$$C'^{(3)}_1 = \frac{\left\{\begin{array}{l} \frac{-3m^2.(10+19m+8m^2)+120m^2.A_1^{(9)}}{16.(2c-2+m)} + \frac{3m^2.(2+m)}{4.(2-m-c)} + \frac{9m^2}{16.(2-m)} \\ -2A'^{(3)}_1+3A_1^{(6)}-3A_2^{(3)}+3A_1^{(9)}.A_1^{(1)}. \end{array}\right\}}{2c-2+m} \quad \begin{array}{l}1\\2\end{array} \qquad [5264]$$

Reducing these formulas to numbers, we find,*

The function [4904], or $a\delta u$, contains the two terms [5259b], and these produce, in the first term of [5090p line 2], the terms,

$$-2.\frac{a^2.dv}{\sqrt{a_{\prime}}}.c^2e'.\left\{A'^{(2)}_1.\cos.S+A'^{(3)}_1.\cos.D\right\}; \qquad [5261q]$$

which are the same as the terms depending on $A'^{(2)}_1$, $A'^{(3)}_1$ [5261c, e].

The next of the functions enumerated in [5090p], is the function [5090g]; and we have, in line 25, the neglected term $3A_1^{(7)}.c^2e'.\cos.S$, corresponding to the second term of [5261c line 2]; and, in [5090g line 24], the neglected term $3A_1^{(6)}c^2e'.\cos.D$, as in the second term of [5261e line 2]. Again, the term $-2A_2^{(4)}e'.\cos.(2v-2mv-c'mv)$, in the first column of [5090g], being combined with $3c^2.\cos.2cv$, in the second column, gives $-3A_2^{(4)}c^2e'.\cos.S$; corresponding to the term depending on $A_2^{(4)}$ [5261c]. In like manner, the term $-2A_2^{(3)}e'.\cos.(2v-2mv+c'mv)$ [5090g col. 1], being combined with the same term $3c^2.\cos.2cv$, in column 2, gives $-3A_2^{(3)}c^2e'.\cos.D$; corresponding to $A_2^{(3)}$ [5261e]. [5261r] [5261s] [5261t]

The last of the functions [5090p], is that in the table [5090i]; and we have, in the first column of this table, the term $A_1^{(1)}c.\cos.(2v-2mv-cv)$; in the second column, the omitted term $3A_1^{(8)}ee'.\cos.(cv+c'mv)$, which produce, in the third column, the term $\frac{3}{2}A_1^{(1)}.A_1^{(8)}c^2e'.\cos.S$, neglecting the common factor $\frac{a^2.dv}{\sqrt{a_{\prime}}}$. In like manner, we have, in the first column, the term $A_1^{(8)}cc'.\cos(cv+c'mv)$; in the second column, $3A_1^{(1)}e.\cos(2v\text{-}2mv\text{-}cv)$: these produce also, in the third column, an equal term $\frac{3}{2}A_1^{(1)}.A_1^{(8)}c^2e'.\cos.S$. Adding this to the preceding term, we get $3A_1^{(1)}.A_1^{(8)}c^2e'.\cos.S$, corresponding to the last term of [5261c]. In exactly the same way, we find, that the terms of $a\delta u$, depending on $A_1^{(1)}c.\cos.(2v-2mv-cv)$, $A_1^{(9)}cc'.\cos.(cv-c'mv)$, produce, in the third column of [5090i], the expression $3A_1^{(1)}.A_1^{(9)}c^2e'.\cos.D$, corresponding the last term of [5261e]. [5261u] [5261v] [5261w] [5261x]

* (2936) Substituting the values [5117,5191,5157,&c.] in [5260,5261,5363,5264], we get $A'^{(2)}_1$, $A'^{(3)}_1$, $C'^{(2)}_1$, $C'^{(3)}_1$ [5265]; and then, [5262] becomes as in [5266], [5265a]

[5265]
$$A_1'^{(2)} = 0,744932; \quad 1$$
$$A_1'^{(3)} = -0,0153320; \quad 2$$
$$C_1'^{(2)} = 0,563137; \quad 3$$
$$C_1'^{(3)} = -0,0235572. \quad 4$$

Hence we obtain, in $nt+\varepsilon$, the two following inequalities;

[5266]
$$5^s,88. \quad \sin.(2cv-2v+2mv+c'mv-2\varpi-\varpi') \quad 1$$
$$-0^s,25. \quad \sin.(2cv-2v+2mv-c'mv-2\varpi+\varpi'). \quad 2$$

The inequalities depending on the arguments $2cv\pm c'mv-2\varpi\mp\varpi'$, are very easily found, by considering the expression of dt [5081]. This expression gives, in that of $nt+\varepsilon$, the inequalities,*

[5267]
$$\frac{3A_1^{(8)}e^2e'}{2c+m}.\sin.(2cv+c'mv-2\varpi-\varpi')$$
$$+\frac{3A_1^{(9)}e^2e'}{2c-m}.\sin.(2cv-c'mv-2\varpi+\varpi');$$

[5267′] and it is evident, that they are the only terms of the fourth order, depending on these arguments. By reducing them to numbers, we obtain, in $nt+\varepsilon$, the two following inequalities;

[5268]
$$-3^s,16.\sin.(2cv+c'mv-2\varpi-\varpi')$$
$$+4^s,50.\sin.(2cv-c'mv-2\varpi+\varpi').$$

It is evident, from the expression of dt, [5081], that the inequality depending on the argument $4v-4mv-cv+\varpi$, must be sensible.† To

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* (2937) The functions [5090p], which represent the value of dt, give, in [5090g lines 26, 27], the two following terms, which were omitted in that table;

[5267a]
$$dt=\frac{a^2.dv}{\sqrt{a_\prime}}.e^2e'.\{3A_1^{(8)}.\cos.(2cv+c'mv)+3A_1^{(9)}.\cos.(2cv-c'mv)\}.$$

Dividing this by $\frac{1}{n}=\frac{a^2}{\sqrt{a_\prime}}$ [5092c]; and then integrating, we get, in $nt+\varepsilon$, the two terms [5267]. A slight inspection of the functions enumerated in [5090p] shows, that there are no other terms of this form, and of the fourth order.

[5268a] † (2938) This will fully appear, by the inspection of the terms of $nt+\varepsilon$, depending on this argument in [5280, 5281, 5283].

determine it, we shall represent the corresponding term of $a\delta u$, by

$$a\delta u = A'^{(4)}_{3} e.\cos.(4v-4mv-cv+\varpi). \qquad [5269]$$

It is evident, that there cannot be produced such terms in the differential equation in u [4961], except by the variation of the terms of the equation [4754], depending on the disturbing force.* We have developed these variations in § 8. The first is $-\frac{3m'.u'^3.\delta u}{2h^2.u^4}$; and it produces no term of the fourth order, depending on $\cos.(4v-4mv-cv+\varpi)$. The second variation is,† [5269'] [5270]

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* (2939) This is evident, from the examination of the functions [4960e], which compose the equation [4961]; since the terms enumerated between [4866] and [4901] do not contain the angle $4v-4mv-cv$. The next of these functions is that in [4908], which arises from the development of $-\frac{3m'.u'^3.\delta u}{2h^2.u^4}$ [4908g]; and we find, by inspection, that it contains no term of the fourth order, depending on this angle. The same may be observed of the functions [4913, 4918, 4922, 4928, 4942—4960]. The three remaining functions [4911, 4925, 4934], which are derived from the quantities mentioned in [5271, 5273, 5275], produce some important terms, as will be seen in the following notes. [5269a] [5269b]

† (2940) This expression is the same as that in [4910], which is developed in [4911, 4918]. The term depending on the second of these functions, is retained by the author, though it produces only terms of the fifth order [5271c]. Substituting the values [4937n, 5082h'], in [5271], it becomes,

$$-\frac{9\overline{m}^2}{2a_{,}}.a\delta u.\cos.(2v-2mv)+\frac{3\overline{m}^2}{a_{,}}.\delta v'.\sin.(2v-2mv). \qquad [5271a]$$

Now we have, in [4904 line 2, 4917], the terms of $a\delta u$, $\delta v'$, represented by

$$a\delta u = A^{(1)}_{1} e.\cos.(2v-2mv-cv); \qquad \delta v' = -2m.A^{(1)}_{1} e.\sin.(2v-2mv-cv). \qquad [5271b]$$

The first of these quantities produces, in [5271a], the term,

$$-\frac{9\overline{m}^2}{4a_{,}}.A^{(1)}_{1} e.\cos.(4v-4mv-cv);$$

and the second, the term,

$$\frac{3\overline{m}^2}{a_{,}}.A^{(1)}_{1}.mc.\cos.(4v-4mv-cv); \qquad [5271c]$$

the sum of these two expressions is evidently equal to that in [5272].

[5271] $$-\frac{9m'.u'^3}{2h^2.u^4}.\delta u.\cos.(2v-2v')+\frac{3m'.u'^3}{h^2.u^3}.\delta v'.\sin.(2v-2v')\,;$$

it produces the term,

[5272] $$-\frac{3\overset{2}{m}}{4a_{,}}.(3-4m).A_1^{(1)}e.\cos.(4v-4mv-cv+\varpi).$$

The third variation is,*

[5273] $$\frac{6m'.u'^3}{h^2.u^4}\cdot\frac{du}{dv}\cdot\frac{\delta u}{u}.\sin.(2v-2v')-\frac{3m'.u'^3}{2h^2.u^4}\cdot\frac{d\delta u}{dv}.\sin.(2v-2v')$$
$$+\frac{3m'.u'^3.\delta v'}{h^2.u^4}\cdot\frac{du}{dv}.\cos.(2v-2v').$$

It produces the term,

[5274] $$-\frac{3\overset{2}{m}}{4a_{,}}.(2-2m-c).A_1^{(1)}e.\cos.(4v-4mv-cv+\varpi).$$

Lastly, the fourth variation is,†

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* (2941) The three terms of the function [5273] are the same as those in [1924], [5273a] which are developed in [1925]. The first of them is computed in [1923e, &c.], and evidently contains no term of the fourth order, depending on the proposed angle. The same is to be observed of the third term of [5273], which is computed in [4923q]. The second term of [5273] is,
$$-\frac{3m'.u'^3}{2h^2.u^4}\cdot\frac{d\delta u}{dv}.\sin.(2v-2v'),$$
and it becomes, by substituting the values [4937n, 4865],

[5273b] $$-\frac{3\overset{2}{m}}{2a_{,}}\cdot\frac{a.d\delta u}{dv}.\sin.(2v-2mv).$$

Now, $\frac{a.d\delta u}{dv}=\frac{d.(a\delta u)}{dv}$ contains, in [4904 line 2], the term,

[5273c] $$-(2-2m-c).A_1^{(1)}e.\sin.(2v-2mv-cv)\,;$$

hence the preceding expression produces the term [5274], as is evident, by multiplying, and reducing the product by means of [17] Int.

† (2942) The expression [5275] is ths same as that in [4931], which is developed in [5274a] [4934], by means of the functions enumerated in [4932k]; namely, [4931p, u, 4932a, f]. The first of these functions [4931p] contains in its second line, a term of the *fifth* order, depending on $A_2^{(0)}$, which is neglected on account of its smallness. It also contains a term depending on $A_1^{(1)}$, which is omitted in [4931p line 6], but is easily found, by the

$$\frac{12m'}{h^2.a}.\{1+\tfrac{3}{4}\gamma^2.\cos.(2gv-2\theta)\}.\int\frac{u'^3.dv}{u^4}.\left\{\frac{\delta u}{u}.\sin.(2v-2v')+\tfrac{1}{2}.\delta v'.\cos.(2v-2v')\right\}$$
$$-\left\{\frac{dd.\delta u}{dv^2}+\delta u\right\}.\int\frac{3m'.u'^3.dv}{h^2.u^4}.\sin.(2v-2v')-\frac{9m'}{h^2.a}.\int\frac{u'^2.\delta u'}{u^4}.dv.\sin.(2v-2v');$$ [5275]

it produces the term,

$$-\frac{3\overline{m}^2}{a_\prime}.\frac{(2-5m)}{4-4m-c}.A_1^{(1)}e.\cos.(4v-4mv-cv+\varpi).$$ [5276]

Therefore, the differential equation in u becomes, by noticing only these terms,*

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method there used, to be, $$-\frac{6\overline{m}^2}{a_\prime}.\frac{1}{4-4m-c}..A_1^{(1)}e.\cos.(4v-4mv-cv);$$ [5274b]

neglecting e'^2, and putting,

$$k=A_1^{(1)}e;\quad k'=1;\quad i=2-2m-c;\quad i'=2-2m\quad [4931l].$$ [5274b′]

This is the same as the first term of the expression [5276]. The next of the functions [5274a] is [4931u]; which may evidently be neglected on account of its smallness. We then have [4932a], which contains, in its first line, a term depending on $A_1^{(1)}$, which is omitted in the table, but is easily found to be,

$$\frac{6\overline{m}^2}{a_\prime}.\frac{m}{4-4m-c}.A_1^{(1)}e.\cos.(4v-4mv-cv).$$ [5274c]

The last of the functions [5274a] is [4932f]; this also contains a term which is neglected in its sixth line, and is represented by,

$$-\frac{3\overline{m}^2}{2a_\prime}.\left\{\frac{(2-2m-c)^2-1}{2.(1-m)}\right\}.A_1^{(1)}e.\cos.(4v-4mv-cv).$$ [5274d]

This may be reduced to another form, by observing, that, by putting $c=1$, we have, very nearly, $(2-2m-c)^2-1=(1-2m)^2-1=-4m$; so that this term may be represented by,

$$\frac{3\overline{m}^2}{a_\prime}.m.A_1^{(1)}e.\cos.(4v-4mv-cv),\quad\text{or,}\quad\frac{3\overline{m}^2}{a_\prime}.\frac{3m}{4-4m-c}.A_1^{(1)}e.\cos(4v-4mv-cv),\text{ nearly}.$$ [5274e]

Adding this to the term [5274c], we get,

$$\frac{3\overline{m}^2}{a_\prime}.\frac{5m}{4-4m-c}.A_1^{(1)}e.\cos.(4v-4mv-cv),$$ [5274f]

as in the second term of [5276].

* (2913) Adding together the terms [5272, 5274, 5276], we obtain those connected

[5277] $$0=\frac{ddu}{dv^2}+u-\frac{3\overline{m}^2}{4a_{\prime}}\cdot\left\{5-6m-c+\frac{4.(2-5m)}{4-4m-c}\right\}.A_1^{(1)}e.\cos.(4v-4mv-cv+\varpi).$$

If we substitute in it, for $a\delta u$, the term,

[5278] $$a\delta u=A'^{(4)}_3 e.\cos.(4v-4mv-cv+\varpi)\quad[5269],$$

we shall find,*

[5279] $$A'^{(4)}_3=\frac{-\frac{3m^2}{4}.\left\{5-6m-c+\frac{4.(2-5m)}{4-4m-c}\right\}.A_1^{(1)}}{(4-4m-c)^2-1}.$$

Then, if we put

[5280] $$C'^{(4)}_3 e.\sin.(4v-4mv-cv+\varpi),$$

for the corresponding term of $nt+\varepsilon$, we shall have, by § 15,†

[5277a] with $\cos.(4v-4mv-cv)$, in [5277]; to which we must add, as in [5241c], the two terms $\frac{ddu}{dv^2}+u$, to obtain [5277].

[5279a] * (2944) Substituting, in [5277], the assumed value of au, or $a\delta u$ [5278]; and that of m^2 [5082h'], we easily obtain the expression of $A'^{(4)}_3$ [5279].

[5281a] † (2945) Proceeding as in [5261a, &c.], we may take the differential of the term of $nt+\varepsilon$ [5280], and multiply it by $\frac{1}{n}=\frac{a^2}{\sqrt{a_{\prime}}}$ [5092c], and we shall get, in dt, the term,

[5281b] $$dt=(4-4m-c).C'^{(4)}_3 e.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.(4v-4mv-cv).$$

Substituting the assumed value of $C'^{(4)}_3$, we find, that the result is represented by the function [5281d], or the numerator of the expression [5281], multiplied by the common factor

[5281c] $$e.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.(4v-4mv-cv);$$

and, we shall find, upon examination, that if we neglect the consideration of this factor, the corresponding terms of the value of dt [5090p] will agree with the function [5281d].

[5281d] $$\left\{\frac{3m^2}{4.(1-m)}+\frac{3m^2.(1-m)}{4-4m-c}+3A_2^{(0)}\right\}.A_1^{(1)}-2A'^{(4)}_3.$$

[5281e] To prove this, we shall now compare this expression with that which is derived from the functions [5090p]. The first of these functions depends on [5082s], or the value of Q' [5082q]; and this last function contains, in [5082q line 2], the two terms,

$$C''^{(4)}_{3} = \frac{\left\{ \frac{3m^2}{4.(1-m)} + \frac{3m^2.(1-m)}{4-4m-c} + 3A^{(0)}_{2} \right\} . A^{(1)}_{1} - 2A'^{(4)}_{3}}{4-4m-c} . \qquad [5281]$$

Reducing these formulas to numbers, we obtain,

$$A'^{(4)}_{3} = -0,000799351 ; \qquad [5282]$$

$$C''^{(4)}_{3} = 0,00294934 . \qquad [5282']$$

Hence arises, in the expression of $nt+\varepsilon$, the inequality,*

$$33'',38.\sin.(4v-4mv-cv+\varpi). \qquad [5283]$$

The inequality depending on $4v-4mv-2cv+2\varpi$, may also be sensible; the expression of ndt [5081, &c.] contains the following quantity,†

$$-\tfrac{1}{2}a\times\text{function }[4931p], \quad -\tfrac{1}{2}a\times\text{function }[4932a]. \qquad [5281e']$$

Now, the omitted term of $-\frac{1}{2}a\times[4931p \text{ line } 6]$, produces the term,

$$\frac{3\overline{m}^2.a}{a_{\prime}} \cdot \frac{1}{4-4m-c}.A^{(1)}_{1}e.\cos.(4v-4mv-cv); \qquad [5281f]$$

and, that in $-\frac{1}{2}a\times[4932a \text{ line } 1]$, gives,

$$-\frac{3\overline{m}^2.a}{a_{\prime}} \cdot \frac{m}{4-4m-c} . A^{(1)}_{1}e.\cos.(4v-4mv-cv). \qquad [5281f']$$

The sum of these becomes, by using the value of m^2 [5082h'],

$$\frac{3m^2.(1-m)}{4-4m-c}.A^{(1)}_{1}\,e.\cos.(4v-4mv-cv), \qquad [5281g]$$

corresponding to the second term of [5281d]. The next of the functions [5090p] is [5090b]; it produces nothing. The term depending on [4904] produces $-2A'^{(4)}_{3}$, as in the last term of [5281d]. The term omitted in [5090g line 11] gives $\frac{3m^2}{4.(1-m)}A^{(1)}_{1}$ as in the first term of [5281d]. Lastly, the double combination of the terms [5281h]

$$A^{(0)}_{2}.\cos.(2v-2mv), \qquad A^{(1)}_{1}e.\cos.(2v-2mv-cv) \quad [5090i], \qquad [5281i]$$

gives, by a process like that in [5261u, &c.], the term $3.A^{(0)}_{2}.A^{(1)}_{1}$, as in the third term of [5281d].

* (2946) Substituting, in [5280], the values of $C''^{(4)}_{3}$, c [5282', 5194], it becomes as in [5283]. [5283a]

† (2947) If we examine the functions contained in the expression of dt [5090p],

[5284] $$\tfrac{3}{2}.(A_1^{(1)})^2.e^2.dv.\cos.(4v-4mv-2cv+2\varpi).$$

Hence arises, in $nt+\varepsilon$, the term,

[5285] $$\frac{3.(A_1^{(1)})^2.e^2.\sin.(4v-4mv-2cv+2\varpi)}{2.(4-4m-2c)}.$$

It is evident, that it is the only term of the fourth order, depending on the same argument, which enters in the expression of $nt+\varepsilon$. Reducing it to seconds, it becomes,

[5286] $$22^s,26.\sin.(4v-4mv-2cv+2\varpi).$$

We shall see, in [5578 line 10], that the tables of Mason and Burg both
[5286′] agree in making the coefficient of this inequality nearly equal to 15^s; which seems to indicate, that this coefficient is well determined by observation; consequently, the difference 7^s, between this result and the
[5286″] preceding computation, must arise, in a great measure, from the quantities of the fifth order, which we have neglected. To prove this, and to show, at the same time, that a farther approximation will diminish the difference between
[5286‴] the theory and observation, we shall proceed to determine this coefficient, so as to include quantities of the *fifth* order.

We shall denote the corresponding term of $a\delta u$, by

[5287] $$a\delta u = A_3^{\prime(5)}e^2.\cos.(4v-4mv-2cv+2\varpi).$$

It is evident, that terms of this kind are produced in the differential equation [4961], solely, by the variation of the term of the equation [4754], arising from the disturbing force. We have just given the four variations of these
[5287′] terms [5270—5275]. The first variation [5270], produces no term of the fifth order,* depending on $\cos.(4v-4mv-2cv+2\varpi)$. The second variation

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[5284a] we shall find, that the term [5281], with the factor $\frac{a^2}{\sqrt{a,}}$, is omitted in [5090*i* line 7], and this is the only term of the *fourth* order, depending on the angle $4v-4mv-2cv$.
[5284b] The integral of this expression, being divided by $\frac{1}{n}$ [5092*c*], gives the corresponding term of $nt+\varepsilon$ [5285]. Substituting the values [5117, 5194, 5158], we get [5286].

[5286a] * (2948) The computation of the terms of the formula [5290], is made in the same manner as that in [5277]; the former being multiplied by e^2, and the latter by e; so

[5271] produces the term,*

$$\frac{9\,\overline{m}^2}{4a_{\prime}}\cdot\{2A_1^{(1)}-A_1^{(11)}\}.e^2.\cos.(4v-4mv-2cv+2\varpi). \qquad [5288]$$

The terms of the fifth order, depending on $\cos.(4v-4mv-2cv+2\varpi)$, which are produced by the third variation [5273], mutually destroy each other, except in quantities of the sixth order.† Lastly, the fourth variation [5288′]

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that the similar terms of [5290], are of a higher order by unity, than those of [5277]; and, in retaining terms of the fifth order [5286‴], we shall have to notice only the same functions as in [5271, 5273, 5275]; that in [5270] being, as in the former case, insensible. [5286b]

* (2949) The terms of the second variation [5271] are developed in [4911, 4918]. The last of these expressions produces, in [4918, 4918f], terms of the *sixth* order, containing e^2, which may be neglected. The first term of [5271], is found as in [4910l], by multiplying the function [4910k] by $2a\delta u$ [4904]. Now, if we combine the term, [5288a]

$$-\frac{9\overline{m}^2}{4a_{\prime}}.\cos.(2v-2mv) \quad [4910k \text{ line } 1], \quad \text{with} \quad 2A_1^{(11)}e^2.\cos(2cv-2v+2mv) \quad [4904 \text{ line } 12], \qquad [5288b]$$

we get,

$$-\frac{9\overline{m}^2}{4a_{\prime}}.A_1^{(11)}e^2.\cos.(4v-4mv-2cv), \quad \text{as in } [5288]; \qquad [5288b']$$

and, if we combine the term,

$$\frac{9\overline{m}^2}{2a_{\prime}}.e.\cos.(2v-2mv-cv) \quad [4910k \text{ line } 2], \quad \text{with} \quad 2A_1^{(1)}e.\cos.(2v-2mv-cv) \quad [4904 \text{ line } 2], \qquad [5288c]$$

we get,

$$\frac{9\overline{m}^2}{2a_{\prime}}.A_1^{(1)}e^2.\cos.(4v-4mv-2cv), \quad \text{as in } [5288]. \qquad [5288d]$$

The remaining terms of the sixth and higher orders are neglected.

† (2950) The first term of [5273] is represented, in [4923c], by the expression $-4a\delta u\times$ function [4879]; and the only terms of [4904], necessary to be retained, are those depending on $A_1^{(11)}$, $A_1^{(1)}$, which may produce quantities connected with e^2. Now, by retaining only the quantities which are multiplied by $e^2.\cos.(4v-4mv-2cv)$, we find, that the term depending on $A_2^{(0)}$ [4904 line 1], combined with [4879 line 7], produces a term of the sixth order, which may be neglected. The term, [5288e] [5288f]

$$-4A_1^{(1)}e.\cos.(2v-2mv-cv) \quad [4904 \text{ line } 2],$$

multiplied by,

$$\frac{3\overline{m}^2}{4a_{\prime}}.e.\cos.(2v-2mv-cv) \quad [4879 \text{ line } 1], \qquad [5288g]$$

[5275] produces the term,*

[5289] $$\frac{3\overline{m}^2}{2a_{\prime}}\cdot\left\{\frac{5A_1^{(1)}-2A_1^{(11)}}{2-2m-c}+\frac{(1-2m).(3-2m).(10+19m+8m^2)}{4.(2c-2+2m)}.A_2^{(0)}+\frac{A_1^{(11)}}{2-2m}\right\}c^2.\cos\begin{pmatrix}4v-4mv\\-2cv+2\varpi\end{pmatrix}.$$

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produces

[5288h] $$-\frac{3\overline{m}^2}{2a_{\prime}}.A_1^{(1)}e^2.\cos.(4v-4mv-2cv);$$

and this is the only term of [5288c], which is of sufficient importance to be noticed. The second term of [5273] is developed as in [4923x]; and a little attention will show, that the only term of $a\delta u$ [4904], necessary to be noticed, is $A_1^{(1)}e.\cos.(2v-2mv-cv)$,

[5288i] corresponding, in [4923w], to $k=A_1^{(1)}e$, $i=2-2m-c=1$, nearly. Combining this with the term $k'.\cos.v'$ [4923u, 4885 line 2], which is nearly equal to

$$-2e.\cos.(2v-2mv-cv);$$

making $k'=-2e$, $v'=2v-2mv-cv$; we get, for the second term of [4923x], the expression,

[5288k] $$-\frac{3\overline{m}^2}{4a_{\prime}}.ikk'.\cos.(4v-4mv-2cv)=\frac{3\overline{m}^2}{2a_{\prime}}.A_1^{(1)}e^2.\cos.(4v-4mv-2cv).$$

This is equal to the term [5288h], but has a different sign; so that the two terms destroy each other, as in [5288']; therefore, the whole of this function may be neglected.

* (2951) The fourth variation [5275 or 4931] is developed in the functions which are

[5289a] enumerated in [4932k]; namely, [4931p, u, 4932a, f]. Now, the first of these functions [4931p] gives a term, which is produced, by combining, in the manner explained in [4931n], the term $A_1^{(1)}e.\cos.(2v-2mv-cv)$, of the first column of [4931p line 6], with the term $-\frac{5}{2}e.\sin.(2v-2mv-cv)$, in its second column; which give,

[5289b] $$\frac{6\overline{m}^2}{a_{\prime}}.\tfrac{5}{2}A_1^{(1)}.\frac{e^2}{4-4m-2c}.\cos.(4v-4mv-2cv),\text{ as in the first term of [5289].}$$

In like manner, the combination of the term $A_1^{(11)}e^2.\cos.(2cv-2v+2mv)$, in the first column of [4931p line 25], with $\sin.(2v-2mv)$, in its second column, gives,

[5289c] $$-\frac{6\overline{m}^2}{a_{\prime}}.A_1^{(11)}.\frac{e^2}{4-4m-2c}.\cos.(4v-4mv-2cv),\text{ as in the second term of [5289].}$$

The function [4931u] contains nothing of the proposed form and order. The function [4932a] contains a quantity depending on $A_1^{(1)}$ of the sixth order, which is neglected by

[5289d] the author, on account of its smallness. The last of these functions is [4932f]; it contains a term of the proposed form, which is found by combining the term $A_2^{(0)}.\cos.(2v-2mv)$, in column 1 of [4932f line 1], with the term of its second column, corresponding to

[5289e] [4885 line 10], $$-\frac{(10+19m+8m^2)}{4.(2c-2+2m)}.e^2.\cos.(2cv-2v+2mv).$$

This term, found by the method in [4932e'], is,

Therefore, the differential equation [4961] becomes, by noticing only these terms,*

$$0=\frac{ddu}{dv^2}+u+\frac{3\overline{m}^2}{2a_{\prime}}\cdot\left\{\begin{array}{l}3A_1^{(1)}-\tfrac{3}{2}A_1^{(11)}+\dfrac{5A_1^{(1)}-2A_1^{(11)}}{2-2m-c}+\dfrac{A_1^{(11)}}{2-2m}\\ +\dfrac{(1-2m).(3-2m).(10+19m+8m^2)}{4.(2c-2+2m)}\cdot A_2^{(0)}\end{array}\right\}.e^2.\cos.(4v-4mv-2cv+2\varpi). \qquad [5290]$$

Substituting $A_3'^{(5)}e^2.\cos.(4v-4mv-2cv+2\varpi)$ for $a\delta u$, we obtain, [5291]

$$A_3'^{(5)}=\frac{3m^2}{2}\cdot\frac{\left\{\begin{array}{ll}3A_1^{(1)}-\tfrac{3}{2}A_1^{(11)}+\dfrac{5A_1^{(1)}-2A_1^{(11)}}{2-2m-c}+\dfrac{A_1^{(11)}}{2-2m} & 1\\ +\dfrac{(1-2m).(3-2m).(10+19m+8m^2)}{4.(2c-2+2m)}\cdot A_2^{(0)} & 2\end{array}\right\}}{(4-4m-2c)^2-1}. \qquad [5292]$$

If we denote the corresponding term of $nt+\varepsilon$ by

$$C_2''^{(5)}e^2.\sin.(4v-4mv-2cv+2\varpi), \qquad [5293]$$

we shall have, by § 15,†

$$\frac{3\overline{m}^2}{2a_{\prime}}.\{4.(1-m)^2-1\}.\frac{(10+19m+8m^2)}{4.(2c-2+2m)}.A_2^{(0)}e^2.\cos.(4v-4mv-2cv)\,; \qquad [5289f]$$

and, by using the reduction $4.(1-m)^2-1=(1-2m).(3-2m)$ [4961h], it is easily reduced to the form of the term depending on $A_2^{(0)}$ [5289]. Lastly, the term $A_1^{(11)}e^2.\cos.(2cv-2v+2mv)$ [4932f col. 1], being combined with $\frac{\cos.(2v-2mv)}{2-2m}$, in col. 2, [5289$g$] gives $\frac{3\overline{m}^2}{2a_{\prime}}\cdot\frac{A_1^{(11)}}{2-2m}$, as in [5289]; observing, that in this case, the factor $-(i^2-1)$ [4932c'] is nearly equal to unity; since $i=2c-2+2m=2m$, nearly. The remaining terms of these functions are neglected by the author, on account of their smallness.

* (2952) Adding the terms [5288, 5289], and connecting the sum with the two terms $\frac{ddu}{dv^2}+u$ [5241c], we get [5290]. Substituting in it the assumed value of $a\delta u$ [5291], [5290a] and using m^2 [5032h'], we get $A_3'^{(5)}$ [5292].

† (2953) By proceeding in the same manner as in [5245a—c], we find, that the term of $nt+\varepsilon$ [5293], gives, in dt, the term,

$$dt=(4-4m-2c).C_2''^{(5)}e^2.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.(4v-4mv-2cv). \qquad [5294a]$$

Comparing this with the terms of dt [5030p], we get, for $C_2''^{(5)}$, the same expression as in [5294]; or, in other words, the terms of the functions [5030p], being divided by the

[5294] $$C_2'^{(5)} = \frac{\left\{\begin{array}{ll} \frac{-3m^2.(5A_1^{(1)}-2A_1^{(11)})}{4.(2-2m-c)} - \frac{27m^4}{64.(1-m)}.\frac{(10+19m+8m^2)}{2c-2+2m} - 2A_3'^{(5)} & 1 \\ +3A_3'^{(4)} + \frac{3m^2}{4.(1-m)}.A_1^{(11)} - \frac{3m^2.A_1^{(1)}}{4-4m-c} - \frac{3m^2.(10+19m+8m^2)}{8.(2c-2+2m)}.A_2^{(0)} & 2 \\ +\frac{3}{2}.(A_1^{(1)})^2+3A_2^{(0)}.A_1^{(11)}-6A_2^{(0)}.A_1^{(1)} & 3 \end{array}\right\}}{4-4m-2c}.$$

[5294b] common factor $$e^2.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.(4v-4mv-2cv),$$

produce the terms in the three lines of the numerator of the expression [5294], as will appear by the following examination. The first of the functions [5090p] represents the value of Q' [5082s or 5082q]. Now, the last of these expressions [5082q line 2] contains the terms,

[5294c] $$-\tfrac{1}{2}.\text{function}\,[4885]-\tfrac{1}{2}.\text{function}\,[4889]-\tfrac{1}{2}a.\text{function}\,[4931p]-\tfrac{1}{2}a.\text{function}\,[4932a];$$

[5294d] which we shall separately examine. The mere inspection of [4885, 4889], shows, that they produce nothing of the proposed form and order. The next of these functions is

[5294e] $-\frac{1}{2}a\times$function [4931p]; and, as the common factor of the terms of this table is $\frac{6\overline{m}^2}{a_{\prime}}$, we have, by using [5082h'], $-\frac{1}{2}a\times\frac{6\overline{m}^2}{a_{\prime}} = -3m^2$; Then, by combining, as in [4931n], the term $A_1^{(1)}e.\cos.(2v-2mv-cv)$, of the first column of [4931p line 6], with $-\frac{5}{2}c.\sin.(2v-2mv-cv)$, of its second column, we get,

[5294f] $$-3m^2.\tfrac{5}{2}A_1^{(1)}e^2.\frac{1}{4-4m-2c}.\cos.(4v-4mv-2cv),\quad \text{as in the first term of [5294 line 1].}$$

In like manner, the term $A_1^{(11)}e^2.\cos(2cv-2v+2mv)$, of the first column of [4931p line 25], being combined with $\sin.(2v-2mv)$, of its second column, gives,

[5294g] $$3m^2.A_1^{(11)}e^2.\frac{1}{4-4m-2c}.\cos.(4v-4mv-2cv),\quad \text{as in the second term of [5294 line 1].}$$

The last of the functions [5294c] is that depending on [4932a], which upon examination, is found to produce no term of the required form and order. Besides these terms, arising from the value of Q', [5082s or 5082q], we must add a term we have formerly

[5294h] neglected, in finding the value of $\frac{3}{8}M_2^2$, which makes a part of the value of Q' [5082m]. For, it is evident, that in deducing the value of $\frac{3}{8}M_2^2$ [5082o], from that of M_2 [5082n], we have neglected the term,

[5294i] $$\frac{27m^4}{16.(1-m)}.P_{\prime}.\cos.(4v-4mv+V)\quad [5082o\text{ line }2];$$

supposing, as in [5082n, &c.] that $P_{\prime}.\cos.(2v-2mv+V)$ represents any term between the braces in [4885]. Now, if we take this term, in [4885 line 10], we shall have,

[5294i'] $$P_{\prime} = -\frac{(10+19m+8m^2)}{4.(2c-2+2m)}.e^2;$$

Reducing these formulas to numbers, we find, [5294']

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and, by changing the signs of the angle in [4885 line10], to make it conform to [5082n], we get $V=-2cv$; substituting these in [5294i], it becomes, without noticing the factor $\frac{a^2.dv}{\sqrt{a_,}}$,

$$-\frac{27m^4}{64.(1-m)}\cdot\frac{(10+19m+8m^2)}{2c-2+2m}.e^2.\cos.(4v-4mv-2cv)\ ; \qquad [5294k]$$

as in the third term of [5294 line 1]. The next of the functions [5090p], is that in the table [5090b]. This contains a quantity, which is found by combining the term $-2c.\cos.cv$ [5090b col.1], with the term $\frac{3m^2}{4-4m-c}.A_1^{(1)}e.\cos.(4v-4mv-cv)$ of Q', in [5090b col.2]. This term was omitted in [4931p line 6], and also in $-\frac{1}{2}a\times$ function [4931p], in computing the value of Q' [5082q line 2]. The combination of these two terms of the table [5090b], gives $-\frac{3m^2}{4-4m-c}..A_1^{(1)}e^2.\cos.(4v-4mv-cv)$, corresponding to the third term of [5294 line 2]. In the original work the divisor $4-4m-c$ is inaccurately printed, being put equal to $2-2m-c$. [5294l] [5294m]

The term depending on $-[4904]\times 2.\frac{a^2.dv}{\sqrt{a_,}}$, in [5090p line 2], gives $-2.A_3^{\prime(5)}$, by using the term of $a\delta u$ [5287]; this agrees with the last term in [5294 line 1]. The next of the functions [5090p] is that in [5090g], which produces several terms. Thus, by combining the term of $-2a\delta u=-2A_3^{\prime(4)}e.\cos.(4v-4mv-cv)$ [5278], which would occur in the first column of [5090g], with the term $-3e.\cos.cv$, in its second column, we get $3A_3^{\prime(4)}e^2.\cos.(4v-4mv-2cv)$, corresponding to the first term in [5294 line 2]. In the next place, the term $-2A_1^{(11)}e^2.\cos.(2cv-2v+2mv)$, in column 1 [5090g line 28], being combined with $-\frac{3m^2}{4.(1-m)}.\cos.(2v-2mv)$, in column 2, gives, [5294n] [5294o]

$$\frac{3m^2}{4.(1-m)}..A_1^{(11)}e^2.\cos.(4v-4mv-2cv), \qquad [5294p]$$

corresponding to the second term of [5294 line 2]. Again, the term $-2.A_2^{(0)}.\cos(2v-2mv)$, in the first column of [5090g line 1], being combined with that term of its second column, which is contained in the first line of [5090e], by means of the term [5082s line 10]; namely,

$$\frac{3m^2.(10+19m+8m^2)}{8.(2c-2+2m)}.e^2.\cos.(2cv-2v+2mv),$$

produces the term,

$$\frac{-3m^2.(10+19m+8m^2)}{8.(2c-2+2m)}..A_2^{(0)}e^2.\cos.(4v-4mv-2cv), \qquad [5294q]$$

corresponding to the last term in [5294 line 2]. The last of the functions [5090p] is [5090i]. This produces, in [5090i line 7], the term $\frac{3}{2}.(A_1^{(1)})^2.e^2.\cos.(4v-4mv-2cv)$, as in the first term of [5294 line 3]. The combination of the term $A_2^{(0)}.\cos.(2v-2mv)$, in the [5294r]

[5295] $$*A'^{(5)}_3 = 0,00436374;$$

[5295′] $$C'^{(5)}_2 = 0,0249067;$$

which gives, in $nt+\varepsilon$, the inequality,

[5296] $$15^s,46 \,.\, \sin.(4v - 4mv - 2cv + 2\varpi).$$

The difference between this result and that of the tables is insensible; and
[5296′] we see, by this calculation, that, to make the theory agree wholly with observations, relative to all the lunar inequalities, it is only necessary to carry on the approximation to quantities of the fifth order. This appears also from the calculation of the inequality depending on $\sin.(v-mv)$, in which we
[5296″] have noticed quantities of that order. For, we shall hereafter find [5589], that the result of this analysis, compared with that which is obtained by observation, gives nearly the same value of the sun's parallax, as that which is deduced from the transits of Venus over the sun.

The inequality depending on the argument $cv-v+mv-\varpi$ may be sensible, on account of the smallness of the coefficient of v. To determine this inequality, we shall put,

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first column of [5090*i*], with $3A^{(11)}_1 e^2.\cos.(2cv-2v+2mv)$, in the second column,
[5294s] produces $\frac{3}{2} A^{(0)}_2 . A^{(11)}_1 e^2.\cos.(4v-4mv-2cv)$; and the similar combination of $A^{(11)}_1 e^2.\cos.(2cv-2v+2mv)$, in the first column, with $A^{(0)}_2.\cos.(2v-2mv)$, in the second column, gives an equal quantity, $\frac{3}{2} A^{(0)}_2 . A^{(11)}_1 e^2.\cos.(4v-4mv-2cv)$; the sum of these
[5294t] two terms is $3 A^{(0)}_2 . A^{(11)}_1 e^2.\cos.(4v-4mv-2cv)$, corresponding to the second term of [5294 line 3]. In exactly the same way, we find, that the *double* combination of the terms $A^{(0)}_2.\cos.(2v-2mv)$, $A^{(1)}_1 e.\cos.(2v-2mv-cv)$ [5090*i*], produces in that table, or in the value of $3.(a\delta u)^2$, the term $3A^{(0)}_2 . A^{(1)}_1 e.\cos.(4v-4mv-cv)$; and, if we multiply this by $-4e\cos.cv$, which was neglected in [5090*k*], it produces the term,

[5294u] $$-6A^{(0)}_2 . A^{(1)}_1 e^2.\cos.(4v-4mv-2cv),$$

corresponding to the last term of [5294 line 3].

* (2951) Substituting in [5292] the values [5117, 5157 — 5167], we get for $A'^{(5)}_3$, a
[5295a] value which is nearly equal to that in [5295]. Using the same in [5294], we get for $C'^{(5)}_2$ a value which exceeds, by a small quantity, that in [5295′]. This difference is owing
[5295b] to the inaccurate divisor of the term mentioned in [5294*m*]. Substituting, in [5293], the
[5295c] values of $C'^{(5)}_2$, e [5295′, 5194], we get [5296]; the coefficient would be increased about 1^s, by correcting the divisor as in [5294*m*].

$$a\delta u = A_1^{\prime\prime(6)}.\frac{a}{a'}.e.\cos.(cv-v+mv-\varpi)\,; \qquad [5297]$$

and

$$C_1^{\prime\prime(6)}.\frac{a}{a'}.e.\sin.(cv-v+mv-\varpi)\,, \qquad [5298]$$

for the parts of $a\delta u$ and $nt+\varepsilon$, depending on this argument, we shall have, by noticing the perturbations of the earth by the moon,*

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* (2955) In this note we shall put, for brevity,

$$\text{v} = cv - v + mv - \varpi\,; \qquad [5297a]$$

and we shall then examine successively the functions enumerated in [4960e], for the purpose of collecting together the terms of the equation [4961], which depend on the angle v, and correspond to the annexed expression;

$$a\,\delta u = A_1^{\prime\prime(6)}.\frac{a}{a'}.e.\cos.\text{v} \quad [5297]. \qquad [5297b]$$

We shall retain the terms depending on the first power of e, neglecting the higher powers of this quantity, and the terms depending on e', &c. The *first* of the functions [4960e], [5297c] which contains terms depending on v, is [4872], being the development of $\frac{9m'.u'^4}{8h^2.u^4}.\cos.(v-v')$. Now, in retaining only the terms of the order e, we obtain from [4870h],

$$\frac{9m'.u'^4}{8h^2.u^4} = -\frac{9m'.a^4}{8a_{,}.a'^4}.4\,e.\cos.cv. \qquad [5297d]$$

Moreover, by neglecting terms of the order me, we have $\cos.(v-v') = \cos.(v-mv)$ [4837]. Multiplying this by the preceding expression [5297d], and retaining only the terms depending on the angle v, we get, in [4872], the expression,

$$-\frac{9m'.a^4}{8a_{,}.a'^4}\,2e\,.\cos.\text{v}\,; \qquad [5297e]$$

and, by substituting

$$\frac{m'.a^4}{a_{,}.a'^4} = \frac{m'.a^3}{a_{,}a'^3}.\frac{a}{a'} = \frac{\overset{2}{m}}{a_{,}}.\frac{a}{a'} = \frac{m^2}{a}.\frac{a}{a'} \quad [4865, 5082h'], \qquad [5297e']$$

it becomes, by a slight reduction,

$$-\frac{36m^2}{16a}.\frac{a}{a'}.e.\cos.\text{v}. \qquad [5297f]$$

The *second* of the functions, which must be noticed, is

$$-\frac{3m'.u'^4}{8h^2.u^5}.\frac{du}{dv}.\sin.(v-v') \quad [4880]\,; \qquad [5297g]$$

it was neglected in [4881], on account of its smallness, and not inserted in [4960e]. Substituting the values [4937n, 5297e'], it becomes successively,

[5299] $$A_1^{\prime(6)}=\frac{-3m^2.\{\frac{1}{16}.(21-11c-7m-20\mu)-5A_1^{(17)}-\frac{5}{8}A_1^{(1)}\}}{(c-1+m).\{1-(c-1+m)^2-\frac{3}{2}m^2\}};$$

[5297h] $$-\frac{3m'.a^4}{8a_,a'^4}.a.\frac{du}{dv}.\sin.(v-mv)=-\frac{3m^2}{8a}.\frac{a}{a'}.a.\frac{du}{dv}.\sin.(v-mv).$$

[5297i] Now, by putting $c=1$, we have in [4878a] the term $\frac{du}{dv}=-\frac{e}{a}.\sin.cv$. If we substitute this, in the last expression [5297h], it produces the term,

[5297k] $$\tfrac{3}{16}.\frac{m^2}{a}.\frac{a}{a'}.e.\cos.\text{v}.$$

Adding it to the term [5297f] we find, that the sum becomes

[5297l] $$-\frac{33m^2}{16a}.\frac{a}{a'}e.\cos.\text{v}.$$

The *third* of the functions, to be noticed in [1960e], is [4892]. This is found, as in [4892a], by multiplying the sum of the functions [4885, 4889] by the function [4890]. If we retain terms of the order e only, we may put the function [4890] equal

[5297l'] to $\frac{1}{a}$. Multiplying this by the function [4885], it produces nothing of the proposed form and order; so that it is only necessary to notice the terms arising from the other function,

[5297m] or $\frac{1}{a}\times$ function [4889], taking care to insert the terms depending on e, which were neglected in the development of [4889].

If we substitute, in the first member of [4889], the values $h^2=a_,$, $u'=a'^{-1}$ [4937n], we get, by dividing by a,

[5297n] $$\frac{1}{a}\times\text{ function [4889]}=-\frac{3m'}{4a_,a.a'^4}.\int\frac{dv}{u^5}.\sin.(v-v').$$

Now, by retaining only the terms of the first order depending on e, we have, in [4826, 4837],

[5297o] $$u=a^{-1}.(1+e.\cos.cv)\ ;\quad v'=mv-2me.\sin.cv.$$

From the first of these equations we get [5297q]; and from the second, we deduce [5297r]; which is easily reduced to the form [5297s]. Multiplying together the two expressions

[5297p] [5297q, s], and then the product by dv, retaining only the terms depending on $e.\sin.\text{v}$, we get [5297t];

[5297q] $$u^{-5}=a^5.(1-5e.\cos.cv);$$

[5297r] $$\sin.(v-v')=\sin.(v-mv)+2me.\sin.cv.\cos.(v-mv)\quad\text{[60] Int.}$$

[5297s] $$=\sin.(v-mv)+me.\sin.\text{v}+\ \&c.;$$

[5297t] $$\frac{dv}{u^5}.\sin.(v-v')=a^5.\,dv.(\tfrac{5}{2}e+me).\sin.\text{v}.$$

$$C'^{(6)}_1 = \frac{\dfrac{-\frac{3}{2}.m^2.\{\frac{1}{8}.(5+2m-10\mu)-5A_1^{(17)}-\frac{5}{8}A_1^{(1)}\}}{c-1+m}-2A'^{(6)}_1+3A_1^{(17)}+3A_1^{(1)}.A_1^{(17)}+\dfrac{3m^2}{8.(1-m)}}{c-1+m}. \qquad [5300]$$

The integral of [5297t], being substituted in the second member of [5297n], it becomes as in the second member of [5297u], which is easily reduced to the form in the third member, by using [5297e'],

$$\frac{1}{a}\times \text{function [4889]} = \frac{3m'.a^4}{4a_{,}.a'^4}\cdot\frac{(\frac{5}{2}c+mc)}{c-1+m}\cdot\cos.\mathrm{v} = \frac{3m^2}{16a}\cdot\frac{a}{a'}\cdot e\cdot\frac{(10+4m)}{c-1+m}\cdot\cos.\,\mathrm{v}. \qquad [5297u]$$

Adding this to the sum of the terms in [5297l], we obtain,

$$\frac{3m^2}{16a}\cdot\frac{a}{a'}\cdot e\cdot\left\{-11+\frac{(10+4m)}{c-1+m}\right\}\cdot\cos.\mathrm{v} = \frac{3m^2}{16a}\cdot\frac{a}{a'}\cdot e\cdot\left\{\frac{21-11c-7m}{c-1+m}\right\}\cdot\cos.\,\mathrm{v}, \qquad [5297v]$$

being the same as the three first terms, connected with e, in [5298f].

The *fourth* of the functions selected from [4960e], is that in [4908 line 1], which, by using [5082h'], becomes

$$-\frac{3m^2}{2a}\cdot a\delta u = -\frac{3m^2}{2a}\cdot A'^{(6)}_1\cdot\frac{a}{a'}\cdot e\cdot\cos.\mathrm{v} \quad [5297h]\text{; as in the last term of [5298}f\text{].} \qquad [5297w]$$

The *fifth* of these functions [4960e] is [4934], or rather, that part of it which is contained in [4934p]. For, by combining the terms $A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$ [4934p line 31], in its first column, with $-\frac{5}{2}e.\sin.(2v-2mv-cv)$, in its second column, by the method in [4931n], using also m^2 [5082h'], we get the term,

$$-\frac{3m^2}{a}\cdot\frac{a}{a'}\cdot\frac{e}{c-1+m}\cdot 5A_1^{(17)}.\cos.\,\mathrm{v}\,; \qquad [5297x]$$

which is the same as that depending on $A_1^{(17)}$ in [5298f].

The *sixth* of these functions [4960e] is [4946], or, it is rather the part

$$\frac{5\overline{m}^2}{4a_{,}}\cdot\frac{a}{a'}\cdot\int a\delta u\,.\,dv\,.\,3\,.\sin.(v-v'), \qquad [5297y]$$

which is contained in [4945 line 2]. Substituting, for $a\delta u$, the term,

$$A_1^{(1)}.\,e\,.\cos.(2v-2mv-cv) \quad [4904\text{ line }2], \qquad [5297z]$$

and using $v'=mv$ [4837], also m^2 [5082h']; it becomes, by noticing only the part which depends upon the angle v,

[5300′] * From these we deduce,

$$\frac{15\overline{m}^2}{4a_{\prime}}\cdot\frac{a}{a'}\cdot A_1^{(1)}\cdot e\cdot\int dv.\sin.(v-mv).\cos.(2v-2mv-cv)=\frac{15\overline{m}^2}{8a_{\prime}}\cdot\frac{a}{a'}\cdot A_1^{(1)}\cdot e\cdot\int dv\,.\sin.\mathrm{v}$$

[5298a] $$=-\frac{15m^2}{8a}\cdot\frac{a}{a'}\cdot A_1^{(1)}\cdot e\cdot\frac{1}{c-1+m}\cdot\cos.\mathrm{v}.$$

This is the same as the term depending on $A_1^{(1)}$ in [5298f].

The *seventh* of the functions, [4960e] is that in [4957], which is derived from [4956] or [4882], being a term of the function $\frac{2}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}$ [4881′]. This is to be multiplied

[5298b] by the factor $\frac{ddu}{dv^2}+u=\frac{1}{a}$, nearly [5297l]; to obtain the corresponding term of [4754] or [5298f]. Now, the variation of [4956] contains the term,

[5298b′] $$\frac{3m'\mu}{2h^2}\cdot\int\frac{u'^4dv}{u^5}\cdot\sin.(v-v')\ [4956d]\,;$$

and by substituting $h^2=a_{\prime}$, $u'=a'^{-1}$ [4937n], it becomes,

[5298c] $$\frac{3m'.\mu}{2a_{\prime}.a'^4}.\int\frac{dv}{u^5}\,.\sin.(v-v')\,.$$

If we notice only the first term of the second member of [5297t] we easily find, that the term depending on e, in the preceding expression, can be put under the form,

[5298d] $$\frac{3m'.\mu}{2a_{\prime}.a'^4}\cdot\int a^5dv\,.\tfrac{5}{2}e\,.\sin.\mathrm{v}=-\frac{3m'.\mu}{2a_{\prime}.a'^4}\cdot a^5\,.\tfrac{5}{2}\,e\,.\frac{1}{c-1+m}\cdot\cos.\mathrm{v}\,;$$

and, if we reduce it, by means of [5297e′], it becomes,

[5298e] $$-\frac{3m^2}{16a}\cdot\frac{a^2}{a'}\cdot e\cdot\frac{20\mu}{c-1+m}\,.\cos.\mathrm{v}\,.$$

[5298e′] Multiplying this by the factor $\frac{1}{a}$ [5298b], we get the term of [5298f], depending on μ. Hence we find, by adding together the terms [5297v, w, x, 5298a, e′], and connecting them with $\frac{ddu}{dv^2}+u$ [5241c], the following equation, for the determination of this part of u;

[5298f] $$0=\frac{ddu}{dv^2}+u+\frac{3m^2}{a}\cdot\frac{a}{a'}\cdot e.\left\{\frac{\frac{1}{16}(21-11c-7m-20\mu)-5A_1^{(17)}-\frac{5}{8}A_1^{(1)}}{c-1+m}\right\}.\cos.\mathrm{v}-\frac{3m^2}{2a}.A_1'^{(6)}.\frac{a}{a'}.e.\cos.\mathrm{v}.$$

Substituting in this, the assumed value of au or $a\delta u$ [5297b]; namely,

[5298g] $$a\delta u=A_1'^{(6)}\cdot\frac{a}{a'}\cdot e\,.\cos.\mathrm{v}\,,$$

we get, by reduction, the value of $A_1'^{(6)}$ [5299].

* (2956) The differential of the term of $nt+\varepsilon$ [5298], being multiplied by $\frac{1}{n}=\frac{a^2}{\sqrt{a_{\prime}}}$

$$A_1'^{(6)} = -0{,}260496\,;$$ [5301]

$$C_1'^{(6)} = -0{,}293763.$$ [5301']

[5092c], gives in dt the following term, using the abridged symbol v [5297a];

$$dt = (c-1+m).C_1'^{(6)}.\frac{a}{a'}\,.\,e\,.\,\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.\,\mathrm{v}.$$ [5300a]

Substituting the assumed value of $C_1'^{(6)}$ [5300], we find, that this expression of dt is represented by the function [5300c], or the numerator of the expression [5300] multiplied by the *common factor* $\frac{a}{a'}\,.\,e\,.\,\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.\mathrm{v}$; and, that this is correct, will appear by the [5300b] examination in [5300d—s] where we shall find, that the corresponding part of the value of dt [5090p], divided by the same common factor [5300b], is accurately represented by the function [5300c];

$$\frac{-\frac{3}{2}m^2.\{\frac{1}{8}.(5+2m-10\mu)-5A_1^{(17)}-\frac{5}{8}A_1^{(1)}\}}{c-1+m}-2A_1'^{(6)}+3A_1^{(17)}+3A_1^{(1)}.\,A_1^{(17)}+\frac{3\,m^2}{8.(1-m)}.$$ [5300c]

To prove this, we shall observe, that the *first* of the functions [5090p line 1] is

$$[5082s \text{ or } 5082q]\times\frac{a^2.\,dv}{\sqrt{a_{\prime}}} = Q'\times\frac{a^2.dv}{\sqrt{a_{\prime}}}\,;$$ [5300c']

and, by retaining only the terms in [5082q line 2], it becomes,

$$\{-\tfrac{1}{2}\text{function}[4885]-\tfrac{1}{2}\text{function}[4889]-\tfrac{1}{2}a.\text{function}[4931p]-\tfrac{1}{2}a.\text{function}[4932a]\}\times\frac{a^2.dv}{\sqrt{a_{\prime}}}.$$ [5300d]

The inspection of [4885] shows that it produces nothing of the proposed form and order in [5300c]. The next term of [5300d] is,

$$-\tfrac{1}{2}\,\text{function}\,[4889]\times\frac{a^2.\,dv}{\sqrt{a_{\prime}}}\,;$$ [5300e]

and, by substituting [5297u], it becomes,

$$-\frac{3m^2}{16}\,.\,\frac{a}{a'}\,.\,e\,.\frac{(5+2m)}{c-1+m}.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.\mathrm{v}\,;$$ [5300f]

and, if we neglect the common factor [5300b], it produces the two first terms of [5300c], depending on $5+2m$. In the table [4931p] we find a term which is produced by connecting the term $A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv)$, in the first column, with $-\frac{5}{2}e.\sin.(2v-2mv-cv)$, in [5300g] its second column. These give, in the third column of that table, the term,

$$-\frac{6\,\overline{m}^2}{a_{\prime}}\,.\,\tfrac{5}{2}e\,.\,\frac{1}{c-1+m}\,.\,A_1^{(17)}.\,\frac{a}{a'}\,.\cos.\,\mathrm{v}.$$

Substituting this in [5300d], and using the value of m^2 [5082h'], it produces the term,

$$\tfrac{3}{2}\,m^2.\,\frac{5}{c-1+m}\,.A_1^{(17)},\ \text{ as in the fourth term of [5300c].}$$ [5300h]

The last of the functions [5300d] produces nothing of importance. The *second* of

[5301''] Hence we obtain, in $nt+\varepsilon$, the inequality,

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the functions [5090p] is [5090b]; and, by combining the term $-2e.\cos.cv$, in its first column, with the term $-\frac{3}{8}m^2.\frac{a}{a'}.\frac{1}{1-m}.\cos.(v-mv)$, in its second column, or
[5300i] [5082s line 19], we get $\frac{3m^2}{8.(1-m)}$, connected with the common factor [5300b], as in the last term of [5300c]. The *third* of the functions [5090p] is $-$function[4904]$\times 2.\frac{a^2.dv}{\sqrt{a_{,}}}$; and, by substituting the value of $a\delta u$ [5297], and neglecting the common factor [5300b],
[5300k] we get $-2A_1^{(6)}$ [5300c]. The *fourth* of the functions [5090p] is [5090g]; and, by combining the term $-2A_1^{(7)}.\frac{a}{a'}.\cos.(v-mv)$, in the first column, with $-3e.\cos.v$, in the
[5300l] second column, we get $3A_1^{(7)}$, connected with the factor [5300b], as in the seventh term of [5300c]. The last of the functions [5090p] is [5090i]; and, if we combine
[5300m] $A_1^{(1)}e.\cos.(2v-2mv-cv)$, in its first column, with $3A_1^{(7)}.\frac{a}{a'}.\cos.(v-mv)$, in its second column, we get $\frac{3}{2}A_1^{(1)}.A_1^{(7)}$, connected with the factor [5300b]. In like manner, the combination of $A_1^{(7)}.\frac{a}{a'}.\cos.(v-mv)$, in the first column, with $3A_1^{(1)}e.\cos.(2v-2mv-cv)$,
[5300n] in its second column, gives the same term $\frac{3}{2}A_1^{(1)}.A_1^{(7)}$. The sum of these two terms is $3A_1^{(1)}.A_1^{(7)}$, as in the eighth term of [5300c]. We have yet to notice the terms of [5081], or of [5300c], corresponding to the parts of the equation [4961], which are contained in [5298a, c']. These terms of [5081] may be derived, in a very simple
[5300o] manner, from those of [4961], by the same process of derivation which is used in computing [5082l] from [4916f]; namely, by dividing this last expression by $-2a^{-1}$ [5082k—l];
[5300p] or rather, by multiplying it by $-\frac{1}{2}a$; and annexing the common factor $\frac{a^2.dv}{\sqrt{a_{,}}}$ [5081]. The propriety of using this method of derivation is manifest from the consideration, that the *first* of these terms [5298a] is derived from the function,

[5300q] $$\left(\frac{ddu}{dv^2}+u\right).\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}, \quad \text{in [4943—4945 line 2]};$$

and this function is very nearly equal to,

[5300r] $$\frac{1}{a}.\frac{2}{h^2}.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2} \quad [4890].$$

Moreover, the *second* of these terms [5298c'] is derived, as in [5298b, &c.], from the function [4956 or 4882], which is a part of the function [4881'], by multiplying it by
[5300s] $\frac{ddu}{dv^2}+u=\frac{1}{a}$, nearly [5298$b$]; and this last product is evidently equal to the function [5300q], from which the first term is derived. On the other hand, the corresponding terms of dt [5300u, v] arise from the function $Q'\times\frac{a^2.dv}{\sqrt{a_{,}}}$ [5300c], whose chief term, connected

$$* \quad -8'',31.(1+i).\sin.(cv-v+mv-\varpi).$$ [5302]

The inequality depending on the argument $v-mv+cv-\varpi$ is easily obtained from § 15; and it is evidently expressed by,†

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with Q, is $-\frac{a^2.dv}{\sqrt{a_{\prime}}}\cdot\frac{1}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}$ [5082m]; and this is evidently equal to the [5300t] product of the function [5300r] by the factor $-\frac{1}{2}a.\frac{a^2.dv}{\sqrt{a_{\prime}}}$, as in [5300$p$]. Now, if we multiply the expressions [5298a, e'] by the factor $-\frac{1}{2}a.\frac{a^2.dv}{\sqrt{a_{\prime}}}$, they become, respectively, as in [5300u,v];

$$+\frac{15m^2}{16}\cdot\frac{a}{a'}\cdot A_1^{(1)}c.\frac{1}{c-1+m}\cdot\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.\mathrm{v}$$ [5300u]

$$+\frac{3m^2}{16}\cdot\frac{a}{a'}\cdot c\cdot\frac{10\mu}{c-1+m}\cdot\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.\mathrm{v}.$$ [5300v]

Dividing these by the common factor [5300b], we obtain,

$$\frac{15m^2}{16}\cdot\frac{1}{c-1+m}\cdot A_1^{(1)}, \quad \text{and} \quad \frac{3m^2}{16}\cdot\frac{10\mu}{c-1+m};$$ [5300w]

which correspond to the fifth and third terms of [5300c], respectively. Hence it appears, that the value of $C_1''^{(6)}$ [5300] agrees with the preceding calculation. Substituting, in [5299, 5300], the values [5117, 5194, 5158, 5173], also that of $\mu=\frac{1}{59,6}$ [4320, 4948'], [5300x] we get, for $A_1''^{(6)}$, $C_1''^{(6)}$, nearly the same values as in [5301, 5301'].

* (2957) Substituting, in [5298], the values of $C_1''^{(6)}$, c, &c. [5301', 5194, 5221], [5302a] we get nearly the same expression as in [5302].

† (2958) The coefficient [5303] may be computed in the same manner as that in [5298 or 5300]; but the change of the divisor from $c-1+m$, which is of the order m, to $c+1-m$, of the order 2, enables us to neglect, in [5303], all the terms which [5303a] appear in [5300], except $A_1^{(17)}$. The term depending on $A_1^{(17)}$ is found in the same manner as in [5300l], by combining the term

$$-2A_1^{(17)}.\frac{a}{a'}.\cos.(v-mv),$$

in the first column of [5090g], with the term $-3e.\cos.cv$, in its second column; which gives, in the third column, the corresponding term of

$$dt=3A_1^{(17)}.\frac{a}{a'}.e.\frac{a^2.dv}{\sqrt{a_{\prime}}}.\cos.(v-mv+cv).$$ [5303b]

Integrating, and then dividing by $\frac{1}{n}=\frac{a^2}{\sqrt{a_{\prime}}}$ [5092c], we get the expression [5303]; and, by [5303c] using the values of e, m, c, &c. [5117, 5194, 5221], it becomes as in [5304].

[5303] $$\frac{3\,A_{1}^{(17)}}{1-m+c}\cdot\frac{a}{a'}\cdot e\,.\,\sin.(v-mv+cv-\varpi)\;;$$

consequently, it is equal to

[5304] $$-5'',01\,.\,(1+i)\,.\,\sin.(v-mv+cv-\varpi).$$

[5304'] By following the same process, we may determine the other inequalities of the fourth order; but, as they are less than the errors of our approximation, it will be useless to investigate them by the theory, unless we wish to carry on the approximation to quantities of the fifth order.

If we collect together the inequalities of the fourth order, which we have just determined, they will become,*

Terms of $nt+\varepsilon$, of the fourth order.

[5305]
$$\begin{array}{lr}
+\;8'',67\,.\,\sin.(2v-2mv-2gv+cv+2\theta-\varpi) & 1\\
-\;8'',11\,.\,\sin.(2cv+2v-2mv-2\varpi) & 2\\
+10'',17\,.\,\sin.(2v-2mv+cv-c'mv-\varpi+\varpi') & 3\\
+\;5'',88\,.\,\sin.(2cv-2v+2mv+c'mv-2\varpi-\varpi') & 4\\
-\;0'',25\,.\,\sin.(2cv-2v+2mv-c'mv-2\varpi+\varpi') & 5\\
-\;3'',16\,.\,\sin.(2cv+c'mv-2\varpi-\varpi') & 6\\
+\;4'',50\,.\,\sin.(2cv-c'mv-2\varpi+\varpi') & 7\\
+33'',38\,.\,\sin.(4v-4mv-cv+\varpi) & 8\\
+15'',46\,.\,\sin.(4v-4mv-2cv+2\varpi) & 9\\
-\;8'',31\,.\,(1+i)\,.\,\sin.(cv-v+mv-\varpi) & 10\\
-\;5'',01\,.\,(1+i)\,.\,\sin.(v-mv+cv-\varpi). & 11
\end{array}$$

[5306] 18. *We shall now consider the moon's motion in latitude.* We have before determined the tangent of the latitude s; and, as the expression of the arc, by its tangent s, is $s-\frac{1}{3}s^3+\frac{1}{5}s^5-$ &c. [48] Int. we find, that the latitude of the moon is very nearly represented by the following function;†

[5305a] * (2959) If we connect together the quantities contained in [5240, 5248, 5257, 5266, 5268, 5283, 5296, 5302, 5304]; the sum becomes as in [5305].

[5306a] † (2960) From $s=\gamma.\sin.gv+\delta s$ [4897i], we get, by neglecting the second and higher powers of δs; and reducing by means of [1, 2] Int.

$$\gamma.(1-\tfrac{1}{4}\gamma^2).\sin.(gv-\theta)+\delta s.\{1-\tfrac{1}{2}\gamma^2+\tfrac{1}{2}\gamma^2.\cos.(2gv-2\theta)\}+\tfrac{1}{12}\gamma^3.\sin.(3gv-3\theta)\,; \qquad [5307]$$

from which, by using the preceding value of γ [5117], we get the latitude, as in the following expression;*

Moon's latitude.

$$\text{Moon's Latitude} = \left\{ \begin{array}{lr}
18542^s{,}79.\sin.(gv-\theta) & 1 \\
+\ 12^s{,}56.\sin.(3gv-3\theta) & 2 \\
+525^s{,}23.\sin.(2v-2mv-gv+\theta) & 3 \\
+\ \ 1^s{,}14.\sin.(2v-2mv+gv-\theta) & 4 \\
-\ \ 5^s{,}59.\sin.(gv+cv-\theta-\varpi) & 5 \\
+\ 19^s{,}85.\sin.(gv-cv-\theta+\varpi) & 6 \\
+\ \ 6^s{,}46.\sin.(2v-2mv-gv+cv+\theta-\varpi) & 7 \\
-\ \ 1^s{,}39.\sin.(2v-2mv+gv-cv-\theta+\varpi) & 8 \\
-\ 21^s{,}60.\sin.(2v-2mv-gv-cv+\theta+\varpi) & 9 \\
+\ 24^s{,}35.\sin.(gv+c'mv-\theta-\varpi') & 10 \\
-\ 25^s{,}94.\sin.(gv-c'mv-\theta+\varpi') & 11 \\
-\ 10^s{,}20.\sin.(2v-2mv-gv+c'mv+\theta-\varpi') & 12 \\
+\ 22^s{,}42.\sin.(2v-2mv-gv-c'mv+\theta+\varpi') & 13 \\
+\ 27^s{,}40.\sin.(2cv-gv-2\varpi+\theta) & 14 \\
+\ \ 5^s{,}13.\sin.(2cv+gv-2v+2mv-2\varpi-\theta) & 15
\end{array} \right\}. \qquad [5308]$$

$$-\tfrac{1}{3}s^3=-\tfrac{1}{3}\gamma^3.\sin.^3gv-\delta s.\gamma^2.\sin.^2gv$$
$$=-\tfrac{1}{3}\gamma^3.\{\tfrac{3}{4}\sin.gv-\tfrac{1}{4}\sin.3gv\}-\delta s.\gamma^2.\{\tfrac{1}{2}-\tfrac{1}{2}\cos.2gv\}. \qquad [5306b]$$

The sum of the two expressions [5306a, b], is easily reduced to the form,

$$s-\tfrac{1}{3}s^3=\gamma.(1-\tfrac{1}{4}\gamma^2).\sin.gv+\delta s.\{1-\tfrac{1}{2}\gamma^2+\tfrac{1}{2}\gamma^2.\cos.2gv\}+\tfrac{1}{12}.\gamma^3.\sin.3gv. \qquad [5306c]$$

Substituting this in the expression of the arc [5306], and neglecting the terms of a higher order, it becomes as in [5307]. We may remark, that the term $\delta s.\frac{1}{2}\gamma^2.\cos.2gv$ [5306c], produces, by means of the term [4897 line 1], the expression

$$\tfrac{1}{2}B_1^{(0)}\gamma^3.\cos.2gv.\sin.(2v-2mv-gv)\,; \qquad [5306d]$$

from which we obtain the term $\frac{1}{4}B_1^{(0)}.\gamma^3.\sin.(2v-2mv+gv)$, which is of the same form as that which is retained in [4897 line 2]; hence the expression of the latitude [5306c] becomes, very nearly,

$$\gamma.(1-\tfrac{1}{4}\gamma^2).\sin.gv+\tfrac{1}{12}\gamma^3.\sin.3gv+\tfrac{1}{4}B_1^{(0)}.\gamma^3.\sin.(2v-2mv+gv)+(1-\tfrac{1}{2}\gamma^2).\delta s. \qquad [5306e]$$

* (2961) Substituting in [5306c], the expression of δs [4897], and then the values

19. *It now remains to determine the third co-ordinate of the moon, or its parallax. The sine of the moon's horizontal parallax is represented by*

[5309] $\frac{D}{r}=\frac{Du}{\sqrt{1+ss}}$, *D being the earth's radius.** Considering the smallness of this sine, we may take it for the expression of the parallax itself; and, if we substitute the value of u [5309*b*]; namely,

[5310] $$u=\frac{1}{a}\cdot\{1+e^2+\tfrac{1}{4}\gamma^2+e.(1+e^2).\cos.(cv-\varpi)-\tfrac{1}{4}\gamma^2.\cos.(2gv-2\theta)\}+\delta u\ ;$$

neglecting terms of the order $\frac{D}{a}\cdot e^4$, we shall find, that *this parallax is represented by the following formula,*

Moon's parallax.

[5311] $$\frac{D}{r}=\frac{Du}{\sqrt{1+ss}}=\frac{D}{a}.(1+e^2).\{1+e.[1-\tfrac{1}{4}\gamma^2+\tfrac{1}{4}\gamma^2.\cos.(2gv-2\theta)].\cos(cv-\varpi)+a\delta u-s\delta s\}.$$

[5307*a*] of $B_1^{(0)}....B_2^{(15)}$ [5176—5191], also those of m, e, &c. [5117,5194,5221], we get [5308], nearly; a few of the small terms being neglected.

* (2962) If we substitute the value of r [4776], in the well known expression of the

[5309*a*] *sine* of the horizontal parallax $\frac{D}{r}$, it becomes as in [5309], or as in the first member of [5309*f*], and this may be taken for the parallax itself, by neglecting its third power. Now, if we add to the expression of u [4826], the part of δu, [4904, &c.], arising from the perturbations, it becomes, by neglecting terms of the fourth order,

[5309*b*] $$u=\frac{1}{a}\cdot\left\{1+e^2+\tfrac{1}{4}\gamma^2+e.(1+e^2).\cos.cv-\tfrac{1}{4}\gamma^2\cos.2gv+a\delta u\right\}\quad\text{as in [5310]}$$

[5309*c*] $$=\frac{1}{a}\cdot(1+e^2).\left\{1+\tfrac{1}{4}\gamma^2.(1-\cos.2gv)+e.\cos.cv+a\delta u\right\}.$$

Developing the radical $\sqrt{1+s^2}$, neglecting s^4, and substituting for s, its value [5306*a*], we get,

[5309*d*] $$\frac{1}{\sqrt{1+s^2}}=1-\tfrac{1}{2}s^2=1-\tfrac{1}{2}.\{\gamma^2.\sin.^2gv+2\delta s.\gamma.\sin.gv\}$$

[5309*e*] $$=1-\tfrac{1}{4}\gamma^2.(1-\cos.2gv)-\delta s.\gamma.\sin.gv\,.$$

Multiplying together the expressions [5309*c*, *e*], and the product by D, we get, by neglecting γ^4, &c. of the fourth order,

[5309*f*] $$\frac{Du}{\sqrt{1+ss}}=\frac{D}{a}.(1+e^2).\{1+e.(1-\tfrac{1}{4}\gamma^2+\tfrac{1}{4}\gamma^2.\cos.2gv).\cos.cv+a\delta u-\delta s.\gamma.\sin.gv\}.$$

This becomes as in [5311], by substituting, in its last term, the approximate value of

[5309*g*] $$\gamma.\sin.gv=s.\ [5306a].$$

To determine $\frac{D}{a}$, we shall observe, that we have, in [4968],*

$$\frac{1}{a}=\frac{1}{a_{,}}\,.\,0{,}9973020\,;$$ [5312]

and, by [5082, 5090],†

$$\frac{a^2}{\sqrt{a_{,}}}\times 1{,}0003084=\frac{1}{n}.$$ [5313]

Hence we get,‡

$$\frac{1}{a}=\sqrt[3]{\frac{n^2.(1{,}0003084)^2}{0{,}9973020}}\,.$$ [5314]

Let $2\,\varepsilon$ be twice the space which the earth's attraction would make a particle describe in the time t, in the parallel, on which the square of the [5315]
sine of the latitude is $\frac{1}{3}$. This attraction is $\frac{M}{D^2}$ [1812, 1811l],§ the [5316]
earth M being supposed elliptical. But we have before put $M+m=1$ [5317]
[4775″]; m being here the moon's mass; therefore, we have,

$$2\,\varepsilon=\frac{M.t^2}{(M+m).D^2}\,.$$ [5318]

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* (2963) Substituting, in $\overset{2}{\overline{m}}$ [5094], the value of m [5117], we get $\overset{2}{\overline{m}}=0{,}0055796$. With this, and the values of c', m, γ, $A_2^{(0)}$, $B_1^{(0)}$ [5117, 5157, 5176], we find, that [5312a]
the equation [4968] becomes nearly as in [5312].

† (2964) If we substitute, in the coefficient of $\frac{a^2.dv}{\sqrt{a_{,}}}$ [5082], the values of m, $A_2^{(0)}$,
$A_1^{(1)}$, e [5117, 5157, 5158, 5194], it becomes $\frac{a^2.dv}{\sqrt{a_{,}}}.1{,}0003084$. This is to be put equal [5313a]
to $\frac{dv}{n}$ [5090]; hence we get [5313].

‡ (2965) We have, in [5312], $\frac{1}{\sqrt{a_{,}}}=\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{0{,}9973020}}$; multiplying this by
$a^2.1{,}0003084$, and substituting, for the first member of this product, its value $\frac{1}{n}$ [5313], [5314a]
we get $\frac{1}{n}=a^{\frac{3}{2}}.\frac{1{,}0003084}{\sqrt{0{,}9973020}}$; whence we easily deduce [5314].

§ (2966) Changing z into ε, in [67], we get $2\varepsilon=gt^2$; g being the force of [5316a]
gravity [54$^{\text{v}}$ line 7]. Now, in the parallel of latitude, mentioned in [5315], we have

Hence we deduce,*

[5319] $$\frac{D}{a} = \sqrt[3]{\frac{M}{M+m} \cdot D \cdot \frac{n^2 t^2}{2\varepsilon} \cdot \frac{(1,0003084)^2}{0,9973020}}.$$

[5320] If we suppose t to be equal to a centesimal second, and T equal to the number of centesimal seconds of time, during a sideral revolution of the moon,

[5321] we shall have† $n^2 = \frac{4\pi^2}{T^2}$, π being the ratio of the semi-circumference to the radius. If l be the length of a pendulum, vibrating in a centesimal

[5322] second of time, upon the parallel under consideration, we shall have, as in

[5323] [86′], $2\varepsilon = \pi^2 l$,‡ which gives,

[5316b] $\mu^2 - \frac{1}{3} = 0$ [1618‴]; hence the expression of V [1812], becomes $V = \frac{M}{r}$, M *being the mass of the ellipsoidal earth*, and r the distance of the attracted point from its centre, [1616ᵛ, 1616ˣⁱˣ]. Now, the attraction of the earth, in the direction r, is represented

[5316c] by $-\left(\frac{dV}{dr}\right) = \frac{M}{r^2}$ [1811l, 5316b]; and, by changing r into D, to conform to the present notation [5309], it gives very nearly, the expression of the gravity $g = \frac{M}{D^2}$ [5316];

[5316d] hence [5316a] becomes $2\varepsilon = \frac{Mt^2}{D^2}$. We have put, in [4775″], $M+m$ equal to unity,

[5316e] therefore, for the sake of homogeneity, we may change M into $\frac{M}{M+m}$, in the preceding expression of 2ε, and then it becomes as in [5318].

* (2967). Multiplying [5318] by $\frac{D^3}{2\varepsilon}$, and extracting the cube root, we get,

[5319a] $$D = \sqrt[3]{\frac{M}{M+m} \cdot D \cdot \frac{t^2}{2\varepsilon}};$$

the product of this, by the expression [5314], gives [5319].

† (2968) Noticing only the mean motions, we have $nt = v$ [5220]. Now, when v

[5321a] is equal to 2π, t becomes T [5320], and we get $nT = 2\pi$, or $n = \frac{2\pi}{T}$; whose square gives n^2 [5321].

‡ (2969) Changing, in the formula [86′], z into ε, and r into l, to conform

[5323a] to the present notation, we get $2\varepsilon = \pi^2 l$, as in [5323]. Multiplying it by $\frac{4}{lT^2}$, we obtain $\frac{8\varepsilon}{lT^2} = \frac{4\pi^2}{T^2}$; hence $n^2 = \frac{8\varepsilon}{lT^2}$ [5321]; and, as $t = 1$, [5320], we have

[5323b] $\frac{n^2 t^2}{2\varepsilon} = \frac{4}{lT^2}$; substituting this in [5319] and putting $4.(1,0003084)^2 = (2,0006168)^2$, we get [5324].

$$\frac{D}{a} = \sqrt[3]{\frac{M}{M+m} \cdot \frac{D}{l} \cdot \frac{(2{,}0004168)^2}{0{,}9973020.T^2}} \,. \qquad [5324]$$

The length of a pendulum, vibrating in a centesimal second, upon the same parallel, is equal to $0^{\text{met.}},740905$ [2054],* we must increase it by its 434th part, to obtain the length which could obtain independently of the centrifugal force; hence we have, $l = 0^{\text{met.}},742612$. The value of D is equal to $6369374^{\text{met.}}$ [389*b*, nearly]; lastly, we have, by the phenomena of the tides, $m = \frac{M}{58{,}6}$ [4321]; and, by observation, $T = 2732166$ centesimal seconds; hence we have,† [5325] [5326] [5327] [5328]

$$\frac{D}{a} = 0{,}01655101 \,. \qquad [5329]$$

Estimating in seconds the coefficient $\frac{D}{a}.(1+e^2)$, we find it equal to $3424^s,16$. This being premised, we find, for the expression of the moon's parallax, in the proposed parallel; ‡ [5330]

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* (2970) This value corresponds with the formula [2054], putting $\sin.^2\psi = \sin.^2\text{lat.} = \frac{1}{3}$, as in [5316]. This must be increased $\frac{1}{434}$ part, to correct for the centrifugal force [388^{iv}], by which means it becomes $l = 0^{\text{met.}},742612$, as in [5326]. This will be varied a little if we use the corrected value of l [2054*n*, or 2056*p*]. [5325*a*] [5325*b*]

† (2971) Substituting, in [5324], the values of l, m, D, T, [5326 – 5328], we get [5329], nearly. Multiplying this by $1+ee$ [5194], and then by the radius in seconds, we get the expression [5330], nearly. This would be varied a little by correcting the value of l, as in [5325*b*], and also by the change in the value of m [11906, 3380*b*]. [5329*a*]

‡ (2972) Substituting the value of $\frac{D}{a}.(1+ee) = 3424^s,16$ [5330] in [5311], it becomes [5330*a*]

$$\text{Moon's Parallax} = 3424^s,16.\{1+e.(1-\tfrac{1}{4}\gamma^2+\tfrac{1}{4}\gamma^2.\cos.2gv).\cos.cv+a\delta u-s\delta s\}. \qquad [5330b]$$

We may substitute in this $\frac{1}{4}\gamma^2 e.\cos.2gv.\cos.cv = \frac{1}{8}\gamma^2 e.\cos.(2gv-cv)$, neglecting the term depending on the angle $2gv+cv$, because the term is small; and angles of this form are not retained in [5331]. Moreover, the chief term of δs [5308 line 3, or 5307] is $525^s,23.\sin.(2v-2mv-gv)$; and the chief term of s [5307] is [5330*c*] [5330*d*]

$$\gamma.(1-\tfrac{1}{4}\gamma^2).\sin.gv = 0{,}0899.\sin.gv \quad [5117 \text{ line } 5]. \qquad [5330e]$$

Multiplying these two expressions together, we get, in $s\delta s$, the terms,

$$23^s,6.\{\cos.(2gv-2v+2mv)-\cos.(2v-2mv)\} \,; \qquad [5330f]$$

substituting this in [5330*b*], and dividing by the radius in seconds 206265^s, it produces the terms $0^s,39.\{-\cos.(2gv-2v+2mv)+\cos.(2v-2mv)\}$. Hence, [5330*b*] becomes,

Moon's Parallax.

[5331]

$$
\text{Moon's Parallax} = \left\{
\begin{array}{ll}
3424^{s},16 & \\
+187^{s},48.\cos.(cv-\varpi) & \\
+\ 24^{s},68.\cos.(2v-2mv) & \\
+\ 38^{s},07.\cos.(2v-2mv-cv+\varpi) & \\
-\ \ 0^{s},70.\cos.(2v-2mv+cv-\varpi) & \\
-\ \ 0^{s},17.\cos.(2v-2mv+c'mv-\varpi') & \\
+\ \ 1^{s},64.\cos.(2v-2mv-c'mv+\varpi') & \\
-\ \ 0^{s},33.\cos.(c'mv-\varpi') & \\
-\ \ 0^{s},22.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi') & \\
+\ \ 1^{s},63.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi') & \\
-\ \ 0^{s},65.\cos.(cv+c'mv-\varpi-\varpi') & \\
+\ \ 0^{s},87.\cos.(cv-c'mv-\varpi+\varpi') & \\
+\ \ 0^{s},01.\cos(2cv-2\varpi) & \\
+\ \ 3^{s},60.\cos.(2cv-2v+2mv-2\varpi) & \\
+\ \ 0^{s},07.\cos.(2gv-2\theta) & \\
-\ \ 0^{s},17.\cos.(2gv-2v+2mv-2\theta) & \\
-\ \ 0^{s},01.\cos.(2c'mv-2\varpi') & \\
-\ \ 0^{s},95.\cos.(2gv-cv-2\theta+\varpi) & \\
-\ \ 0^{s},06.\cos.(2v-2mv-2gv+cv+2\theta-\varpi) & \\
-\ \ 0^{s},97.(1+i).\cos.(v-mv) & \\
+\ \ 0^{s},16.(1+i).\cos.(v-mv+c'mv-\varpi') & \\
-\ \ 0^{s},04.\cos(2v-2mv+cv-c'mv-\varpi+\varpi') & \\
-\ \ 0^{s},15.\cos.(4v-4mv-cv+\varpi) & \\
+\ \ 0^{s},05.\cos(4v-4mv-2cv+2\varpi) & \\
+\ \ 0^{s},13.\cos(2cv-2v+2mv+c'mv-2\varpi-\varpi') & \\
+\ \ 0^{s},02.\cos.(2cv+2v-2mv-2\varpi) & \\
-\ \ 0^{s},12.(1+i).\cos.(cv-v+mv-\varpi) &
\end{array}
\right\}.
\qquad
\begin{array}{r}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\25\\26\\27
\end{array}
$$

[5330g]

$$\text{Moon's Parallax} = 3424^{s},16.\{1+e.(1-\tfrac{1}{4}\gamma^2).\cos.cv+\tfrac{1}{8}\gamma^2 e.\cos.(2gv-cv)+a\delta u\}$$
$$-\,0^{s},39.\cos.(2gv-2v+2mv)+0^{s},39.\cos.(2v-2mv).$$

We must now substitute the value of $a\delta u$ [4904, 5242, 5251, 5258, 5269, 5287, 5297, &c.], also [5117, 5157—5175, 5221], and we shall get [5331].

CHAPTER II.

ON THE LUNAR INEQUALITIES ARISING FROM THE OBLATENESS OF THE EARTH AND MOON.

20. We shall now consider the terms arising from the oblateness of the earth and moon.* We have seen, in [4773], that the effect of this oblateness is to add to the expression of Q [4756] the quantity,

$$(M+m) \cdot \left\{ \frac{\delta V}{M} + \frac{\delta V'}{m} \right\} = \text{increment of } Q. \qquad [5332]$$

If we put,

$\alpha\rho$ = the ellipticity of the earth ; [5333]

$\alpha\varphi$ = the ratio of the centrifugal force to the gravity at the equator ; [5333']

D = the mean radius of the earth ; [5334]

μ = the sine of the moon's declination ; [5334']

we shall have, as in [1812],†

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* (2973) This subject is treated of, in a more simple and elegant manner, in the appendix to this volume [5937—5971] ; and again, in the fifth volume [12952—12996]; where some very small terms are noticed, which are neglected in this volume ; but they [5332a] have but little effect on the resulting formulas. We shall, in the notes on this chapter, restrict ourselves to the terms here investigated by the author, and shall follow the same [5332b] method of demonstration which he has used.

† (2974) We have, in [1812], for *an ellipsoid of revolution*,

$$V = \frac{M}{r} + (\tfrac{1}{2}\alpha\varphi - \alpha h) \cdot \frac{1}{r^3} \cdot M \cdot (\mu^2 - \tfrac{1}{3}). \qquad [5335a]$$

[5335] $$V = \frac{M}{r} + \left\{ \tfrac{1}{2}\alpha\varphi - \alpha\rho \right\} \cdot \frac{D^2}{r^3} \cdot M.(\mu^2 - \tfrac{1}{3}).$$

If the earth vary from the elliptical form, we shall have, by § 32, 35, of book iii.,*

[5336] $$V = \frac{M}{r} + \left\{ (\tfrac{1}{2}\alpha\varphi - \alpha\rho).(\mu^2 - \tfrac{1}{3}) + \alpha h'.(1-\mu^2).\cos.2\varpi \right\} . M. \frac{D^2}{r^3} ;$$

[5337] *$\alpha\rho$ and $\alpha h'$ being constant quantities, depending on the figure of the terrestrial*
[5338] *spheroid; and ϖ the angle formed by one of the two principal axes of the earth, situated in the plane of the equator, with the terrestrial meridian, passing through the moon's centre* [1746′]. It is evident, by the following analysis,

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[5335b] To conform to the present notation, we must put $\alpha h = \alpha\rho$ [1795′, 5333]; and, in the
[5335c] second term, we must change $\frac{1}{r^3}$ into $\frac{D^2}{r^3}$, to render it homogeneous with the first term; observing, that the radius of the earth D [5334], is supposed to be nearly equal to unity in [1795″, 1812]. Making these changes in [5335a], it becomes as in [5335].

* (2975) Neglecting the attraction of foreign bodies, and the terms depending on r^{-4}, in [1811], we get, for an ellipsoid of a general form,

[5336a] $$V = \frac{4\pi}{3r} . \int_0^1 \rho . d.a^3 + \frac{4\alpha.\pi}{3r^3} . Y^{(2)} . \int_0^1 \rho.d.a^3 - \frac{\alpha}{r^3} . Z^{(2)}.$$

[5336b] To render this homogeneous, we must multiply the two last terms by D^2, as in [5335c], and substitute, for $\frac{4}{3}\pi\int_0^1 \rho.d.a^3$, its value M [1811′, 4757]; by which means it becomes,

[5336c] $$V = \frac{M}{r} + \left\{ \alpha Y^{(2)} - \frac{\alpha}{M}.Z^{(2)} \right\} .M. \frac{D^2}{r^3}.$$

If we substitute this value of M, in $\alpha Z^{(2)}$ [1793], we get,

[5336c′] $$-\frac{\alpha}{M} . Z^{(2)} = \tfrac{1}{2}\alpha\varphi . (\mu^2 - \tfrac{1}{3}) ;$$

and, from [1763], we have, by changing αh into $\alpha\rho$, as in [5335b], also h'''' into h', to conform to the present notation,

[5336d] $$Y^{(2)} = -\rho.(\mu^2 - \tfrac{1}{3}) + h'.(1-\mu^2).\cos.2\varpi ;$$

[5336e] the earth being supposed to revolve about one of its principal axes [1762′, &c.]. Substituting
[5336f] these in [5336c], it becomes as in [5336]. The radius of an ellipsoid is represented by $1+\alpha Y^{(2)}$ [1775, or 1503a]; and, if we substitute the value of $Y^{(2)}$ [5336d], we get,

[5336g] $$1 - \alpha\rho.(\mu^2 - \tfrac{1}{3}) + \alpha h'.(1-\mu^2).\cos.2\varpi = \text{radius of the spheroid.}$$

[5336h] At the pole, where $\mu = 1$, this becomes $1 - \frac{2}{3}\alpha\rho$; subtracting this from [5336g], and

that the term depending on $\cos.2\varpi$ has no sensible influence on the
lunar motions on account of the rapidity with which the angle ϖ varies; [5338']
so that the value of V, which we shall here use, is the same as in the [5339]
elliptical hypothesis, with an ellipticity equal to $\alpha\rho$; but, in the general
case of any spheroid whatever, $\alpha\rho$ *does not express the ellipticity* [5336k].
We may, therefore, suppose, in this general case, that the value of Q [4756] [5339']
is increased, on account of the oblateness of the earth, by the function,

$$(\tfrac{1}{2}\alpha\varphi-\alpha\rho)\cdot\frac{D^2}{r^3}\cdot(\mu^2-\tfrac{1}{3}) \;=\; \text{increment of } Q \text{ [4756]}; \qquad [5340]$$

$M+m$ *being taken for the unity of mass* [4775'', 5336l]. [5340']

*We shall, in the first place, consider the variation of the orbit, or the moon's
motion in latitude, depending on this cause. If we put* λ *for the obliquity* [5341]
of the ecliptic to the equator, and fix the origin of the angle v *in the vernal* [5342]
*equinox, at a given epoch; we shall have, very nearly,** [5343]

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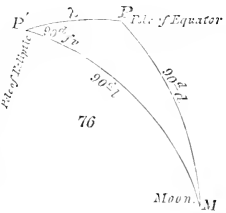
putting $\mu=0$, we get the excess of the equatorial radius, or,

$$\text{the ellipticity} = \alpha\rho + \alpha h'.\cos.2\varpi. \qquad [5336i]$$

Hence it appears, that the ellipticity of the different meridians varies, with the different
values of 2ϖ, from $\alpha\rho-\alpha h'$ to $\alpha\rho+\alpha h'$; instead of being generally represented by [5336k]
$\alpha\rho$, as in [5333, 5339']. From [4767, 5336] we get δV, and then the first term of [5336l]
[5332] becomes as in [5340, 5340'].



* (2976) In the annexed figure, P is the pole of the moveable equator; P' the pole of the ecliptic; M the place of the moon; so that, if the moon's latitude be represented by l, and the declination by d, we shall have $PM=90^d-d$; $P'M=90^d-l$; $PP'=\lambda$; $PP'M=90^d-fv$ [5345]. Substituting these symbols in the formula [5344c], which is the same as [1345[8]], we get [5344d], using the symbol $\mu=\sin.d$ [5334']. This is reduced to the form [5344e], by means of the expressions of $\sin.l$, and $\cos.l$ [4776b]; [5344a] [5344b] [5344b']

$$\cos.PM = \sin.P'P.\sin.P'M.\cos.PP'M + \cos.P'P.\cos.P'M; \qquad [5344c]$$

$$\mu = \sin.\lambda.\cos.l.\sin.fv + \cos.\lambda.\sin.l; \qquad [5344d]$$

$$\mu = \sin.\lambda.\frac{1}{\sqrt{1+ss}}.\sin.fv + \cos.\lambda.\frac{s}{\sqrt{1+ss}}. \qquad [5344e]$$

[5344] $$\mu = \sin.\lambda.\sqrt{1-ss}.\sin.fv + s.\cos.\lambda;$$

[5345] *fv being the apparent longitude of the moon, referred to the moveable vernal equinox.* We must, therefore, add to the value of Q a quantity, which we shall represent by,*

Terms of Q.

[5346] $$Q = (\tfrac{1}{2}\alpha\rho - \alpha\rho).\frac{D^2}{r^3}.\{\sin.^2\lambda.(1-s^2).\sin.^2 fv + 2s.\sin.\lambda.\cos.\lambda.\sin.fv + s^2.\cos.^2\lambda - \tfrac{1}{3}\}.$$

This being premised, we shall resume the equation [4755]. We have developed, in [5018—5049], the different terms of this equation, depending on the sun's action. It is evident, that the preceding function adds to the equation [4755] the following quantity,†

[5347] $$2.(\alpha\rho - \tfrac{1}{2}\alpha\varphi).\frac{D^2.u}{h^2}.\sin.\lambda.\cos.\lambda.\sin.fv + (g^2-1).H.\sin.fv;$$

[5344f] If we neglect the third and higher powers of s, we may change $\frac{1}{\sqrt{1+s^2}}$ into $\sqrt{1-s^2}$, and $\frac{s}{\sqrt{1+s^2}}$ into s; by which means, the formula [5344e] becomes as in [5344].

* (2977) Substituting μ [5344] in [5340], and putting $2s$, for $2s.\sqrt{1-s^2}$, in the coefficient of $\sin.fv$, we get [5346]. Now, we have $\frac{1}{r} = \frac{u}{\sqrt{1+s^2}}$ [4776],

[5346a] which is nearly equal to $u.\sqrt{1-s^2}$; substituting this in [5346], neglecting s^3, &c., we get, for this part of Q, the following expression;

[5346b] $$Q = (\tfrac{1}{2}\alpha\rho - \alpha\rho).D^2.u^3.\{\sin.^2\lambda.(1-s^2)^{\frac{5}{2}}.\sin.^2 fv + 2s.\sin.\lambda.\cos.\lambda.\sin.fv + s^2.\cos^2\lambda - \tfrac{1}{3}(1-s^2)^{\frac{3}{2}}\}.$$

† (2978) The substitution of the value of Q [5346b], in [4755], produces an

[5347a] equation of the same kind as [5037], in which Γ is composed of a series of terms, of the form $k_{\prime}.\sin.(i_{\prime}t+\varepsilon_{\prime})$, depending on Q. When $i_{\prime}$ is very nearly equal to unity,

[5347b] the corresponding term of s will be very much increased by the divisors introduced by the integration; as in the similar case of the equation treated of in [4819], as will be seen in

[5347c] [5347r—t]. Now, $f-1$ is of the order $\frac{1}{250000}$ [5347q]; therefore, the term depending on $\sin.fv$ must be particularly noticed; and, in fact, it is the only one the author considers as necessary to retain in this calculation. In making the substitution of the value of Q

[5347d] [5346b], in [4755], we may neglect the quantities $\left(\frac{dQ}{dv}\right)$, $\left(\frac{dQ}{du}\right)$; because they are

[5347e] multiplied by s, or $\frac{ds}{dv}$, of the order $\gamma.\sin.gv$, or $\gamma.\cos.gv$, and produce only terms of small value, in which $i_{\prime}$ differs considerably from unity. We may also neglect

supposing the inequality of δs, depending on the angle fv, to be [5347']

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the term $-\frac{s^2}{h^2.u^2}.\left(\frac{dQ}{ds}\right)$ [4755], because it is multiplied by s^2. Hence the equation [4755] becomes,

$$0=\frac{dds}{dv^2}+s-\frac{1}{h^2.u^2}.\left(\frac{dQ}{ds}\right). \qquad [5347f]$$

Now, by noticing only the terms depending on $\sin.fv$, we get, from [5346b],

$$\left(\frac{dQ}{ds}\right)=2.(\tfrac{1}{2}\alpha\varphi-\alpha\rho).D^2u^3.\sin.\lambda.\cos.\lambda.\sin.fv. \qquad [5347g]$$

Substituting this in [5347f], it produces the term,

$$-\frac{1}{h^2.u^2}.\left(\frac{dQ}{ds}\right)=2.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2u}{h^2}\sin.\lambda.\cos.\lambda.\sin.fv; \qquad [5347h]$$

which is the same as the first term of [5347], or the first term of the function Γ [5037]. The other term of Γ is deduced from [5040 line 1], $\frac{3}{2}m^2.\frac{a}{a_{,}}.\delta s$, by the successive [5347i]
substitution of [5082h′, 4828e]; by which means, it becomes,

$$\tfrac{3}{2}m^2.\delta s=(g^2-1).\delta s, \quad \text{nearly}; \qquad [5347k]$$

and, if we use the value of δs [5348], it produces $(g^2-1).H.\sin.fv$, as in the last term of [5347]. Hence, the equation [5347f], by retaining only the terms depending on [5347l]
the angle fv, is reduced to the following form;

$$0=\frac{dds}{dv^2}+s+2.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2u}{h^2}.\sin.\lambda.\cos.\lambda.\sin.fv+(g^2-1).H.\sin.fv. \qquad [5347m]$$

Substituting the assumed value of δs, or $s=H.\sin.fv$ [5348], and dividing by $\sin.fv$, we get,

$$0=(-f^2+1).H+2.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2u}{h^2}.\sin.\lambda.\cos.\lambda+(g^2-1).H. \qquad [5347n]$$

Connecting together the terms depending on H, and dividing by its coefficient, we get,

$$H=\frac{-2.(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g^2-f^2}.\frac{D^2u}{h^2}.\sin.\lambda.\cos.\lambda. \qquad [5347n']$$

The moon's longitude v, is counted from the *fixed* axis x, or the *fixed* vernal [5347o]
equinox [4760′]; and fv [5345] is the same longitude, counted from the *moveable* vernal equinox; hence, $f-1$ is of the same order as the ratio of the precession of the equinoxes to the moon's mean motion. Now, the annual precession is nearly 50^s [4614], and
the moon's annual motion is $\frac{360^d}{m}=4813^d$, nearly [5117, 5117a]. These quantities [5347p]
are to each other in the ratio of 1 to 340000, nearly; hence, $f-1$ is of the order
$\frac{1}{340000}$; which is very small, in comparison with $g-1=\frac{3}{4}m^2=\frac{1}{250}$, nearly [5117 line 3]; [5347q]

represented by,

[5348] $$\delta s = H.\sin.fv.$$

We may, moreover, easily satisfy ourselves, that this quantity is the only sensible one which results from the function Q [5346]. Adding it to the differential equation [5049], and observing, that $f-1$ is extremely small,
[5349] in comparison with $g-1$, we get, by integration,

[5350] $$H = \frac{-2.(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g^2-1} \cdot \frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda.$$

Hence we obtain in s, or in the moon's motion in latitude, the inequality,*

[5351] $$\delta s = -\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1} \cdot \frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda.\sin.fv.$$

[5351'] *Which is the only sensible inequality of the moon's motion in latitude, arising from the oblateness of the earth. This inequality is equivalent to the supposition that the moon's orbit, instead of moving on the plane of the ecliptic, with a*
[5352] *constant inclination, moves, with the same condition, upon a plane passing always through the equinoxes, between the equator and the ecliptic, and inclined to this*

[5347q'] so that we may put $f=1$, in [5347n'], and we shall get,

[5347r] $$H = \frac{-2.(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g^2-1} \cdot \frac{D^2 u}{h^2}.\sin.\lambda.\cos.\lambda.$$

[5347s] Substituting $u=\frac{1}{a}$, $h^2=a_{\prime}=a$, [4937n, 5312], it becomes as in [5350]. We may observe, that if f differ considerably from unity, it will make the corresponding value of H, deduced from [5347n'], very small; because the divisor, in finding H, will be a large number of the order g^2-f^2, instead of the very small one of the order g^2-1 [5347r]; and, for this reason, most of these terms of Q
[5347t] may be neglected, considering that they are multiplied by the very small factor $(\alpha\rho-\frac{1}{2}\alpha\varphi).\frac{D^2}{a^2}$; which can become sensible only by means of a small divisor.

* (2979) We have $g^2-1=(g+1).(g-1)=2.(g-1)$, nearly [4828e],
[5351a] substituting this in [5350], and then the resulting value in $\delta s = H.\sin.fv$ [5348], we get [5351].

*last plane, by an angle, which may be represented as follows,**

$$\text{Angle of inclin. of the equator and fixed plane} = \frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}\cdot\frac{D^2}{a^2}\cdot\sin.\lambda.\cos.\lambda. \qquad [5353]$$

Inclination of the equator to the fixed plane.

We have found in [5329, 5117],

$$\frac{D}{a}=0{,}01655101\,; \qquad g-1=0{,}00402175\,; \qquad [5354]$$

also at the epoch, in 1750,

$$\lambda = 23^d\ 28^m\ 17^s{,}9 \quad [4353''] . \qquad [5355]$$

Lastly, $\alpha\varphi=\frac{1}{289}$ [1594*a*, &c.]; therefore, by supposing $\alpha\rho=\frac{1}{334}$ [2034] [5356]
the preceding inequality becomes,†

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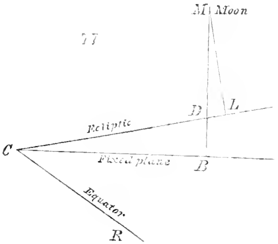
* (2980) The angle of inclination of the ecliptic to the fixed plane, given in [5353], being put, for brevity, equal to A, we shall have $A=-H$ [5353, 5350, 5351*a*]; and [5352a] $\delta s=-A\,.\sin.fv$ [5348]. Suppose, in the annexed figure, that CR represents the equator, CB the fixed plane, CL the [5352b] ecliptic, M the place of the moon, ML a circle of latitude, perpendicular to the ecliptic, MDB the arc perpendicular to the fixed plane; then the difference of the arcs ML, [5352c] MB, will be very nearly represented by



$$BD=\text{angle } BCD\times\sin.CD=A\,.\sin.fv \quad [5352a, 5345]. \qquad [5352d]$$

Hence it is evident, that if the moon's latitude, above the fixed plane, be expressed by $BM=s$, its latitude, counted from the ecliptic, will be very nearly represented by

$$ML=MB-BD=s-A\,.\sin.fv=s+\delta s \quad [5352b]\,;\ \text{as in [5352].} \qquad [5352e]$$

† (2981) The expression of A [5352*a*, 5353], is,

$$A=\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}\cdot\frac{D^2}{a^2}\,.\sin.\lambda\,.\cos.\lambda. \qquad [5357a]$$

Substituting the values [5354, 5355], and that of $\alpha\varphi$ [5356], we obtain,

$$A=5132^s{,}9\,.\,\alpha\rho-8^s{,}88\,; \qquad [5357b]$$

hence,

$$\alpha\rho=\frac{A+8^s{,}88}{5132^s{,}9}\,. \qquad [5357c]$$

Inequality in latitude [5357] depending on the oblateness of the earth.

[5357] $$\delta s = -6^s{,}487 \,.\, \sin. fv \,.$$

[5358] It would be $-13^s{,}436.\sin.fv$, if the oblateness be $\frac{1}{230}$, which corresponds to the supposition that the earth is homogeneous [1592a]. *Therefore, if*

[5358'] *this inequality be carefully observed, it will be very useful in ascertaining the oblateness of the earth.*

[5358''] *We shall now consider the variations in the radius vector, and in the moon's longitude arising from the oblateness of the earth.* We may deduce them from the equations [4753, 4754]; but it is more simple and accurate, to use the formulas [919, 923]. For this purpose, we shall suppose, that the

[5359] differential characteristic δ refers to the quantity $\frac{1}{2}\alpha\varphi - \alpha\rho$. We shall then observe, that the functions R, rR', [913, 928'], are represented by, *

[5360] $$R = -Q + \frac{1}{r} \quad [4774a]; \qquad rR' = r\,.\left(\frac{dR}{dr}\right).$$

Hence, the equation [919] becomes,†

[5361] $$0 = \frac{d^2.\,r\delta r}{dt} + \frac{r\delta r}{r^3} + 2\,.\int \delta.\mathrm{d}R + \delta.r\,.\left(\frac{dR}{dr}\right).$$

[5362] We have, in R, the term,‡ $R = 2.(\alpha\rho - \frac{1}{2}\alpha\varphi).\frac{D^2}{r^3}\,.\sin.\lambda.\cos.\lambda.s.\sin.fv$. This contains the following term,

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[5357d] If $\alpha\rho = \frac{1}{334}$, the first of these equations gives $A = 6^s{,}487$, as in [5357]; and, if $\alpha\rho = \frac{1}{230}$, it becomes $A = 13^s{,}436$, as in [5358].

* (2982) If we substitute the expression of Q [5346b], in R [5360], we shall obtain,

$$R = -Q + \frac{1}{r}$$

[5360a] $$= \frac{1}{r} + (\alpha\rho - \tfrac{1}{2}\alpha\varphi).D^2u^3.\left\{\sin.^2\lambda.(1-s^2)^{\frac{5}{2}}.\sin.^2 fv + 2s.\sin\lambda.\cos\lambda.\sin fv + s^2.\cos^2\lambda - \tfrac{1}{3}(1-s^2)^{\frac{3}{2}}\right\};$$

which will be used hereafter.

† (2983) The equation [5361], is the same as [919], using the expression of rR'

[5361a] [5360], and that of $\mu = M + m = 1$ [914', 5340'].

‡ (2984) The term of R, retained in [5362], is the same as that in [5360a],

[5362a] depending upon $s.\sin.fv$. Substituting in it the chief term of s; namely, $s = \gamma.\sin.gv$

$$\delta R = (\alpha_{\rho} - \tfrac{1}{2}\alpha_{\phi}) \cdot \frac{D^2}{r^3} \cdot \gamma \cdot \sin.\lambda . \cos.\lambda . \cos.(gv - fv - \theta).$$ [5363]

This term of δR gives, in $f\delta.dR$, an expression which is exactly similar and equal to δR. For the differential characteristic d [916'], refers only to the moon's co-ordinates; and, we have, by noticing only the preceding term, [5363']

$$f\ \delta.dR = \delta R.$$ [5364]

Then we obtain,*

$$\delta.r.\left(\frac{dR}{dr}\right) = -3 \cdot (\alpha_{\rho} - \tfrac{1}{2}\alpha_{\phi}) \cdot \frac{D^2}{r^3} \cdot \gamma . \sin.\lambda . \cos.\lambda . \cos.(gv - fv - \theta).$$ [5365]

If we substitute these values of $f\delta.dR$, and $\delta.r.\left(\frac{dR}{dr}\right)$ [5364, 5365], in the differential equation [5361], we shall find, that the expression δr contains a term, depending on $\cos.(gv - fv - \theta)$, but it is insensible, not having $g-1$ for a divisor, which the corresponding term of δs has. [5366]

[4897i], it produces the term given in [5363], which depends on the angle $(g-f).v$; $\frac{1}{r}$ being used instead of u. Now, the coefficient of this angle, is of the order $g-1$, or m^2 [5347q], and the integration of $d\delta v$, in [5387], introduces $g-1$ as a divisor; and it is on this account, that the terms depending on the angle $gv-fv$ are retained by the author. [5362b] [5362c]

* (2985) The partial differential of δR [5363], taken relatively to r, and multiplied by $\frac{r}{dr}$, gives [5365], as is evident from the nature of the symbol δ [5359]. If we substitute the values [5364, 5365] in [5361], they will produce in it an expression, which we shall represent by $\frac{\Pi}{r^3}$. Then, if we put, for a moment $r\delta r = u$, the equation [5361], will become $0 = \frac{ddu}{dt^2} + \frac{u}{r^3} + \frac{\Pi}{r^3}$. Multiplying this by r^3, and putting for dt, its chief term $\frac{a^2.dv}{\sqrt{a_{,}}}$ [5081], or $r^{\frac{3}{2}}dv$, nearly, it becomes, $0 = \frac{ddu}{dv^2} + u + \Pi$; which is of the same form as [4845], supposing $N = 1$. Its integral [4847] introduces the divisor $i^2 - N^2 = i^2 - 1$, which is nearly equal to -1; because, in the present case, $i = g - f$ [4846, 5363] is of the order m^2 [5347q]. Hence it is evident, that $u = r\delta r$, is not increased by the introduction of a small divisor in the integration. This agrees with [5366]. [5366a] [5366b] [5366c] [5366d] [5366e] [5366f]

It is not the same with the expression of the longitude. The formula [923] gives, in $d\delta v$, the following terms;*

[5367] $$d\delta v = \frac{3dt^2.\int\delta.\mathrm{d}R+2dt^2.\delta.r.\left(\frac{dR}{dr}\right)}{r^2.dv}.$$

Substituting the value of δR [5363], we obtain, in $d\delta v$, the following term;†

[5368] $$d.\delta v = -\frac{3dt^2.(\alpha\gamma-\frac{1}{2}\alpha\gamma)}{r^2.dv}.\frac{D^2}{r^3}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta).$$

But, this is not the only term of the same kind, in the expression of $d\delta v$.
[5369] The sun's action gives, in Q [4806], the term $Q=\frac{m'.u'^3}{4u^2}.(1-2s^2)$.
[5370] Substituting in it the value of u [4776], we obtain, in $R=-Q+\frac{1}{r}$ [5360], the expression,‡

[5371] $$R=-\tfrac{1}{4}m'u'^3.r^2.(1-3s^2);$$

which gives, in δR, the term,

[5372] $$\delta R=\tfrac{3}{2}m'u'^3.r^2.s\delta s;$$

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* (2986) Noticing only the terms depending on the angle $gv-fv-\theta$, or those
[5367a] which produce the factor $s\delta s$ in [5373, &c.], we may neglect δr, and then we obtain from [923],

[5367b] $$d\delta v = \frac{3dt^2.\int\delta.\mathrm{d}R+2dt^2.r\delta R'}{r^2.dv}.$$

Now, we evidently have $r\delta R'=\delta.(rR')-R'\delta r$; and, for the same reason as in [5367a],
[5367c] we may reject $R'\delta r$; then, using the value of rR' [5360], we get $r\delta R'=\delta.r.\left(\frac{dR}{dr}\right)$; hence, the preceding expression of $d\delta v$ becomes as in [5367].

[5368a] † (2987) Substituting $\int\delta.\mathrm{d}R=\delta R$ [5364], in [5367]; and then using the values [5363, 5365], we obtain the expression [5368], by a slight reduction.

[5370a] ‡ (2988) From u [4776], we deduce $\frac{1}{u^2}=\frac{r^2}{1+s^2}=r^2.(1-s^2)$, nearly; multiplying this by $\frac{1}{4}m'u'^3.(1-2s^2)$, we get the value of the term of Q [5369]; and, by the substitution in R [5370], we obtain the term [5371], neglecting quantities
[5370b] of the order s^4. The variation of [5371], relative to the characteristic δ, putting $\delta r=0$ [5367a], gives δR [5372].

from which we easily deduce the following expression ;*

$$3f\delta.\mathrm{d}R+2\delta.r.\left(\frac{dR}{dr}\right) = \tfrac{21}{2}.m'u'^3.r^2.s\delta s.$$ [5373]

We have, very nearly,† $m'u'^3.r^3=m^2$, also $g=1+\frac{3}{4}m^2$ [5117 or 4828e]; hence the function [5373] becomes, [5374]

$$3f\delta.\mathrm{d}R+2\delta.r.\left(\frac{dR}{dr}\right) = \frac{14.(g-1).s\delta s}{r}.$$ [5375]

Substituting in it $\delta s=-\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}.\frac{D^2}{r^2}.\sin.\lambda.\cos.\lambda.\sin.fv$ [5351,5374a]; and $s=\gamma.\sin.(gv-\theta)$ [5050], we obtain, in [5375], the following term ;‡ [5376]

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* (2989) From [5371], we get, by differentiation,

$$r.\left(\frac{dR}{dr}\right)=-\tfrac{1}{2}m'u'^3.r^2.(1-3s^2).$$ [5373a]

Its variation relative to the characteristic δ, neglecting δr [5367a], gives,

$$\delta.r.\left(\frac{dR}{dr}\right)=3m'u'^3.r^2.s\delta s;$$ [5373b]

and, from [5364, 5372], we have $f\,\delta.\mathrm{d}R=\frac{3}{2}m'.u'^3r^2.s\delta s$. Substituting these values in the first member of [5373], it becomes as in its second member.

† (2990) We have nearly $r=a$, $u'=\frac{1}{a'}$ [4937n, &c.]; substituting these in $m'.u'^3.r^3$, we get $m'.u'^3.r^3=\frac{m'a^3}{a'^3}=\bar{m}^2$ [4865]; and, from [5094], it appears that this is equal to m^2 nearly, as in [5374]. If we use the value of g, [5374], it becomes $m'.u'^3.r^3=\frac{4}{3}.(g-1)$. Substituting this in [5373], we get [5375]. [5374a] [5374b]

‡ (2991) Multiplying together the values of s and δs, [5376]; reducing, and retaining only the term depending on the angle $gv-fv-\theta$, we get,

$$s\delta s=-\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{2(g-1)}.\frac{D^2}{r^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta).$$ [5376a]

Multiplying this by $\frac{14.(g-1)}{r}$, we obtain the expression of the second member of the equation [5375], as in [5377]. Multiplying this last function by $\frac{dt^2}{r^2.dv}$, we get the term of $d\delta v$, corresponding to the second member of [5367]; namely, [5376b]

$$-\frac{7dt^2.(\alpha\rho-\frac{1}{2}\alpha\varphi)}{r^2.dv}.\frac{D^2}{r^3}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta):$$ [5376c]

adding this to the term [5368], we get [5378].

[5377] $$3\int\delta.dR+2\delta.r.\left(\frac{dR}{dr}\right)=-7.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{r^3}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta).$$

Multiplying this by $\frac{dt^2}{r^2.dv}$, we obtain, in the expression of $d\delta v$ [5367], a term which is to be added to that in [5368]; and the sum becomes,

[5378] $$d\delta v=-\frac{10dt^2.(\alpha\rho-\frac{1}{2}\alpha\varphi)}{r^2.dv}.\frac{D^2}{r^3}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta).$$

[5378′] We may substitute in it, a for r, dv for ndt [603, 4828], and $n^2a^3=1$ [3709′]; by which means, it becomes,*

[5379] $$d\delta v=-10dv.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta).$$

This value of $d\delta v$ corresponds to the angle contained between the two
[5380] consecutive radii vectores r and $r+dr$, as in [923—925]. Now, if we put
[5380′] this angle equal to $dv_{,}$, dv will represent its projection upon the plane of the ecliptic, and we shall have, as in [925],†

[5381] $$dv=dv_{,}.\frac{\sqrt{(1+s^2)^2-\frac{ds^2}{dv^2}}}{\sqrt{1+s^2}};$$

or, very nearly,

[5382] $$dv=dv_{,}.\left\{1+\tfrac{1}{2}s^2-\tfrac{1}{2}.\frac{ds^2}{dv^2}\right\}.$$

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* (2992) Substituting in the factor $\frac{dt^2}{r^2.dv}.\frac{D^2}{r^3}$, which occurs in [5378], the values
[5379a] $dt=\frac{dv}{n}$, $r=a$, and $n^2a^3=1$ [5378′], it becomes,

[5379b] $$\frac{dv}{n^2a^3}.\frac{D^2}{a^2}=dv.\frac{D^2}{a^2};$$

hence [5378] changes into [5379].

† (2993) The expression [5381] is nearly the same as that in [925], changing v into $v_{,}$, and $v_{,}$ into v in order to adapt it to the notation in [5380], which is different
[5381a] from that in [923′], observing that, on account of the smallness of s, we may change $\frac{ds}{dv_{,}}$ into $\frac{ds}{dv}$. Developing [5381], according to the powers and products of s, $\frac{ds}{dv}$; neglecting the fourth dimension of these quantities, it becomes as in [5382].

Substituting for s the expression,*

$$s = \gamma . \sin. (gv-\theta) - \frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1} . \frac{D^2}{a^2} . \sin. \lambda . \cos. \lambda . \sin. fv ; \qquad [5383]$$

we get,†

$$dv = dv_{,} . \left\{ 1+\tfrac{1}{2} . (\alpha\rho-\tfrac{1}{2}\alpha\varphi) . \frac{D^2}{a^2} . \gamma . \sin.\lambda . \cos.\lambda . \cos.(gv-fv-\theta)+\&c. \right\} . \qquad [5384]$$

Hence we see, that to obtain the value of $d\delta v$, relative to the angle v, formed by the projection of the radius vector r, upon the ecliptic, with a

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* (2991) Substituting in [5050], the values of δs [5351], we get [5383]; which by using the value of A [5357a], becomes as in [5383c], omitting for brevity the symbol θ. Its differential gives [5383d], observing that f is nearly equal to unity [5347q]. [5383a]
Squaring these expressions, retaining only the products $\sin.gv.\sin.fv$, $\cos.gv.\cos.fv$, [5383b]
which produce the term depending on $\cos.(gv-fv)$, we get [5383e,f], whose sum is as in [5383g]; this is used in the following note;

$$s = \gamma.\sin.gv - A.\sin.fv ; \qquad [5383c]$$

$$\frac{ds}{dv} = g\gamma.\cos.gv - A.\cos.fv ; \qquad [5383d]$$

$$\tfrac{1}{2}s^2 = -A\gamma.\sin.gv.\sin.fv+\&c. = -\tfrac{1}{2}A\gamma.\cos.(gv-fv)+\&c. ; \qquad [5383e]$$

$$-\tfrac{1}{2}.\frac{ds^2}{dv^2} = +gA\gamma.\cos.gv.\cos.fv+\&c. = \tfrac{1}{2}gA\gamma.\cos.(gv-fv)+\&c. ; \qquad [5383f]$$

$$\tfrac{1}{2}s^2-\tfrac{1}{2}.\frac{ds^2}{dv^2} = \tfrac{1}{2}.(g-1).A\gamma.\cos.(gv-fv) . \qquad [5383g]$$

† (2995) Substituting [5383g], in [5382], we get,

$$dv = dv_{,} . \{1+\tfrac{1}{2}(g-1) . A\gamma . \cos.(gv-fv)\} ; \qquad [5386a]$$

and by using the value of A [5357a], it becomes as in [5384]. Hence it appears that this reduction, adds to the value of dv, the term $dv_{,} . \frac{1}{2} (g-1) . A\gamma . \cos.(gv-fv)$,
or $dv . \frac{1}{2} (g-1) . A\gamma.\cos.(gv-fv)$ nearly; which by the substitution of A [5357a], [5386b]
becomes as in the second member of [5385]. This term of dv, is a part of that
depending on $\alpha\rho-\frac{1}{2}\alpha\varphi$, which is denoted by $d\delta v$ in [5359, 5379, 5385, &c.]. Adding [5386c]
together the two parts of $d\delta v$ [5379, 5385], we get the complete value [5386], and its
integral, putting $f=1$, gives δv [5387]. This expression is obtained, to a somewhat [5386d]
greater degree of accuracy, in [12995]; where small terms are computed, of the order
$\frac{3m}{76}$, in comparison with those which are here investigated. [5386e]

fixed right line: we must add to the preceding expression of $d\delta v$ [5379], the term,

[5385] $$d\delta v=\tfrac{1}{2}dv.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta) \quad [5386b];$$

which gives,

[5386] $$d\delta v=-\tfrac{19}{2}dv.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta) \quad [5386d];$$

and, by integration,

[5387] Inequality in longitude depending on the oblateness of the earth. $$\delta v=-\tfrac{19}{2}.\frac{(\alpha\rho-\tfrac{1}{2}\alpha\varphi)}{g-1}.\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\sin.(gv-fv-\theta).$$

This is the only sensible inequality in the moon's motion in longitude, arising from the oblateness of the earth. It may be observed, that $fv-gv+\theta$ *

[5388] expresses the longitude of the ascending node of the orbit, counted from the moveable vernal equinox; hence it follows, that the expression of the true longitude, in terms of the mean longitude, contains the following inequality;

[5389] $$\delta v=\tfrac{19}{2}.\frac{(\alpha\rho-\tfrac{1}{2}\alpha\varphi)}{g-1}.\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\sin.(\text{longitude of the ascending node}).$$

[5390] The coefficient of this inequality is† $5^s,552$, if $\rho=\frac{1}{334}$; it becomes $11^s,499$, if $\rho=\frac{1}{230}$.

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* (2996) It is evident, from [4813, 4817], that $gv-\theta$ represents nearly the moon's

[5388a] distance from the ascending node on the fixed ecliptic, counted according to the order of the signs; and fv [5345], the moon's distance from the *moveable* equinox, counted in the same order. Subtracting the first of these expressions from the second, we obtain

[5388b] $fv-gv+\theta$, which must evidently represent the distance of the node from the equinox, or its longitude. Hence,

[5388c] $$-\sin.(gv-fv-\theta)=\sin.(fv-gv+\theta)=\sin.(\text{longitude of the ascending node}).$$

Substituting this in [5387], we get [5389].

† (2997) Substituting A [5357a], in [5389], it becomes,

[5390a] $$\delta v=\tfrac{19}{2}.A\gamma.\sin.(\text{longitude of the ascending node}).$$

The values of A, corresponding to the ellipticities $\frac{1}{334}$, $\frac{1}{230}$, have already been computed in [5357, 5358], and found to be $6^s,487$, $13^s,436$, respectively. Multiplying

[5390b] these by $\frac{19}{2}.\gamma=0,855767$ [5117 line 5], we get the values [5390]. If we put the coefficient of [5389] equal to A', we shall have, by comparing it with [5357a],

[5390c] $$A'=\tfrac{19}{2}.A\gamma, \quad \text{or,} \quad A=\frac{2}{19\gamma}.A';$$

The oblateness of the earth affects also the motions of the perigee and nodes of the lunar orbit. For, the value of Q is, by this means, increased by the quantity,* [5390′]

$$Q=(\alpha\rho-\tfrac{1}{2}\alpha\varphi).(1-\tfrac{3}{2}s^2).\{\tfrac{1}{3}-(1-s^2).\sin.^2\lambda.\sin.^2 fv-2s.\sin\lambda.\cos\lambda.\sin fv-s^2.\cos^2\lambda\}.D^2u^3. \quad [5391]$$

This produces, in the equation [4754], the following term ;†

$$-\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi).D^2u^2}{h^2}.(1-\tfrac{3}{2}.\sin.^2\lambda); \quad [5392]$$

and, by substituting for u, its approximate value,

$$u=\frac{1}{a}.\{1+e.\cos.(cv-\varpi)\} \quad [4826], \quad [5393]$$

and observing, that h^2 is very nearly equal to a [4859], we obtain, in the differential equation [4961, or 5392a], the terms,

Terms of the equation [4754].

$$-\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{a}.\frac{D^2}{a^2}.(1-\tfrac{3}{2}.\sin.^2\lambda) \quad [5394]$$

$$-\frac{2.(\alpha\rho-\frac{1}{2}\alpha\varphi)}{a}.\frac{D^2}{a^2}.(1-\tfrac{3}{2}.\sin.^2\lambda).e.\cos.(cv-\varpi). \quad [5395]$$

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substituting this in [5357b, c], and reducing, we get the following equations, which may be used hereafter ;

$$A'=4392^s,6.\alpha\rho-7^s,6; \quad [5390d]$$

$$\alpha\rho=\frac{A'+7^s,6}{4392^s,6}. \quad [5390e]$$

* (2998) If we change the signs of the two factors of Q [5346b], which does not alter its value ; and then vary the place of its last term, we get,

$$Q=(\alpha\rho-\tfrac{1}{2}\alpha\varphi).D^2.u^3.\{\tfrac{1}{3}(1-s^2)^{\frac{3}{2}}-(1-s^2)^{\frac{5}{2}}.\sin.^2\lambda.\sin.^2 fv-2s.\sin\lambda.\cos\lambda.\sin.fv-s^2.\cos^2\lambda\}. \quad [5391a]$$

Dividing the last factor by $(1-s^2)^{\frac{3}{2}}$, and then multiplying by the equivalent expression $1-\frac{3}{2}s^2$, neglecting terms of the order s^3, we get [5391]. If we neglect also the terms depending on s, and substitute $\sin.^2 fv=\frac{1}{2}-\frac{1}{2}.\cos.2fv$, it becomes,

$$Q=(\alpha\rho-\tfrac{1}{2}\alpha\varphi).D^2.u^3.\{\tfrac{1}{3}-\tfrac{1}{2}.\sin.^2\lambda+\tfrac{1}{2}.\sin.^2\lambda.\cos.2fv\}; \quad [5391b]$$

which is used in the next note.

† (2999) Upon the same principles, by which we have obtained the equation [4755] under the form [5347f], we may reduce [4754], to the following form,

Hence we easily find, that the motion of the perigee is increased by the following quantity nearly;*

[5396] Increment of the motions of the perigee and node.
$$\delta\varpi = (\alpha\rho - \tfrac{1}{2}\alpha\varphi) \cdot \frac{D^2}{a^2} \cdot v \cdot \{1 - \tfrac{3}{2} \cdot \sin.^2\lambda\}.$$

It is evident, from the equation [4755], that *the retrograde motion of the node, will be increased by the same quantity.* If we reduce it to numbers,

[5397] we obtain,† $0{,}0000026384.v$; which is insensible.

[5392a]
$$0 = \frac{ddu}{dv^2} + u - \frac{1}{h^2} \cdot \left(\frac{dQ}{du}\right).$$

This contains the most important part of the terms now under consideration depending on Q; the neglected quantities being of a different form and order from those which are retained in [5394, 5395]. Now, the expression of Q [5391b], gives in [5392a], the terms,

[5392b]
$$-\frac{1}{h^2} \cdot \left(\frac{dQ}{du}\right) = -\frac{(\alpha\rho - \frac{1}{2}\alpha\varphi)}{h^2} \cdot D^2u^2 \cdot \{1 - \tfrac{3}{2}.\sin.^2\lambda + \tfrac{3}{2}.\sin.^2\lambda.\cos.2fv\}.$$

If we neglect the part depending on the angle $2fv$, it becomes as in [5392]. If we use the values [4937n], and put, for brevity,

[5392c]
$$B = (\alpha\rho - \tfrac{1}{2}\alpha\varphi) \cdot \frac{D^2}{a^2} \cdot (1 - \tfrac{3}{2}.\sin.^2\lambda),$$

we get,

[5392d]
$$-\frac{1}{h^2} \cdot \left(\frac{dQ}{du}\right) = -B\,a\,u^2.$$

Substituting $u^2 = \frac{1}{a^2} \cdot (1 + 2e.\cos.cv)$ [5393], and neglecting e^2, it becomes,

[5392e]
$$-\frac{B}{a} - 2B \cdot \frac{e}{a} \cdot \cos.cv, \quad \text{as in [5394, 5395]};$$

hence the equation [5392a], is reduced to the following form,

[5392f]
$$0 = \frac{ddu}{dv^2} + u - \frac{B}{a} - 2B \cdot \frac{e}{a} \cdot \cos.cv.$$

* (3000) Neglecting terms of the order e^2, e'^2, we find that the coefficient of

[5396a] $\frac{e}{a} \cdot \cos.cv$, in the equation [4961], is represented by $-p$ [4975]; and, it is evident,

[5396b] that the terms depending on B, in [5392f, or 4961], augment the value of p by the quantity $\delta p = 2B$. Now the motion of the perigee is represented, in [4984b], by

[5396c] $(1 - \sqrt{1-p}).v$, which is very nearly equal to $\frac{1}{2}pv$; so, that if p be augmented by

[5396d] δp, the motion of the perigee will be increased by $\frac{1}{2}\delta p.v = Bv$, as in [5396, 5392c].

† (3001) If we neglect terms of the order e'^2, e^2, &c., and also, for brevity, the

We shall now make an interesting remark, upon the preceding inequality of the moon's motion in latitude. This inequality is nothing more than the reaction of the nutation of the earth's axis, discovered by Bradley. To prove this, we shall put γ for the inclination of the lunar orbit to the plane we have spoken of in [5352], which passes always through the equinoxes, and is inclined to the ecliptic by an angle [5353], equal to $\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}\cdot\frac{D^2}{a^2}\cdot\sin.\lambda\cdot\cos.\lambda$. [5398] [5398'] [5399]

This inequality is the reaction of the nutation of the earth.

The inclination of the lunar orbit to the ecliptic, will be,

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symbol θ, we shall find, that the retrograde motion of the nodes is,

$$\{\sqrt{1+p''}-1\}.v=\tfrac{1}{2}p''.\,v,\quad \text{nearly [5059]};$$ [5397a]

observing, that $p''\gamma.\sin.gv$ [5053], is the term of [5049, or 4755], depending on $\sin.gv$. The inspection of the value of Q [5391a], shows, that the quantity $\left(\frac{dQ}{dv}\right)$ produces nothing of importance in [4755]. If we neglect s^2, and put $h^2=a$, $u=a^{-1}$, in the other terms of [4755 line 2], we find, that this equation becomes, [5397b]

$$0=\frac{dds}{dv^2}+s-s\cdot\left(\frac{dQ}{du}\right)-a\cdot\left(\frac{dQ}{ds}\right).$$ [5397c]

Multiplying the equation [5392d], by h^2s, and substituting the preceding values of h^2, u, we get $-s\cdot\left(\frac{dQ}{du}\right)=-Bs$. Again, if we take the partial differential of Q, [5391a], relative to s, and multiply it by $-a$, putting $u=a^{-1}$, we shall get [5397g]. Neglecting s^2, putting $\sin.^2fv=\frac{1}{2}-\frac{1}{2}.\cos.2fv$, and omitting the terms depending on fv, $2fv$, we get [5397h]. Substituting $\cos.^2\lambda=1-\sin.^2\lambda$, and reducing successively, using B [5392c], it becomes as in [5397i]; [5397d] [5397e] [5397f]

$$-a\cdot\left(\frac{dQ}{ds}\right)=(\alpha\rho-\tfrac{1}{2}\alpha\varphi)\cdot\frac{D^2}{a^2}\cdot\left\{\begin{array}{c} s.(1-s^2)^{\frac{1}{2}}-5s.(1-ss)^{\frac{3}{2}}\sin.^2\lambda\,\sin.^2fv \\ +2.\sin.\lambda.\cos.\lambda.\sin.fv+2s.\cos.^2\lambda \end{array}\right\}$$ [5397g]

$$=(\alpha\rho-\tfrac{1}{2}\alpha\varphi)\cdot\frac{D^2}{a^2}.s.\left\{1-\tfrac{5}{2}.\sin.^2\lambda+2.\cos.^2\lambda\right\}=(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.s.\left\{3-\tfrac{9}{2}.\sin.^2\lambda\right\}$$ [5397h]

$$=3Bs.$$ [5397i]

Substituting the values [5397d, i], in [5397c], we get,

$$0=\frac{dds}{dv^2}+s+2Bs,\qquad\text{or}\qquad 0=\frac{dds}{dv^2}+s+2B.\gamma.\sin.gv,\quad\text{nearly [5383]};$$ [5397k]

hence the value of p'' [5053], is increased by the quantity $2B$, nearly; consequently the motion of the node $\frac{1}{2}p''v$ [5397a] is augmented by the quantity Bv, being the same as that of the perigee, [5396d], as in [5397]. [5397l]

Substituting, in [5396], the values [5351—5356], we get,

[5400] $$*\gamma-\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}\cdot\frac{D^2}{a^2}\cdot\sin.\lambda\cdot\cos.\lambda\cdot\cos.(gv-fv-\theta) = \text{inclination of orbit to the ecliptic.}$$

[5401] Now, the area described by the moon about the earth's centre of gravity, is $\frac{1}{2}r^2.dv$ [372a]. This area, projected upon the ecliptic, is decreased in the
[5401′] ratio of the cosine of the inclination of the moon's orbit [5400] to the radius; therefore, it is represented by,

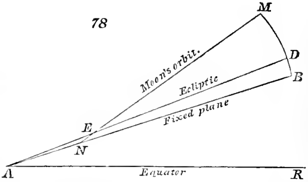
[5402] $$\tfrac{1}{2}r^2.dv.\cos.\left\{\gamma-\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}\cdot\frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta)\right\} = \text{projec. of the area } \tfrac{1}{2}r^2.dv.$$

Hence, the expression of this area contains the inequality,†

[5397m] $$\delta\varpi=\delta\theta=0,0000026384.v, \quad \text{as in [5397]};$$

[5397n] and, by putting $v=360^d$, it becomes $\delta\varpi=0^s,3$, corresponding to one revolution of the moon. This part of the motion of the perigee is insensible, in comparison with its
[5397o] whole motion $0,00845199.v$ [5117 line 2]; being only $\frac{1}{32000}$ part of it.

* (3002) In the annexed figure, let
[5400a] AR be the equator, ANB the fixed plane, AED the ecliptic, NEM the moon's orbit; then, if we make arc NM=arc $NB=90^d$,
[5400b] and describe about N, as a pole, the arc MDB, we shall have arc $MB=\gamma$ [5398′], angle $DAB=A$ [5357a]. Moreover, we have, very nearly, in the



[5400c] triangle DAB, arc $DB=A.\sin.AB=A.\sin.(AN+90^d)=A.\cos.AN$; and, as AN is nearly equal to $AE=fv-gv+\theta$ [5388], we have $DB=A.\cos.(fv-gv+\theta)$; hence,

[5400d] $$MD=MB-BD=\gamma-A.\cos.(fv-gv+\theta)=\gamma-A.\cos.(gv-fv-\theta).$$

Now, from the extreme smallness of the arcs DB, EN, it is evident, that the arc MD represents very nearly the value of the angle MED, or the inclination of the moon's
[5400e] orbit to the ecliptic. This agrees with [5400]. We may moreover remark, that the angle $gv-fv-\theta$, or $fv-gv+\theta$, corresponding to the distance of the node from the equinox varies only about 3^d, in a periodical revolution of the moon; consequently, the angle of
[5400f] inclination [5400] alters but little, during that revolution; and the factor of $\frac{1}{2}r^2.dv$, in the inequality [5403], is nearly constant in the whole of that period.

[5403a] † (3003) Putting, for brevity, $A'=A.\cos.(gv-fv-\theta)$ [5357a], in the expression of the projection of the area [5402], and then developing, as in [61] Int.,

$$\tfrac{1}{2}r^2.dv.\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}.\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta) = \text{a term of the projection of } \tfrac{1}{2}r^2.dv\,;$$ [5403]

and, as we have, very nearly,* $r^2.dv = a^2.dt$, *dt denoting the moon's mean motion*, this inequality will be represented by, [5404]

$$\tfrac{1}{2}D^2.dt.\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta) = \text{a term of the projection of } \tfrac{1}{2}r^2.dv.$$ [5405]

Multiplying this expression by the moon's mass, which we shall represent by L; then, dividing the product by $\frac{1}{2}dt$, we obtain the momentum of [5406] the moon's force about the centre of gravity of the earth, arising from the oblateness of the earth.† Hence we get, for this momentum, the following [5406'] expression;

$$L.D^2.\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta) = \text{momentum of the moon.} \quad (i)$$ [5407]

Momentum of the moon corresponding to the oblateness of the earth.

In consequence of the equality between the action and reaction, the same cause

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neglecting the second and higher powers and products of A', it becomes,

$$\tfrac{1}{2}r^2.dv.\cos.(\gamma-A') = \tfrac{1}{2}r^2.dv.\{\cos.\gamma+A'.\sin.\gamma\} = \tfrac{1}{2}r^2.dv.\cos.\gamma+\tfrac{1}{2}r^2.dv.A'\gamma, \quad \text{nearly.}$$ [5403b]

Re-substituting the value of A', in the last part of this expression, we obtain the term [5403].

* (3004) We have $r^2.dv = a^2.ndt.\sqrt{1-e^2}$ [1057]; and, by neglecting e^2, changing also the mean motion ndt into dt, so as to correspond to the notation in [5404], [5404a] it becomes $r^2.dv = a^2.dt$, as in [5404]; substituting this in [5403], we get [5405]. In this process, we neglect the consideration of the perturbations of the moon's motion by the [5404b] sun's action, using the elliptical value of $r^2.dv$ [5404a]; observing, that the rejected terms are of a different form or order, from that in [5405].

† (3005) The arc which the moon describes in her orbit, in the time dt, being [5406a] resolved in a direction perpendicular to the radius r, is evidently represented by $r.dv$; consequently, the velocity, in that direction, is $r.\frac{dv}{dt}$; and the force is proportional to it. Multiplying this by the radius r, and by the mass L, we get the corresponding *momentum* of the moon [29'], [5406b]

$$r^2.\frac{dv}{dt}.L, \qquad \text{or} \qquad \frac{\frac{1}{2}r^2.dv}{\frac{1}{2}dt}.L, \quad \text{as in [5406'].}$$ [5406c]

Substituting, in this last expression, for $\frac{1}{2}r^2.dv$, the term given in [5405], we obtain the corresponding part of the moon's momentum, as in [5407].

must produce, in the particles of the earth, a momentum which is equal and contrary to the preceding. This momentum is indicated by the nutation of the earth's axis, and we may determine its value by means of the formulas
[5408] in book v. § 6. For, we see, in [3101], that if we put V for the obliquity of the ecliptic to the equator, the moon's action upon the earth produces, in consequence of the oblate form of the earth, an increment in the angle V, which is represented by,*

[5409] $$\frac{l\lambda}{(1+\lambda).(g-1)}\cdot\gamma\cdot\cos.(gv-fv-\theta) = \text{increment of the obliquity } V\,;$$

l and λ being the same as in that article. The element of the rotatory
[5409′] motion of the earth being supposed ndt [3015]; the sum of the momenta of the forces acting upon each particle of the earth, multiplied by the mass
[5410] of the particle, is equal to nC; C being the momentum of inertia of the earth, relative to its axis of rotation.† To reduce this momentum to

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* (3006) Of the five terms which compose the value of θ [3101], and of θ'
[5409a] [3360 or 3378], or that of V, in the notation [5408], the *first* is constant; the *second* is secular; the *fourth* and *fifth* are small, and depend on the places of the sun and moon. The *third* is that upon which the nutation depends; namely,

[5409b] $$\frac{l\lambda c'}{(1+\lambda).f'}\cdot\cos.(f't+\beta')\;;$$

c' [3086] being nearly the same as γ [5398′]; and,

[5409c] $$-f't-\beta' = fv-gv+\theta \quad [3086', 5388],$$

representing the longitude of the moon's ascending node, counted from the moveable vernal equinox. Substituting these values in [5409b], it becomes,

[5409d] $$\frac{l\lambda\gamma}{(1+\lambda).f'}\cdot\cos.(gv-fv-\theta).$$

Now, the mean increment of v, in the time t, being represented by t [5404], it
[5409e] will follow, from the equation [5409c], that $-f'=f-g=1-g$, nearly [5347q], or $f'=g-1$; substituting this in [5409d], we get the increment of the inclination V
[5409f] [5409]. We may remark, that this use of the symbol V is restricted to § 20 [5408] to [5422]; in other parts of this chapter, V denotes the function [5336, &c.].

† (3007) The angular velocity of a particle of the earth about its axis of revolution
[5410a] being n [5409′], its actual velocity, at any distance $r_{,}$ from the axis, is $nr_{,}$. Multiplying this by the same radius $r_{,}$, and by the mass of the particle dm, we get

the ecliptic, we must multiply it by the cosine of its obliquity, or by,*

$$\cos.\left\{ V + \frac{l\lambda}{(1+\lambda).(g-1)}.\gamma.\cos.(gv-fv-\theta) \right\};$$ [5411]

we shall, therefore, have the following inequality, in the momentum of the earth [5411b];

$$-\frac{l\lambda.nC.\sin.V}{(1+\lambda).(g-1)}.\gamma.\cos.(gv-fv-\theta) = \text{inequality in the earth's momentum.}$$ [5412]

We have, in [3098],

$$l = \frac{3m^2}{4n}.\frac{(2C-A-B)}{C}.(1+\lambda).\cos.V;$$ [5413]

mt denoting the mean motion of the earth [3059]; also† $\lambda.m^2 = \frac{L}{a^3}$; [5414]
a being the moon's mean distance from the earth; and, since we represent the moon's mean motion by t [5404], and the mass of the earth by M [5415]
[4757]; we have, very nearly,‡ $\frac{M}{a^3} = 1$, which gives $\lambda.m^2 = \frac{L}{M}$; [5416]

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the momentum of this particle, equal to $n.r_{\prime}^2.dm$ [29']. Integrating this, relative to the whole mass of the earth, it becomes $n.\int r_{\prime}^2.dm$; in which $r_{\prime}^2$ is represented by $x''^2+y''^2$, of the formula [229], the axis of rotation being z''; consequently, this expression becomes, [5410b]

$$n.\int r_{\prime}^2.dm = n.\int(x''^2+y''^2).dm = n.C \quad [229], \text{ as in } [5410].$$ [5410c]

* (3008) Putting the function [5409] equal to δV, the whole obliquity will become $V+\delta V$. Its cosine, by [61] Int. is represented by $\cos.V-\delta V.\sin.V$, nearly. Multiplying this, by the momentum nC, it produces the term, $-nC.\sin.V.\delta V$; and, by substituting the value of δV [5409], it becomes as in [5412]. [5411a] [5411b]

† (3009) This is easily deduced from $\lambda.m^2 = \frac{L'}{a'^3}$ [3079], changing L' into L, and a' into a, to conform to the alterations in the notation, which is used in [3078, 5406, 5414]. We may also observe, that in deducing the value of l [5413], from [3098], we must change h into V, to conform to [3357, 5408]. [5414a]

‡ (3010) The mean increment of v, in the time t, is very nearly represented by nt [5095]; consequently that of dv is ndt; and as this is put equal to dt, in [5404], we shall have $n=1$. Substituting this and $\mu = M+m$ [4775''], in [3700], [5415a]

thus, the preceding inequality becomes,*

[5417] $$-\frac{3L}{4M}\cdot\frac{(2C-A-B)}{g-1}\cdot\rho.\sin.V.\cos.V.\cos.(gv-fv-\theta)=\text{inequality in the earth's momentum.}$$

We have, from [2960—2962],†

[5418] $$2\,C-A-B=\tfrac{16}{9}\cdot\pi\cdot(\alpha\,\rho-\tfrac{1}{2}\,\alpha\,\varphi)\cdot D^2.\int 3\,\Pi\cdot R^2\cdot d\,R\,;$$

[5419] ρ being the oblateness of the earth; D its semi-diameter; R the radius
[5420] of one of its particles, whose density is Π; and π the semi-circumference,
[5421] whose radius is unity. The mass of the earth is ‡ $M=\frac{4}{3}\pi.\int 3\,\Pi.R^2.d\,R$;

[5415b] we get $\frac{M+m}{a^3}=1$; which, by neglecting the mass of the moon m, in comparison
[5415c] with that of the earth M, becomes $\frac{M}{a^3}=1$, as in [5416]. This gives $a^3=M$, and, by substituting it in [5414], we obtain the expression of $\lambda.m^2$ [5416].

* (3011) From [5416] we get $m^2=\frac{L}{\lambda M}$; hence [5413] becomes,

[5416a] $$l=\frac{3L}{4M}\cdot\frac{(2C-A-B)}{nC}\cdot\frac{(1+\lambda)}{\lambda}\cdot\cos.V\,;$$

substituting this in [5412], we get [5417].

† (3012) Subtracting the sum of the values of A, B [2960, 2961], from $2C$ [2962], we get,

[5418a] $$2C-A-B=\frac{48}{27}\cdot\alpha\,\pi.(h-\tfrac{1}{2}\varphi).\textstyle\int_0^1\rho\cdot d.a^3=\frac{16}{9}\cdot\pi\cdot(\alpha h-\tfrac{1}{2}\alpha\varphi).\int_0^1\rho.3a^2.da\,;$$

in which φ [2951], is the same as in [5333'], and $h=\rho$ [5335$^{\prime\prime}$]. Moreover, we
[5418b] must change a, ρ [2947], into R, Π [5419, 5420], to conform to the present notation; hence the last expression [5418a] becomes,

[5418c] $$2C-A-B=\tfrac{16}{9}\cdot\pi\cdot(\alpha\rho-\tfrac{1}{2}\alpha\varphi)\cdot\textstyle\int_0^1 3\Pi\cdot R^2\cdot dR.$$

The two members of this equation are not homogeneous; for in the first member, A, B, C [2920—2922], are of the *fifth* order in R, and the second member is only of the
[5418d] *third* order; we must, therefore, multiply the second member, by the square of the mean radius of the earth D [5334], which is taken for unity in [2947']; and then it becomes
[5418e] as in [5418]. In the original work, the factor 3, under the sign $\int$, is accidentally omitted.

‡ (3013) This is similar to the expression [1506a], changing the notation, as in

which is to be substituted in [5418]; and then the resulting value in [5417], changing also the obliquity of the ecliptic V [5408j, into λ [5341]; [5422]
hence the inequality [5417] becomes,

$$-L.D^2.\frac{(\alpha\rho-\frac{1}{2}\alpha\varphi)}{g-1}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv-\theta) = \text{inequality in the earth's momentum.} \quad [5423]$$

This expression is the same as that in [5407], *with a contrary sign. Hence it follows, that the preceding inequality of the moon's motion in latitude, is the reaction of the nutation of the earth's axis; and, that there would be an equilibrium about the centre of gravity of the earth, by means of the forces which produce these two inequalities, supposing all the particles of the earth and moon to be firmly connected with each other;* since the moon compensates for the smallness of the forces which act on it, by the length of the lever to which it is attached. [5424]

The inequality in the moon's latitude is the reaction of the nutation of the earth's axis.

21. To notice the effect of the moon's figure, which is not exactly spherical, we shall observe, that it introduces into Q [4756], the term,

$$(M+m).\frac{\delta V'}{m} \quad [4773], \text{ or more simply, } \quad \frac{\delta V'}{m}; \quad [5425]$$

because, we have put $M+m=1$ [4775″]. Now, from [1505, 1809′, 4770′], we obtain,*

$$\delta V' = \frac{4\alpha\pi}{5r^3}.\int_0^{a}\rho.d.(a^5.Y^{(2)}); \quad [5426]$$

[5418b]. Substituting the value of M in [5418], we get,

$$2C-A-B = \tfrac{4}{3}.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).D^2M; \quad [5421i]$$

and, by using this expression, and that of $V=\lambda$ [5422]; we may reduce the inequality [5417], to the form [5423].

* (3014) We may neglect the terms of V [1505], which are divided by r^4, on account of their smallness; also those depending on $Y^{(0)}$, $Y^{(1)}$, as is done in [1809′, 1811], and then it becomes, by accenting the letter V', so as to conform to the notation [4769],

$$V' = \frac{4\pi}{3r}.\int_0^1\rho.d.a^3 + \frac{4\alpha\pi}{5r^3}.\int_0^1\rho.d.(a^5Y^{(2)}) = \frac{m}{r} + \frac{4\alpha\pi}{5r^3}.\int_0^1\rho.d.(a^5Y^{(2)}) \; [5429]. \quad [5425a]$$

Substituting this in [4770′], we get $\delta V'$ [5426]; the limits of the integral being changed from 0, 1, to 0, a. Multiplying the expression of $\delta V'$ [5426], by $M+m=1$ [5425b]
[4775″]; and then dividing by m [5429], we get [5430].

[5427] the integral being taken from $a=0$, to a, *equal to the moon's semi-*
[5428] *diameter, which we shall denote by* a, *and* ρ *being the density of the*
[5429] *stratum of the moon corresponding to* a. We have $m=\frac{4}{3}\pi.\int_0^{\mathrm{a}}\rho.d.a^3$ [1506a]; hence we deduce,

[5430]
$$(M+m).\frac{\delta V'}{m}=\frac{3\alpha.\int_0^{\mathrm{a}}\rho.d.(a^5.Y^{(2)})}{5r^3.\int_0^{\mathrm{a}}\rho.d.a^3}.$$

To determine $\int_0^{\mathrm{a}}\rho.d.(a^5.Y^{(2)})$, we shall observe that we have, in [1761], for $Y^{(2)}$, an expression of the following form,*

[5431]
$$Y^{(2)}=h'.(\tfrac{1}{3}-\mu^2)+h''.\mu.\sqrt{1-\mu\mu}.\sin.\varpi+h'''.\mu.\sqrt{1-\mu\mu}.\cos.\varpi$$
$$+h''''.(1-\mu\mu).\sin.2\varpi+h^{\mathrm{v}}.(1-\mu\mu).\cos.2\varpi.$$

Then, the properties of the axes of rotation [1753—1757], give,†

[5432]
$$0=\int_0^{\mathrm{a}}\rho.d.(a^5h'');\qquad 0=\int_0^{\mathrm{a}}\rho.d.(a^5h''');\qquad 0=\int_0^{\mathrm{a}}\rho.d.(a^5h^{\mathrm{i}});$$

and then, from [2948—2950], we obtain,‡

[5431a] * (3015) The expression of $Y^{(2)}$ [5431], is the same as in [1761], increasing the accents on h, by unity.

† (3016) Substituting the expression of $Y^{(2)}$ [5431], in [1757], we get,

[5432a]
$$U^{(2)}=\alpha.(\tfrac{1}{3}-\mu^2).\int_0^{\mathrm{a}}\rho.d.(a^5h')+\alpha\mu.\sqrt{1-\mu^2}.\sin.\varpi.\int_0^{\mathrm{a}}\rho.d.(a^5h'')+\alpha\mu.\sqrt{1-\mu^2}.\cos.\varpi.\int_0^{\mathrm{a}}\rho.d.(a^5h''')$$
$$+\alpha.(1-\mu^2).\sin.2\varpi.\int_0^{\mathrm{a}}\rho.d.(a^5h'''')+\alpha.(1-\mu^2).\cos.2\varpi.\int_0^{\mathrm{a}}\rho.d.(a^5h^{\mathrm{v}}).$$

Comparing this, with the value of $U^{(2)}$ [1753], we get,

[5432b]
$$H=-\alpha.\int_0^{\mathrm{a}}\rho.d.(a^5h');\quad H'=\alpha.\int_0^{\mathrm{a}}\rho.d.(a^5h'');\quad H''=\alpha.\int_0^{\mathrm{a}}\rho.d.(a^5h''');$$
$$H'''=\alpha.\int_0^{\mathrm{a}}\rho.d.(a^5h'''');\quad H''''=\alpha.\int_0^{\mathrm{a}}\rho.d.(a^5h^{\mathrm{v}}).$$

[5432c] Now, the properties of the principal axes give, in [1754], $H'=0$, $H''=0$, $H'''=0$; substituting these in [5432b], and dividing by α, we get, from the second, third and fourth equations, the values [5432].

‡ (3017) Substituting the values of A, B, C [2948—2950], in $2C-A-B$, we get the expression [5133b], by putting $\cos.^2\varpi+\sin.^2\varpi=1$. This is easily reduced to the form [5433c], by introducing the value of $Y^{(2)}$ [5431], and neglecting the
[5433a] terms depending on h'', h''', h'''', on account of the integrals [5432]. We may also neglect the term depending on $\cos.2\varpi$; because, at the limits of the integral $\varpi=0$ $\varpi=2\pi$, it has the same value; and the integral taken between these limits vanishes. Hence we have,

$$2C-A-B=\tfrac{16}{15}.\alpha\pi.\textstyle\int_0^a\rho.d.(a^5h'); \qquad [5433]$$

$$B-A=\tfrac{16}{15}.\alpha\pi.\textstyle\int_0^a\rho.d.(a^5h^{v}). \qquad [5433']$$

Thus, we have,*

$$2C-A-B=3\alpha.\textstyle\int.\rho.d.(a^5Y^{(2)}).(\tfrac{1}{3}-\mu^2).d\mu.d\varpi \qquad [5433b]$$

$$=3\alpha.\textstyle\int.\rho.d.(a^5h').(\tfrac{1}{3}-\mu^2)^2.d\mu.d\varpi+3\alpha.\int.\rho.d.(a^5h^{v}).(\tfrac{1}{3}-\mu^2).(1-\mu^2).\cos.2\varpi.d\mu.d\varpi \qquad [5433c]$$

$$=3\alpha.\textstyle\int.\rho.d.(a^5h').(\tfrac{1}{3}-\mu^2)^2.d\mu.d\varpi. \qquad [5433d]$$

Now we have, by the usual rules of integration,

$$\int_0^{2\pi}d\varpi=2\pi\,; \qquad \int_{-1}^{1}(\tfrac{1}{3}-\mu^2)^2.d\mu=\tfrac{8}{45} \quad [2933i,l,\text{ or }3569e]\,; \qquad [5433e]$$

substituting these in [5433d], we get [5433]. In like manner, if we substitute the values of A, B [2948, 2949], in $B-A$, we get the first expression [5433g]. Substituting in this, the value of $Y^{(2)}$ [5431], and neglecting, as above, h'', h''', h'''', we get [5433h]; reducing also, by means of $\cos.^2\varpi-\sin.^2\varpi=\cos.2\varpi$; $\cos.^2 2\varpi=\tfrac{1}{2}+\tfrac{1}{2}.\cos.4\varpi$; and neglecting, as in [5133a], the terms depending on $\cos.2\varpi$, $\cos.4\varpi$, we obtain [5433i]; [5433f]

$$B-A=\alpha\textstyle\int\rho.d.(a^5Y^{(2)}).d\mu.d\varpi.(1-\mu^2).(\cos.^2\varpi-\sin.^2\varpi)$$

$$=\alpha\textstyle\int\rho.d.(a^5Y^{(2)}).d\mu.d\varpi.(1-\mu^2).\cos.2\varpi \qquad [5433g]$$

$$=\alpha\textstyle\int\rho.d.(a^5h').d\mu.d\varpi.(\tfrac{1}{3}-\mu^2).(1-\mu^2).\cos.2\varpi+\alpha\int\rho.d.(a^5h^{v}).(1-\mu^2)^2.\cos.^2 2\varpi.d\mu.d\varpi \qquad [5433h]$$

$$=\tfrac{1}{2}\alpha\textstyle\int\rho.d.(a^5h^{v}).(1-\mu^2)^2.d\mu.d\varpi. \qquad [5433i]$$

Substituting the integrals

$$\int_0^{2\pi}d\varpi=2\pi, \qquad \int_{-1}^{1}(1-\mu^2)^2.d\mu=\frac{16}{15} \qquad [1754e,f], \qquad [5433k]$$

in this last expression, it becomes as in [5433'].

* (3018) Substituting the value of $Y^{(2)}$ [5431], in $\int\rho.d.(a^5Y^{(2)})$, and neglecting the terms depending on h'', h''', h'''', on account of the equations [5132], we get [5434a]. The integrals of this expression are easily obtained from [5433, 5433'], and, by substitution, we get [5434b];

$$\textstyle\int_0^a\rho.d.(a^5Y^{(2)})=(\tfrac{1}{3}-\mu^2).\int_0^a\rho.d.(a^5h')+(1-\mu^2).\cos.2\varpi.\int_0^a\rho.d.(a^5h^{v}) \qquad [5434a]$$

$$=(\tfrac{1}{3}-\mu^2).\frac{2C-A-B}{\tfrac{16}{15}\alpha\pi}+(1-\mu^2).\cos.2\varpi.\frac{B-A}{\tfrac{16}{15}\alpha\pi}. \qquad [5434b]$$

Substituting this in [5430], and making a slight reduction, we get [5434]. Multiplying this by the second member of [5435], and dividing by its first member C, we obtain [5436].

[5434] $$(M+m).\frac{\delta V'}{m}=\frac{9}{16\pi}.\frac{1}{r^3.\int_0^a\rho.d.a^3}.\{(2C-A-B).(\tfrac{1}{3}-\mu^2)+(B-A).(1-\mu^2).\cos.2\varpi\}.$$

We have, very nearly, in [2962],

[5435] $$C=\frac{8\pi}{15}.\int_0^a\rho.d.a^5;$$

therefore,

Terms of Q.

[5436] $$(M+m).\frac{\delta V'}{m}=\tfrac{3}{10}.\frac{\int_0^a\rho.d.a^5}{\int_0^a\rho.d.a^3}.\frac{1}{r^3}.\left\{\begin{array}{l}\frac{(2C-A-B)}{C}.(\tfrac{1}{3}-\mu^2)\\ +\frac{(B-A)}{C}.(1-\mu^2).\cos.2\varpi\end{array}\right\}=\text{terms of } Q.$$

In this expression, ϖ is the angle formed by the principal axis of the moon,
[5436′] directed towards the earth, and the *plane which passes through the earth's centre, and the axis of the moon's equator*;* μ is the sine of the earth's

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* (3019) The notation which is here used, is similar to that for the earth [5338, 5334′];
[5436a] and corresponds also with [2910, 3435, &c.]. In defining the angle ϖ, in the original work, the words, *line connecting the centres of the earth and moon*, are inadvertently used, instead of the part printed in italics in [5436′]. If we suppose the line connecting the
[5436b] centres of the moon and earth to be projected upon the plane of the lunar equator, then
[5436c] ϖ will represent the angle formed by this projected line, or *radius vector*, and the moon's longest axis, which is directed nearly towards the earth; this axis being taken as the origin
[5436c′] of the angle ϖ; hence we have, by supposing the angular and rotatory motion to commence together, when $\varpi=0$;

[5436d] $\varpi=$ angular motion of this radius vector — moon's rotatory motion.

Now, in [3440, 3433f], v represents the apparent motion of the earth in longitude, seen
[5436e] from the moon; and φ the rotatory motion of the moon; so that, if we neglect the terms arising from the reduction of v to the plane of the lunar equator, we may put v for the
[5436f] angular motion of the radius vector, and φ for the rotatory motion; and by this means [5436d] becomes,

[5436g] $$\varpi=v-\varphi.$$

The differential, relative to the characteristic d, affects only the moon's co-ordinates [5363′], in its relative motion about the earth; and, as φ depends on the rotatory motion, we shall
[5436h] get, for the differential of the equation [5436g], the expression $\mathrm{d}\varpi=dv$; therefore,

[5436i] $$\mathrm{d}.\cos.2\varpi=-2dv.\sin.2\varpi, \quad \text{as in [5437, 5437′]}.$$

If we substitute the expression of $v-\varphi$ [5436g], in [3447c], we get,

declination, seen from the moon, and referred to the moon's equator [2909, 3435,&c.]. It is evident, that, by increasing v by dv, ϖ increases by dv; therefore, we have $\mathrm{d}.\cos.2\varpi = -2dv.\sin.2\varpi$ [5436i]; the differential symbol d referring only to the co-ordinates of the moon; moreover, we have, as in [5360], [5437] [5437′]

$$R = -Q + \frac{1}{r}. \qquad [5438]$$

The part of $\mathrm{d}R$, relative to the spheroidal form of the moon, produces the following expression, neglecting the square of μ;* [5438′]

$$\mathrm{d}R = \frac{3}{5r^3}\cdot\frac{\int_0^a \rho.d.a^5}{\int_0^a \rho.d.a^3}\cdot\frac{(B-A)}{C}\cdot dv.\sin.2\varpi. \qquad [5439]$$

Hence we get, in δv, or in the moon's true longitude, the following term of the formula [931];†

$$\delta v = \tfrac{9}{5}\cdot\frac{\int_0^a \rho.d.a^5}{\int_0^a \rho.d.a^3}\cdot\frac{1}{r^2}\cdot\frac{(B-A)}{C}\cdot\int\int dv^2.\sin.2\varpi. \qquad [5440]$$

$$\varpi = -u + H.\sin.\Pi + \&c.; \qquad [5436k]$$

u being the moon's libration in longitude [3164″]; so that any inequality which occurs in u, may occur also in ϖ, but with a different sign, as in [5111, &c.]. If we substitute, in [5436k], the value of u [3456], we get, [5436l]

$$\varpi = -Q.\sin.\left\{mt.\sqrt{3.\left(\frac{B-A}{C}\right)} + F\right\} - \&c.; \qquad [5436m]$$

and, if we change Q into K, it produces the term mentioned in [5441]. [5436n]

* (3020) The part of Q mentioned in [5125], and developed in [5436], produces in R [5438] the following quantity;

$$R = -\tfrac{3}{10}\cdot\frac{\int_0^a \rho.d.a^5}{\int_0^a \rho.d.a^3}\cdot\frac{1}{r^3}\cdot\left\{\frac{(2C-A-B)}{C}\cdot(\tfrac{1}{3}-\mu^2) + \frac{(B-A)}{C}\cdot(1-\mu^2).\cos.2\varpi\right\}. \qquad [5439a]$$

If we neglect the square of μ, as in [5438′]; then take its differential relative to d, using the expression [5437′], we get [5439].

† (3021) We have, in δv [931], the term $\frac{3a}{\mu}.\int\int\frac{ndt.\mathrm{d}R}{\sqrt{1-e^2}}$; and, if we neglect e^2, putting $a=r$, nearly; also $\mu=1$, as in [4775″]; it becomes $3r.\int\int ndt.\mathrm{d}R$. [5440a]
Now, ndt is nearly equal to dv [5095]; therefore, δv contains the term $3r.\int\int dv.\mathrm{d}R$; and, by substituting the value of $\mathrm{d}R$ [5439], it becomes as in [5440].

The angle ϖ is always very small [3468, 5436′]; so that we may suppose $\sin.2\varpi = 2\varpi$. Moreover, from [3456], we find, that ϖ contains a term of the

[5441] form $-K.\sin.\left\{v.\sqrt{\frac{3.(B-A)}{C}}+F\right\}$ [5436m, n]. This term, taken with a contrary sign, represents, in [3456, 5436m], the real libration of the moon. As it increases very slowly, it would seem, that it ought to become

[5441′] sensible by double integration: this is the only term of the expression of ϖ, which it is necessary to notice. It produces, in δv, the term,*

[5442] Inequality in the moon's longitude, arising from the spheroidal form of the moon.

$$\delta v = \tfrac{6}{5}.\frac{\int_0^{a}\rho.d.a^5}{\int_0^{a}\rho.d.a^3}.\frac{K}{r^2}.\sin.\left\{v.\sqrt{3.\frac{(B-A)}{C}}+F\right\}.$$

The libration $K.\sin.\left\{v.\sqrt{3.\frac{(B-A)}{C}}+F\right\}$ being insensible, we cannot suppose, that it amounts to a centesimal degree. Moreover, the coefficient

[5443] $\frac{6}{5}.\frac{1}{r^2}.\frac{\int_0^{a}\rho.d.a^5}{\int_0^{a}\rho.d.a^3}$ is extremely small. If the moon be homogeneous, it becomes†

[5444] $\frac{6}{5}.\frac{a^2}{r^2}$; now, $\frac{a}{r}$ is the sine of the moon's apparent semi-diameter; hence,

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* (3022) Substituting 2ϖ for $\sin.2\varpi$, in the integral expression of $\int\int dv^2.\sin.2\varpi$, which occurs in [5440], and then the term of ϖ [5441], we obtain, by successive

[5442a] integrations, the expression [5442c], retaining only the most important term, having the divisor $B-A$, arising from the double integration;

[5442b] $$\int\int dv^2.\sin.2\varpi = 2\int\int dv^2.\varpi = -2K.\int\int dv^2.\sin.\left\{v.\sqrt{\frac{3.(B-A)}{C}}+F\right\}$$

[5442c] $$=\frac{2C.K}{3.(B-A)}.\sin.\left\{v.\sqrt{\frac{3.(B-A)}{C}}+F\right\}.$$

Substituting this in [5440], we get [5442].

† (3023) The moon being supposed homogeneous, and $\rho=1$, we have,

[5443a] $$\int_0^{a}\rho.d.a^5 = a^5;\qquad \int_0^{a}\rho.d.a^3 = a^3;\qquad \text{hence,}\quad \frac{\int_0^{a}\rho.d.a^5}{\int_0^{a}\rho.d.a^3} = a^2.$$

Substituting this, in [5443], we get,

[5443b] $$\tfrac{6}{5}.\frac{1}{r^2}.\frac{\int_0^{a}\rho.d.a^5}{\int_0^{a}\rho.d.a^3} = \tfrac{6}{5}.\frac{a^2}{r^2} = \tfrac{6}{5}.\sin.^2(\text{moon's semi-diameter}) = \tfrac{6}{5}.(0{,}0045)^2 = 0{,}000024;$$

[5443c] and, if we suppose $K=1^\circ=54^m=3240^s$, we shall get $0{,}000024.K=0^s{,}07$, for the coefficient of the correction [5442]; which is insensible.

the product of K, by this coefficient, is wholly insensible. If the moon be not homogeneous, its density must increase from the surface to the centre; then, this coefficient is yet less.* Hence it follows, *that the preceding inequality of the moon's longitude is insensible; and, that the variation from a spherical form does not produce any sensible inequality in the motion in longitude.* [5444'] [5445]

As to the latitude, we must observe, that μ is the sine of the earth's declination, seen from the moon [5437], and referred to the lunar equator; moreover, the ascending node of the moon's orbit always coincides with the descending node of its equator [3433]; therefore, we shall have,† [5445']

$$\mu^2 = \{s + \lambda . \sin.(gv - \theta)\}^2 ; \qquad [5446]$$

λ being here the inclination of the lunar equator to the ecliptic. Hence we get, ‡ [5446']

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* (3024) Changing R into a, in [277'] and multiplying by $\frac{5}{3}$, we get

$$\frac{5 \int_0^a \rho . a^4 . da}{3 \int_0^a \rho . a^2 . da} < a^2 ; \qquad \text{or} \qquad \frac{\int_0^a \rho . d . a^5}{\int_0^a \rho . d . a^3} < a^2 ; \qquad [5444a]$$

being less than its value a^2, corresponding to $\rho = 1$ [5443a].

† (3025) It is found by observation, that the *descending* node of the lunar equator always coincides with the *ascending* node of the lunar orbit [3133]; and the inclination of the lunar orbit to the ecliptic is nearly equal to γ [5100], also the inclination of the equator to the ecliptic is λ [5446']; therefore, the inclination of the lunar orbit to the lunar equator, is nearly equal to $\gamma + \lambda$. Now from [5383], we find, that the moon's latitude, or the angular *elevation* of the moon above the ecliptic, is nearly represented by $s = \gamma . \sin.(gv - \theta)$; hence the corresponding angular *depression* of the earth, as seen from the moon, is $-\gamma . \sin.(gv - \theta)$; and it is evident, that by changing the inclination γ into $\gamma + \lambda$ [5446b], we get the angular depression of the earth below the lunar equator $-(\gamma + \lambda) . \sin.(gv - \theta)$. This may be put equal to its sine μ, and by using the value of s [5446c], we get, [5446a] [5446b] [5446c] [5446d]

$$\mu = -(\gamma + \lambda) . \sin.(gv - \theta) = - s - \lambda . \sin (gv - \theta) = - \{s + \lambda . \sin.(gv - \theta)\} ; \qquad [5446e]$$

whose square is the same as [5446].

‡ (3026) The partial differential of μ^2 [5446], relative to s, being divided by $2ds$, gives the first of the expressions [5447]; substituting in this, the value $\sin.(gv - \theta) = \frac{s}{\gamma}$ [5446c] we get the second form of that equation. [5447a]

[5447] $$\mu.\left(\frac{d\mu}{ds}\right) = s+\lambda.\sin.(gv-\theta) = \frac{(\lambda+\gamma)}{\gamma}.s;$$

therefore, the spheroidal form of the moon, adds to the expression of

[5447'] $-\frac{1}{h^2u^2}.\left(\frac{dQ}{ds}\right)$, in the equation [4755], the term,*

[5448] $$\tfrac{3}{5}.\frac{\int_0^a\rho.d.a^5}{\int_0^a\rho.d.a^3}.\frac{1}{r^2}.\frac{(\lambda+\gamma)}{\gamma}.s.\left\{\frac{2C-A-B}{C}+\frac{B-A}{C}.\cos.2\varpi\right\}.$$

[5448'] Now, as we have very nearly, $\cos.2\varpi=1$, it adds to [4755], the quantity,†

[5449] $$\tfrac{6}{5}.\frac{\int_0^a\rho.d.a^5}{\int_0^a\rho.d.a^3}.\frac{1}{r^2}.\frac{(\lambda+\gamma)}{\gamma}.\frac{(C-A)}{C}.s = \text{term of [4755]}.$$

It is evident, from [5397k,l], that this term adds to the motion of the node, the quantity,‡

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* (3027) Substituting, in $-\frac{1}{h^2u^2}.\left(\frac{dQ}{ds}\right)$ [5447'], the terms of Q, given in [5436], we get,

[5448a] $$-\frac{1}{h^2u^2}.\left(\frac{dQ}{ds}\right)=\tfrac{3}{5}.\frac{\int_0^a\rho.d.a^5}{\int_0^a\rho.d.a^3}.\frac{1}{h^2u^2.r^3}.\mu.\left(\frac{d\mu}{ds}\right).\left\{\frac{2C-A-B}{C}+\frac{B-A}{C}.\cos.2\varpi\right\};$$

[5448b] substituting the last of the expressions [5447], we get [5448]; observing that h^2, and u^{-1}, are nearly equal to a, or r [4937n,&c.].

† (3028) Since ϖ is very small, we have nearly $\cos.2\varpi=1$; hence we get,

[5449a] $$\frac{2C-A-B}{C}+\frac{B-A}{C}.\cos.2\varpi=\frac{2C-A-B}{C}+\frac{B-A}{C}=2.\frac{C-A}{C};$$

substituting this in [5448], we get [5449].

‡ (3029) Substituting, in [5449], the value of s [5416c], it produces, in the equation [4755] or in its development [5347f], the quantity,

[5450a] $$\tfrac{6}{5}\frac{\int_0^a\rho.d.a^5}{\int_0^a\rho.d.a^3}.\frac{1}{r^2}.\frac{(\lambda+\gamma)}{\gamma}.\frac{(C-A)}{C}.\gamma.\sin.(gv-\theta).$$

[5450b] This is similar to the term which is computed in [5397k]; and, by making the calculation as in [5397k,l], we find, that the preceding term [5450a), produces in p'', the term,

[5450c] $$\delta p''=\tfrac{6}{5}.\frac{\int_0^a\rho.d.a^5}{\int_0^a\rho.d.a^3}.\frac{1}{r^2}.\frac{\lambda+\gamma}{\gamma}.\frac{C-A}{C};$$

[5450d] and the corresponding motion of the node, computed as in [5397k,l], is $\frac{1}{2}\delta p''.v$ as in [5450].

Motion of the node arising from the spheroidal figure of the moon.

$$\tfrac{3}{5} \cdot \frac{\int_0^a \rho \,.\, d.a^5}{\int_0^a \rho \,.\, d.a^3} \cdot \frac{v}{r^2} \cdot \frac{(\lambda+\gamma)}{\gamma} \cdot \frac{(C-A)}{C} = \text{term of } \delta\theta\,. \qquad [5450]$$

In [3545] we have* $\dfrac{C-A}{C} = 0{,}000599$; hence it is evident, that the preceding quantity is insensible. [5451]

We find, likewise, that the spheroidal form of the moon adds to the term $\dfrac{-s}{h^2u} \cdot \left(\dfrac{dQ}{du}\right)$ of the equation [4755], the term,† [5452]

$$-\tfrac{3}{5} \cdot \frac{\int_0^a \rho \,.\, d.a^5}{\int_0^a \rho \,.\, d.a^3} \cdot \frac{1}{r^2} \cdot \frac{(C-2A+B)}{C} \,.\, s = \text{term of [4755]}. \qquad [5453]$$

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* (3030) We have $\dfrac{C-A}{A} = 0{,}000599$ [3545]; hence it follows, that C is nearly equal to A; and we may, therefore, change A into C, in the denominator; by this means we shall get $\dfrac{C-A}{C} = 0{,}000599$ [5451]. Moreover, $\lambda = 1^d\,29^m$ [5446'] [3434]; $\gamma = 5^d\,8^m\,50^s$ [5117]; and if we suppose the moon to be homogeneous, we shall have $\dfrac{6}{5} \cdot \dfrac{1}{r^2} \cdot \dfrac{\int_0^a \rho.d.a^5}{\int_0^a \rho.d.a^3} = 0{,}000024$ [5443b]. Substituting these in [5450], it becomes, $0{,}00000001.v$ nearly. Now, in one lunar month, $v = 1296000^s$; substituting it, we get $0^s{,}01$, for the motion of the node in a lunar month, arising from this cause. This is wholly insensible. [5451a] [5451b] [5451c] [5451d]

† (3031) Substituting in the term of Q [5436] the value of $r = \dfrac{1}{u}$ nearly [4776], we get [5452b]. Neglecting μ, on account of its smallness, and putting $\cos.2\varpi = 1$ [5448'], we get [5452c], [5452a]

$$Q = \tfrac{3}{10} \cdot \frac{\int_0^a \rho.d.a^5}{\int_0^a \rho.d.a^3} \cdot u^3 \cdot \left\{ \frac{2C-A-B}{C} \cdot (\tfrac{1}{3}-\mu^2) + \frac{(B-A)}{C} \cdot (1-\mu^2).\cos.2\varpi \right\} \qquad [5452b]$$

$$= \tfrac{1}{5} \cdot \frac{\int_0^a \rho.d.a^5}{\int_0^a \rho.d.a^3} \cdot u^3 \cdot \frac{(C-2A+B)}{C}. \qquad [5452c]$$

This gives,

$$\left(\frac{dQ}{du}\right) = \tfrac{3}{5} \cdot \frac{\int_0^a \rho.d.a^5}{\int_0^a \rho.d.a^3} \cdot u^2 \cdot \frac{(C-2A+B)}{C}\,; \qquad [5452d]$$

and by multiplying it by $-\dfrac{s}{h^2u}$; using also $h^2 = a$, $u^{-1} = a = r$ nearly [4937n], we get [5453].

This adds to the motion of the node, the term,*

[5454] $$-\tfrac{3}{10}\cdot\frac{\int_0^a\rho\,.d.a^5}{\int_0^a\rho\,.d.a^3}\cdot\frac{v}{r^2}\cdot\frac{(C-2A+B)}{C}=\text{term of }\delta\theta;$$

a quantity which is wholly insensible.

[5454a] * (3032) The expression [5454] may be derived from [5453], in the same manner as [5450] is from [5449]; namely, by changing s into $\frac{1}{2}v$. To estimate roughly the value of the expression [5454], we may observe, that in the case of homogeneity, we have,

[5454b] $$\frac{B-A}{C}=\frac{15\lambda'}{4r_{\prime}^3};\qquad \frac{C-A}{A}=\frac{5\lambda'}{r_{\prime}^3}\quad [3576].$$

Their sum is,

[5454c] $$\frac{C-2A+B}{C}=\frac{35\lambda'}{4r_{\prime}^3}=\tfrac{7}{4}\cdot\frac{(C-A)}{A}=0{,}001\quad [5451a], \text{ nearly };$$

hence it is evident, that the term [5454] is insensible, like the corresponding term [5450] which is computed in [5451d].

CHAPTER III.

ON THE INEQUALITIES OF THE MOON, DEPENDING ON THE ACTION OF THE PLANETS.

22. It now remains to consider the action of the planets upon the moon. We shall put,

$P =$ the mass of a planet ; [5455]

$X, Y, Z =$ the rectangular co-ordinates of the planet, referred to the centre of the earth ; [5455′]

$f =$ the distance of the planet from the earth's centre. [5455″]

Then, it is evident, that *the action of the planet* P, *will increase the value of* Q [4756], *by the quantity*,*

Terms of Q.

$$Q = -\frac{P.(xX+yY+zZ)}{f^3} + \frac{P}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}; \qquad [5456]$$

or,†

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* (3033) The *disturbing force* of the planet P, upon the moon, in her relative motion about the earth, is computed by the same differential formulas which are used for the *disturbing force* of the sun. We must, in this case, change the mass m' of the sun [4757″], into that of the planet P ; and the co-ordinates x', y', z' of the sun [4758′], into those of the planet X, Y, Z [5455′]; by which means, the distance r' of the sun from the earth [4759′], changes into f [5455″], which represents the distance of the planet from the earth. Making these alterations in the two last terms of Q [4756], we obtain the part of Q [5456], upon which the disturbing force of the planet P depends. [5456a] [5456b] [5456c] [5456d] [5456e]

† (3034) The development of [4774] is given in [4775], and, if we multiply this by

[5457] $$Q = \frac{P}{f} - \frac{\frac{1}{2}P.r^2}{f^3} + \tfrac{3}{2}P.\frac{(Xx+Yy+Zz)^2}{f^5} + \&c.$$

Let

[5458] X', Y', Z', be the co-ordinates of the planet P, refered to the sun's centre;

[5458'] x', y', z', the co-ordinates of the earth, referred to the sun's centre;

then we shall have,

[5459] $$X = X' - x'; \qquad Y = Y' - y'; \qquad Z = Z' - z'.$$

Hence, the function [5457] becomes,*

[5460] $$Q = \frac{P}{f} - \frac{\frac{1}{2}P.r^2}{f^3} + \tfrac{3}{2}P.\frac{(X'x+Y'y+Z'z-xx'-yy'-zz')^2}{f^5} + \&c.$$

[5461] *We shall take the ecliptic for the fixed plane*, which makes $z'=0$, and, we shall put,

Symbols.

[5462] R = the radius vector of the planet P, projected upon this plane;

[5463] U = the angle formed by the projection of the radius, and by a fixed right line, taken in the same plane;

[5464] S = the tangent of the heliocentric latitude of the planet P;

[5465] r' = the radius vector of the earth;

[5465'] v' = the angle formed by the earth's radius and the fixed line.

Then, we shall have,

[5457a] P; changing also x', y', z', r', into X, Y, Z, f, respectively, as in [5456b–d], we get,

[5457b] $$\frac{P}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} = \frac{P}{f} + \frac{P.(xX+yY+zZ-\frac{1}{2}r^2)}{f^3} + \tfrac{3}{2}\frac{P(xX+yY+zZ-\frac{1}{2}r^2)^2}{f^5} + \&c.$$

Substituting this in [5456]; reducing and neglecting terms of the order Xf^{-5}, or f^{-4};

[5457c] we get [5457]; observing, that the terms depending on the first power of $(xX+yY+zZ)$, mutually destroy each other.

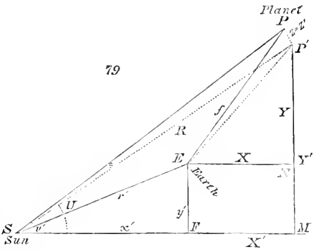
* (3035) Substituting, in [5457], the values of X, Y, Z [5459], we get

[5460a] [5460].

$$* f = \sqrt{R^2.(1+SS)+r'^2-2Rr'.\cos.(U-v')}. \qquad [5466]$$

Hence, the part of Q, relative to the action of P upon the moon, will be,†

$$Q = \frac{P}{f} - \frac{\frac{1}{2}P.(1+ss)}{u^2.f^3} + \frac{3}{2}P.\frac{\{R.\cos.(v-U)-r'.\cos.(v-v')+R.sS\}^2}{u^2.f^5} + \&c. \qquad [5467]$$



* (3036) In the annexed figure, S is the place of the sun; E that of the earth; P the place of the planet; and P' its projection on the plane of the ecliptic SMP'. Then, $z'=0$ gives $Z=Z'$ [5461, 5459]; and the rectangular co-ordinates of E, P, referred to the sun, are $SF=x'$; $FE=y'$; $SM=X'$; $MP'=Y'$; $P'P=Z'$; and, by drawing EN parallel to SM, we have $EN=X$; $NP'=Y$; $P'P=Z$; $SE=r'$; $SP'=R$; $EP=f$; angle $FSE=v'$; angle $FSP'=U$; tang. $PSP'=S$. Fron these symbols we easily obtain, [5466a] [5466b] [5466c] [5466d] [5466d'] [5466e] [5466e']

$$X'=R.\cos.U; \qquad Y'=R.\sin.U; \qquad Z'=RS;$$
$$x'=r'.\cos.v'; \qquad y'=r'.\sin.v'; \qquad z'=0. \qquad [5466f]$$

The values of the co-ordinates of the moon x, y, z, and of the radius r, referred to the earth's centre, are given in [4776—4779]. Now, the distance $EP=f$, is evidently equal to $\sqrt{(X^2+Y^2+Z^2)}$; and, if we substitute the values [5459], we get, by development, [5466g]

$$f=\sqrt{(X^2+Y^2+Z^2)}=\sqrt{\{(X'-x')^2+(Y'-y')^2+(Z'-z')^2\}}$$
$$=\sqrt{\{(X'^2+Y'^2+Z'^2)+(x'^2+y'^2+z'^2)-2(X'x'+Y'y'+Z'z')\}}. \qquad [5466h]$$

Substituting in this, the values of

$$X'^2+Y'^2+Z'^2=SP^2=R^2.(1+S^2); \qquad r'^2=x'^2+y'^2+z'^2; \qquad [5466i]$$
$$X'x'+Y'y'+Z'z'=Rr'.\{\cos.U.\cos.v'+\sin.U.\sin.v'\}=Rr'.\cos.(U-v'); \qquad [5466k]$$

it becomes as in [5466].

† (3037) Substituting the values [5466f, 4776—4779], in the first members of [5467a, b], and making the usual reductions by means of [24] Int., we get the second

or, by neglecting the square of S,*

[5468] $$Q=\frac{P}{f}+\frac{P.(1-2s^2)}{4u^2.f^3}+3P.\frac{\{R^2.\cos(2v-2U)+r'^2.\cos(2v-2v')-2Rr'.\cos(2v-U-v')\}}{4u^2.f^5}$$
$$+3P.\frac{R.sS.\{R.\cos.(v-U)-r'.\cos.(v-v')\}}{u^2.f^5}+\&c.$$

[5468'] As the term $\frac{P}{f}$ does not contain either u, v, or s, it will not enter into the equations [4753—4755]. The term $\frac{P}{4u^2.f^3}$ gives, by its

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members of these expressions;

[5467a] $$X'x+Y'y+Z'z=\frac{R}{u}.\{\cos.U.\cos.v+\sin.U.\sin.v+Ss\}=\frac{R}{u}.\{\cos.(U-v)+Ss\};$$

[5467b] $$-xx'-yy'-zz'=-\frac{r}{u}.\{\cos.v'.\cos.v+\sin.v'.\sin.v\}=-\frac{r}{u}.\cos.(v-v').$$

[5467c] Substituting these, and $r^2=\frac{1+ss}{u^2}$ [4776], in [5460], we get [5467].

* (3038) If we develop the numerator of the last term of [5467], and neglect the
[5468a] square of S, we shall find, that the terms containing the first power of S are the same as in the second line of [5468]. The remaining part of this numerator of [5467] is
[5468b] as in the first member of [5468c]; and, by developing, using [20] Int., it becomes as in [5468d]; and, by the substitution of f^2 [5166], we finally obtain [5468e];

[5468c] $$\{R.\cos.(v-U)-r'.\cos.(v-v')\}^2$$
$$=R^2.\cos.^2(v-U)+r'^2.\cos.^2(v-v')-2Rr'.\cos.(v-U).\cos.(v-v')$$

[5468d] $$=\tfrac{1}{2}.\{R^2+r'^2-2Rr'.\cos(U-v')\}+\tfrac{1}{2}\{R^2.\cos(2v-2U)+r'^2.\cos(2v-2v')-2Rr'.\cos(2v-U-v')\}$$

[5468e] $$=\tfrac{1}{2}.f^2+\tfrac{1}{2}.\{R^2.\cos.(2v-2U)+r'^2.\cos.(2v-2v')-2Rr'.\cos.(2v-U-v')\}.$$

The part of this expression between the braces, being substituted in the numerator of the last term of Q [5167], produces the third term of [5468 line 1]; the other part of
[5468f] [5468e] is $\tfrac{1}{2}f^2$; which gives, in [5468], the term $\tfrac{3}{2}P.\frac{\frac{1}{2}f^2}{u^2.f^5}=\frac{3P}{4u^2.f^3}$. Connecting this with the second term of [5467], which may be put under the form $\frac{P.(-2-2s^2)}{4u^2.f^3}$, we
[5468g] get $\frac{P.(1-2s^2)}{4u^2.f^3}$, as in the second term of [5468]. Finally, the first term $\frac{P}{f}$ [5467], is the same as in [5468]; and we may observe, as in [5468'], that this term may be neglected; for, f [5466] does not contain r, s, v; and its partial differentials, relative to these
[5468h] quantities, will vanish from the general formulas [4753—4755], which are used in this chapter, in finding the perturbations.

development, a function of this form,*

$$\frac{P}{4u^2.f^3}=\frac{P}{4u^2}.\{\tfrac{1}{2}A^{(0)}+A^{(1)}.\cos.(U-v')+A^{(2)}.\cos.2(U-v')+\&c.\}=\text{terms of } Q. \quad [5469]$$

Hence, the term $-\frac{1}{h^2}.\left(\frac{dQ}{du}\right)$, of the equation [4754], produces the following function;

$$\frac{P}{2h^2.u^3}.\{\tfrac{1}{2}A^{(0)}+A^{(1)}.\cos.(U-v')+A^{(2)}.\cos.2.(U-v')+\&c.\}=\text{terms of } -\frac{1}{h^2}.\left(\frac{dQ}{du}\right); \quad [5470]$$

and it is evident, from § 9, 10, that there will result from it, in the expression of au, the quantity,†

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* (3039) If we substitute the value of f [5466], in the term $\frac{P}{4u^2.f^3}$, of the expression [5468], we may develop it, in the usual manner, in a series of the form [5469]. This part of Q gives, in $-\frac{1}{h^2}.\left(\frac{dQ}{du}\right)$, the expression [5470]; as is evident by [5469a] differentiation. The next term of [5468] is $-\frac{2Ps^2}{4u^2.f^3}$; and, as it is of the order s^2, in comparison with [5469], it may be neglected. The next terms of [5468] contain the angle [5469b] $2v$; but these quantities do not produce, by integration in $nt+\varepsilon$ [5474], any term of importance, arising from a small divisor like $i-m$. The same remark may be made relative to the terms of [5468] containing $v-U$, $v-v'$; and, as they are also multiplied [5469c] by the small quantity Ss, they may be neglected. Moreover, a little attention will show, that the substitution of Q [5468], in the four first lines of [5081], will produce no terms of the like kind, depending on angles having a small coefficient except they are multiplied [5469d] by quantities of the order of the excentricities, &c.; and, by neglecting such quantities as in [5486′, &c.], we shall find, that the first term of importance is that in [5081 line 5], which gives in dt the term $-\frac{a^2.dv}{\sqrt{a_{\prime}}}.\dot{2}a\delta u$. Multiplying this part of dt by $n=\frac{\sqrt{a_{\prime}}}{a^2}$ [5469e] [5092c], we get, in ndt, the term $ndt=-dv.2a\delta u$; which will be used hereafter. [5469f]

† (3040) Substituting $U=iv$ [5463, 5472], $v'=mv$, $h^2=a$, $u=a^{-1}$ [4937n], [5470a] in [5470], and then connecting it with the two terms $\frac{ddu}{dv^2}+u$ [4754], and with the term of the same equation, which is developed in [4908 line 1]; namely,

$$-\frac{3\overset{2}{m}}{2\,a_{\prime}}.\,a\delta u=-\tfrac{3}{2}m^2.\delta u \quad [5082h'], \text{ nearly}; \quad [5470b]$$

[5471] $$-\tfrac{1}{2}Pa^3.\left\{\frac{A^{(1)}.\cos.(i-m).v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{A^{(2)}.\cos.2(i-m).v}{1-\frac{3}{2}m^2-4(i-m)^2}+\frac{A^{(3)}.\cos.3(i-m).v}{1-\frac{3}{2}m^2-9(i-m)^2}+\&c.\right\}=\text{terms of } a\delta u;$$

[5472] *i being the ratio of the mean motion of the planet P to that of the moon.* Hence arises, in ndt [5081, &c.], the function,*

[5473] $$Pa^3.dv.\left\{\frac{A^{(1)}.\cos.(i-m).v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{A^{(2)}.\cos.2(i-m).v}{1-\frac{3}{2}m^2-4(i-m)^2}+\frac{A^{(3)}.\cos.3(i-m).v}{1-\frac{3}{2}m^2-9(i-m)^2}+\&c.\right\}=\text{terms of } ndt;$$

consequently, we have, in $nt+\varepsilon$, the following expression;

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it becomes,

[5470c] $$0=\frac{ddu}{dv^2}+u+\tfrac{1}{2}Pa^2.\left\{\tfrac{1}{2}A^{(0)}+A^{(1)}.\cos.(i-m).v+A^{(2)}.\cos.2(i-m).v+\&c.\right\}-\tfrac{3}{2}m^2.\delta u.$$

Now, supposing any term of u, or δu, to be represented by,

[5470d] $$\delta u=B^{(n)}.\cos.n.(i-m).v,$$

and substituting it in [5470c], we find, by retaining only the terms depending on this angle, and dividing by $\cos.n.(i-m).v$,

[5470e] $$0=-n^2.(i-m)^2.B^{(n)}+B^{(n)}+\tfrac{1}{2}Pa^2.A^{(n)}-\tfrac{3}{2}m^2.B^{(n)}.$$

Hence we get,

[5470f] $$B^{(n)}=\frac{-\frac{1}{2}Pa^2.A^{(n)}}{1-\frac{3}{2}m^2-n^2.(i-m)^2};$$

and, the term of $a\delta u$ [5470d], corresponding to that in [5470c], which contains $A^{(n)}$, is,

[5470g] $$a\delta u=-\tfrac{1}{2}Pa^3.\frac{A^{(n)}.\cos.n.(i-m).v}{1-\frac{3}{2}m^2-n^2.(i-m)^2}.$$

From this formula we may easily deduce any term of $a\delta u$ [5471], from that which
[5470h] depends on the same angle $n.(i-m).v$ in [5470c], by multiplying the term of [5470c], depending on $A^{(n)}$, by the factor $\dfrac{-a}{1-\frac{3}{2}m^2-n^2.(i-m)^2}$.

* (3041) Substituting, in $ndt=-dv.2a\delta u$ [5469f], the value of $a\delta u$ [5471], we
[5473a] get [5473], whose integral gives [5474]. Now, from [5475], we have $a^3=\dfrac{m^2.a'^3}{m'}$; substituting this in [5474], we get [5476].

$$\frac{Pa^3}{i\text{-}m}\cdot\left\{\frac{A^{(1)}.\sin.(i-m).v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}A^{(2)}.\sin.2(i-m).v}{1-\frac{3}{2}m^2-4(i-m)^2}+\frac{\frac{1}{3}A^{(3)}.\sin.3(i-m).v}{1-\frac{3}{2}m^2-9(i-m)^2}+\&c.\right\}=\text{terms of } nt+\varepsilon. \qquad [5474]$$

Now, we have $\frac{m'.a^3}{a'^3}=m^2$ [5374a—b]; m' being the sun's mass. Hence, [5475]
the preceding function becomes,

$$\frac{\frac{P}{m'}.m^2.a'^3}{i-m}\cdot\left\{\frac{A^{(1)}.\sin.(i-m).v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}A^{(2)}.\sin.2(i-m).v}{1-\frac{3}{2}m^2-4(i-m)^2}+\frac{\frac{1}{3}A^{(3)}.\sin 3(i-m).v}{1-\frac{3}{2}m^2-9(i-m)^2}+\&c.\right\}=\text{terms of } nt+\varepsilon. \qquad (A) \qquad [5476]$$

In the case of a planet, inferior to the earth, we have, by putting α *for the ratio of the mean distance of the planet from the sun, to that of the earth from the sun*, and retaining the denominations of chap. vi. of the sixth book,* [5477]

$$a'^3.A^{(1)}=b_{\frac{3}{2}}^{(1)}; \qquad a'^3.A^{(2)}=b_{\frac{3}{2}}^{(2)}; \qquad a'^3.A^{(3)}=b_{\frac{3}{2}}^{(3)}, \quad \&c.; \qquad [5478]$$

which changes the function [5476] into the following;

Terms of $nt+\varepsilon$, from the direct action of an inferior planet.

$$\frac{\frac{P}{m'}.m^2}{i-m}\cdot\left\{\frac{b_{\frac{3}{2}}^{(1)}.\sin.(i-m).v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}b_{\frac{3}{2}}^{(2)}.\sin.2(i-m).v}{1-\frac{3}{2}m^2-4(i-m)^2}+\frac{\frac{1}{3}b_{\frac{3}{2}}^{(3)}.\sin.3(i-m).v}{1-\frac{3}{2}m^2-9(i-m)^2}+\&c.\right\}=\text{terms of } nt+\varepsilon; \qquad (B) \qquad [5479]$$

in which *we may take, for $(i-m).v$, the mean longitude of the planet, minus that of the earth.*

With respect to a superior planet, α *denotes the ratio of the mean distance* [5479′]

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* (3042) Changing a, a' [956], into R, r' [5462, 5465], respectively, in order to conform to the present notation; also, the angle $n't-nt+\varepsilon'-\varepsilon$, into $U-v'$, [5478a]
it becomes, by neglecting, for brevity, the consideration of the excentricities,

$$\{R^2+r'^2-2Rr'.\cos.(U-v')\}^{-\frac{3}{2}}=\tfrac{1}{2}\Sigma.B^{(i)}.\cos.i.(U-v'). \qquad [5478b]$$

If we neglect S^2, the first member of this expression becomes equal to f^{-3} [5466].
Multiplying this by $\frac{P}{4u^2}$, we get,

$$\frac{P}{4u^2.f^3}=\frac{P}{4u^2}.\{\tfrac{1}{2}\Sigma.B^{(i)}.\cos.i.(U-v')\}. \qquad [5478c]$$

Comparing this with the development in [5469, 956], we get $B^{(i)}=A^{(i)}$. Substituting [1006], and multiplying by a'^3, we obtain $a'^3.A^{(i)}=b_{\frac{3}{2}}^{(i)}$, as in [5478]. Substituting these in [5476], we get [5479]. [5478d]

of the earth from the sun, to that of the planet; so that we have,*

[5480] $a'^3.A^{(1)}=\alpha^3.b^{(1)}_{\frac{3}{2}}$; $a'^3.A^{(2)}=\alpha^3.b^{(2)}_{\frac{3}{2}}$; $a'^3.A^{(3)}=\alpha^3.b^{(3)}_{\frac{3}{2}}$; &c.

Terms of $nt+\varepsilon$, from the direct action of a superior planet.

This changes the function [5476] into the following form,

[5481] $$\frac{\frac{P}{m'}.m^2\alpha^3}{i-m}.\left\{\frac{b^{(1)}_{\frac{3}{2}}.\sin.(i-m).v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}b^{(2)}_{\frac{3}{2}}.\sin.2(i-m).v}{1-\frac{3}{2}m^2-4(i-m)^2}+\frac{\frac{1}{3}b^{(3)}_{\frac{3}{2}}.\sin.3(i-m).v}{1-\frac{3}{2}m^2-9(i-m)^2}+\&c.\right\}=\text{terms of }(nt+\varepsilon). \qquad (C)$$

These are the only sensible terms which can result from the direct action of the planet P on the moon.

But, the sun's action upon the moon may render sensible, in the motion of that satellite, the perturbations of the radius vector of the earth's orbit, arising from the action of the planet P upon the earth, and may produce, in the moon's motions, inequalities of the same order as those we have just considered. To prove

Indirect action of the planets.

[5482] it, we shall consider the term $\frac{m'.u'^3}{2h^2.u^3}$ [4866], which is a part of the equation

[5483] [4754]. We shall suppose $\frac{\delta r'}{a'}=\frac{P}{m'}.K.\cos.(\beta'n''t-\beta n't+B)$, to be any

[5484] term of $\frac{\delta r'}{a'}$, arising from the action of the planet P upon the earth;† $n''t$ denoting the mean motion of P, and $n't$ that of the earth; the corresponding term of $\frac{\delta u'}{u'}$ will be,

[5485] $$\frac{\delta u'}{u'}=-\frac{P}{m'}.K.\cos.(\beta'n''t-\beta n't+B).$$

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* (3043) The equation [5478d] holds good for a superior planet, by merely changing, in the factor $a'^{\frac{3}{2}}$, the quantity a', corresponding to the earth's distance from the

[5480a] sun, into $\frac{a'}{\alpha}$ [5179'], which represents that of the superior planet from the sun; by which means, it becomes,

[5480b] $$\left(\frac{a'}{\alpha}\right)^3.A^{(i)}=b^{(i)}_{\frac{3}{2}}, \qquad \text{or} \qquad a'^3.A^{(i)}=\alpha^3.b^{(i)}_{\frac{3}{2}}, \qquad \text{as in [5480].}$$

Substituting this in [5476], we get [5481].

† (3044) This form is the same as is used in various places; as, for example, in

Hence, the term $\frac{m'.u'^3}{2h^2.u^3}$ produces the following;

$$-\frac{3P.u'^3}{2h^2.u^3}.K.\cos.(\beta'n''t-\beta n't+B).$$ [Term of 4754] [5486]

*If we consider only those inequalities of $\frac{\delta r'}{a'}$, which are independent of the excentricities of the orbit, and represent them by the series,** [5486']

$$\frac{P}{m'}.\{K^{(1)}.\cos(n''t-n't+\varepsilon''-\varepsilon')+K^{(2)}.\cos2(n''t-n't+\varepsilon''-\varepsilon')+K^{(3)}.\cos3(n''t-n't+\varepsilon''-\varepsilon')+\&c\}=\text{terms of }\frac{\delta r'}{a'};$$ [5487]

the term $\frac{m'.u'^3}{2h^2.u^3}$ will produce, in [4754 or 4961], the function,†

$$-\frac{3m^2}{2a}.\frac{P}{m'}.\{K^{(1)}.\cos.(i-m).v+K^{(2)}.\cos.2(i-m).v+K^{(3)}.\cos.3(i-m).v+\&c.\};$$ [Terms of 4754] [5488]

whence results, in $a\delta u$, the function,

$$\frac{3m^2}{2}.\frac{P}{m'}.\left\{\frac{K^{(1)}.\cos(i-m).v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{K^{(2)}.\cos.2(i-m).v}{1-\frac{3}{2}m^2-4(i-m)^2}+\frac{K^{(3)}.\cos.3(i-m).v}{1-\frac{3}{2}m^2-9(i-m)^2}+\&c.\right\}=\text{terms of } a\delta u.$$ [5489]

This gives, in $nt+\varepsilon$ [5095], the following terms;

[1023, 4306, 4308, &c.]. Now we have, very nearly, in [4777e], $u'=\frac{1}{r'}$; and, the differential of its logarithm gives, [5486a]

$$\frac{\delta u'}{u'}=-\frac{\delta r'}{r'}=-\frac{\delta r}{a'}, \text{ nearly};$$ [5486b]

substituting this in [5483], we get [5485]. If we vary u', by the quantity $\delta u'$, it produces, in $\frac{m'.u'^3}{2h^2.u^3}$, the term,

$$\frac{3\,m'.u'^2.\delta u'}{2h^2.u^3}=\frac{3m'.u'^3}{2h^2.u^3}.\frac{\delta u'}{u'};$$ [5486c]

and, by substituting the value of $\frac{\delta u'}{u'}$ [5485], it becomes as in [5486].

* (3015) The form assumed in [5487], is the same as that in [4306 lines 9—11]; decreasing the accents on n''', n'', ε''', ε'', &c. by unity, so as to conform to the notation here used. [5487a]

† (3016) Substituting, in [5486], the values $u=a^{-1}$ $u'=a'^{-1}$, $h^2=a$ [4937n], also $\frac{a^3}{a'^3}=\frac{m^2}{m'}$ [5475], it becomes, [5487b]

Terms of $nt+\varepsilon$ arising from the indirect action of a planet.

[5490] $$-\frac{3m^2}{i-m}\cdot\frac{P}{m'}\cdot\left\{\frac{K^{(1)}\sin.(i-m).v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}K^{(2)}.\sin.2(i-m).v}{1-\frac{3}{2}m^2-4(i-m)^2}+\frac{\frac{1}{3}K^{(3)}.\sin.3(i-m).v}{1-\frac{3}{2}m^2-9(i-m)^2}+\&c.\right\}=\text{terms of } nt+\varepsilon. \quad (D)$$ *

This function is of the same order as that which results from the direct action of the planets upon the moon [5479,5481]. We shall now compute these several inequalities for Venus, Mars, and Jupiter.

Relatively to Venus, we have, in [4126,4132],

Action of Venus.

$$\alpha = 0,72333230\,; \qquad 1$$

$$b_{\frac{3}{2}}^{(0)} = 9,992539\,; \qquad 2$$

[5491] $$b_{\frac{3}{2}}^{(1)} = 8,871894\,; \qquad 3$$

$$b_{\frac{3}{2}}^{(2)} = 7,386580\,; \qquad 4$$

$$b_{\frac{3}{2}}^{(3)} = 5,953940\,. \qquad 5$$

Hence we deduce, by means of [974],†

[5488a] $$-\frac{3m^2}{2a}\cdot\frac{P}{m'}\cdot K.\cos.(\beta'n''t-\beta n't+B)\,;$$

which is the same as the product of the assumed value of $\frac{\delta r'}{a'}$ [5183], by the quantity

[5488b] $-\frac{3m^2}{2a}$. Therefore, if we multiply the assumed value of $\frac{\delta r'}{a'}$ [5487], by the same factor $-\frac{3m^2}{2a}$, we shall obtain the corresponding expression, which arises from the variation of

[5488c] $\frac{m'.u'^3}{2h^2.u^3}$, as in [5488]. This term forms a part of the equation [4754, or 4961]; and, we may find the corresponding part of u, or rather of $a\delta u$, as in [5170g, h], by multiplying

[5488d] any term of [5188], depending on $K^{(n)}.\cos.n.(i-m).v$, by $-\frac{a}{1-\frac{3}{2}m^2-n^2.(i-m)^2}$; hence we obtain [5489].

* (3047) Substituting the terms of $a\delta u$ [5489], in $ndt=-2dv.a\delta u$ [5469f],

[5490a] and integrating, we get the terms of $nt+\varepsilon$ [5490]. We may remark, that in the original work, by a typographical error, the terms of [5490], are made to depend on $\cos.(i-m).v$;

[5490b] $\cos.2.(i-m).v$, &c. instead of $\sin.(i-m).v$; $\sin.2.(i-m).v$, &c.

† (3048) Putting $s=\frac{3}{2}$, in [974], and then, successively, $i=0$, $i=1$, we get;

$$b_{\frac{5}{2}}^{(0)} = 85{,}77422\,;$$ [5492]

$$b_{\frac{5}{2}}^{(1)} = 83{,}40760\,.$$

By observations, we have $i-m = 0{,}0467900$;* therefore, by supposing, as in [4061, line 3],† [5492']

$$\frac{P}{m'} = \frac{1}{383130}\,;$$ [5493]

we find, that the function [5479], reduced to seconds, becomes,

Direct action of Venus.

$$\begin{array}{lcr} +\,0^s{,}577273.\sin.(i-m).v & & 1 \\ +\,0^s{,}241919.\sin.2.(i-m).v & [\text{Terms of } nt+\varepsilon] & 2 \\ +\,0^s{,}131463.\sin.3.(i-m).v & & 3 \\ \&c. & & \end{array}$$ [5494]

What we have here represented by $\frac{\delta r'}{a'}$, is denoted by $\delta r''$, in [4306, line 1, &c.], and we have, in that article, by means of the action of Venus,

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$$b_{\frac{5}{2}}^{(0)} = \frac{(1+\alpha^2).b_{\frac{3}{2}}^{(0)} + \frac{2}{3}.\alpha.b_{\frac{3}{2}}^{(1)}}{(1-\alpha^2)^2}\,; \qquad b_{\frac{5}{2}}^{(1)} = \frac{\frac{5}{3}.(1+\alpha^2).b_{\frac{3}{2}}^{(1)} - \frac{2}{3}.\alpha.b_{\frac{3}{2}}^{(2)}}{(1-\alpha^2)^2}\,.$$ [5492a]

With these formulas, we may compute the values [5492], by using the expressions [5491].

* (3049) If we use the same notation as in [4077], we shall find, that the mean motion of Venus, in comparison with that of the earth, is represented by $\frac{n'}{n''}$. Multiplying this by $m = 0{,}0748013$ [5117], which expresses the ratio of the sun's mean motion to that of the moon, we get the expression of i [5472], or the ratio of the mean motion of Venus to that of the moon; consequently, [5493a] [5493b]

$$i = 0{,}0748013\,.\frac{n'}{n''}\,. \qquad \text{Hence,} \quad i-m = 0{,}0748013\,.\frac{(n'-n'')}{n''}\,;$$ [5493c]

and, by substituting the values of n', n'' [4077], it becomes as in [5492'].

† (3050) In the present notation, P is the mass of the planet [5455], m' that of the sun [4757″]; hence $\frac{P}{m'}$, of the present notation, is the same as m' [4061 line 3], [5494a]

[5495]
$$\frac{\delta r'}{a'} = 0{,}0000015553 \qquad 1$$
$$- 0{,}0000060012.\cos.(i-m).v \qquad 2$$
$$+ 0{,}0000171431.\cos.2.(i-m).v \qquad 3$$
$$+ 0{,}0000027072.\cos.3.(i-m).v \qquad 4$$
$$+ \&c.$$

The function [5490], reduced to seconds, becomes,*

Indirect action of Venus.

[5496]
$$+ 0^s{,}448818.\sin.(i-m).v \qquad 1$$
$$- 0^s{,}645333.\sin.2.(i-m).v \qquad 2$$
$$- 0^s{,}068705.\sin.3.(i-m).v \quad \text{[Terms of } nt+\varepsilon] \qquad 3$$
$$\&c.$$

If we connect this with the preceding expression [5495], *we shall have for the lunar inequalities, depending on the direct and indirect actions of Venus, upon the moon*;

Whole action of Venus on $nt+\varepsilon$.

[5497]
$$+ 1^s{,}026091.\sin.(i-m).v \qquad 1$$
$$- 0^s{,}403414.\sin.2.(i-m).v \qquad 2$$
$$+ 0^s{,}062758.\sin.3.(i-m).v \quad \text{[Terms of } nt+\varepsilon] \qquad 3$$
$$\&c.$$

We must increase these inequalities in the ratio of 1,0743 to 1 [4605].

[5494b] and by putting $\mu' = 0$, it becomes as in [5493]. Substituting, in [5479], the values [5491—5493], also that of m [5117], it becomes nearly as in [5494].

* (3051) Comparing [5487, 5495], we get,

[5496a]
$$\frac{P}{m'} \cdot K^{(0)} = 0{,}0000015553\,; \qquad \frac{P}{m'} \cdot K^{(1)} = -\,0{,}0000060012\,;$$
$$\frac{P}{m'} \cdot K^{(2)} = 0{,}0000171431, \quad \&c.$$

[5496b] Substituting these, and m [5117], also $i-m$ [5492'], in [5490], we get the terms of $nt+\varepsilon$, arising from the indirect action of the planet Venus on the moon, as in [5496].

Relatively to Mars, we have from [4159, 4165],

Action of Mars.

$$\alpha = 0{,}65630030\,; \qquad 1$$

$$b_{\frac{3}{2}}^{(0)} = 6{,}856336\,; \qquad 2$$

$$b_{\frac{3}{2}}^{(1)} = 5{,}727893\,; \qquad 3$$

$$b_{\frac{3}{2}}^{(2)} = 4{,}404530\,; \qquad 4$$ [5498]

$$b_{\frac{3}{2}}^{(3)} = 3{,}255964\,; \qquad 5$$

&c.

Hence we deduce,*

$$b_{\frac{5}{2}}^{(0)} = 38{,}00346\,;$$

$$b_{\frac{5}{2}}^{(0)} = 36{,}20013\,.$$ [5499]

Observations give $i-m = -0{,}0350306$;† therefore, by supposing, as in [4061 line 5], [5500]

$$\frac{P}{m'} = \frac{1}{1846082}\,;$$ [5501]

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* (3052) Substituting the values [5498], in [5492a], we get [5499]. [5498a]

† (3053) Changing n' into n''' in [5493c], we get the value of

$$i - m = 0{,}0748013\,.\,\frac{(n'''-n'')}{n''}\,,$$ [5500a]

corresponding to Mars; and, by using the values [4077], it becomes as in [5500]. Substituting these, and $\frac{P}{m}$, [5501], in [5481], we get [5502]. The expression [5504], is deduced from [5490, 5503], in the same way as [5496] is obtained from [5490, 5495], in [5496a,b]. In the original work, the coefficient of [5505 line 2], is erroneously printed, [5500b]

$$1'',201491 = 0^{s},389283\,, \quad \text{instead of} \quad 1'',211491 = 0^{s},392523\,.$$ [5500c]

the function [5481], becomes,

Direct action of Mars.

[5502]

$$
\begin{aligned}
&-0^{s},029177.\sin.(i-m).v\\
&-0^{s},011260.\sin.2.(i-m).v\\
&-0^{s},005584.\sin.3.(i-m).v \qquad \text{[Terms of } nt+\varepsilon]\\
&\&c.
\end{aligned}
$$

We have, in [4306 lines 8—11], from the action of Mars,

[5503]

$$
\begin{aligned}
\frac{\delta r'}{a'} = &-0,0000000478 && 1\\
&+0,0000005487.\cos.(i-m).v && 2\\
&+0,0000080620.\cos.2.(i-m).v && 3\\
&-0,0000006475.\cos.3.(i-m).v && 4\\
&\&c.
\end{aligned}
$$

The formula [5490], reduced to seconds, becomes,

Indirect action of Mars.

[5504]

$$
\begin{aligned}
&+0^{s},054760.\sin.(i-m).v && 1\\
&+0^{s},403783.\sin.2.(i-m).v \qquad \text{[Terms of } nt+\varepsilon] && 2\\
&-0^{s},021753.\sin.3.(i-m).v && 3\\
&\&c.
\end{aligned}
$$

If we connect together the terms in [5502, 5504], we shall obtain *the lunar inequalities depending on the direct and indirect actions of Mars upon the Moon ;*

Complete action of Mars on $nt+\varepsilon$.

[5505]

$$
\begin{aligned}
&+0^{s},025583.\sin.(i-m).v && 1\\
&+0^{s},392523.\sin.2.(i-m).v \qquad \text{[Terms of } nt+\varepsilon] && 2\\
&-0^{s},027337.\sin.3.(i-m).v && 3\\
&\&c.
\end{aligned}
$$

We must decrease these inequalities, in the ratio of 0,725 to 1 [4608].

Relatively to Jupiter, we have, as in [4167, 4173],

Action of Jupiter.

$$\alpha = 0{,}19226461\,;$$ 1

$$b_{\frac{3}{2}}^{(0)} = 2{,}176460\,;$$ 2

$$b_{\frac{3}{2}}^{(1)} = 0{,}619063\,;$$ 3 [5506]

$$b_{\frac{3}{2}}^{(2)} = 0{,}148198\,;$$ 4

$$b_{\frac{3}{2}}^{(3)} = 0{,}032439$$ 5

&c.

Hence we deduce, from [5492a],

$$b_{\frac{5}{2}}^{(0)} = 2{,}51906\,;$$ 1 [5507]

$$b_{\frac{5}{2}}^{(1)} = 1{,}13310\,.$$ 2

We have, by observation, $i - m = -\,0{,}0684952\,;$* therefore, by supposing, as in [4061 line 6], [5508]

$$\frac{P}{m'} = \frac{1}{1067{,}09}\,,$$ [5509]

the function [5481] becomes,

Direct action of Jupiter.

$$-\;0^{s}{,}070391.\sin.(i-m).v$$ 1

$$-\;0^{s}{,}008547.\sin.2.(i-m).v$$ 2 [5510]

$$-\;0^{s}{,}001275.\sin.3.(i-m).v \qquad \text{[Terms of } nt+\varepsilon]$$ 3

&c.

---

* (3054) Changing n' into n^{iv}, in [5493c], we get the value of

$$i - m = 0{,}0748013\,.\frac{(n^{\text{iv}} - n'')}{n''}\,,$$ [5508a]

corresponding to Jupiter; and, by using the values [4077], it becomes as in [5508]. Substituting this and $\frac{P}{m'}$ [5509], in [5481], we get [5510]. The expression [5512] is deduced from [5490, 5511], in the same manner as [5504] is found, in the last note.

We have, from [4306, lines 13—16], by means of Jupiter's action,

$$\frac{\delta r'}{a'} = -0,0000011581 \qquad 1$$

[5511]
$$+0,0000159384.\cos.(i-m).v \qquad 2$$
$$-0,0000090986.\cos.2.(i-m).v \qquad 3$$
$$-0,0000006550.\cos.3.(i-m).v \qquad 4$$
&c.

The formula [5490], reduced to seconds, becomes,

Indirect action of Jupiter.

$$+0^{s},816336.\sin.(i-m).v \qquad 1$$

[5512]
$$-0^{s},236377.\sin.2.(i-m).v \qquad 2$$
$$-0^{s},011625.\sin.3.(i-m).v \qquad \text{[Terms of } nt+\varepsilon] \qquad 3$$
&c.

If we connect it with the preceding [5510], we obtain for *the lunar inequalities depending on the direct and indirect actions of Jupiter upon the moon,*

Whole action of Jupiter on $nt+\varepsilon$.

$$+0^{s},745945.\sin.(i-m).v \qquad 1$$
$$-0^{s},244924.\sin.2.(i-m).v \qquad 2$$

[5513]
$$-0^{s},012900.\sin.3.(i-m).v \qquad \text{[Terms of } nt+\varepsilon] \qquad 3$$
&c.

[5513′] *If we take, with a contrary sign,*all the inequalities resulting from the actions of the planets upon the moon,* [5497, 5505, 5513], *we shall obtain the inequalities produced by this action, in the expression of the moon's true*
[5514] *longitude;* we may, therefore, reduce them to tables, observing, that $(i-m).v$ may be supposed equal to the mean longitude of the planet, *minus* that of the earth. It would be useful to introduce these inequalities into the lunar tables, considering the precision to which these tables have been carried.

---

* (3055) The inequalities of the expression of $nt+\varepsilon$ [5095], arising from the actions
[5515a] of the planets, are given in [5497, 5505, 5513]; and to obtain the corresponding terms of v [5095], we must evidently change their signs.

The term $\frac{PA^{(0)}}{4h^2.u^3}$, of the expression of $-\frac{1}{h^2}.\left(\frac{dQ}{du}\right)$, gives, in the equation [4961], the term,* [5515]

$$-\tfrac{3}{4}.Pa^2.A^{(0)}e\,.\cos.(cv-\varpi).$$ [Term of 4961] [5516]

Hence it is evident, that the value of c *is decreased by the action of an inferior planet, by the quantity*,†

$$\tfrac{3}{8}.\frac{P}{m'}.m^2.b_{\frac{3}{2}}^{(0)};$$ [Decrement of c] [5517]

and, *by the action of a superior planet, by the quantity*,

$$\tfrac{3}{8}.\frac{P}{m'}.m^2.\alpha^3.b_{\frac{3}{2}}^{(0)}.$$ [Decrement of c] [5518]

---

* (3056) The equation [4751, or 4961], contains the term $-\frac{1}{h^2}.\left(\frac{dQ}{du}\right)$, which is developed in [5470], and contains the term $\frac{PA^{(0)}}{4h^2.u^3}$. Substituting $h^2=a$ [4937n], and u [5393], which gives $\frac{1}{u^3}=a^3.\{1-3e.\cos.(cv-\varpi)\}$, nearly; we obtain the term [5516], depending on c. [5516a]

† (3057) Neglecting e^2, e'^2, and also, for brevity, the symbol ϖ, we have, as in [5396a], $-p$ for the coefficient of $\frac{e}{a}.\cos.cv$, in [4961]; hence it is evident, that the quantity [5516] increases p, by the term $\delta p=\frac{3}{4}P.a^3.A^{(0)}$. The corresponding increment of the motion of the perigee is $\frac{1}{2}\delta p.v=\frac{3}{8}P.a^3.A^{(0)}.v$ [5396d]. Substituting the value of a^3 [5473a], it becomes, [5516a']

$$\tfrac{3}{8}.\frac{P}{m'}.m^2.a'^3.A^{(0)}.v.$$ [5516b]

Now, the motion of the perigee is represented by $(1-c).v$ [4817]; hence it is evident, that the preceding expression decreases the value of c by the quantity,

$$\tfrac{3}{8}.\frac{P}{m'}.m^2.a'^3.A^{(0)}.$$ [5516c]

If we substitute the value $a'^3.A^{(0)}=b_{\frac{3}{2}}^{(0)}$ [5478d], corresponding to an inferior planet, it becomes as in [5517]; and, if we use the value $a'^3.A^{(0)}=\alpha^3.b_{\frac{3}{2}}^{(0)}$ [5480b], corresponding to a superior planet, the decrement of c becomes as in [5518]. [5516d]

[5519] Likewise, the term $\frac{m'.u'^3}{2h^2.u^3}$ [4865'] gives, in the equation [4961], the quantity,*

[5520] $$\frac{9m'.u'^3}{2h^2}.a^3.\frac{\delta r'}{a'}.e.\cos.(cv-\varpi);$$

[5521] $\frac{\delta r'}{a'}$ representing the constant part of the perturbations of the radius vector

[5521'] of the earth's orbit, given in [4306]. Hence, the value of c is increased by this means, by the quantity,†

[5522] $$\frac{9m^2}{4}.\frac{\delta r'}{a'}.$$ [Increment of c]

[5520a] * (3058) The variation of the term [5519] is given in [5486c], namely, $\frac{3m'.u'^3}{2h^2.u^3}.\frac{\delta u'}{u'}$; and, by substituting,

[5520b] $$\frac{\delta u'}{u'}=-\frac{\delta r'}{a'}\quad[5486b],\quad\text{it becomes}\quad -\frac{3m'.u'^3}{2h^2.u^3}.\frac{\delta r'}{a'}.$$

If we use the value of u^{-3} [5516a], it will produce the term,

[5520c] $$\frac{9m'.u'^3}{2h^2}.a^3.\frac{\delta r'}{a'}.e.\cos.(cv-\varpi),\quad\text{depending on } c.$$

In the original work it is erroneously printed,

[5520d] $$-\frac{9m'.u'^3}{2h^2.u^3}.\frac{\delta r'}{a'}.e.\cos.(cv-\varpi);$$

the sign being wrong, and a^3 changed into u^{-3}.

† (3059) Substituting $u'=a'^{-1}$, $h^2=a$ [4937n], and then $\frac{m'.a^3}{a'^3}=m^2$ [5475], in [5520], it becomes

[5522a] $$\tfrac{9}{2}m^2.\frac{\delta r'}{a'}.\frac{e}{a}.\cos.(cv-\varpi).$$

This produces, in p [5396a], the term,

[5522b] $$\delta p=-\tfrac{9}{2}m^2.\frac{\delta r'}{a'};$$

and, in the motion of the perigee $\frac{1}{2}\delta p.v$ [5396d], the term,

[5522c] $$-\tfrac{9}{4}m^2.\frac{\delta r'}{a'}.v.$$

Now, the motion of the perigee being $(1-c).v$ [4817], it is evident, that this produces

[5522d] an *increment* in the value of c, which is represented by the function [5522]. In the original work, the word *increased* [5521'], is printed *decreased*.

It is easy to prove, that all these quantities are insensible.* [5522']

We shall now consider the perturbations of the moon's motions in latitude. The sum of the terms,

$$-\frac{s}{h^2.u}.\left(\frac{dQ}{du}\right)-\frac{(1+ss)}{h^2.u^2}.\left(\frac{dQ}{ds}\right), \qquad \text{[Terms of 4755]} \qquad [5523]$$

Perturbations of the moon's latitude, by the action of the planets.

which make a part of the equation [4755], acquires, by the action of the planet P, the quantity,†

$$\frac{3P.s}{2h^2.u^4.f^3}+\frac{3P.Rr'.S.\cos.(v-v')-3P.R^2.S.\cos.(v-U)}{h^2.u^4.f^5}. \qquad \text{[Terms of 4755]} \qquad [5524]$$

This function contains, relatively to *an inferior planet*, the term,‡

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* (3060) That these quantities are insensible, is evident by computing any one of the terms; for example, that in [5517], corresponding to Venus. Substituting, in this, the values of

$$\frac{P}{m'}=\tfrac{1}{383130}\ \ [5193]\,; \qquad m^2=0,0055\ \ [5117d]\,; \qquad b_{\frac{3}{2}}^{(0)}=10, \text{ nearly } [5491]\,; \qquad [5523a]$$

we get,

$$\tfrac{3}{8}.\frac{P}{m'}.m^2.b_{\frac{3}{2}}^{(0)}=0,00000006\,; \qquad [5523b]$$

which is insensible, in comparison with the whole coefficient of the motion of the perigee $c-1=0,00815199$ [5117 line 2]. [5523c]

† (3061) Taking the partial differentials of Q [5168], relative to u, s, we get, by neglecting terms of the order s^2,

$$s.\left(\frac{dQ}{du}\right)=-\frac{P.s}{2u^3.f^3}-3P.s.\frac{\{R^2.\cos.(2v-2U)+r'^2.\cos.(2v-2v')-2Rr'.\cos.(2v-U-v')\}}{2u^3.f^5}; \qquad [5524a]$$

$$\left(\frac{dQ}{ds}\right)=-\frac{2P.s}{2u^2.f^3}+3P.\frac{RS.\{R.\cos.(v-U)-r'.\cos.(v-v')\}}{u^2.f^5}. \qquad [5524b]$$

Multiplying [5524a], by $-\frac{1}{h^2.u}$, and [5524b], by $-\frac{(1+ss)}{h^2.u^2}$, or simply, by $-\frac{1}{h^2.u^2}$, and adding the products, we get the value of the function [5523], as in [5524], nearly: [5524c] neglecting the terms depending on the angles $2v-2U$, $2v-2v'$, $2v-U-v'$; because they do not produce, by the integrations, any term of s, having the small divisor $g-1$: [5524d] which the other terms [5527, 5528] acquire, as will be seen in the following note.

‡ (3062) We shall notice the effect of the first term of [5524] $\frac{3P.s}{2h^2.u^4.f^3}$, in [5531a].

[5525] $$-\tfrac{3}{4}P.\alpha.\frac{a^3}{a'^3}.\left\{\alpha.b^{(0)}_{\frac{5}{2}}-b^{(1)}_{\frac{5}{2}}\right\}\lambda.\sin.(v-\theta);$$ [Terms of 4755]

[5526] λ *being the inclination of the orbit of the planet* P *to the ecliptic, and* θ *the*

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and shall consider the rest of this function in the present note. If we divide the equation

[5525a] [5469], by $\frac{P}{4u^2}$, and substitute [5478], we shall obtain, successively, the values of f^{-3} [5525b, c]; and, by using the same notation for f^{-5}, we get its value [5525d];

[5525b] $$\frac{1}{f^3}=\tfrac{1}{2}A^{(0)}+A^{(1)}.\cos.(U-v')+A^{(2)}.\cos.2.(U-v')+\&c.$$

[5525c] $$=\frac{1}{a'^3}.\left\{\tfrac{1}{2}b^{(0)}_{\frac{3}{2}}+b^{(1)}_{\frac{3}{2}}.\cos.(U-v')+b^{(2)}_{\frac{3}{2}}.\cos.2.(U-v')+\&c.\right\};$$

[5525d] $$\frac{1}{f^5}=\frac{1}{a'^5}.\left\{\tfrac{1}{2}b^{(0)}_{\frac{5}{2}}+b^{(1)}_{\frac{5}{2}}.\cos.(U-v')+b^{(2)}_{\frac{5}{2}}.\cos.2.(U-v')+\&c.\right\}.$$

The first of these developments is used in [5534b]; the second in this note [5525h].

[5525e] Now, as λ is very small [5526, 4082], we shall have, very nearly, $S=\lambda.\sin.(U-\theta)$ [5526, 5463, 679]; hence we get,

[5525f] $$S.\cos.(v-v')=\tfrac{1}{2}\lambda.\sin.(U-v'+v-\theta)+\tfrac{1}{2}\lambda.\sin.(U+v'-v-\theta);$$

[5525g] $$S.\cos.(v-U)=\tfrac{1}{2}\lambda.\sin.(v-\theta)+\tfrac{1}{2}\lambda.\sin.(2U-v-\theta).$$

We shall now multiply these two last expressions by the value of f^{-5} [5525d], and

[5525h] reduce the product by formula [18] Int.; neglecting the terms in which the coefficient of the angle v differs considerably from unity; because they are not much increased by integration; whilst the terms depending on $\sin.(v-\theta)$, are considerably augmented by

[5525i] the divisor of the order $g-1$, as in [5347b or 5527, 5528]; hence we get, by making the usual reductions;

[5525k] $$\frac{1}{f^5}.S.\cos.(v-v')=\frac{\lambda}{4a'^5}.b^{(1)}_{\frac{5}{2}}.\sin.(v-\theta)+\&c.;$$

[5525l] $$\frac{1}{f^5}.S.\cos.(v-U)=\frac{\lambda}{4a'^5}.b^{(0)}_{\frac{5}{2}}.\sin.(v-\theta)+\&c.$$

Substituting [5525k, l] in the two last terms of [5524], they produce *the following term of* [4755];

[5525m] $$\frac{3P.Rr'.S.\cos.(v-v')-3P.R^2.S.\cos.(v-U)}{h^2.u^4.f^5}=\frac{3PR}{4h^2.u^4.a'^5}.\left\{r'.b^{(1)}_{\frac{5}{2}}-R.b^{(0)}_{\frac{5}{2}}\right\}.\lambda.\sin.(v-\theta).$$

Substituting, in this second member, the approximate values $h^2=a$, $u=a^{-1}$,

[5525n] $r'=a'$ [5470a, &c.]; and, for an inferior planet, $R=\alpha a'$, nearly [5462, 5477, &c.], we get the expression [5525].

longitude of its ascending node. This produces in s, *for an inferior planet*, the term,* [5526']

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* (3063) If we put, for brevity,

$$H' = -\frac{3PR\lambda}{4h^2.u^4.a'^5}.\left\{R.b_{\frac{5}{2}}^{(0)} - r'.b_{\frac{5}{2}}^{(1)}\right\},$$ [5527a]

in the second member of [5525m], it becomes $H'.\sin.(v-\theta)$. This represents a term of the equation [4755], or of the similar equations [5347f, m]; and may be integrated as in [5347f—n']. If we suppose the term of δs, corresponding to [5527b], to be represented by $\delta s = H''.\sin.(v-\theta)$, which is similar to [5348], the equation corresponding to [5347m], will become, [5527b] [5527c]

$$0 = \frac{dds}{dv^2} + s + H'.\sin.(v-\theta) + (g^2-1).H''.\sin.(v-\theta).$$ [5527d]

Substituting, in this, the assumed value of s, or δs, [5527c], we find, that the two first terms mutually destroy each other. Dividing the rest by $\sin.(v-\theta)$, we get the following equation, which is similar to that in [5347n]; [5527e]

$$0 = H' + (g^2-1).H''.$$ [5527f]

Dividing by $g^2-1=(g+1).(g-1)=2.(g-1)$, nearly [5351a], we get $H'' = -\frac{H'}{2.(g-1)}$. [5527g]
Substituting this in δs [5527c], and then resuming the value of H' [5527a], we get,

$$\delta s = \frac{3PR}{8.(g-1).h^2.u^4.a'^5}.\left\{R.b_{\frac{5}{2}}^{(0)} - r'.b_{\frac{5}{2}}^{(1)}\right\}.\lambda.\sin.(v-\theta).$$ [5527h]

Substituting the values [5525n], corresponding to *an inferior planet*, we get [5527i]; and, by using the value of a^3 [5173a], it becomes as in [5527k];

$$\delta s = \frac{3P\alpha}{8.(g-1)}.\frac{a^3}{a'^3}.\left\{\alpha.b_{\frac{5}{2}}^{(0)} - b_{\frac{5}{2}}^{(1)}\right\}.\lambda.\sin.(v-\theta)$$ [5527i]

$$= \frac{3.\frac{P}{m'}.\alpha.m^2}{8.(g-1)}.\left\{\alpha.b_{\frac{5}{2}}^{(0)} - b_{\frac{5}{2}}^{(1)}\right\}.\lambda.\sin.(v-\theta).$$ [5527k]

This agrees with [5527]; observing, that *we have corrected this formula, for a mistake in the original work, where it is printed with the prefix of a negative sign.* [5527l]

In making the calculation for *a superior planet*, we must change the factor $\frac{1}{a'^5}$, in the second member of [5525d], into $\frac{1}{R^5}$; and the same change must be made in [5527a, h]; by which means, this last formula becomes, for a superior planet, [5527m]

Terms of δs, arising from the action of an inferior planet ;

[5527] $$\delta s = \frac{\frac{3}{8}\cdot\frac{P}{m'}.\alpha m^2.\left\{\alpha.b^{(0)}_{\frac{5}{2}}-b^{(1)}_{\frac{5}{2}}\right\}}{g-1}.\lambda.\sin.(v-\theta) \;;$$ [Inferior planet]

and, *for a superior planet*, this inequality becomes,

and, from that of a superior planet.

[5528] $$\delta s = \frac{\frac{3}{8}\cdot\frac{P}{m'}.\alpha^3 m^2.\left\{b^{(0)}_{\frac{5}{2}}-\alpha.b^{(1)}_{\frac{5}{2}}\right\}}{g-1}.\lambda.\sin.(v-\theta).$$ [Superior planet]

Reducing these inequalities to numbers, by using the masses of Venus, Mars and Jupiter [4605, 4608, 4065], we get, for Venus,*

[5529] $$\delta s = -0'',276468.\sin.(v-\theta') \;;$$ [Action of Venus]

[5527n] $$\delta s = \frac{3PR}{8.(g-1).h^2.u^4.R^5}\cdot\left\{R.b^{(0)}_{\frac{5}{2}}-r'.b^{(1)}_{\frac{5}{2}}\right\}.\lambda.\sin.(v-\theta).$$

[5527o] Now, substituting as in [5525n], $h^2=a$, $u=a^{-1}$, $r'=a'$, and then, $R=\frac{a'}{\alpha}$, we get [5527p] ; and, by using a^3 [5473a], it becomes as in [5527q] ;

[5527p] $$\delta s = \frac{3P\alpha^3}{8.(g-1)}\cdot\frac{a^3}{a'^3}\cdot\left\{b^{(0)}_{\frac{5}{2}}-\alpha\,.\,b^{(1)}_{\frac{5}{2}}\right\}$$

[5527q] $$= \frac{3\,.\frac{P}{m'}.\,\alpha^3.\,m^2}{8.(g-1)}\cdot\left\{b^{(0)}_{\frac{5}{2}}-\alpha\,.\,b^{(1)}_{\frac{5}{2}}\right\}.$$

[5527r] This agrees with [5528] ; *the expression being corrected as in* [5527l], *for the mistake of prefixing the negative sign.* The terms we have here computed [5527k, q], have the small divisor $g-1$, of the order m^2 [4828e]; and, even with this divisor, they amount
[5527s] only to a fraction of a second, as appears in [5529—5531] ; hence it is manifest, that the terms of this kind, which have large divisors, must be wholly insensible.

* (3064) Substituting, in [5527], the values of $\frac{P}{m'}=\frac{1}{356632}$ [4605], α [4126], also
[5529a] $b^{(0)}_{\frac{5}{2}}$, $b^{(1)}_{\frac{5}{2}}$, deduced from [5492], g, m [5117], $\lambda=\varphi'$ [4082], it becomes, as in [5529].
[5529b] In like manner we obtain from, [5528], the expressions [5530, 5531] ; using the mass of Mars [4608], and that of Jupiter [4065] ; also the other elements as in [4159—4173, 4082] ;
[5529c] m, g, being as before. *We have corrected the signs of the expressions* [5529, 5530, 5531], *for the error* [5527l, r], *which is found in the original work;* the numeral coefficients given by the author being,

and, for Mars,

$$\delta s = +0^s{,}005497.\sin.(v-\theta''') ;$$ [Action of Mars] [5530]

also, for Jupiter,

$$\delta s = +0^s{,}037925.\sin.(v-\theta^{iv}) ;$$ [Action of Jupiter] [5531]

θ', θ''', θ^{iv}, *being the longitudes of the ascending nodes of the orbits of Venus, Mars, and Jupiter.* [5532]

Finally, it is evident, that the value of g, *is increased by the action of the planet* P, by the quantity,

$$\tfrac{3}{8}\cdot\frac{P}{m'}.\, m^2.\, b_{\frac{3}{2}}^{(0)},$$ relative *to an inferior planet;* [5533]

Increment of g, by the direct action of the planets.

and, by the quantity,

$$\tfrac{3}{8}\cdot\frac{P}{m'}.\, m^2.\, \alpha^3.\, b_{\frac{3}{2}}^{(0)},$$ relative *to a superior planet.** [5534]

$$+0'',853296 = +0^s{,}276468 ; \quad -0'',016966 = -0^s{,}005497 ; \quad -0'',117051 = -0^s{,}037925.$$ [5529d]

* (3065) If we substitute, in the first term of [5524], $\frac{3P.s}{2h^2u^4.f^3}$, which was neglected [5534a] in [5525a], the value of $\frac{1}{f^3}$ [5525b], and retain only the part which is independent of $U-v'$, we obtain the expression $\frac{3P.s}{2h^2.u^4}\cdot\tfrac{1}{2}A^{(0)}$. Substituting the values $h^2=a$, [5534b] $u=a^{-1}$, and a^3 [5473a], it becomes successively,

$$\tfrac{3}{4}.Pa^3.A^{(0)}.s = \tfrac{3}{4}\cdot\frac{P}{m'}\cdot m^2.a'^3.A^{(0)}.s\,.$$ [5534c]

This term of [4755], increases the value of p'' [5397k, l], by the quantity,

$$\delta p'' = \tfrac{3}{4}\cdot\frac{P}{m'}\cdot m^2.\,a'^3.\,A^{(0)},$$ [5534d]

and the corresponding increment of the motion of the node [5397l], is,

$$\tfrac{1}{2}\,\delta p''.\,v = \tfrac{3}{8}\cdot\frac{P}{m'}\cdot m^2.\,a'^3.\,A^{(0)}.\,v\,.$$ [5534e]

Now, the motion of the node is represented by $(g-1).v$ [4817]; hence the increment of g is represented by,

The term $\frac{3m'.u'^3.s}{2h^2.u^4}$, which forms a part of the equation [4755], and is
[5535] developed in [5021], *decreases* the value of g, by the quantity, $\frac{9m^2}{4}\cdot\frac{\delta r'}{a'}$;*

Decrement of g, by the indirect action of the planets.

$\frac{\delta r'}{a'}$ being the constant part of the perturbations of the radius vector of the earth's orbit. *Hence, the value of g is decreased by the action of the planets, by the same quantity that c [5522] is increased by the same action.*
[5536] But these quantities are insensible [5535f].

The direct action of the planet P upon the moon, introduces in the equation [4961], a quantity of the form,†

[5534f]
$$\delta g = \tfrac{3}{8}\cdot\frac{P}{m'}\cdot m^2.a'^3.A^{(0)}.$$

[5534g] For an inferior planet, we have $a'^3.A^{(0)} = b^{(0)}_{\frac{3}{2}}$ [5178d]; substituting this in [5534f],
[5534h] we get δg [5533]. For a superior planet $a'^3.A^{(0)} = \alpha^3.b^{(0)}_{\frac{3}{2}}$ [5480b]; hence δg [5534f], becomes as in [5534].

[5535a] * (3066) The variation of the term $\frac{3m'.u'^3.s}{2h^2.u^4}$, taken relatively to u', becomes
[5535b] as in the *first* or *second* member of [5535c]. Substituting $\frac{\delta u'}{u'} = -\frac{\delta r'}{a'}$ [5520b], it becomes as in its *third* member; and, by successive substitutions, using the values [5534b], we finally obtain [5535d];

[5535c]
$$\frac{9m'.u'^2\delta u'.s}{2h^2.u^4} = \frac{9m'.u'^3.s}{2h^2.u^4}\cdot\frac{\delta u'}{u'} = -\frac{9m'}{2}\cdot\frac{u'^3.s}{h^2.u^4}\cdot\frac{\delta r'}{a'} = -\frac{9m'}{2}\cdot\frac{a^3}{a'^3}\cdot\frac{\delta r'}{a'}\cdot s$$

[5535d]
$$= -\frac{9m^2}{2}\cdot\frac{\delta r'}{a'}\cdot s.$$

Now, proceeding as in [5531e, &c.] we find, that the expression [5535d], produces in p''
[5535e] the term $\delta p'' = -\frac{9}{2}m^2.\frac{\delta r'}{a'}$; and therefore in g the *increment*, $\delta g = -\frac{9}{4}m^2.\frac{\delta r'}{a'}$, as in [5535]; being the same as that of c [5522], except in its sign. The quantities
[5535f] thus computed, in [5533, 5534, 5535], are of nearly the same order as that in [5522], and must be insensible, as in [5522'].

† (3067) As an example of the manner in which terms of the form [5537], or such as are free from the sines and cosines of the periodical angles, are introduced into [4961],
[5537a] by means of the function Q, we may mention, those which arise from the substitution of f [5466], in Q [5467]. For, in [669 line 1], we have, relative to the earth,

$$M.\frac{P}{m'}.m^2.e'^2+M'.\frac{P}{m'}.m^2.e'e''+M''.\frac{P}{m'}.m^2.e''^2+\&c.; \qquad [5537]$$

e'' being the ratio of the excentricity to the semi-major axis, in the orbit of P. Hence, there arises in the moon's mean longitude, a secular equation analogous to that we have found in [5095*d*],

Direct action of the planets on the secular equation.

$$\tfrac{3}{2}m^2.\textstyle\int(e'^2-E'^2).dv. \qquad [5538]$$

This last expression arises from the development of the term $\frac{m'.u'^3}{2h^2.u^3}$ [4866 line 1, 5083, &c.]; and it is incomparably superior to the former, on account of the small factor $\frac{P}{m'}$, connected with the first expression. *Thus, the indirect action of the planet P upon the moon, transmitted by means of the sun, is, as it regards this inequality, much more important than the direct action, which may be neglected, without any sensible error.* [5539]

$$r'=a'.\{1+\tfrac{1}{2}e'^2-e'.\cos.v'+\&c.\}; \qquad [5537b]$$

and for the attracting planet P,

$$R=R''.\{1+\tfrac{1}{2}e''^2-e''.\cos.U+\&c.\}; \qquad [5537c]$$

R'' being its mean distance. From these values, we easily perceive, that r'^2 contains a term depending upon e'^2; R^2 a term, depending on e''^2; Rr' a term, depending on $e'e''.\cos.(U-v')$; therefore, $Rr'.\cos.(U-v')$ contains a term depending on $e'e''$. [5537d] Substituting these in [5466], we find, that f contains such terms, free from periodical angles, and depending on e'^2, $e'e''$, e''^2; which are, by this means, introduced into Q [5467], and finally into [4961]. [5537e] If we proceed with the function [5537], by the method which is used in [5083—5089], it will produce terms of the form [5087], or rather like [5095*d*, or 5538]; but they will be much less than those in [5538], by reason of the small factor $\frac{P}{m'}.m^2$, which attaches, as in [5476], to the terms depending on the direct action of the planet P. [5537f] [5537g]

CHAPTER IV.

COMPARISON OF THE PRECEDING THEORY WITH OBSERVATION.

23. In the first place, *we shall consider the mean motions of the moon, of the perigee, and of the nodes.* The expression of the moon's mean longitude, in a function of its true longitude, contains, in [5095], the secular inequality,

[5540] $$\tfrac{3}{2}\, m^2.\!\int(e'^2-E'^2).dv.$$

Secular inequality of the longitude; Hence, the expression of the true longitude, in a function of the mean longitude, contains the secular inequality,*

[5541] $$\delta v = -\tfrac{3}{2}m^2.\!\int(e'^2-E'^2).ndt.$$

[5541′] *If we represent the number of Julian years elapsed since* 1750, *by* t, we shall have, as in [4611],

[5542] $$2e' = 2E' - t.0'',171793 - t^2.\,0'',0000068194.$$

reduced to seconds. Therefore, the inequality [5541] is represented by,†

[5543] $$\delta v = 10'',181621.i^2 + 0'',01853844.i^3\,; \qquad \left[\begin{smallmatrix}\text{Secular equation}\\ \text{in longitude.}\end{smallmatrix}\right]$$

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* (3068) From [5096*b*] we obtain,

[5541*a*] $$v = nt + \varepsilon - \tfrac{3}{2}m^2.\!\int(e'^2-E'^2).dv - \Sigma C.\sin.(iv+\beta).$$

[5541*b*] In the secular part of this expression $-\tfrac{3}{2}m^2.\!\int(e'^2-E'^2).dv$, we may substitute ndt for dv, and it will become as in [5541].

† (3069) If we put $2a$, $2b$, for the coefficients of t, t^2 [5542], divided by

[5543*a*] the radius in seconds 206265^s, to reduce them to parts of unity, we shall have,

i being the number of centuries elapsed since the epoch of 1750. This [5543′]
secular equation was found by observation, before I discovered the cause of it by the theory of gravity. It is ascertained, by the comparison of a great number of eclipses, which were observed by the Chaldeans, Greeks, and Arabs, that *the moon's mean motion has increased, from the most remote period to the present day; and the observed acceleration is very nearly conformable to the preceding theory.* [5544] This secular equation is placed beyond doubt, by Mr. Bouvard, by a profound discussion of the ancient eclipses, which were known to astronomers; and also of those he has obtained from an Arabian manuscript of Ibn Junis.

We have seen, in [5231], that the sideral motion of the moon's perigee, deduced from the preceding theory, differs from its true value, but by a four [5544′]
hundred and forty-fifth part.* According to the theory, this motion is subjected to a secular equation equal to $-3{,}00052.k$; k being that of the [5545]

$$a = \frac{0{,}171793}{2\times 206265} = 0{,}000000416438\ ;$$
$$b = \frac{0{,}0000068194}{2\times 206265} = 0{,}0000000000165307\,. \qquad [5543b]$$

We also have $E' = 0{,}01681395$ [4080 line 3], corresponding to t' [5117]; hence, [5543c]
[5542] gives, by neglecting terms of the order t^3,

$$e' = E' - at - bt^2\ ; \quad \text{and,} \quad e'^2 = E'^2 - 2E'.(at+bt^2)+a^2t^2. \qquad [5543d]$$

Substituting this expression of e'^2, in the secular equation [5541], it becomes as in [5543f]; whose integral is in [5543g]; and, by putting $t=100.i$ [5541′, 5543′], we [5543e]
get [5543h];

$$\delta v = \tfrac{3}{2}m^2n\,.\,f\{2E'.atdt+(2E'b-a^2).t^2dt\} \qquad [5543f]$$
$$= \tfrac{3}{2}m^2n\,.\{E'.at^2+(\tfrac{2}{3}E'b-\tfrac{1}{3}a^2).t^3\} \qquad [5543g]$$
$$= \tfrac{3}{2}.100^2.m^2n\,.E'a.i^2+\tfrac{3}{2}.100^3.m^2n.(\tfrac{2}{3}E'b-\tfrac{1}{3}a^2).i^3. \qquad [5543h]$$

This last expression is easily reduced to the form [5543], by the substitution of the values of a, b, E' [5543b, c], also m [5117]; and, for n, the motion of the moon in a Julian year, which is taken for the unit of time in [5541′], making

$$m\,n = 129577^s{,}349 \quad [4077 \text{ line } 3,\ 4835]. \qquad [5543i]$$

* (3070) This is erroneously quoted in the original work, as a five hundred and sixtieth part. [5544a]

moon's mean motion [5232, 5541]; so that the secular equation of the
[5546] anomaly [5238] is $4{,}00052.k$, or very nearly four times that of the mean motion. The preceding equation was discovered by me, by means of the theory of gravity; and, I have found, from the theory, that *the motion of the moon's perigee decreases from age to age; and, that it is now less, by about*
[5547] *fifteen centesimal minutes in a century, than in the time of Hipparchus.** This result of the theory has been confirmed by the discussion of the ancient and the modern observations.

The motion of the moon's perigee is decreasing.

We have seen, in [5233], that the sideral motion of the nodes of the lunar
[5548] orbit, upon the apparent ecliptic, deduced from the preceding analysis, differs from its true value only by a three hundred and fiftieth part. *The secular equation of the longitude of the node is, by the same article, equal to*
[5549] $0{,}735452.k$ [5234, 5541]. *This is also confirmed by the ancient eclipses.*

Secular equation of the node.

24. *We shall now consider the periodical inequalities of the moon's motion*
[5549'] *in longitude.* In order to compare with observation, the preceding results of the theory, we shall consider, as the result of observation, the coefficients of the last lunar tables of Mason, and those of the new tables of Burg. The coefficients of Mason's tables have been determined by the comparison of a very great number of Bradley's observations; and, those of Burg, by
[5550] means of more than three thousand observations of Maskelyne. These tables have been arranged in a manner, which is quite convenient for calculation; so as to diminish the number of the arguments, making them
[5550'] depend, the one upon the other. The following is the process for determining, by Mason's tables, the equations of the moon's true longitude. This method I have developed, in a series of sines of angles, increasing in proportion to v.

Periodical inequalities of the moon.

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* (3071) If we put successively, in [5513], $i=-20$, $i=-19$, we shall find, that the difference of the two results is $6^m\,16^s$; which represents nearly the acceleration
[5547a] of the moon's motion, in a century, since the time corresponding to the mean of these two values of i, or 1950 years before the epoch of 1750, which is about the time of Hipparchus. Multiplying the preceding expression by $-3{,}00052$ [5515], we get
[5547b] nearly 19^m for the secular decrement of the motion of the perigee; instead of $15'$, given by the author in [5547].

We must first compute the following terms, in which *the anomalies are counted from the perigee*; [5550'']

Tables of Mason and Burg.

Coefficients of Burg's Tables.	Coefficients of Mason's Tables.		
$-671^s,8$....	$-668^s,6$.sin.(☉'s mean anom.)		1
$-6^s,0$....	$-8^s,9$.sin.(2.☉'s mean anom.)		2
$+53^s,9$....	$+55^s,9$.sin.(2.☽'s mean long.—2.☉ true long.+☉ mean anom.)		3
$+76^s,5$....	$+75^s,3$.sin.(2.☽ mean long.—2.☉ true long.—☉ mean anom.)		4
$-57^s,8$....	$-57^s,8$.sin.(2.☽ mean long.—2.☉ true long.+☽ mean anom.)		5
$+4829^s,5$....	$+4828^s,4$.sin.(2.☽ mean long.—2.☉ true long.—☽ mean anom.)	[Evection.]	6
$+35^s,4$....	$+35^s,0$.sin.(4.☽ mean long.—4.☉ true long.—2.☽ mean anom.)		7
$+124^s,6$....	$+123^s,5$.sin.(2.☽ mean long.—2.☉ true long.—☽ mean anom.+☉ mean anom.)		8
$+47^s,6$....	$+46^s,5$.sin.(2.☽ mean long.—2.☉ true long.—☽ mean anom.—☉ mean anom.)		9
$+39^s,3$....	$+42^s,0$.sin.(☽ mean anom.—☉ mean anom.)		10
$-21^s,4$....	$-22^s,7$.sin.(☽ mean long.—☉ true long.—☽ mean anom.)		11
$-58^s,6$....	$-57^s,4$.sin.(2.☽ mean long.—2.☉ true long.—2.☽ mean anom.)		12
$+62^s,5$....	$+60^s,4$.sin.(2.mean long. of ☽'s node—2.☉ true long.)	(M)	13
$+11^s,5$....	$+17^s,0$.sin.(☽ mean long.—☉ true long.+☉ mean anom.)		14
$+4^s,9$....	$+3^s,1$.sin.(☽ mean long.—☉ true long.—☉ mean anom.)		15
$-4^s,6$....	$-3^s,7$.sin.(2.☽ mean long.—2.☉ true long.+2.☽ mean anom.)		16
$-10^s,6$....	$-12^s,4$.sin.(4.☽ mean long.—4.☉ true long.—☽ mean anom.)		17
$-6^s,4$....	$-6^s,3$.sin.(2.☽ mean long.—2.mean long. ☽'s node—2.☽ mean anom.)		18
$-8^s,8$....	$-8^s,3$.sin.(2.mean long. ☽'s node—2.☉ true long.+☽ mean anom.)		19
$+6^s,9$....	$+5^s,3$.sin.(2.mean long. ☽'s node—2.☉ true long.—☽ mean anom.)		20
$+6^s,8$....	$+7^s,7$.sin.(mean long. ☽'s node)		21
$+2^s,6$....	$+0^s,0$.sin.(2.☽ mean long.—2.☉ true long.—2.☉ mean anom.)		22
$-2^s,6$....	$-0^s,0$.sin.(☽ mean long.—☉ true long.+☽ mean anom.)		23
$+2^s,1$....	$+0^s,0$.sin.(3.☽ mean anom.—2.☽ mean long.+2.☉ true long.)		24
$+2^s,2$....	$+0^s,0$.sin.(2.☽ mean long.—2.☉ true long.+☽ mean anom.+☉ mean anom.)		25
$+1^s,3$....	$+0^s,0$.sin.(2.☽ mean long.—2.☉ true long.+☽ mean anom.—☉ mean anom.)		26
$+1^s,1$....	$+0^s,0$.sin.(4.☽ mean long.—4.☉ true long.—3.☽ mean anom.)		27
$+1^s,2$....	$+0^s,0$.sin.(2.☽ mean long.—2.☉ true long.—2.☽ mean anom.+☉ mean anom.)		28
$+1^s,1$....	$+0^s,0$.sin.(☽ mean long.—☉ true long.—☽ mean anom.+☉ mean anom.).		29

Inequalities in the moon's longitude. [5551]

Tables of Mason and Burg. [5552] The sum of all these terms must be added to the moon's mean anomaly, to which we must also add the function A, given by the equation,

Correction of the mean anomaly. [5553]

$$\begin{array}{llll} & \text{By Burg} & \text{By Mason.} & \\ A = & -1337^s,30 \ldots\ldots & -1302^s,0.\sin.(\odot \text{ mean anom.}) & 1 \\ & -\ \ 11^s,00 \ldots\ldots & -\ \ 14^s,0.\sin.(2.\odot \text{ mean anom.})\,; & 2 \end{array}$$

[5553'] Corrected anomaly. and we shall obtain the moon's *corrected anomaly*, by means of which we must compute the following terms;

Equation of the centre. [5554]

$$\begin{array}{llll} \text{Burg.} & \text{Mason.} & & \\ +22692^s,2 \ldots\ldots & +22695^s,3.\sin.(\text{☽ corrected anom.}) & & 1 \\ +\ \ \ 776^s,4 \ldots\ldots & +\ \ \ 777^s,0.\sin.(2.\text{☽ corrected anom.}) & & 2 \\ +\ \ \ \ \ 37^s,3 \ldots\ldots & +\ \ \ \ \ 37^s,2.\sin.(3.\text{☽ corrected anom.}) & (N) & 3 \\ +\ \ \ \ \ \ \ 2^s,0 \ldots\ldots & +\ \ \ \ \ \ \ 2^s,0.\sin.(4.\text{☽ corrected anom.}). & & 4 \end{array}$$

[5555] First corrected longitude. The sum of the terms in [5551, 5554] must be added to the moon's mean longitude, and we shall obtain the *moon's corrected longitude*, which must be used in computing the following terms;

Variation. [5556]

$$\begin{array}{llll} \text{Burg.} & \text{Mason.} & & \\ -\ \ 122^s,1 \ldots\ldots & -\ \ 116^s,4.\sin.(\text{☽ corrected long.}-\odot \text{ true long.}) & & 1 \\ +2141^s,7 \ldots\ldots & +2141^s,1.\sin.(2.\text{☽ corrected long.}-2.\odot \text{ true long.}) & & 2 \\ +\ \ \ \ \ 3^s,3 \ldots\ldots & +\ \ \ \ \ 5^s,2.\sin.(3.\text{☽ corrected long.}-3.\odot \text{ true long.}) & (P) & 3 \\ +\ \ \ \ \ 7^s,3 \ldots\ldots & +\ \ \ \ \ 8^s,8.\sin.(4.\text{☽ corrected long.}-4.\odot \text{ true long.}). & & 4 \end{array}$$

Second corrected longitude. [5556'] [5557] We must connect the terms [5556] with the corrected longitude of the moon [5555], and thus, form a *second corrected longitude*, to which we must add the supplement of the node, or the whole circumference, *minus* the longitude of the node. We must also add to it the function B, determined by the equation,

Correction of the node. [5558]

$$\begin{array}{ll} & \text{Burg.} \qquad\qquad \text{Mason.} \\ B = & +540^s,0 \ldots\ldots +552^s,0.\sin.(\odot \text{ corrected anom.})\,; \end{array}$$

[5558'] and we shall obtain *the moon's distance from the corrected node.* We must subtract the moon's corrected anomaly from the double of this distance, and
[5559] multiply the sine of this argument by $-84^s,4$, according to Burg; or, by $-84^s,1$, according to Mason; and we shall get another inequality, which

we must add to the inequalities [5551, 5554, 5556]. Lastly, we must add the same inequality to the preceding distance of the moon from the corrected node, in order to form *the argument of latitude*; and, we must multiply the sine of double this argument by $-406'',8$, according to Burg, or, by $-407'',7$, according to Mason, and we shall obtain the inequality called *the reduction to the ecliptic*; which must be added to the preceding inequalities, to obtain the longitude of the moon, counted from the mean vernal equinox. We must here observe, that the *mean longitudes of the moon, of its node, and of its mean anomaly, must be corrected for the secular inequalities.*

Tables of Mason and Burg.

[5560] Argument of latitude

[5561]

[5562] Reduction to the ecliptic.

[5563]

From this process I have deduced the following expression of the periodical inequalities of the moon's mean longitude, developed in terms of the true longitude, counted upon the ecliptic. This development requires particular attention, to prevent the omission of any sensible term.* We [5564]

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* (3072) We shall here point out the general principles of the method of developing the functions [5551—5573], in the forms given in [5574—5579], without entering into any minute numerical details, which would be inconsistent with the limits of the present work. In the first place, we shall show how the functions [5551, &c.], or the expression of the true longitude, may be reduced, so as to depend wholly on the mean motions $nt+\varepsilon$, $n't+\varepsilon'$, &c.; noticing the secular inequalities, as in [5563], but omitting any particular reference to them in the present note; and then, by inverting the series, we can obtain the expression of the mean longitude $nt+\varepsilon$, in terms of the *true* longitude v, so as to conform to the present theory [5095]. Several of the functions in the table [5551] do not require any reductions; as, for example, those in [5551 lines 1, 2, 10, &c.], which depend on the mean motions; but, in those inequalities which contain the sun's *true* longitude, we must substitute its value, deduced from [668], by accenting the symbols v, e, &c. to conform to the notation used in this theory [4779']. Hence we have,

[5564a]

[5564b]

[5564c]

[5564d]

$$\text{Sun's true longitude } v' = \text{sun's mean longitude } (n't+\varepsilon')+\mathrm{e}'; \qquad [5564e]$$

e' being used for brevity, to denote the periodical terms of the values of v' [668], or those which depend on coefficients, containing the excentricity e' and its powers, multiplied by sines of the periodical angles; and, it may be represented in the following manner; [5564f] [5564g]

$$\mathrm{e}' = \Sigma\, \alpha'.\sin.(i'n't+\beta'). \qquad [5564h]$$

Now, if we put a, for the coefficient of any one of the inequalities [5551]; T', for the part of the argument which depends on the *mean* motions; and $i\mathrm{e}'$, for the part of the [5564i]

Tables of Mason and Burg. have neglected those inequalities which are less than a centesimal second, or $0^s,324$. A part of the inequalities of this expression arise merely from the
[5564'] development of the formula, corresponding to the process in Mason's tables,

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same argument, depending on e'; it becomes of the same form as in the first member of
[5564k] [5564l]. Developing this, by [21] Int., we get the second member of [5564l]; and, by substituting the values of $\sin.ie'$, $\cos.ie'$, deduced from [43, 44] Int., we obtain [5564m];

[5564l] $$a.\sin.(T'+ie') = a.\cos.ie'.\sin.T'+a.\sin.ie'.\cos.T'$$

[5564m] $$= a.\{1-\tfrac{1}{2}.i^2e'^2+\tfrac{1}{24}.i^4e'^4-\&c.\}.\sin.T'+a.\{ie'-\tfrac{1}{6}.i^3e'^3+\&c.\}.\cos.T'.$$

Substituting, in this last expression, the value of $e'=\Sigma\alpha'.\sin.(i'n't+\beta')$ [5564h], and its powers; then reducing the products, by means of [17—20] Int., we finally get the value
[5564n] of $a.\sin.(T'+ie')$, under the form of a series of terms, depending exclusively on the *mean* motions; and the whole function [5551] may be included in a general expression of the form,

[5564o] $$\Sigma\alpha.\sin.(\beta t+\gamma);$$

in which the angles depend wholly on the mean motions. If we substitute this, in the expression of the moon's corrected anomaly [5553'], we get,

[5564p] ☽'s corrected anomaly = ☽'s mean anomaly + function [5553] + $\Sigma\alpha.\sin.(\beta t+\gamma)$.

The sine of this expression, or the sine of any multiple of it i, which occurs in [5554], may be developed, as in the general formula [5564l, m], by putting,

[5564q] $T = i.$(☽'s mean anomaly); $ie' = i.\{$function [5553]$+\Sigma\alpha.\sin.(\beta t+\gamma)\}$.

By this means the function [5554] may be made to depend on the *mean* motions; therefore,
[5564r] the *corrected* longitude of the moon [5555] will also be given in terms of the *mean* motions. Substituting these in [5556], and reducing, by a similar process to that we have used, we
[5564s] get, as in [5556'], the moon's longitude *twice* corrected; whence, by using B [5558], we easily obtain the corrected distance from the node [5558'], which gives the correction [5559]. In like manner, we get the *reduction* [5562]; and, finally, obtain the true longitude v, expressed in terms depending on the *mean* motions; and, if we denote the
[5564t] *mean longitude* by $nt+\varepsilon=T$, the expression of the true longitude v, may be put under the general form,

[5564u] $$v = T+\Sigma.B.\sin.(iT+\gamma);$$

in which the angles $(iT+\gamma)$ correspond to the mean motions.

[5564u'] This last formula may be inverted, by means of La Grange's theorem [629c], which, by changing $\downarrow x$ into x, then x into T, and t into v, becomes,

which we have just explained; so that they cannot be considered as the result of observation. To distinguish the different inequalities, we have marked with *an asterisk* those computed by Mason, by the comparison of Bradley's observations, and which have all been again determined by Burg, by means of a very great number of Maskelyne's observations. *We shall commence with the great inequality of the first order; and then shall give, successively, the five inequalities of the second order, the fifteen inequalities of the third order, and all the inequalities of the fourth and of higher orders,* [5565] Inequalities in the moon's longitude. [5566]

$$T = v + F(T); \qquad \text{or} \qquad v = T - F(T); \qquad [5564v]$$

$$T = v + F(v) + \frac{1}{1.2} \cdot \frac{d.\{F(v)^2\}}{dv} + \frac{1}{1.2.3} \cdot \frac{d^2.\{F(v)^3\}}{dv^2} + \&c. \qquad [5564w]$$

Comparing together the values of v [5564u, v], we get,

$$\Sigma B.\sin.(iT+\gamma) = -F(T); \qquad \text{whence,} \qquad F(v) = -\Sigma B.\sin.(iv+\gamma). \qquad [5564x]$$

Substituting this last expression in [5564w], and making the necessary reductions, we finally obtain the values of T, or $nt+\varepsilon$, under the following form;

$$nt+\varepsilon = v + \Sigma C.\sin.(iv+\beta); \qquad [5564y]$$

which is the same as in [5096b], neglecting, as in [5564b], the consideration of the secular inequalities. This corresponds with the results in [5574—5579].

A similar process must be used, in reducing the expressions of the latitude [5595] to the form [5596]; or, that of the horizontal parallax [5603] to the form [5605]. There are no other difficulties in performing these operations, than those which arise from the great length of the calculations, in consequence of the numerous equations, which require attention, in order to procure accurate results. [5564z]

In applying the formula [5564m] to most of the small inequalities in [5551, &c.], we may neglect the square and higher powers of e'. For, e' is nearly equal to $\frac{1}{60}$ [5565a] [5117 line 4]; hence we have $a e'^2 = \frac{a}{3600}$; and, if $a < 100^s$, as is the case with twenty-six out of twenty-nine of the inequalities in the table [5551], it becomes [5565b] $a e'^2 < 0^s,03$, which is insensible. Moreover, in the equations which do not exceed 12^s [5521 lines 2, 14—29], we have $a e' = \frac{a}{60} < 0^s,2$; and the coefficient of the corresponding term of $ac'.\cos.T'$ [5564m] is so small, that it may be frequently neglected; [5565c] and then we may put simply $a.\sin.T'$, for $a.\sin.(T'+ie')$.

which have been compared with observations; lastly, all the other inequalities.
[5566′] We shall place, in the *second* column, the results of this analysis; and, in the *third* column, the excess of the numbers in the second column above those in the first. In the *fourth* column, we shall give the excess of the coefficients of Burg's new tables, reduced to the same form as in this theory, over those of Mason's tables in the first column. Burg retains, in his tables, the same forms of the arguments as in Mason's tables, which had been adopted from
[5567] the tables of Mayer. It will be sufficiently accurate, in reducing Burg's tables to the forms of the present theory, to apply to the coefficients of Mason's tables, thus reduced, as in the first column, the difference of the corresponding inequalities in the two primitive tables, taken with a contrary
[5567′] sign.* The functions A, B [5553, 5558], differ a little in these two tables, and we have noticed this difference. We may also remark, on this point, that, by introducing in the primitive tables, an inequality in the longitude, depending on

[5568] $$\sin.(☽ \text{ mean anom.} + ☉ \text{ mean anom.});$$

and, in the latitude, an inequality, depending on

[5569] $$\sin.(\text{argument of lat.} + ☉ \text{ mean anom.});$$

and, making the necessary changes in the coefficients of the inequalities, depending on

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* (3073) If we suppose, that the equation [5564u] corresponds to Mason's tables; and,
[5567a] that, in Burg's tables, one of the coefficients B, is changed into $B+\delta B$; it will increase the second member of the equation [5564u] by the quantity $\delta B.\sin.(iT+\gamma)$,
[5567b] which is very nearly equal to $\delta B.\sin.(iv+\gamma)$. Transposing this to the first member of the same equation, we find, that the equation [5564u], corresponding to Burg's tables, becomes,

[5567c] $$v-\delta B.\sin.(iv+\gamma)=T+\Sigma B.\sin.(iT+\gamma);$$

which may be derived from that of Mason [5564u], by merely changing v into
[5567d] $v-\delta B.\sin.(iv+\gamma)$; and, if we make the same change in [5564y], which results from Mason's tables, we get, for Burg's tables, the following expression;

[5567e] $$nt+\varepsilon=v-\delta B.\sin.(iv+\gamma)+\Sigma C.\sin.(iv+\beta).$$

This agrees with the remarks in [5567′].

$$\sin.(☽ \text{ mean anom.} - ☉ \text{ mean anom.});$$ [5570]

and, on

$$\sin.(\text{argument of lat.} - ☉ \text{ mean anom.}),$$ [5571]

we can dispense with the functions A and B: which will give to the tables a greater degree of uniformity.* *Burg has introduced in his tables of the* [5571']

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* (3074) If we put, for a moment, the sun's mean anomaly equal to s, and the moon's mean anomaly, corrected for the equations [5551], equal to m; we shall have $m+A$ for the moon's corrected anomaly, which is to be used in the formulas [5554]. Now, if we put $C = 22692^s,2$, we find, that the first, or chief term of [5554], becomes as in the first member of [5571c]; and, by development, using [21, 43, 44] Int., we get, successively, the expressions in the second members of [5571c, d]; [5571a] [5571b]

$$C.\sin.(m+A) = C.\cos.A.\sin.m + C.\sin.A.\cos.m$$ [5571c]

$$= C.\{1-\tfrac{1}{2}A^2+\tfrac{1}{24}A^4-\&c.\}.\sin.m + C.\{A-\tfrac{1}{6}A^3+\&c.\}.\cos.m.$$ [5571d]

This last expression may be considerably simplified, by observing, that the chief term of A [5553 line 1], expressed in parts of the radius, gives, very nearly,

$$A = -0{,}006.\sin.s; \quad \text{hence} \quad \tfrac{1}{2}A^2 = 0{,}000018.\sin.^2 s = 0{,}000009 - 0{,}000009.\cos.2s;$$ [5571e]

and $\tfrac{1}{2}A^2C = 0^s,2 - 0^s,2.\cos.2s.$

This last expression, being multiplied by $\sin.m$, becomes insensible; consequently, the equation [5571d] may be put under the form,

$$C.\sin.(m+A) = C.\sin.m + C.A.\cos.m.$$ [5571f]

If we suppose $A' = 1337^s,3$, $A'' = 11^s,0$ the expression of A [5553] becomes, [5571g]

$$A = -A'.\sin.s - A''.\sin.2s;$$ [5571h]

substituting this in [5571f], and reducing by [18] Int., we obtain,

$$C.\sin.(m+A) = C.\sin.m - \tfrac{1}{2}A'C.\{\sin.(m+s) - \sin.(m-s)\} \quad 1$$ [5571i]

$$- \tfrac{1}{2}A''C.\{\sin.(m+2s) - \sin.(m-2s)\}. \quad 2$$

The terms in the second line of this equation may be neglected; for, $\tfrac{1}{2}C$ [5571b], expressed in parts of the radius, is nearly equal to $\tfrac{1}{18}$, and $A'' = 11^s$ [5571g]; hence, $\tfrac{1}{2}A''C = 0^s,6$; which is nearly insensible, especially when multiplied by $\sin.(m \pm 2s)$; therefore, the expression [5571i] becomes, [5571k] [5571l]

$$C.\sin.(m+A) = C.\sin.m - \tfrac{1}{2}A'C.\sin.(m+s) + \tfrac{1}{2}A'C.\sin.(m-s).$$ [5571m]

[5572] *motion in longitude, eight new inequalities, which are not given in the reduced tables of Mason, except by their development.* We have distinguished them by *a double asterisk.* Lastly, he has compared with observation, several inequalities, which he has found to be insensible; so that their coefficients, given by the development of Mason's tables, may now be considered as the
[5573] results of observation; we have distinguished these by *a triple asterisk.* We may thus know, by mere inspection, the inequalities which yet remain to be compared with observation. The differences between the two tables being small, enables us to deduce the development of the one from that of the other; and we may, by the inverse method, reduce the inequalities of this theory to the form of Mayer's tables.

Inequalities in the moon's longitude.

(Col. 1.) Inequalities deduced from Mason's tables.	(Col. 2.) Coefficients of this theory.	(Col. 3.) Excess of these coefficients over those of Mason's tables.	(Col. 4.) Excess of the coefficients of Burg's tables over those of Mason.

Inequality of the first order.

[5574] $-22677^s{,}5\,.\sin.(cv-\varpi)^*$ ...... $-22677^s{,}5$ .... $+0^s{,}0$ .... $+3^s{,}1$

[5571n] The terms of this equation, depending on the arguments $m\pm s$, are as in [5568, 5570]. The substitution of the values of the multiples of $m+A$, in [5554 lines 2—4], produces only some small, or insensible inequalities. The function B [5558] being small, its effects on the equations [5559, 5560] are nearly insensible; but, they might be noticed, in a similar manner to that in [5571m, &c.].

[5571o] In like manner, if we suppose the argument of the latitude to be represented by $m'+B$, and the coefficient of the first term of the expression of the latitude by C''; so that the term itself becomes $C''.\sin.(m'+B)$ [5595 line 1]; we may develop it in the same form as in [5571i]; namely,

[5571p] $$C''.\sin.(m'+B)=C''.\sin.m'-\tfrac{1}{2}BC''.\{\sin.(m'+s)-\sin.(m'-s)\};$$

[5571q] in which the terms depending on the angles $m'\pm s$, are as in [5569, 5571]. The effect of the rest of the terms depending on B, is so small, that they are hardly deserving of notice.

[5574a] * (3075) The author remarks in a note upon this part of the work, that the coefficient of the inequality [5574], is one of the arbitrary terms of the theory, and he has thought it best to adopt the result of Burg.

Tables of Mason and Burg.

(Col. 1.) Inequalities deduced from Mason's tables.	(Col. 2.) Coefficients of this theory.	(Col. 3.) Excess of these coefficients over those of Mason's tables.	(Col. 4.) Excess of the coefficients of Burg's tables over those of Mason.	

Inequalities of the second order.

$+\ 462^s,5.\sin.(2cv-2\varpi)^*$	$+\ 467^s,4$	$+4^s,9$	$+0^s,6$	1
$-1903^s,4.\sin.(2v-2mv)^*$	$-1897^s,4$	$+6^s,0$	$-0^s,6$	2
$-4681^s,5.\sin.(2v-2mv-cv+\varpi)^*$	$-4685^s,5$	$-4^s,0$	$-1^s,1$	3 [5575]
$+\ 672^s,5.\sin.(c'mv-\varpi')^*$	$+\ 682^s,4$	$+9^s,9$	$+3^s,2$	4
$+\ 407^s,1.\sin.(2gv-2\theta)^*$	$+\ 406^s,9$	$-0^s,2$	$-0^s,9$	5

Inequalities in the moon's longitude reduced to the form of the present theory.

Inequalities of the third order.

$-\ 10^s,7.\sin.(3cv-3\varpi)^*$	$-\ 11^s,4$	$-0^s,7$	$-0^s,1$	1
$+\ 61^s,1.\sin.(2gv-cv-2\theta+\varpi)^*$	$+\ 66^s,4$	$+5^s,3$	$+0^s,3$	2
$-\ 22^s,4.\sin.(2gv+cv-2\theta-\varpi)^{***}$	$-\ 23^s,0$	$-0^s,6$	$+0^s,0$	3
$+146^s,0.\sin.(2v-2mv+cv-\varpi)^*$	$+147^s,0$	$+1^s,0$	$+0^s,0$	4
$+\ 14^s,5.\sin.(2v-2mv+c'mv-\varpi')^*$	$+\ 13^s,6$	$-0^s,9$	$+2^s,0$	5
$-136^s,5.\sin.(2v-2mv-c'mv+\varpi')^*$	$-134^s,5$	$+2^s,0$	$-1^s,2$	6
$+\ 21^s,7.\sin.(2v-2mv-cv+c'mv+\varpi-\varpi')^*$	$+\ 24^s,3$	$+2^s,6$	$-1^s,1$	7
$-205^s,8.\sin.(2v-2mv-cv-c'mv+\varpi+\varpi')^*$	$-205^s,8$	$-0^s,0$	$-1^s,1$	8 [5576]
$+\ 68^s,6.\sin.(cv+c'mv-\varpi-\varpi')^*$	$+\ 71^s,0$	$+2^s,4$	$+1^s,9$	9
$-116^s,8.\sin.(cv-c'mv-\varpi+\varpi')^*$	$-117^s,3$	$-0^s,5$	$+0^s,8$	10
$+178^s,6.\sin.(2cv-2v+2mv-2\varpi)^*$	$+169^s,1$	$-9^s,5$	$-1^s,2$	11
$+\ 55^s,8.\sin.(2gv-2v+2mv-2\theta)^*$	$+\ 56^s,6$	$+0^s,8$	$+2^s,1$	12
$+\ \ 8^s,9.\sin.(2c'mv-2\varpi')^*$	$+\ 10^s,1$	$+1^s,2$	$-2^s,9$	13
$+116^s,7.\sin.(v-mv)^*$	$+122^s,0.(1+i)$		$+5^s,7$	14
$-\ 19^s,0.\sin.(v-mv+c'mv-\varpi')^*$	$-\ 18^s,8.(1+i)$		$+5^s,5$	15

Tables of Mason and Burg.

[5577]

(Col. 1.) Inequalities deduced from Mason's tables.	(Col. 2.) Coefficients of this theory.	(Col. 3.) Excess of these coefficients over those of Mason's tables.	(Col. 4.) Excess of the coefficients of Burg's tables over those of Mason.	

Inequalities of the fourth order, and of higher orders, which have been compared with observations.

Inequalities in the moon's longitude reduced to the form of the present theory.

	Col. 1	Col. 2	Col. 3	Col. 4	
	$-\ 0^s,3.\sin.(4cv-4\varpi)^*$			$+0^s,0$	1
	$-\ 2^s,0.\sin.(2gv-2cv-2\theta+2\varpi)^*$			$+0^s,1$	2
	$+\ 7^s,7.\sin.(gv-v-\theta)^*$	$+\ 5^s,6$	$-2^s,1$	$-0^s,9$	3
	$-\ 7^s,0.\sin.(3v-3mv)^*$			$+1^s,9$	4
	$+\ 5^s,7.\sin.(4v-4mv)^*$			$+1^s,5$	5
	$+\ 0^s,8.\sin.(cv+2c'mv-\varpi-2\varpi')^*$			$-0^s,2$	6
	$-\ 0^s,8.\sin.(cv-2c'mv-\varpi+2\varpi')^*$			$+0^s,2$	7
	$-\ 8^s,9.\sin.(2cv+2v-2mv-2\varpi)^*$	$-\ 8^s,1$	$+0^s,8$	$+0^s,9$	8
	$+28^s,9.\sin.(4v-4mv-cv+\varpi)^*$	$+33^s,4$	$+4^s,5$	$+1^s,8$	9
	$+15^s,2.\sin.(4v-4mv-2cv+2\varpi)^*$	$+15^s,5$	$+0^s,3$	$-0^s,4$	10
	$-17^s,0.\sin.(cv-v+mv-\varpi)^*$	$-\ 8^s,3.(1+i)$		$+1^s,3$	11
	$-\ 1^s,1.\sin.(v-mv-c'mv+\varpi')^*$			$-1^s,8$	12
	$+\ 9^s,5.\sin.(2v-2mv-2gv+cv+2\theta-\varpi)^*$	$+\ 8^s,7$	$-0^s,8$	$+0^s,5$	13
	$+\ 1^s,2.\sin.(2gv+cv-2v+2mv-2\theta-\varpi)^*$			$+1^s,6$	14
[5578]	$-\ 3^s,5.\sin.(2v-2mv-2c'mv+2\varpi')^{**}$			$-2^s,6$	15
	$-\ 5^s,9.\sin.(cv+v-mv-\varpi)^{**}$	$-\ 5^s,0.(1+i)$		$+2^s,6$	16
	$+\ 1^s,0.\sin.(3cv-2v+2mv-2\varpi)^{**}$			$-2^s,1$	17
	$+\ 0^s,6.\sin.(2v-2mv+cv+c'mv-\varpi-\varpi')^{**}$			$-2^s,2$	18
	$+12^s,8.\sin.(2v-2mv+cv-c'mv-\varpi+\varpi')^{**}$	$+10^s,2$	$-2^s,6$	$-1^s,3$	19
	$+\ 0^s,8.\sin.(4v-4mv-3cv-3\varpi)^{**}$			$-1^s,1$	20
	$+\ 1^s,0.\sin.(2cv-2v+2mv-c'mv-2\varpi+\varpi')^{**}$	$-\ 0^s,2$	$-1^s,2$	$+1^s,2$	21
	$+\ 1^s,3.\sin.(cv-v+mv-c'mv-\varpi+\varpi')^{**}$			$+1^s,1$	22
	$+\ 6^s,4.\sin.(2cv-2v+2mv+c'mv-2\varpi-\varpi')^{***}$	$+\ 5^s,9$	$-0^s,5$		23
	$-\ 1^s,2.\sin.(4v-4mv+cv-\varpi)^{***}$				24
	$+\ 0^s,2.\sin.(4cv-4v+4mv-4\varpi)^{***}$				25
	$-\ 3^s,9.\sin.(2v-2mv+2gv-2\theta)^{***}$				26
	$\pm\ 1^s,1.\sin.(2gv\pm c'mv-2\theta\mp\varpi')^{***}$				27
	$-\ 0^s,3.\sin.(2gv+2cv-2v+2mv-2\theta-2\varpi)^{***}$				28
	$\pm\ 2^s,0.\sin.(2gv-2v+2mv\pm c'mv-2\theta\mp\varpi')^{***}$.				29

(Col. 1.) Inequalities deduced from Mason's tables.	(Col. 2.) Coefficients of this theory.	(Col. 3.) Excess of these coefficients over those of Mason's tables.	Tables of Mason and Burg.

Inequalities of the fourth order, and of a higher order, deduced from Mason's tables, which have not been compared with observations.

Inequalities in the moon's longitude reduced to the form of the present theory.

$+5^{s},0.\sin.(2cv-c'mv-2\varpi+\varpi')$	$+4^{s},5$	$-0^{s},5$	1
$-2^{s},8.\sin.(2cv+c'mv-2\varpi-\varpi')$	$-3^{s},2$	$-0^{s},4$	2
$+4^{s},7.\sin.(4v-4mv-cv-c'mv+\varpi+\varpi')$			3
$-4^{s},5.\sin.(2v-2mv+2gv-cv-2\theta+\varpi)$			4
$-0^{s},4.\sin.(2v-2mv-2gv+2cv+2\theta-2\varpi)$			5
$+1^{s},9.\sin.(4v-4mv-2cv+c'mv+2\varpi-\varpi')$			6
$+1^{s},6.\sin.(4v-4mv-2cv-c'mv+2\varpi+\varpi')$			7
$-1^{s},2.\sin.(3v-3mv-cv+\varpi)$			8 [5579]
$+0^{s},8.\sin.(4gv-4\theta)$			9
$+3^{s},0.\sin.(2v-2mv-cv+2c'mv+\varpi-2\varpi')$			10
$-5^{s},8.\sin.(2v-2mv-cv-2c'mv+\varpi+2\varpi')$			11
$+0^{s},5.\sin.(4v-4mv-2gv-cv+2\theta+\varpi)$			12
$0^{s},5.\sin.(4v-4mv-c'mv+\varpi')$			13
$+0^{s},7.\sin.(6v-6mv-3cv+3\varpi)$			14
$-0^{s},4.\sin.(cv-v+mv+c'mv-\varpi-\varpi')$			15
$+0^{s},3.\sin.(4v-4mv+c'mv-\varpi')$.			16

We see by this table, that the greatest difference between the coefficients of Mason's tables and those of the theory, is $9^{s},9$; and, there is only $8^{s},3$ between the theory and Burg's tables. We might make this difference vanish by carrying on the approximations to terms of a higher order ; *but, the preceding comparison is sufficient to establish incontestibly, that the general law of gravitation is the only cause of all the moon's inequalities.* [5580] [5580']

Two of these inequalities, on account of their importance, must be determined with particular care. The first is that which is called the parallactic inequality, whose argument is $v-mv$. It depends on the sun's parallax. It has been determined by carrying on the approximation to quantities of the fifth order inclusively ; so that we have reason to suppose, [5581]

Parallactic inequality.

that the value which we have obtained, is very accurate. According to Mason's tables, reduced to the form of the present theory, this
[5582] inequality is equal to $116^s,7$ [5576 line 14]; but Burg, who has determined it by the comparison of a very great number of observations, finds it to be
[5583] greater by $5^s,7$ [5576]; therefore, it is equal to $122^s,4$.* Putting this
[5584] last result equal to the coefficient $(1+i).122^s,0$, which is given by the theory in [5220 line 21], we obtain,†

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* (3076) In the *Monatliche Correspondenz*, vol. 28, page 101, is given an extract of a letter from Burckhardt, containing some remarks on the effect of an erroneous estimate
[5583a] of the moon's semi-diameter, in determining the value of the coefficient of the *parallactic inequality*. The usual method of determining the moon's place, by observation, is, by ascertaining the difference between the time of the transit of the moon's *enlightened limb*, over the meridian, and that of some well known fixed star. In this method, the moon's *western* limb is observed, when the angle $v-mv$ is less than 180^d, or $\sin.(v-mv)$ is *positive*; but, the *eastern* limb is observed, when $v-mv$ exceeds 180^d, or
[5583b] $\sin.(v-mv)$ is *negative*. Now, it is evident, that, if there be an error in the estimated value of the moon's semi-diameter, and, that it be taken, for example, *too great* by 1^s, the longitude of the moon's centre, resulting from this observation, will be *increased* by nearly the same quantity, when $\sin.(v-mv)$ is *positive*, and *decreased*, when $\sin.(v-mv)$ is
[5583c] *negative*; consequently, the error of the moon's longitude, arising from this source, will always have the *same sign* as the parallactic inequality, and it will be impossible to separate these two quantities. From this we easily perceive, that it is of great importance,
[5583d] in ascertaining the coefficient of the parallactic inequality, to have the moon's semi-diameter, to the utmost degree of accuracy. Burckhardt supposes, that it is owing, in
[5583e] some measure, to this circumstance, that Mayer's first estimate, given in his lunar theory, which was published by the Commissioners of Longitude of Great Britain, in 1757,
[5583f] makes this coefficient only 115^s; being less by $7^s,4$, than the late accurate determination of Burg.

† (3077) We have, in [5584], $(1+i).122^s,0=122^s,4$. Dividing this by $122^s,0$,
[5584a] we get $1+i=1,003$ nearly, as in [5585]; the slight difference arises from the use of the centesimal division to two or three more places of decimals; hence, [5221] becomes,

[5584b] $$\frac{a}{a'}=\frac{1,003}{400}, \quad \text{as in [5586].}$$

[5584c] Now, the moon's mean horizontal parallax is nearly $\frac{D}{a}$ [5309]; and, in like manner, the sun's horizontal parallax is,

$$1+i = 1,002985\,;$$ [5585]

therefore,

$$\frac{a}{a'} = \frac{1,002985}{400}.$$ [5586]

Now, the sun's parallax is $\frac{D}{a'}$, or $\frac{D}{a}\cdot\frac{a}{a'}$; therefore, it may be represented by, [5587]

$$\frac{D}{a}\cdot\frac{1,002985}{400} = \text{sun's parallax}.$$ [5588]

Determination of the sun's parallax, by means of this lunar inequality.

Substituting for $\frac{D}{a}$, its value $0,01655101$ [5329], we get $8'',56$ *for the sun's mean parallax upon the parallel, in which the square of the sine of the latitude is* $\frac{1}{3}$; which is very nearly the same as has been found by astronomers, from the last transit of Venus [5589*i*, *k*]. *Hence it appears, that the lunar theory furnishes a very accurate method of determining the sun's parallax.** [5589] [5589']

$$\frac{D}{a'} = \frac{D}{a}\cdot\frac{a}{a'} = \frac{D}{a}\cdot\frac{1,003}{400}, \quad \text{as in [5588], nearly.}$$ [5584d]

Substituting in this, the value of $\frac{D}{a}$ [5329], and, multiplying by the radius in seconds 206265^s, it becomes $8^s,56$, as in [5489].

* (3078) We may observe, that the author, in vol. 5 [12737], states the well known fact, that this method of determining the sun's parallax, by means of the parallactic inequality, was given by Mayer, in page 50 of his lunar theory [5583*e*], almost fifty years before the first publication of this volume of the Mécanique Céleste. According to Mayer's calculations, from the theory, the sun's parallax $10^s,8$ corresponds to a parallactic coefficient of $158^s,6$; consequently, the parallax $8^s,56$ corresponds to $125^s,7$, instead of $122^s,4$, which is used by La Place. Mayer supposes this coefficient to be, by observation, only 115^s, corresponding to the parallax $7^s,8$. If he had used the same coefficient $122^s,4$ as La Place, the result of his theory would make the parallax $8^s,3$; which differs but little from the truth; and proves, that Mayer had carried on his approximations to a considerable degree of accuracy, in computing the value of this inequality by the theory. [5589a] [5589b] [5589c] [5589d]

Before closing this note we may remark, that Messrs. Carlini and Plana have given, in Zach's *Correspondance Astronomique*, for the year 1820, page 26, a calculation of the

The second inequality is that which depends on the longitude of the node of
[5590] *the lunar orbit*, or, on the argument $gv-v-\theta$. Its coefficient, according to Mason, is $7^s,7$ [5578 line 3]; but Burg, who has just determined it,
[5590′] by a very great number of observations, reduces it to $6^s,8$ [5578 line 3].
[5591] The theory gives $5^s,552$ [5390], if we suppose the earth's oblateness to
[5592] be $\frac{1}{334}$; or, $11^s,499$ if the oblateness be $\frac{1}{230}$. Hence it is evident, that
[5593] Burg's computation corresponds to the oblateness* $\frac{1}{305,05}$. *This inequality*
Oblateness of the earth. *is determined with great precision by the theory:* and, we have no reason to suppose, that there is, with respect to it, the same degree of uncertainty,

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[5589e] sun's parallax, by this method, making it $8^s,719$. On the other hand, La Place, in reviewing his calculations, in a paper presented to the Board of Longitude of France, January 19,
[5589f] 1820, and printed in the *Connoissance des Tems*, for 1823, page 230, makes it $8^s,65$. In his fifth and last edition of the *Système du Monde*, page 230, he finally adopts the value $26'',58 = 8^s,61$. This differs but very little from the value $8^s,62$ given by Burg, in a
[5589g] late investigation, published in 1826, in vol. iv. page 24, of Schumacher's *Astronomische Nachrichten*. Finally, we may remark, that these results differ but very little from those
[5589h] which are obtained from the transits of Venus in 1761 and 1769. These observations have been lately discussed with great care, by Encke, using the most approved tables; and, in a work entitled, "*Die Entfernung der Sonne von der Erde aus dem Venusdurchgange*
[5589i] *von* 1761," page 143, he gives the parallax from $8^s,43$ to $8^s,55$; by combining, in the best manner, the different observations of the transit of 1761. The results of the
[5589k] observations of the transit of 1769, are given from $8^s,56$ to $8^s,65$, in vol. iv. page 25, of Schumacher's *Astronomische Nachrichten*; and the final result $8^s,5776$ is used in computing the Nautical Almanac for 1834. In conclusion, we may observe, that, in the year 1763, a work was published by Doctor Matthew Stewart, entitled, "*The Distance*
[5589l] *of the Sun from the Earth Determined by the Theory of Gravity, &c.*," by means of the observed motion of the moon's apsides. This method, though it has been approved by Horsley, Playfair, Hutton, and others, is essentially erroneous and defective; as is
[5589m] shown in a paper presented by me to the American Academy of Arts and Sciences, and published in the fourth volume of the first series of their Memoirs.

* (3079) This is easily deduced from the formula [5390e], by substituting $A'=6^s,8$ [5590′, 5390a—e], which gives,

[5593a]
$$\alpha\rho = \frac{6^s,8+7^s,6}{4392^s,6} = \frac{1}{305,05}, \quad \text{nearly.}$$

This result is finally retained by the author, in page 230 of the fifth edition of his *Système du Monde*.

which prevails in most of the other coefficients of the lunar theory, by reason of the slow convergency of the approximations. *As this inequality is proportional to the oblateness of the earth, it deserves the greatest attention of astronomers. It follows incontestibly, from the values assigned to it by Mason and Burg, that the earth is less flattened, than in the case of homogeneity. This is conformable to what has been deduced from other phenomena*, in books iii., iv., v. [5593']

Determination of the oblateness of the earth, by means of an inequality in the longitude.

[5594]

25. *We shall now consider the moon's motion in latitude.* It is found by the tables in the following manner. If we call the *moon's corrected longitude*, the mean longitude added to all the inequalities, except the inequality of the reduction, we shall find that *the moon's latitude is represented by the following expression ;* [5594]

Tables of the moon's latitude, by Burg and Mason.

Burg.	Mason.		
$+18520^s,8$ ...	$+18524^s,5$	. sin.(argument of latitude)	1
$-\ 5^s,0$ ...	$-\ 4^s,4$	. sin.(3.argument of latitude)	2
$+\ 528^s,4$ ...	$+\ 528^s,4$	. sin.(2☽ corrected long.—2☉ true long.—arg. lat.)	3
$-\ 3^s,1$ ...	$-\ 3^s,1$	. sin.(arg. lat.—☉ mean anom.)	4
$+\ 17^s,6$ ...	$+\ 17^s,6$	. sin.(arg. lat.—☽ mean anom.)	5
$+\ 25^s,1$ ...	$+\ 25^s,1$	. sin.(2☽ mean anom.—arg. lat.)	6
$+\ 1^s,9$ ...	$+\ 1^s,9$	. sin.(3☽ mean anom.—arg. lat.) [Moon's latitude.]	7
$+\ 9^s,0$ ...	$+\ 9^s,0$	. sin.(2☽ corr. long.–2☉ true long.–arg. lat.+☉ mean anom.)	8
$+\ 3^s,7$ ...	$+\ 3^s,7$	. sin.(2☽ corr. long.–2☉ true long.–arg. lat.–☉ mean anom.)	9
$+\ 2^s,2$ ...	$+\ 2^s,2$	. sin.(2☽ corr. long.–2☉ true long.–arg.lat.+☽ mean anom.)	10
$+\ 15^s,9$ ...	$+\ 15^s,9$	. sin.(arg. lat.+☽ mean anom.–2☽ corr. lon.+2☉ true long.)	11
$+\ 5^s,2$ ...	$+\ 5^s,2$	. sin.(arg. lat.+2☽ mean anom –2☽ cor. lon.+2☉ true long.)	12
$-\ 8^s,0$ ...	$-\ 0^s,0$	. sin.(☽ corrected longitude).	13

[5595]

Reducing these formulas to sines of angles, which vary in proportion to r, we obtain the following results ;

Inequalities in the moon's latitude, reduced to the form of the present theory.

	(Col. 1.) Inequalities deduced from Mason's tables.	(Col. 2.) Coefficients of this theory.	(Col. 3.) Excess of these coefficients over those of Mason's tables.	(Col. 4.) Excess of the coefficients of Burg's tables over those of Mason.	
	$18543^s,9.\sin.(gv-\theta)^*$	$18542^s,8$	$-1^s,1$	$-3^s,7^*$	1
	$+13^s,9.\sin.(3gv-3\theta)^*$	$+12^s,6$	$-1^s,3$	$-0^s,6$	2
	$+527^s,2.\sin.(2v-2mv-gv+\theta)^*$	$+525^s,2$	$-2^s,0$	$+0^s,0$	3
	$+0^s,7.\sin.(2v-2mv+gv-\theta)$	$+1^s,1$	$+0^s,4$	$+0^s,0$	4
	$-4^s,1.\sin.(gv+cv-\theta-\varpi)^*$	$-5^s,6$	$-1^s,5$	$+0^s,0$	5
	$+19^s,8.\sin.(gv-cv-\theta+\varpi)^*$	$+19^s,8$	$+0^s,0$	$+0^s,0$	6
	$+21^s,7.\sin.(gv+cv-2v+2mv-\theta-\varpi)^*$	$+21^s,6$	$-0^s,1$	$+0^s,0$	7
	$-0^s,8.\sin.(2v-2mv+gv-cv-\theta+\varpi)$	$-1^s,4$	$-0^s,6$	$+0^s,0$	8
	$+6^s,0.\sin.(2v-2mv-gv+cv+\theta-\varpi)^*$	$+6^s,5$	$+0^s,5$	$+0^s,0$	9
	$+24^s,8.\sin.(gv+c'mv-\theta-\varpi')^*$	$+24^s,3$	$-0^s,5$	$-0^s,5$	10
	$-27^s,9.\sin.(gv-c'mv-\theta+\varpi')^*$	$-25^s,9$	$+2^s,0$	$+0^s,5$	11
	$-9^s,5.\sin.(2v-2mv-gv+c'mv+\theta-\varpi')^*$	$-10^s,2$	$-0^s,7$	$-0^s,0$	12
[5596]	$+22^s,2.\sin.(2v-2mv-gv-c'mv+\theta+\varpi')^*$	$+22^s,4$	$+0^s,2$	$+0^s,0$	13
	$+25^s,7.\sin.(2cv-gv-2\varpi+\theta)^*$	$+27^s,4$	$+1^s,7$	$+0^d,0$	14
	$+4^s,3.\sin.(2cv+gv-2v+2mv-2\varpi-\theta)^*$	$+5^s,1$	$+0^s,8$	$+0^s,0$	15
	$-0^s,9.\sin.(3cv-gv-3\varpi+\theta)^*$			$-0^s,0$	16
	$+1^s,0.\sin.(3gv-2v+2mv-3\theta)$			$+0^s,0$	17
	$+0^s,4.\sin.(4v-4mv-gv+\theta)$			$+0^s,0$	18
	$+0^s,6.\sin.(3cv-gv-2v+2mv-3\varpi+\theta)$			$+0^s,0$	19
	$\pm 0^s,6.\sin.(cv+gv-2v+2mv\pm c'mv-\varpi-\theta\mp\varpi')$			$+0^s,0$	20
	$\mp 0^s,6.\sin.(2cv+gv-2v+2mv\pm cv-2\varpi-\theta\mp\varpi)$				21
	$+0^s,9.\sin.(4v-4mv-gv-cv+\theta+\varpi)$			$+0^s,0$	22
	$-0^s,0.\sin.($☽'s true longitude$)^{**}$	$-6^s,5$		$-8^s,0$	23

Here the theory agrees better with observation, than it does in the case relative to the moon's motion in longitude. This happens, in consequence

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* (3080) In a note on this table, the author remarks, that the coefficient of the inequality [5596 line 1], is one of the arbitrary quantities of the theory; and, that he gives the preference to the result of Burg's calculation.

[5596a]

of the greater simplicity in the approximations of the motions in latitude, which renders the results more accurate. For this reason, I have thought it best to compute the tables of the motion in latitude, strictly by the theory; so as to reduce, as much as is possible, the whole science of astronomy to the single principle of universal gravitation. The inequality, [5596']

$$-6^s{,}487 \,.\, \sin.\,(☽\text{'s true longitude}) \quad [5357], \qquad [5597]$$

is not introduced into Mason's tables, but was discovered by me, by the theory; and is now confirmed by observation, in an incontestible manner. Burg found it to be equal to,

$$-8^s{,}0 \,.\, \sin.\,(☽\text{'s true longitude}), \qquad [5598]$$

by the comparison of a very great number of Maskeline's observations. This coefficient is,

$$-6^s{,}487 \quad [5357], \qquad [5599]$$

if we suppose the oblateness of the earth to be $\frac{1}{334}$; it will become,

$$-13^s{,}436 \quad [5358], \qquad [5600]$$

if the oblateness be $\frac{1}{230}$, as in the case of the homogeneity of the earth. Hence it is evident, that *the coefficient* -8^s, *which is found by Burg,* [5601]
corresponds to the oblateness $\frac{1}{304{,}6}$.* *It is very remarkable, that this* [5602]
inequality gives the same oblateness as the inequality in the motion in longitude, depending on the sine of the longitude of the node, which we have given in [5593]. *These two inequalities, which, by the light they throw on the figure of the earth, deserve the utmost attention of observers, unite in the exclusion of the homogeneity of the earth.* [5602']

Determination of the oblateness of the earth, by means of an inequality in the latitude.

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* (3081) Substituting in [5357c], the value of $A=8^s$ [5601], it becomes,

$$a\rho = \frac{8^s+8^s{,}88}{5132^s{,}9} = \frac{1}{304{,}1}\,; \qquad [5602a]$$

which is nearly the same as in [5602]; the slight difference arises from the use of centesimal seconds to a greater number of decimals. The result given in [5602], is used by the author, in page 229, of the fifth edition of his *Système du Monde.*

26. *It now remains to consider the moon's horizontal parallax.* The following is the expression of that parallax, at the equator, according to the tables of Mason and Burg;

Tables of the moon's horizontal parallax, by Burg, Mason and Mayer.

(Col. 1.) Burg.	(Col. 2.) Mason and Mayer.		
$3421^s,0$ ...	$3431^s,4$		1
$-\ 0^s,3$ ...	$-\ 0^s,3$	.cos.(☉'s mean anom.)	2
$+\ 0^s,7$ ...	$+\ 0^s,7$	.cos.(2.☽'s mean long.—2.☉ true long.+☉ mean anom.)	3
$+\ 0^s,8$ ...	$+\ 0^s,8$	.cos.(2.☽ mean long.—2.☉ true long.—☉ mean anom.)	4
$-\ 0^s,1$ ...	$-\ 0^s,1$	.cos.(2.☽ mean long.—2.☉ true long.+☽ mean anom.)	5
$+\ 37^s,3$ ...	$+\ 37^s,3$	.cos.(2.☽ mean long.—2.☉ true long.—☽ mean anom.) [Evection.]	6
$+\ 0^s,3$ ...	$+\ 0^s,3$	.cos.(4.☽ mean long.—4.☉ true long.—2.☽ mean anom.)	7
$+\ 1^s,0$ ...	$+\ 1^s,0$	.cos.(2.☽ mean long.—2.☉ true long.—☽ mean anom.+☉ mean anom.)	8
$+\ 0^s,6$ ...	$+\ 0^s,6$	.cos.(2.☽ mean long.—2.☉ true long.—☽ mean anom.—☉ mean anom.)	9
$+\ 0^s,2$ ...	$+\ 0^s,2$	.cos.(☽ mean anom.—☉ mean anom.)	10
$+\ 0^s,2$ ...	$+\ 0^s,2$	.cos.(☽ mean long.—☉ true long.—☽ mean anom.)	11
[5603] $+\ 2^s,0$ ...	$+\ 2^s,0$	.cos.(2.☽ mean long.—2.☉ true long.—2.☽ mean anom.)	12
$+\ 0^s,4$ ...	$+\ 0^s,4$	.cos.(2.mean long. of ☽'s node—2.☉ true long.)	13
$+187^s,3$ ...	$+187^s,7$	.cos.(☽ corrected anom.)	14
$+\ 10^s,0$ ...	$+\ 10^s,0$	.cos.(2.☽ corrected anom.)	15
$+\ 0^s,2$ ...	$+\ 0^s,3$	.cos.(3.☽ corrected anom.)	16
$+\ 26^s,0$ ...	$+\ 26^s,0$	.cos.(2.☽ corrected long.—2.☉ true long.)	17
$-\ 1^s,0$ ...	$-\ 1^s,0$	.cos.(☽ corrected long.—☉ true long.)	18
$+\ 0^s,2$ ...	$+\ 0^s,2$	.cos.(3.corrected long.—3.☉ true long.)	19
$-\ 0^s,8$ ...	$-\ 0^s,8$	.cos.(2.☽'s true distance from node—☽ corrected anom.).	20

To obtain the moon's horizontal parallax for any latitude: Burg supposes the ellipticity of the earth to be $\frac{1}{330}$, and Mayer uses $\frac{1}{230}$. We have supposed it to be $\frac{1}{305}$, in conformity with the calculations in the
[5604] preceding article; and, we must multiply the coefficients of the table [5603, or 5605], by unity, *minus* the product of the ellipticity by the square of the sine of the latitude [1795″]. This being premised, we have, for the moon's equatorial horizontal parallax, expressed in terms

depending on the cosines of angles, which vary in proportion to the longitude v;*

(Col. 1.) Inequalities deduced from the tables of Mason and Mayer.	(Col. 2.) Coefficients of this theory.	(Col. 3.) Excess of these coefficients over those of Mason's tables.	(Col. 4.) Excess of the coefficients of Burg's tables over those of Mason.	
$+3442^s,4$	$+3427^s,9$	$-14^s,5$	$-10^s,4$	1
$+188^s,5.\cos.(cv-\varpi)$	$+187^s,7$	$-0^s,8$	$-0^s,4$	2
$-0^s,5.\cos.(2cv-2\varpi)$	$+0^s,0$	$+0^s,5$	$+0^s,0$	3
$-0^s,3.\cos.(3cv-3\varpi)$			$+0^s,0$	4
$+0^s,1.\cos.(4cv-4\varpi)$			$+0^s,0$	5
$+24^s,2.\cos.(2v-2mv)$	$+24^s,7$	$+0^s,5$	$+0^s,0$	6
$+38^s,4.\cos.(2v-2mv-cv+\varpi)$	$+38^s,1$	$-0^s,3$	$+0^s,0$	7
$-1^s,2.\cos.(2v-2mv+cv-\varpi)$	$-0^s,7$	$+0^s,5$	$+0^s,0$	8
$-0^s,2.\cos.(2v-2mv+c'mv-\varpi')$	$-0^s,2$	$+0^s,0$	$+0^s,0$	9
$+1^s,7.\cos.(2v-2mv-c'mv+\varpi')$	$+1^s,6$	$-0^s,1$	$+0^s,0$	10
$-0^s,3.\cos.(c'mv-\varpi')$	$-0^s,3$	$-0^s,0$	$+0^s,0$	11
$-0^s,1.\cos.(2v-2mv-cv+c'mv+\varpi-\varpi')$	$-0^s,2$	$-0^s,1$	$+0^s,0$	12
$+1^s,7.\cos.(2v-2mv-cv-c'mv+\varpi+\varpi')$	$+1^s,6$	$-0^s,1$	$+0^s,0$	13
$-0^s,3.\cos.(cv+c'mv-\varpi-\varpi')$	$-0^s,6$	$-0^s,3$	$+0^s,0$	14
$+0^s,5.\cos.(cv-c'mv-\varpi+\varpi')$	$+0^s,9$	$+0^s,4$	$+0^s,0$	15
$+3^s,9.\cos.(2cv-2v+2mv-2\varpi)$	$+3^s,6$	$-0^s,3$	$+0^s,0$	16
$+0^s,4.\cos.(2gv-2v+2mv-2\theta)$	$-0^s,2$	$-0^s,6$	$+0^s,0$	17
$-1^s,0.\cos.(v-mv)$	$-1^s,0.(1+i)$		$+0^s,0$	18
$-0^s,1.\cos.(4v-4mv)$			$+0^s,0$	19
$-0^s,1.\cos.(4v-4mv-2cv+2\varpi)$	$+0^s,0$	$+0^s,1$	$+0^s,0$	20
$-0^s,1.\cos.(4v-4mv-cv+\varpi)$	$-0^s,1$	$+0^s,0$	$+0^s,0$	21
$-0^s,2.\cos.(3cv-2v+2mv-3\varpi)$			$+0^s,0$	22
$-1^s,0.\cos.(2gv-cv-2\theta+\varpi)$	$-1^s,0$	$+0^s,0$	$+0^s,0$	23
$+0^s,2.\cos.(2gv+cv-2\theta-\varpi)$			$+0^s,0$	24
$-0^s,2.\cos.(cv-v+mv-\varpi)$	$-0^s,1.(1+i)$		$+0^s,0$	25
$-0^s,1.\cos.(2cv+2v-2mv-2\varpi)$	$+0^s,0$	$+0^s,1$	$+0^s,0$	26

Tables of the moon's horizontal parallax reduced to the form of the present theory.

[5605]

* (3082) The expression of the parallax [5331] is, for a latitude whose sine is $\sqrt{\frac{1}{3}}$

The equations of the horizontal parallax, in the tables of Mayer, Mason, and Burg, are derived from Mayer's theory, and we see, by the preceding table, that there is but very little difference between the coefficients of these equations, and those of the preceding analysis. We have, however, reason
[5606] to believe, that the present analysis is the most accurate, since this theory represents, better than Mayer's does, the moon's motion in longitude. This is however a mere nicety in analysis; because observations cannot be made, with sufficient accuracy, to determine such slight differences. With respect to the constant term of the parallax, it was determined by observation, both by Mayer and Burg. This last astronomer has grounded his calculations chiefly upon a very great number of Maskelyne's observations, and he has
[5607] found, that this constant part is less than in Mayer's tables, by $10^s,4$. We have deduced this quantity, in [5330], from the experiments upon the length of a pendulum, vibrating in a second; and from the measures of the degrees of the meridian: by this means, we have found, that we must still farther
[5608] decrease, by $4^s,1$ [5605 line 1], the constant part of the parallax given in Burg's tables. The question then arises, whether this difference depends on the errors of the observations, or on those of the elements which we have used in the calculation? This can be ascertained, by a long continued series of observations. The only element which appears to be liable to any considerable degree of uncertainty, is the moon's mass. We have seen, in [4628, 4629'], that to make the result of the theory coincide with the calculations of Burg,
[5609] we must decrease the moon's mass, from $\frac{1}{58,6}$ to $\frac{1}{74,2}$. This diminution appears rather too great, to accord with the phenomena of the tides, with the nutation of the earth's axis, and with the inequality in the solar tables,

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[5330, 5316]; and, by supposing the oblateness equal to $\frac{1}{305}$, we shall obtain the
[5604a] equatorial parallax, by multiplying the function [5331], by $1+\frac{1}{3}\times\frac{1}{305}$ nearly, [1795'']; or by increasing their coefficients $\frac{1}{915}$ part. This process being applied to the numbers in [5331], gives those in [5605 col. 2], corresponding to the present theory; those in
[5604b] [5605 col. 1], being deduced from [5603 col. 2], by a method of inversion similar to that which is used in finding [5575, &c. col. 1], from [5551, &c.].

which depends on the moon's mass. Upon a full consideration of the subject, it appears that we must still farther diminish, by two or three centesimal [5610'] seconds, the constant term of the moon's parallax, as it is given by Burg; who, by the comparison of a very great number of observations, had already diminished the constant term, adopted by other astronomers, and, by that means, obtained very nearly its true value.

CHAPTER V.

ON AN INEQUALITY OF A LONG PERIOD, WHICH APPEARS TO EXIST IN THE MOON'S MOTION.

27. We have remarked, in [4733—4736], that the moon's mean motion, deduced from a comparison of the observations of Flamsteed and Bradley, is sensibly greater than that which results from the observations of
[5611] Bradley, compared with those of Maskelyne; moreover, the observations made within fifteen or twenty years, indicate, in this motion, a still greater diminution. This seems to prove, that there is, in the theory of the moon's motion, one or more inequalities, of a long period; and, it is important to ascertain the law which regulates any such inequality.* If we examine

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* (3083) The propriety of introducing an inequality of this kind, into the lunar theory, has been much discussed by astronomers. It is very apparent, that the theory gives such
[5611a] an inequality; but, the result of the latest observations leads to the belief, that its coefficient is insensible; and, it is not used in Damoiseau's tables, as we have already observed in [4746a]. This correction was proposed by D'Alembert, about sixty years
[5611b] ago, to account for the acceleration of the moon's motion; before La Place had discovered the real cause of that acceleration. To estimate the periods of the arguments of the inequalities treated of in this chapter, we have taken, from the third edition of La Lande's
[5611c] astronomy, the following mean motions, in one hundred years; supposing nt, $n't$, to represent, respectively, the mean motions of the moon and sun, during that time.

[5611d] Motion of ☽'s perigee $= (1-c).nt = 4069^d,2$ [4817];

[5611e] Motion of ☽'s node $= (1-g).nt = -1934^d,2$ [4817];

[5611f] Motion of ☉'s perigee $= (1-c').n't = 1^d,7$ [4817, 4831];

[5611g] Precession of the equinoxes $= (f-1).nt = 1^d,4$ [5347o, 4359].

Hence we obtain the increments of the arguments of the first members of the following

the lunar theory, with the most scrupulous attention, we shall find, that the action of the *planets* produces nothing of this kind. This is made quite evident, by the analysis, given in [5455—5539]. But, *the sun's attraction produces, in the expression of* $nt+\varepsilon$, *an inequality, proportional to the sine of the following angle* ;* [5612]

Inequality in r, depending on the sun's action.

$$3v-3mv+3c'mv-2gv-cv+2\theta+\varpi-3\varpi'. \qquad [5613]$$

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expressions, in one hundred years; also, the times of the periodical revolutions of these arguments, respectively, or the number of years requisite to complete the whole circumference 360^d; [5611h]

			Years	
2☽ node+☽ perigee	$=3nt-2gnt-cnt$	$=200^d,8$	179	[5611i]
2☽ node+☽ perigee—☉ perigee	$=3nt-n't+c'n't-2gnt-cnt$	$=199^d,1$	181	[5611k]
2☽ node+☽ perigee—3☉ perigee	$=3nt-3n't+3c'n't-2gnt-cnt$	$=195^d,7$	184	[5611l]
2☽ node+☽ per.-☉ per.+2.precession	$=2fnt+nt-n't+c'n't-2gnt-cnt$	$=201^d,9$	178	[5611m]
2☽ node+☽ perigee +3.precession	$=3fnt-2gnt-cnt$	$=205^d,0$	175	[5611n]

The arguments [5611l, m, n] correspond, respectively, to [5627, 5633, 5639]. The author commenced with the use of the first of these arguments, as in [5665]; but, he afterwards proposed to change it into the form [5611n]. Burckhardt uses the argument 2☽ node+☽ perigee, in his tables, published in 1812. Several papers were published by LaPlace, Burckhardt and Burg, upon this subject, in the *Connaissance des Tems,* for 1813, 1823, 1824, &c.; and in the *Monatliche Correspondenz,* vols. 24, 26, 28; also by Carlini and Plana, in Zach's *Correspondance Astronomique,* vol. 4, page 26, &c. La Place resumes the subject in the fifth volume of this work [12755']; but does not there speak with much confidence relative to the existence of this inequality. Finally, he omits it altogether, in the last edition of his *Système du Monde,* which was published a short time before his decease. [5611o] [5611p] [5611q]

* (3084) As an example of the production of such quantities, we shall observe, that the function $\left(\frac{dQ}{dv}\right)$ [1809] contains the term,

$$-\frac{15m'.u'^4}{8u^3}.\sin.(3v-3v') ; \qquad [5613a]$$

and, we have, in u'^4, a term of the form,

$$A'.e'^3.\cos.(3c'mv-3\varpi') \quad [4838, \&c.] ; \qquad [5613b]$$

also, in u^{-3}, a term of the form,

The terms which compose this inequality are very small, in the differential equations; but, some of them acquire, by successive integrations, the
[5614] divisor $(3-3m+3c'm-2g-c)^2$; and this can render them sensible, by its extreme smallness. To determine this divisor, we shall observe, that we have, by using the values [5117],

[5615] $$3-2g-c=0{,}00040849.$$

[5616] Moreover, the annual motion of the sun's perigee is $11^s{,}949588$ [4244line1]; hence we have,*

[5617] $$1-c'=0{,}00000922035.$$

From this we get,

[5618] $$3-3m+3c'm-2g-c=0{,}00040642\,;$$

consequently, we have,

$$A''.e\gamma^2.\cos.(2gv+cv-2\theta-\varpi)\;;$$

which is similar to that in [4904 line 16]. The product of these two term gives, by reduction,

$$\tfrac{1}{2}A'A''.e'^3.e\gamma^2.\cos.(3c'mv-2gv-cv+2\theta+\varpi-3\varpi').$$

Multiplying this by the factor,

[5613c] $$-\frac{15m'}{8}.\sin.(3v-3v')=-\frac{15m'}{8}.\sin.(3v-3mv),\quad\text{nearly}\,;$$

[5613d] and reducing, we obtain a term of $\left(\frac{dQ}{dv}\right)$, depending on the sine of the angle mentioned in [5613]. Substituting this in [4753, or 5620'], we find that it will suffer two integrations, which will introduce the divisor [5614].

* (3085) The motion of the sun's perigee is $(1-c').n't$ [5611f]; and, if we put this equal to $11^s{,}949588$ [5616], and $n't=1295977^s{,}349$ [4077 line 3], we shall get,

[5616a] $$(1-c').1295977^s{,}349=11^s{,}949588\,;$$

whence, we easily deduce [5617]. Multiplying this by $3m$ [5117], we obtain,

[5616b] $$3m-3c'm=0{,}00000207\,;$$

subtracting it from [5615], we get [5618]; whose square is as in [5619]. We have
[5616c] corrected the numbers [5615, 5618, 5619], for a small mistake, made by putting [5615] equal to 0,00040859.

$$(3-3m+3c'm-2g-c)^2 = 0{,}00000016518.$$ [5619]

We have seen, however, in [4853', &c.], that the square of the coefficient of the angle v, cannot become a divisor of the corresponding inequality, by means of the successive integrations, when we notice only the *first* power of the disturbing force; but this restriction does not obtain in the terms depending on the *square* of that force; and, the inequality depending on $3v-3mv+3c'mv-2gv-cv+2\theta+\varpi-3\varpi'$, can arise only from these terms. To prove this, we shall consider the term $3a.\int\int ndt.\mathrm{d}R$, of the expression of δv, given by the formula [931]. This term appears to be that upon which the inequality in question must chiefly depend. The development of R gives some terms of the form,* [5619'] [5620] [5620']

$$R = H.\cos.(3nt-3n't+3c'n't-2gnt-cnt+2\theta+\varpi-3\varpi').$$ [5621]

If these terms depend only on the first power of the disturbing force, $n't$ and $c'n't$ will depend on the sun's co-ordinates; and then, the differential $\mathrm{d}R$, which only affects the moon's co-ordinates [5363'], will become, [5622]

$$\mathrm{d}R = -(3-2g-c).ndt.H.\sin.(3nt-3n't+3c'n't-2gnt-cnt+2\theta+\varpi-3\varpi').$$ [5623]

The double integral $3a.\int\int ndt.\mathrm{d}R$ acquires the divisor,

$$(3-3m+3c'm-2g-c)^2;$$ [5624]

m being equal to $\frac{n'}{n}$ [4835]; but, it has for a factor $3-2g-c$, which is very nearly equal to $3-3m+3c'm-2g-c$ [5615, 5618]; so that it must be considered as having only the divisor $3-3m+3c'm-2g-c$, which does not appear to be small enough to render the result sensible. If the preceding term of the expression of R depend on the square of the [5625] [5625']

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* (3086) This is evident, by comparing the value of R [949] with its development [957, &c.]. It also appears, by a process similar to that in [5613a—d], from which we easily perceive, that $-Q$, or R [5360] contains a term of the form, [5621a]

$$H.\cos.(3v-3mv+3c'mv-2gv-cv+2\theta+\varpi-3\varpi').$$ [5621b]

Now, substituting nt for v, and $mn=n'$ [4835], it becomes as in [5621]. Its differential, relative to d, supposing it not to affect $n't$, becomes as in [5623]; but, if we suppose it to affect the part $-2n't$ of the term $-3n't$, and put $n'=mn$, as above, it becomes as in [5626]. [5621c] [5621d]

[5625″] disturbing force; or, in other words, if it arise from the substitution of the parts of r, v, which depend on the first power of that force; then, the moon's co-ordinates will contain the angles $n't$ and $c'nt$. For example, if we suppose, that the part $-2n't$, of the angle $-3n't$, in this term of R, depends on the moon's co-ordinates; we shall have,

[5626] $$\mathrm{d}R = -(3-2m-2g-c).H.ndt.\sin.(3nt-3n't+3c'n't-2gnt-cnt+2\theta+\varpi-3\varpi');$$

and, the term $3a.\int\int n\,dt.\mathrm{d}R$ [5620′] gives, in the expression of the moon's longitude, the following term,

[5627] $$\delta v = \frac{3a.(3-2m-2g-c).n^2.H.\sin.(3nt-3n't+3c'nt-2gnt-cnt+2\theta+\varpi-3\varpi')}{(3-3m+3c'm-2g-c)^2};$$

which may become sensible by the extreme smallness of its divisor. The terms of this kind, are very numerous, and it is difficult to determine all of

[5628] them, with accuracy; but it is sufficient for the present purpose, to prove the possibility of such an inequality; since we may then refer directly to

[5628′] observations to determine its magnitude. *This inequality must be applied to the mean motion, and, therefore, also to the mean anomaly.*

The theory also indicates an inequality, depending upon the oblateness of the earth, and having very nearly the same period as the preceding [5627]. We have seen, in [5340], that the expression of Q contains the term,

[5629] $$Q = (\tfrac{1}{2}\alpha\varphi - \alpha\rho)\,.\,\frac{D^2}{r^3}\,.\,(\mu^2 - \tfrac{1}{3});$$

now, we have, as in [5344],

[5630] $$\mu = s.\cos.\lambda + \sqrt{1-ss}\,.\sin.\lambda.\sin.fv;$$

moreover, we have, in [4776],

[5631] $$r = \frac{\sqrt{1+ss}}{u}.$$

This gives in R, or in $-Q$, [5438], the function,*

* (3087) The square of μ [5630], or rather the square of the last term of that

[5632a] expression, produces $(1-s^2).\sin.^2\lambda.\sin.^2 fv = (1-s^2).\sin.^2\lambda.(\tfrac{1}{2}-\tfrac{1}{2}.\cos.2fv)$; so that μ^2 or $\mu^2-\tfrac{1}{3}$ contains the term $-\tfrac{1}{2}(1-s^2).\sin.^2\lambda.\cos.2fv$. Substituting this in [5629], we obtain in Q, the term,

$$R = -Q = (\tfrac{1}{2}\alpha\varphi - \alpha\rho).\tfrac{1}{2}D^2.\,u^3.\,(1-\tfrac{5}{2}s^2).\sin.^2\lambda.\cos.2fv\,.$$ [5632]

Inequality in v, depending on the oblateness of the earth.

This function produces, by its development, some terms depending on the following angle,*

$$2fnt + nt - n't + c'n't - 2gnt - cnt + 2\theta + \varpi - \varpi'\,.$$ [5633]

They are analogous to those produced by the function R [5626, &c.], relative to the sun's action, which depends on the angle [5621],

$$3nt - 3n't + 3c'n't - 2gnt - cnt + 2\theta + \varpi - 3\varpi'\,.$$ [5634]

The coefficient of the time t, is very nearly the same in both these angles, which differ from each other about 180^d, in the present situation [5635]

$$Q = -(\tfrac{1}{2}\alpha\varphi - \alpha\rho)\,.\,\tfrac{1}{2}D^2.\,\frac{(1-s^2)}{r^3}\,.\,\sin.^2\lambda.\cos.2fv\,.$$ [5632b]

Now the expression of r [5631] gives,

$$\frac{1}{r^3} = u^3.\,(1+s^2)^{-\frac{3}{2}} = u^3.\,(1-\tfrac{3}{2}s^2)\,,\quad\text{and}\quad \frac{1-s^2}{r^3} = u^3.\,(1-\tfrac{5}{2}s^2),\ \text{nearly.}$$ [5632c]

Substituting this in the preceding value of Q, it gives in $-Q$, or R [5438], the term [5632].

* (3088) This angle is produced by the development of the term $u^3.\,s^2.\cos.2fv$, which occurs in [5632]. For the value of s, or rather of $s+\delta s$, [4818, 4896, 4897], contains the terms, [5633a]

$$s = \gamma.\sin.(gv-\theta) + B_2^{(2)}.\ c\gamma.\sin.(gv+cv-\theta-\varpi)\,;$$ [5633b]

whose square produces the term,

$$2B_2^{(2)}.\ c\gamma^2\,.\sin.(gv-\theta)\,.\sin.(gv+cv-\theta-\varpi)\,;$$

or, by reduction,

$$s^2 = -B_2^{(2)}.\ c\gamma^2.\cos.(2gv+cv-2\theta-\varpi).$$ [5633c]

In like manner, the value of u, or that of $u+\delta u$ [4826, 4904], contains the following terms,

$$u = \frac{1}{a}\,.\left\{1 + A_0^{(18)}.\,\frac{a}{a'}\,.\,c'.\cos.(v - mv + c'mv - \varpi')\right\};$$ [5633d]

therefore, u^3 contains the term,

$$3A_0^{(18)}.\,\frac{1}{a^3}\,.\,\frac{a}{a'}\,.\ e'.\cos.(v - mv + c'mv - \varpi')\,.$$ [5633d']

of the sun's perigee.* All the terms of R [5632], depend entirely upon the co-ordinates of the moon ;† so that if we represent by,

[5636] $$R = K.\sin.(2fnt + nt - n't + c'n't - 2gnt - cnt + 2\theta + \varpi - \varpi'),$$

the term of the development of R [5632], which depends upon the
[5637] preceding angle; we shall find, that this term acquires, in the differential $\mathrm{d}R$, the factor $(2f+1-m+c'm-2g-1).n$; therefore, it will have for a divisor,
[5637′] in the double integral $3a.\iint ndt.\mathrm{d}R$, only the *first* power, and not the *square* of this quantity; hence, it is evident, that this term must be insensible.

[5638] Term of v, depending on $Y^{(3)}$. *The term of the form $Y^{(3)}$, which, as we have seen in the third book, may occur in the expression of the radius of the earth, can also introduce into the*

Multiplying this, by the term of s^2 [5633c], and reducing, we get, in $u^3.s^2$, a term of the following form,

[5633e] $$2K'.\cos.(v - mv + c'mv - 2gv - cv + 2\theta + \varpi - \varpi').$$

Lastly, multiplying this by $\cos.2fv$, and reducing, we obtain, in R, a term of the form,

[5633f] $$K.\cos.(2fv + v - mv + c'mv - 2gv - cv + 2\theta + \varpi - \varpi').$$

Now, changing, as in [5621c], v into nt, and putting $mn = n'$, it becomes as in [5633, 5636].

* (3089) Subtracting the angle [5634] from that in [5633], we get for their difference,

[5635a] $$2.(f-1).nt + 2.(1-c').n't + 2\varpi';$$

and, as we have, very nearly, $f=1$, $c'=1$ [5347q, 5617], the preceding expression
[5635b] is, very nearly, equal to $2\varpi'$, which differs but little from 180^d [4081, line 3], as in [5635].

† (3090) The variable quantities which occur in R [5632], are u^3, s^2, $\cos.2fv$;
[5636a] *all* of which refer to the moon; so that for this term of R, the differential $\mathrm{d}R$ [5632] changes into the *complete* differential dR; and by taking the complete differential of [5636], we get,

[5636b] $$\mathrm{d}R = (2fn + n - n' + c'n' - 2gn - cn).dt.\cos.(2fnt + nt - n't + c'n't - 2gnt - cnt + 2\theta + \varpi - \varpi').$$

Substituting $n' = mn$ [5621c] in the factor of this expression, it becomes,

[5636c] $$(2f + 1 - m + c'm - 2g - c).ndt,$$

corresponding to [5637].

*expression of the moon's true longitude, an inequality depending on,**

$$\sin.(3fnt - 2gnt - cnt + 2\theta + \varpi) ; \qquad [5638']$$

which is now nearly confounded with the two preceding ones [5638f]. If [5639] this inequality become sensible, it will furnish new data on the figure of the earth; but some calculations, which I have made for this object, induce me to believe, *that this inequality, like the former, is insensible.* The lapse of [5639'] ages, and new improvements in analysis, will throw light on this delicate and important part of the lunar theory.

28. *We shall now proceed to establish by observations, the existence of the inequality depending on the sine of the angle,*

$$3nt - 3n't + 3c'n't - 2gnt - cnt + 2\theta + \varpi - 3\varpi' \quad [5627]. \qquad [5640]$$

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* (3091) In the same manner as the term $\mu^2 - \frac{1}{3}$ of $Y^{(2)}$ [1528c], introduces into V [1811 or 5336], the term,

$$(\tfrac{1}{2}\alpha\varphi - \alpha\rho) . \mu^2 . M . \frac{D^2}{r^3} ; \qquad [5638a]$$

the term $Y^{(3)}$ [1811, 1528d], produces a term, which contains the factor $\frac{\mu^3}{r^4}$ or $\mu^3 . u^4$ [4776]. Now, the last term of μ [5630], gives in μ^3 the term,

$$(1 - ss)^{\frac{3}{2}} . \sin.^3\lambda . \sin.^3 fv , \quad \text{or} \quad \sin.^3\lambda . \sin.^3 fv \ ;$$

which, by means of [2] Int. gives $-\frac{1}{4} . \sin.^3\lambda . \sin.3fv$. Moreover, the complete value [5638b] of u or $u + \delta u$ [4826, 4904], contains terms of the form,

$$\frac{1}{a} . \left\{ 1 + A_{,} . \cos.(2gv + cv - 2\theta - \varpi) \right\} ; \qquad [5638c]$$

therefore, u^4 produces $4a^{-4} . A_{,} . \cos.(2gv + cv - 2\theta - \varpi)$. Multiplying this by the term, $-\frac{1}{4} . \sin.^3\lambda . \sin. 3fv$ [5638b], and reducing by [18] Int. we obtain,

$$-\tfrac{1}{2} . a^{-4} . A_{,} . \sin.^3\lambda . \sin.(3fv - 2gv - cv + 2\theta + \varpi) ; \qquad [5638d]$$

which, by changing v into nt, produces the angle mentioned in [5638']. The difference between this angle [5638'], and that in [5626], is represented by,

$$3.(f-1) . nt - 3.(c'-1) . n't + 3\varpi' ; \qquad [5638e]$$

which, by reason of the smallness of $f-1$, $c'-1$ [5635b], is now nearly equal to $3\varpi'$; and as this varies slowly, the periods of the inequalities [5627, 5638'], are nearly equal to [5638f] each other, and to that in [5633], as in [5635b, 5639], or in [5611l, m, n].

If we represent this angle by E, we shall evidently have,

[5641] $$E = 2.\text{long.} ☽ \text{node} + \text{long.} ☽ \text{perigee} - 3.\text{long.} ☉ \text{perigee} \quad [5611l];$$

and we shall now proceed to show, that the law of the variations of $\sin.E$, is the same as that of the variations which have been observed in the moon's mean motion.

In the lunar tables, inserted in the third edition of La Lande's astronomy, it is supposed, that in the interval of 100 Julian years, the moon's motion relative
[5642] to the equinoxes, exceeds a whole number of circumferences, by $307^d\,53^m\,12^s$;
[5643] and that the epoch of 1750 is $188^d\,17^m\,14^s,6$. The correction of the epoch of these tables, in 1691, has been determined by Bouvard and Burg; by means of more than two hundred observations of La Hire and Flamsteed;
[5644] they have both found this correction equal to $-4^s,4$.

The correction of the epoch of the same tables, in 1756, has been determined by Mason and Bouvard, by means of a very great number of
[5644′] Bradley's observations; and they have found it to be $0^s,0$. Thus, in the interval from 1691 to 1756, the moon's mean motion was greater than the
[5645] tables, by $4^s,4$, which gives $6^s,8$* for the increment of the mean motion of the same tables in a century.

Burg has found, by a great number of Maskelyne's observations, that the correction of the epoch of these tables is equal to $-3^s,0$, in 1766, and
[5646] $-9^s,1$, in 1779.

Bouvard has found, by a great number of Maskelyne's observations,
[5647] $-17^s,6$, for the correction of the epoch of these tables, in 1789.

Lastly, by a considerable number of observations made at Greenwich, Paris and Gotha, it has been found, that the correction of the epochs of the
[5648] same tables, in 1801, is $-28^s,5$.

[5648′] Hence it appears, that, from 1756 to 1801, the moon's mean motion has decreased in a sensible manner; and, that this diminution is now increasing.

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* (3092) In the interval from 1691 to 1756, which is 65 years, this correction varies
[5645a] $4^s,4$ [5645], which is at the rate of $6^s,8$ in a century, as in [5645].

For, in the interval between 1756 and 1779, which is twenty-three years, this motion was less than by the tables, by $9^s,1$ [5644′, 5646]; and, from [5649] 1779 to 1801, that is, in twenty-two years, it was less by $19^s,4$.* The epoch of 1756, compared with that of 1779, gives $39^s,5$,† for the decrease [5650] of the tabular motion in a century; whilst the epoch from 1756 to 1801 gives $63^s,3$, for this diminution. Therefore, the combination of all these [5651] observations evidently indicates the three following results. *First.* A mean motion greater than that of the tables, from 1691 to 1756 [5644′]. *Second.* A less mean motion from 1756 to the present time [5651]. *Third.* A [5652] diminution which becomes more and more rapid.

These results are conformable to the march of the preceding inequality. For, at the epoch of 1691, the sine of E was negative;‡ and, it was [5653] positive in 1756; therefore, this inequality increases the moon's mean motion, in that interval. In 1756, this sine was positive, and near its *maximum*; and [5654] since that epoch, it has always been decreasing; therefore, the inequality decreases the moon's mean motion. Lastly, this sine was nearly equal to

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* (3093) This is the difference of the two corrections $-9^s,1$, $-28^s,5$ [5646, 5648]. [5648a]

† (3094) The difference of the numbers $0^s,0$, $-9^s,1$ [5644′, 5646] is $9^s,1$, corresponding to the interval $1779-1756=23$ years. This is at the rate of $39^s,5$, in [5650a] 100 years; as in [5650]. If, instead of $-9^s,1$, we had used $-28^s,5$ [5648], corresponding to 1801, the variation would be $28^s,5$, in 45 years; corresponding to $63^s,3$, in a century. These differ a little from the results of the author in the original [5650b] work; who gives $126''=40^s,8$, and $172'',5=55^s,9$, instead of $39^s,5$ and $63^s,3$, respectively.

‡ (3095) According to the tables in La Lande's astronomy, the values of E, at the different epochs, are nearly as follows;

Years,	1691	1750	1756	1801	[5652a]
Values of E,	320^d	76^d	87^d	176^d.	[5652b]

The signs of the angles change from negative to positive, in 1750, &c., as in [5653, &c.]; [5652c] and, in 1756, $\sin.E$ attains nearly its maximum value, or $\sin.90^d$. Moreover, if we represent, as in [5658], by $y.\sin.E$, the part of this correction which depends on E, [5652d] and suppose E to increase by the quantity dE, the corresponding increment of $y.\sin.E$ becomes $y.dE.\cos.E$; which has, evidently, its greatest negative value when $E=180^d$, [5652e] or $\sin.E=0$; as in [5654′].

[5654′] nothing, in 1801 [5652b]; and then, the diminution of the mean motion was the greatest [5652e]. The decrement of the mean motion must, therefore, be greatest about the year last mentioned.

We shall now determine the coefficient of this inequality. It is evident, that it must produce a change, both in the epoch of the tables in 1750, and
[5655] in the mean motion of the tables in a hundred years. We shall put ε for the correction of the epoch of the tables in 1750; x for the diminution
[5656] of the mean motion in a century: and, y for the coefficient of the preceding inequality. The formula for the correction of the epochs of
[5657] the tables, will be, by putting i for the number of centuries elapsed since 1750,

[5658] $$\varepsilon - x \,.\, i + y \,.\, \sin. E. \qquad \text{[Correction of the epoch]}$$

To determine the three unknown quantities ε, x and y; we have
[5658′] compared this formula with the results of observation, at the three epochs 1691, 1756 and 1801; and, by this means, have obtained the three following equations;*

[5659]
$$\varepsilon + x.0{,}59 - y.0{,}63660 = -4^{s}{,}4; \qquad \text{[Year 1691]} \qquad 1$$
$$\varepsilon - x.0{,}06 + y.0{,}99898 = \;\; 0^{s}{,}0; \qquad \text{[Year 1756]} \qquad 2$$
$$\varepsilon - x.0{,}51 + y.0{,}08199 = -28^{s}{,}5. \qquad \text{[Year 1801]} \qquad 3$$

These three equations give,

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* (3096) The coefficients of x, in the equation [5658, or 5659], are represented by
[5659a] $\frac{1}{100}.(1750 - \text{years})$; those of y are the values of $\sin. E$, corresponding to the respective years; similar to those in [5652b], but taken to a greater degree of accuracy. Lastly, the constant terms of the second members, are the quantities computed in
[5659b] [5644, 5644′, 5648]. The equations [5659] give the values of ε, x, y [5660]; as we can easily prove, by substituting them in [5659]. With these values, we find, that the formula [5658] becomes,

[5659c] $$-13^{s}{,}46 - 31^{s}{,}96.i + 15^{s}{,}39.\sin. E\;;$$

from this we obtain the values [5661], using the values of E, corresponding to the different epochs. We may observe, that the quantities $-3^{s}{,}0$, $-9^{s}{,}1$, $-17^{s}{,}6$
[5659d] [5646, 5647] furnish three additional equations, of the form [5659]; and, we can determine the value of ε, x, y, by combining *all* these equations, by the method of the least squares [815e—l].

$$\varepsilon = -13^s,46 \; ; \qquad 1$$
$$x = 31^s,96 \; ; \qquad 2 \qquad [5660]$$
$$y = 15^s,39 \; . \qquad 3$$

By means of these values, we find $-4^s,4$, $+0^s,0$, $-3^s,8$, $-11^s,3$, [5661]
$-18^s,7$, and $-28^s,5$, for the corrections of the six epochs of 1691, 1756, 1766, 1779, 1789, 1801. The sum of these six corrections is $-66^s,7$; [5662]
and the sum of the six corrections determined by observations is $-62^s,6$; [5663]
the whole of these corrections taken together, indicate, therefore, that we must increase the preceding value of ε by $0^s,7$;* and then the formula [5664]
for correcting the tables becomes,

$$-12^s,8 - 31^s,96 \,.\, i + 15^s,39 \,.\, \sin. E \,. \qquad [5665]$$

Calculating by this formula, the corrections for the six epochs, we have,

(Col. 1.)	(Col. 2.) Corrections of the tables by observations.		(Col. 3.) Corrections by the formula.	(Col. 4.) Excess of these corrections above the first.		
1691 . . .	$-\ 4^s,4$	[5644]	$-\ 3^s,7$	$+\ 0^s,7$	1	
1756 . . .	$+\ 0^s,0$	[5644']	$+\ 0^s,7$	$+\ 0^s,7$	2	
1766 . . .	$-\ 3^s,0$	[5646]	$-\ 3^s,1$	$-\ 0^s,1$	3	[5666]
1779 . . .	$-\ 9^s,1$	[5646]	$-10^s,6$	$-\ 1^s,5$	4	
1789 . . .	$-17^s,6$	[5647]	$-18^s,0$	$-\ 0^s,4$	5	
1801 . . .	$-28^s,5$	[5648]	$-27^s,8$	$+\ 0^s,7$	6	

The difference between the results of observation and those of the formula, are within the limits of the errors to which these last results are liable; they may in part depend on the formula itself, which can be rectified by new observations.

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* (3097) If we suppose the expression ε [5658], to be increased by the quantity ε', it will augment each of the six numbers [5661], by the same quantity ε', [5665a]
and the sum of all of them will become $-66^s,7+6\varepsilon'$. Putting this equal to the sum $-62^s,6$ of the corrections by observation, as they are given in the second column [5665b]
of the table [5666], we get $-66^s,7+6\varepsilon' = -62^s,6$; whence $\varepsilon' = 0^s,7$; as in [5664]. Adding this to each of the values [5661], we get the numbers in the third column of [5666]. Subtracting the terms in the second column of this table, from those in [5665c]
the third, we get the corrections in the fourth column.

CHAPTER VI.

ON THE SECULAR VARIATIONS OF THE MOTIONS OF THE MOON AND EARTH, WHICH CAN BE PRODUCED BY THE RESISTANCE OF AN ETHEREAL FLUID SURROUNDING THE SUN.

29. It is possible, that there may be an extremely rare fluid surrounding
[5666] the sun, which alters the motions of the planets and satellites;* it is, therefore, interesting to know its influence on the motions of the moon and earth. To determine it, we shall put,

[5667] x, y, z, for the rectangular co-ordinates of the moon, referred to the centre of gravity of the earth;

[5668] x', y', z', for the rectangular co-cordinates of the earth, referred to the sun's centre.

The moon's absolute velocity about the sun, will be expressed by the following function;†

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* (3098) The existence of such a resisting medium is now considered as highly probable, in consequence of the observed decrease of the times of revolution of Encke's comet, in its
[5667a] successive appearances between the years 1786 and 1829. Encke has given an important paper on this subject, in the ninth volume of Schumacher's *Astronomische Nachrichten*,
[5667b] pag. 317—348; to which we may have occasion to refer, in treating of the perturbations of comets. We shall here merely remark, that the extreme rarity of the mass of this comet, makes it peculiarly well adapted to the discovery of the effects of such a resisting
[5667c] ethereal fluid; which cannot, however, produce any sensible effect on the large and dense bodies of the planets and satellites.

† (3099) The rectangular co-ordinates of the moon, referred to the sun's centre,
[5669a] are represented by $x+x'$, $y+y'$, $z+z'$, as in [5667, 5668]. Their differentials, divided by dt, are,

$$\frac{\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2}}{dt} = \text{the moon's velocity.}$$ [5669]

We shall suppose, that the resistance which the moon suffers, is represented by the product of the square of the velocity by a coefficient K, depending upon the density of the ether, and upon the surface and density of the moon. If we resolve it, in directions parallel to the axes x, y, z, we shall obtain the three following forces;* [5670]

Hypothesis for the resistance of the ether.

$$-\frac{K.(dx'+dx)}{dt^2}\cdot\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2};$$ [Force parallel to x] 1

$$-\frac{K.(dy'+dy)}{dt^2}\cdot\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2};$$ [Force parallel to y] 2 [5671]

$$-\frac{K.(dz'+dz)}{dt^2}\cdot\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2}.$$ [Force parallel to z] 3

Expressions of the resistance of the moon.

In the lunar theory, the earth is supposed to be at rest; we must, therefore, apply to the moon, *in a contrary direction, the resistance which the earth suffers.* This resistance being resolved, in directions parallel to the same [5671']

$$\frac{dx+dx'}{dt}; \qquad \frac{dy+dy'}{dt}; \qquad \frac{dz+dz'}{dt};$$ [5669b]

which evidently represent the velocity of the moon about the sun, resolved in directions parallel to the axes x, y, z. The square root of the sum of the squares of the three partial velocities [5669b], gives the whole velocity [5669]; as is evident from [40a—b]. [5669c]

* (3100) Putting, for brevity,

$$d\text{w} = \sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2},$$ [5671a]

we find, that the absolute velocity of the moon is $\frac{d\text{w}}{dt}$ [5669]; consequently, the resistance is $-K.\frac{d\text{w}^2}{dt^2}$ [5670], in the direction of the described arc $d\text{w}$. The negative sign being prefixed, because the resistance tends to decrease this arc. To resolve this force, in directions parallel to the axes x, y, z, we must multiply it by the expressions, [5671b]

$$\frac{dx'+dx}{d\text{w}}; \qquad \frac{dy'+dy}{d\text{w}}; \qquad \frac{dz'+dz}{d\text{w}};$$ [5671c]

respectively; as is apparent from [40b]. Hence we obtain the expressions [5671].

axes, gives the three following forces.*

Resistance of the earth.

$$-K'.\frac{dx'}{dt^2}\cdot\sqrt{dx'^2+dy'^2+dz'^2}\,; \qquad 1$$

[5672] $$-K'.\frac{dy'}{dt^2}\cdot\sqrt{dx'^2+dy'^2+dz'^2}\,; \qquad 2$$

$$-K'.\frac{dz'}{dt^2}\cdot\sqrt{dx'^2+dy'^2+dz'^2}\,; \qquad 3$$

K' being a coefficient, which differs from K, and depends upon the resistance which the earth suffers. Now having represented the forces, which act upon the moon, parallel to the axes of x, y, and z, by,

[5672'] $$\left(\frac{dQ}{dx}\right),\quad\left(\frac{dQ}{dy}\right),\quad\left(\frac{dQ}{dz}\right),\quad [498b—499a],$$

we shall have, by noticing only the preceding forces,

Relative forces on the moon, considered as moving about the earth at rest.

[5673]

$$\left(\frac{dQ}{dx}\right)=K'.\frac{dx'}{dt^2}\cdot\sqrt{dx'^2+dy'^2+dz'^2} \qquad 1$$

$$-K.\frac{(dx'+dx)}{dt^2}\cdot\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2}\,; \qquad 2$$

$$\left(\frac{dQ}{dy}\right)=K'.\frac{dy'}{dt^2}\cdot\sqrt{dx'^2+dy'^2+dz'^2} \qquad 3$$

$$-K.\frac{(dy'+dy)}{dt^2}\cdot\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2}\,; \qquad 4$$

$$\left(\frac{dQ}{dz}\right)=K'.\frac{dz'}{dt^2}\cdot\sqrt{dx'^2+dy'^2+dz'^2} \qquad 5$$

$$-K.\frac{(dz'+dz)}{dt^2}\cdot\sqrt{(dx'+dx)^2+(dy'+dy)^2+dz'+dz)^2}. \qquad 6$$

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* (3101) The resistances [5671] corresponding to the moon, will evidently give those relative to the earth, by taking the co-ordinates, so as to correspond to the earth, and changing
[5673a] the factor K into K'. This requires that we should put $x=0$, $y=0$, $z=0$; in [5671]. Hence we obtain the forces relative to the earth, as in [5672]. The signs of
[5673b] the forces [5672], must be changed, as in [5671'], and then they must be added to the corresponding quantities in [5671], to obtain the forces of resistance of the ether, supposing the moon to revolve about the earth considered as at rest. These forces are represented, in
[5673c] [498a'—499a], by $\left(\frac{dQ}{dx}\right)$, $\left(\frac{dQ}{dy}\right)$, $\left(\frac{dQ}{dz}\right)$; hence, we easily obtain the expressions [5673].

Now we have, by supposing the moon's co-ordinates only to be variable,

$$dQ = dx \cdot \left(\frac{dQ}{dx}\right) + dy \cdot \left(\frac{dQ}{dy}\right) + dz \cdot \left(\frac{dQ}{dz}\right). \qquad [5674]$$

If we substitute the values,

$$x = \frac{\cos.v}{u}; \qquad y = \frac{\sin.v}{u}; \qquad z = \frac{s}{u}; \qquad [5674']$$

which are given in [4777—4779], we shall obtain,*

$$\begin{aligned} dQ = & -\frac{du}{u^2} \cdot \left\{ \cos.v \cdot \left(\frac{dQ}{dx}\right) + \sin.v \cdot \left(\frac{dQ}{dy}\right) + s \cdot \left(\frac{dQ}{dz}\right) \right\} && 1 \\ & -\frac{dv}{u} \cdot \left\{ \sin.v \cdot \left(\frac{dQ}{dx}\right) - \cos.v \cdot \left(\frac{dQ}{dy}\right) \right\} && 2 \qquad [5675] \\ & +\frac{ds}{u} \cdot \left(\frac{dQ}{dz}\right). && 3 \end{aligned}$$

Then we have,†

$$dQ = \left(\frac{dQ}{du}\right) \cdot du + \left(\frac{dQ}{dv}\right) \cdot dv + \left(\frac{dQ}{ds}\right) \cdot ds; \qquad [5676]$$

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* (3102) The expressions of Q [4756, 5673], may be considered as functions of x, y, z, x', y', z'; but if we suppose the moon's co-ordinates x, y, z, to be the only variable quantities, we shall get for dQ the expression [5674]. Now, the differentials of x, y, z [5674'] give, [5675a]

$$dx = \frac{-dv.\sin.v}{u} - \frac{du.\cos.v}{u^2}; \quad dy = \frac{dv.\cos.v}{u} - \frac{du\,\sin.v}{u^2}; \quad dz = \frac{ds}{u} - \frac{sdu}{u^2}. \qquad [5675b]$$

Substituting these in [5674], and connecting the terms depending on du, dv, ds, we get [5675].

† (3103) Considering the co-ordinates of the moon as the only variable quantities, we shall have the two expressions of dQ [5674, 5676]. In the first of these expressions, the moon's co-ordinates are x, y, z, and in the second u, v, s; and if we substitute, in the first, the values of dx, dy, dz [5675b], it becomes equal to the second, and, by this substitution, produces the function [5675]. Hence it evidently follows, that the expressions [5675, 5676] must be *equivalent*. Now, by comparing together the coefficients of du, dv, ds, in these two last expressions of dQ, we get the equations [5677—5679]. [5676a] [5676b]
Multiplying [5677], by -1, and [5679], by $-\frac{s}{u}$; then, taking the sum of the two products, we get [5680]. [5676c]

and, by comparing these two values of Q, we shall obtain,

[5677] $$\left(\frac{dQ}{du}\right)=-\frac{1}{u^2}\cdot\left\{\cos.v\cdot\left(\frac{dQ}{dx}\right)+\sin.v\cdot\left(\frac{dQ}{dy}\right)+s\cdot\left(\frac{dQ}{dz}\right)\right\};$$

[5678] $$\left(\frac{dQ}{dv}\right)=-\frac{1}{u}\cdot\left\{\sin.v\cdot\left(\frac{dQ}{dx}\right)-\cos.v\cdot\left(\frac{dQ}{dy}\right)\right\};$$

[5679] $$\left(\frac{dQ}{ds}\right)=\frac{1}{u}\cdot\left(\frac{dQ}{dz}\right).$$

Hence we deduce,

[5680] $$-\left(\frac{dQ}{du}\right)-\frac{s}{u}\cdot\left(\frac{dQ}{ds}\right)=\frac{1}{u^2}\cdot\left\{\cos.v\cdot\left(\frac{dQ}{dx}\right)+\sin.v\cdot\left(\frac{dQ}{dy}\right)\right\}\quad[5676c].$$

Now we have, as in [4777f—h],

[5681] $$x'=\frac{\cos.v'}{u'};\qquad y'=\frac{\sin.v'}{u'};\qquad z'=\frac{s'}{u'};$$

v' *denoting the longitude of the earth seen from the sun.* If we take for a
[5682] fixed plane, that of the ecliptic in 1750, we may suppose $s'=0$. We shall represent by $r'dq'$, the small arc which is described by the earth in the time dt, and we shall have,

[5683] $$\sqrt{(dx'^2+dy'^2+dz'^2)}=r'dq'\quad[40a,\&c.].$$

This arc is to that which is described by the moon, in its relative motion
[5684] about the earth, very nearly in the ratio of* $\frac{a'm}{a}$ to 1; therefore, it is at
[5484′] least, thirty times as great; and we shall have, very nearly,†

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* (3104) Whilst the moon describes the angle dv, with the radius a, the sun
[5684a] describes the angle mdv, with the radius a', nearly; as is evident from [4837], [4838, &c.]; so that the space described by the moon is adv, and the space
[5684b] described by the sun $a'mdv$, nearly. The ratio of the second of these expressions,
[5684c] to the first, is denoted by $\frac{a'm}{a}$, as in [5684]. Substituting $\frac{a'}{a}=400$, $m=0{,}0748$, [5221, 5117], it becomes nearly equal to 30, as in [5684′].

† (3105) Developing the terms in the first member of [5685], and substituting $r'dq'$ [5683], it becomes,

[5685a] $$\sqrt{\{r'^2dq'^2+2dx'dx+2dy'dy+2dz'dz+dx^2+dy^2+dz^2\}}.$$

$$\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2} = r'dq' + \frac{dx'.dx}{r'dq'} + \frac{dy'.dy}{r'dq'}.$$ [5685]

If we neglect the excentricity of the earth's orbit, we shall have $dq' = mdt$; the time t being represented by the moon's mean motion. Then we have,* [5686]

$$\frac{dx'}{r'dq'} = -\sin.v'; \qquad \frac{dy'}{r'dq'} = \cos.v';$$ [5687]

consequently,

$$\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2} = ma'.dt - dx.\sin.v' + dy.\cos.v'.$$ [5688]

Hence, we easily obtain,†

Putting now $s'=0$ [5682], we have $z'=0$ [5681]; substituting this, and neglecting also $dx^2+dy^2+dz^2$, in comparison with the other terms, we easily reduce it to the form in the second member of [5685]. [5685b]

* (3106) Neglecting the excentricity of the earth's orbit, we may put $r'=a'=\frac{1}{u'}$ [4937n], and the described arc [5683] becomes $r'dq'=a'dv'$; moreover, the values of x', y' [5681], become $x'=a'.\cos.v'$, $y'=a'.\sin.v'$; whose differentials are $dx'=-a'dv'.\sin v'$, $dy'=a'dv'.\cos.v'$. Dividing these by the above expression $r'dq'=a'dv'$, we obtain the values [5687]; substituting these, and $dq'=mdt$ [5686], in [5685], we get [5688]. The expressions [5683, 5688] may be put under the forms [5687e,f], by merely changing, as above, r' into $\frac{1}{u'}$, and dq', or dv' into mdt. Lastly, the expressions of $r'dq'$, dx', dy' [5687b, c], may be put under the forms [5687g], which will be of use hereafter; [5687a] [5687b] [5687c] [5687d]

$$\sqrt{dx'^2+dy'^2+dz'^2} = \frac{mdt}{u'};$$ [5687e]

$$\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2} = \frac{mdt}{u'} - dx.\sin.v' + dy.\cos.v';$$ [5687f]

$$r'dq' = \frac{mdt}{u'}; \qquad dx' = -\frac{mdt.\sin.v'}{u'}; \qquad dy' = \frac{mdt.\cos.v'}{u'}.$$ [5687g]

† (3107) Multiplying [5687e] by the value of $\frac{K'dr'}{dt^2}$ [5687g], we get [5687h];

[5689] $$\left(\frac{dQ}{dx}\right)=\frac{-(K'-K).m^2.\sin.v'}{u'^2}-\frac{3K.m}{2u'}.\frac{dx}{dt}+\frac{Km}{2u'}.\frac{dx}{dt}.\cos.2v'+\frac{Km}{2u'}.\frac{dy}{dt}.\sin.2v';$$

[5690] $$\left(\frac{dQ}{dy}\right)=\frac{(K'-K).m^2.\cos.v'}{u'^2}-\frac{3K.m}{2u'}.\frac{dy}{dt}+\frac{Km}{2u'}.\frac{dx}{dt}.\sin.2v'-\frac{Km}{2u'}.\frac{dy}{dt}.\cos.2v';$$

[5691] $$\left(\frac{dQ}{dz}\right)=-\frac{Km}{u'}.\frac{dz}{dt}.$$

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again, multiplying [5687f] successively by $-\frac{Kdx'}{dt^2}$, $-\frac{Kdx}{dt^2}$, and neglecting, in the last product, the terms of the order $dx\,.\,dy$, we get [5687i, k];

[5687h] $$K'.\frac{dx'}{dt^2}.\sqrt{dx'^2+dy'^2+dz'^2}=-\frac{K'.m^2.\sin.v'}{u'^2};$$

[5687i] $$-K.\frac{dx'}{dt^2}.\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2}=\frac{Km^2.\sin.v'}{u'^2}-\frac{Km}{u'}.\frac{dx}{dt}.\sin.^2v'+\frac{Km}{u'}.\frac{dy}{dt}.\sin.v'.\cos.v';$$

[5687k] $$-K.\frac{dx}{dt^2}.\sqrt{(dx'+dx)^2+(dy'+dy)^2+(dz'+dz)^2}=-\frac{Km}{u'}.\frac{dx}{dt}.$$

Adding together the expressions [5687h, i, k], we find, that the first member of the sum
[5687l] becomes the same as the expression of $\left(\frac{dQ}{dx}\right)$ [5673 lines 1, 2]; and the second member
[5687m] of this sum is easily reduced to the form [5689], by substituting $\sin.^2v'=\frac{1}{2}-\frac{1}{2}.\cos.2v'$, $\sin.v'.\cos.v'=\frac{1}{2}.\sin.2v'$, and making a slight reduction.

We may obtain [5690], from [5673 lines 3, 4], by a similar process; or, we may find it more readily by derivation. For, if we change, reciprocally, x into y, and x' into
[5687n] y', we shall find, that $\left(\frac{dQ}{dx}\right)$ [5673 lines 1, 2] changes into $\left(\frac{dQ}{dy}\right)$ [5673 lines 3, 4]. Now, this change in the values of x', y', is made by putting $\sin.v'$ for $\cos.v'$, and $\cos.v'$ for $\sin.v'$, in [5681]. This does not alter the value of $\sin.v'.\cos.v'=\frac{1}{2}.\sin.2v'$
[5687o] [5687i], but it changes $\sin.^2v'$ [5687i] into $\cos.^2v'=\frac{1}{2}+\frac{1}{2}.\cos.2v'$; so that we must write $+\frac{1}{2}.\cos.2v'$, instead of $-\frac{1}{2}.\cos.2v'$, in [5687m]. Hence it appears, that we may
[5687p] obtain [5690] from [5689], by writing dx for dy, reciprocally, also $\cos.v'$ for $\sin.v'$, and changing the sign of $\cos.2v'$.

Lastly, if we substitute $z'=0$ [5685b], in [5673 lines 5, 6], we find, that the term
[5687q] in [5673 line 5] vanishes, and the factor of the radical in [5673 line 6] becomes $-K.\frac{dz}{dt^2}$.
[5687r] Multiplying this by the value of the radical [5687f], and neglecting terms of the second power in dx, dy, dz, we get [5691].

If we substitute the values of x, y, and neglect the square of the excentricity of the moon's orbit, we shall get,*

$$-\left(\frac{dQ}{du}\right)-\frac{s}{u}\cdot\left(\frac{dQ}{ds}\right)=\frac{(K'-K)\,m^2.\sin.(v-v')}{u^2.\,u'^2}+\frac{3Km.du}{2u^4.u'.dt} \quad 1$$

[5692]

$$-\frac{Km}{2u^3.u'}\cdot\frac{dv}{dt}\cdot\sin.(2v-2v')-\frac{Km}{2u^4.u'}\cdot\frac{du}{dt}\cdot\cos.(2v-2v'). \quad 2$$

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* (3108) Multiplying [5689] by $\cos.v$, also [5690] by $\sin.v$, and reducing the sum of these products, by means of the formulas [5692b,c], which are deduced from [22, 24] Int., we get the equation [5692d]. In like manner, we may obtain [5692e]; or, it may be more simply derived from [5692d], by changing v into $v+90^d$, where it explicitly occurs, in both members; [5692a]

$$-\sin.v'.\cos.v+\cos.v'.\sin.v=\sin.(v-v')\,; \qquad [5692b]$$

$$\cos.2v'.\cos.v+\sin.2v'.\sin.v=\cos.(v-2v')\,;$$

[5692c]

$$\sin.2v'.\cos.v-\cos.2v'.\sin.v=-\sin.(v-2v')\,;$$

$$\cos.v.\left(\frac{dQ}{dx}\right)+\sin.v.\left(\frac{dQ}{dy}\right)=\frac{(K'-K).m^2.\sin.(v-v')}{u'^2}-\frac{3Km}{2u'}\cdot\left\{\frac{dx}{dt}.\cos.v+\frac{dy}{dt}.\sin.v.\right\}$$

[5692d]

$$+\frac{Km}{2u'}\cdot\left\{\frac{dx}{dt}.\cos.(v-2v')-\frac{dy}{dt}.\sin.(v-2v')\right\};$$

$$-\left\{\sin.v.\left(\frac{dQ}{dx}\right)-\cos.v.\left(\frac{dQ}{dy}\right)\right\}=\frac{(K'-K).m^2.\cos.(v-v')}{u'^2}-\frac{3Km}{2u'}\cdot\left\{-\frac{dx}{dt}.\sin.v+\frac{dy}{dt}.\cos.v\right\}$$

[5692e]

$$+\frac{Km}{2u'}\cdot\left\{-\frac{dx}{dt}.\sin.(v-2v')-\frac{dy}{dt}.\cos.(v-2v')\right\}.$$

We must substitute, in [5692d,e], the values of dx, dy [5675b]; and, in performing this operation, we may use the following theorems, supposing W to be any angle whatever;

$$dx.\cos.W+dy.\sin.W=-\frac{dv}{u}.\sin.(v-W)-\frac{du}{u^2}.\cos.(v-W)\,; \qquad [5692f]$$

$$dx.\sin.W-dy.\cos.W=-\frac{dv}{u}.\cos.(v-W)+\frac{du}{u^2}.\sin.(v-W). \qquad [5692g]$$

The equation [5692f] may be easily proved to be correct, by substituting, in the first member, the values of dx, dy [5675b], and developing the second member, by means of [22, 24] Int. The equation [5692g] may be found in the same manner; or, it may [5692h]

[5693]
$$\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}=\frac{(K'-K).m^2.dv}{u'^2.u^3}.\cos.(v-v')-\frac{3Km}{2u'.u^4}.dv.\frac{dv}{dt}$$
$$-\frac{Km}{2u'.u^4}.dv.\frac{dv}{dt}.\cos.(2v-2v')+\frac{Km}{2u'.u^5}.dv.\frac{du}{dt}.\sin.(2v-2v');$$

[5694]
$$\left(\frac{dQ}{dv}\right)\cdot\frac{du}{u^2.dv}=\frac{(K'-K).m^2}{u'^2.u^3}\cdot\frac{du}{dt}.\cos.(v-v')-\frac{3Km}{2u'.u^4}\cdot\frac{du}{dt}$$
$$-\frac{Km}{2u'.u^4}\cdot\frac{du}{dt}.\cos.(2v-2v').$$

Density of the ether, supposed to be represented by a function of the distance from the sun.

The value of K is not constant. If we suppose the density of the ether to be proportional to a function of the distance from the sun, and denote this function by,

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be more easily derived from [5692f], by changing the arbitrary angle W into $W-90^d$. If we now put $W=v$, in [5692f, g], we shall get the two equations [5692i, k]; and, if we put $W=-(v-2v')$, or $v-W=2v-2v'$, we shall get [5692l, m], respectively, making some slight reductions;

[5692i] $$dx.\cos.v+dy.\sin.v=-\frac{du}{u^2};$$

[5692k] $$dx.\sin.v-dy.\cos.v=-\frac{dv}{u};$$

[5692l] $$dx.\cos.(v-2v')-dy.\sin.(v-2v')=-\frac{dv}{u}.\sin.(2v-2v')-\frac{du}{u^2}.\cos.(2v-2v');$$

[5692m] $$-dx.\sin.(v-2v')-dy.\cos.(v-2v')=-\frac{dv}{u}.\cos.(2v-2v')+\frac{du}{u^2}.\sin.(2v-2v').$$

Substituting the expressions [5692i, l], in [5692d], and then, the result in [5680], we get [5692]. In like manner, if we substitute [5692k, m], in [5692e], and then, the result in
[5692n] [5678], we get, by multiplying by $\frac{dv}{u^2}$, the expression [5693]. Lastly, multiplying [5693] by $\frac{du}{dv^2}$, we get [5694]; observing, that the term of this expression, having the factor $\frac{du}{dv}\cdot\frac{du}{dt}$, may be neglected, as a quantity of the order e^2. For, $\frac{du}{dv}$ is of the
[5692o] order e [4826]; and the same may be observed of $\frac{du}{dt}$, which is evidently of the same order as $\frac{du}{dv}$ [5686].

$$\varphi(u') = \text{density of the ether near the earth ;}$$ [5695]

it will become, for the moon, in which u' changes into $u'-\frac{u'^2}{u}.\cos.(v-v')$,* [5696]

$$\varphi(u')-\frac{u'^2}{u}.\varphi'(u').\cos.(v-v') = \text{density of the ether near the moon ;}$$ [5697]

$\varphi'(u')$ *being the differential of* $\varphi(u')$, *divided by* du' ; so that we may suppose, [5698]

$$K = H.\varphi(u')-\frac{H.u'^2}{u}.\varphi'(u').\cos.(v-v').$$ [5699]

This being premised, if we neglect those periodical inequalities, which do

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* (3109) Substituting $s'=0$ [5682], in the expression of the distance r' of the earth from the sun [4777e], it becomes $u'=\frac{1}{r'}$. If the quantities r', u', corresponding to the earth, be increased by $\delta r'$, $\delta u'$, for the moon; we shall have, by taking the variation of the preceding expression, [5696a]

$$\delta u' = -\frac{1}{r'^2}.\delta r' = -u'^2.\delta r'.$$ [5696b]

The radius vector r, drawn from the earth to the moon, makes, with the continuation of the radius r', an angle which is represented by $v-v'$; and, it is evident, on account of the great distance of the sun, in comparison with that of the moon, that the moon's distance from the sun must exceed that of the earth, by the quantity $r.\cos.(v-v')$, nearly ; hence, $\delta r' = r.\cos.(v-v')$. Substituting this, in [5696b], and putting $r=\frac{1}{u}$, nearly [4776], we get, [5696c] [5696d]

$$\delta u' = -\frac{u'^2}{u}.\cos.(v-v'), \quad \text{as in [5696]}.$$ [5696e]

Now, the function $\varphi(u')$ [5695], corresponding to the earth, changes into $\varphi(u'+\delta u')$, for the moon; and, if we develop it, according to the powers of $\delta u'$, by Taylor's theorem [617], neglecting the square and higher powers of $\delta u'$, it becomes,

$$\varphi(u')+\delta u'.\varphi'(u').$$ [5696f]

Substituting the value of $\delta u'$ [5696e], it becomes as in [5697]. Lastly, multiplying this by the constant quantity H, we get the expression of the resistance [5699]. Encke, in making the calculation of the orbit of the comet [5667b], supposed the function $\varphi(u')$, or, $\varphi\left(\frac{1}{r'}\right)$, to be represented by $\varphi\left(\frac{1}{r'}\right) = \frac{1}{r'^2}$. [5696g]

not depend on the sine, or cosine, of $cv-\varpi$, we shall have,*

[5700] $$\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}=\frac{H.m^2.dv}{2u^4}\cdot\varphi'(u')-\frac{3Hm}{2u'.u^4}\cdot\varphi(u')\cdot dv\cdot\frac{dv}{dt}.$$

If we substitute the values,†

[5701] $$u=\frac{1}{a}.\{1+e.\cos.(cv-\varpi)\};\qquad dt=dv.\{1-2e.\cos.(cv-\varpi)\};$$

we shall obtain,

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* (3110) Substituting the value of K [5699], in [5693], and neglecting the terms which depend on the sine or cosine of $v-v'$, or its multiples, we get [5700]. For, the

[5700a] first term of [5699] $H.\varphi(u')$, being combined with the second term of [5693], produces the second of [5700]; and, the second term of [5699],

[5700b] $$-\frac{Hu'^2}{u}\cdot\varphi'(u')\cdot\cos.(v-v'),$$

being substituted for K, in the first term of [5693], produces,

[5700c] $$\frac{H\,m^2.dv}{u^4}.\varphi'(u').\cos.^2(v-v')=\frac{Hm^2.dv}{u^4}.\varphi'(u').\{\tfrac{1}{2}+\tfrac{1}{2}.\cos.2.(v'-v)\};$$

which gives the first term of [5700].

† (3111) If we neglect the second and higher powers of e, we get, from [4826],

[5701a] the expression of u [5701]. Moreover, the mean motion of the moon being represented by t [5686], we get, from [4828] $n=1$; and then,

[5701b] $$t+\varepsilon=v-\frac{2e}{c}\cdot\sin.(cv-\varpi);$$

whose differential is the same as the value of dt [5701]. These values of u, dt [5701] give,

[5701c] $$\frac{1}{u^4}=a^4.\{1-4e.\cos.(cv-\varpi)\};\qquad dv\cdot\frac{dv}{dt}=dv.\{1+2e.\cos.(cv-\varpi)\}.$$

Substituting these, in the second member of [5700], it becomes,

[5701d] $$\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}=\tfrac{1}{2}H.m^2a^4.dv.\varphi'(u').\{1-4e.\cos.(cv-\varpi)\}-\frac{3Hm}{2u'}.a^4.dv.\varphi(u').\{1-2e.\cos.(cv-\varpi)\}$$
$$=-\tfrac{1}{2}H.ma^4.dv.\left\{\frac{3\varphi(u')}{u'}-m.\varphi'(u')\right\}+Hma^4.\left\{\frac{3}{u'}.\varphi(u')-2m.\varphi'(u')\right\}.edv.\cos(cv-\varpi).$$

Integrating this, we get [5702].

$$f\left(\frac{dQ}{dv}\right).\frac{dv}{u^2} = -\tfrac{1}{2}.H.m.a^4.v.\left\{\frac{3\,\varphi(u')}{u'} - m.\varphi'(u')\right\} \qquad 1$$

$$+ H.m.a^4.\left\{\frac{3}{u'}.\varphi(u') - 2m.\varphi'(u')\right\}.e.\sin.(cv-\varpi). \qquad 2$$ [5702]

Then we shall have,*

$$-\left(\frac{dQ}{du}\right) - \frac{s}{u}.\left(\frac{dQ}{ds}\right) = -\tfrac{3}{2}.H.m.a^3.\frac{\varphi(u')}{u'}.e.\sin.(cv-\varpi)\,;$$ [5703]

$$\left(\frac{dQ}{dv}\right).\frac{du}{u^2.dv} = \tfrac{1}{2}.H.m.a^3.\left\{\frac{3\varphi(u')}{u'} - m.\varphi'(u')\right\}.e.\sin.(cv-\varpi).$$ [5704]

Now, if we put,

$$\alpha = H.m.a^3.\left\{\frac{3\varphi(u')}{u'} - m.\varphi'(u')\right\}\,;$$ [5705]

$$\beta = H.m.a^3.\left\{\frac{6\varphi(u')}{u'} - \tfrac{9}{2}m.\varphi'(u')\right\}\,;$$ [5706]

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* (3112) In substituting the value of K [5699], in the second member of [5692], and neglecting the terms depending on the sine or cosine of $v-v'$, or its multiples [5700a], it will be only necessary to retain the term $\frac{3Km.du}{2u^4.u'.dt}$ [5692]. Now, the differential of u [5701], being divided by dt [5701], gives, by neglecting terms of the order e^2, and observing, that $c=1$, nearly; [5703a]

$$\frac{du}{dt} = -\frac{e}{a}.\sin.(cv-\varpi).$$ [5703b]

Hence, the term [5703a] becomes,

$$-\frac{3Km}{2u^4.u'}.\frac{e}{a}.\sin.(cv-\varpi)\,;$$ [5703c]

and, if we substitute $\frac{1}{u^4}=a^4$, nearly; also, the first term of K [5699], namely, $H.\varphi(u')$; we shall get the value of the first member of [5692, or 5703], as in the second member of [5703]. By similar substitutions, we may obtain [5704]; but, it is more easily obtained, by multiplying the differential of [5702], by [5703d]

$$\frac{du}{dv} = -\frac{e}{a}.\sin.(cv-\varpi) \quad [5701]\,;$$ [5703e]

and then, dividing the product by dv; observing, that we need only notice the first line of [5702], because the second line produces terms of the order e^2.

we must add to the second member of the equation [4754], or to the second member of [4961] the following function;*

[5707] $$-\frac{\alpha v}{a_{\prime}}+\beta\cdot\frac{e}{a_{\prime}}\cdot\sin.(cv-\varpi).$$

[5708] The value of $\frac{1}{a}$ [4968] will, by this means, be increased by the quantity†

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* (3113) We have, very nearly,

$$\frac{ddu}{dv^2}+u=\frac{1}{a}\quad[4890,4892d],\quad\text{and}\quad h^2=a_{\prime}\quad[4863];$$

hence,

[5707a] $$\left(\frac{ddu}{dv^2}+u\right)\cdot\frac{2}{h^2}=\frac{2}{a\,a_{\prime}};$$

multiplying this by [5702], we get,

[5707b] $$\left(\frac{ddu}{dv^2}+u\right)\cdot\frac{2}{h^2}\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}=-H.m.a^3.\left\{\frac{3\varphi(u')}{u'}-m.\varphi'(u')\right\}\cdot\frac{v}{a_{\prime}}$$
$$+H.m.a^3.\left\{\frac{6\varphi(u')}{u'}-4m.\varphi'(u')\right\}\cdot\frac{e}{a_{\prime}}.\sin.(cv-\varpi).$$

[5707c] Now, dividing the sum of the expressions [5703, 5704] by h^2, or $a_{\prime}$, and adding the quotient to [5707b], we find, that the sum becomes,

[5707d] $$-H.m.a^3.\left\{\frac{3\varphi(u')}{u'}-m.\varphi'(u')\right\}\cdot\frac{v}{a_{\prime}}+H.m.a^3.\left\{\frac{6\varphi(u')}{u'}-\tfrac{9}{2}.m.\varphi'(u')\right\}\cdot\frac{e}{a_{\prime}}.\sin.(cv-\varpi).$$

Substituting in this, the abridged symbols α, β [5705, 5706], we get [5707]; which represents the sum of all the terms of [4754], depending on the part of Q now under
[5707e] consideration; as is evident by observing, that the first members of the three expressions mentioned in [5707c] contain all the terms of Q [4754]. If we now connect the function [5707], with the two first terms of [4754], we obtain the following equation, for the determination of u;

[5707f] $$0=\frac{ddu}{dt^2}+u-\frac{\alpha v}{a_{\prime}}+\beta\cdot\frac{e}{a_{\prime}}.\sin.(cv-\varpi).$$

In which we may change $a_{\prime}$ into a [4968].

† (3114) If we put, for a moment, $A=-\frac{\alpha v}{a_{\prime}}$, and neglect the part of [5707f], depending on e, the equation becomes as in [4963a]; whence, we get, as in [4963a, b],

[5708a] $u=-A=\frac{\alpha v}{a_{\prime}}$, for the part of u which corresponds to A. Now we have, very

[5708b] nearly, $u=\frac{1}{a}$ [4937n], whose variation gives $\delta u=-\frac{\delta a}{a^2}$, or $\delta a=-a^2.\delta u$; and,

$\frac{\alpha v}{a_{,}}$; consequently, the value of a will be decreased by $\alpha a_{,}.v$. We shall [5709]
then have, as in [4973], very nearly,*

$$\frac{-2\,d.\frac{e}{a}}{d\,v} + \beta\,.\,\frac{e}{a_{,}} = 0. \qquad [5710]$$

This gives,†

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by substituting, for δu, the value of u [5708a], and putting also $a_{,}=a$, it becomes $\delta a = -\alpha a.v$, as in [5709]. [5708ϵ]

* (3115.) The term $\beta.\frac{e}{a}.\sin.(cv-\varpi)$ [5707f], is to be added to the second member of [4973c]; therefore also, to that of [4973g], which is deduced from [4973c]. Now, it is evident, from [4973h], that the effect of this will be to add to the second member of the equation [4973], the term $\beta.\frac{e}{a_{,}}$, or $\beta.\frac{e}{a}$; without altering [4974]. To find the effect of this additional term of [4973], it is only necessary to notice it, together with the chief term of that equation, [5710a]

$$-2\,.\left(c-\frac{d\varpi}{dv}\right).\frac{d.\left\{c.\frac{(1+ee)}{a}\right\}}{dv}; \qquad [5710b]$$

neglecting the other small term, which depends on $\frac{e}{a}.\frac{dd\varpi}{dv^2}$; observing also, that ϖ [4980], *is deduced from* [4978], *which is not altered*, by the introduction of the terms [5707]. Moreover, it follows, from [4978a, 5228q], that $c-\frac{d\varpi}{dv}$ is nearly equal to unity. Substituting this in [5710b], and neglecting the terms of the order e^3, it becomes $-2\,.\frac{d.\frac{e}{a}}{dv}$; to which we must add the term $\beta.\frac{e}{a}$ [5710a]; and we shall obtain [5710], representing the equation [4973], adapted to the present case. [5710c] [5710d] [5710e]

† (3116). Putting, for a moment, $\frac{e}{a}=x$, and also $a_{,}=a$, we find that [5710] may be put under the form [5711a]

$$-\frac{2dx}{dv}+\beta.x=0, \quad \text{or} \quad \frac{dx}{x}=\tfrac{1}{2}\,\beta dv; \qquad [5711b]$$

whose integral is $\log.\frac{x}{f}=\frac{1}{2}\beta\,v$, f being a constant quantity. Now, βv being very small, we have very nearly $\frac{1}{2}\beta v=\log.(1+\frac{1}{2}\beta v)$ [58] Int.; hence [5711b']

[5711] $$\frac{e}{a} = \text{constant}.\{1+\tfrac{1}{2}\beta v\};$$

consequently,

[5712] $$e = \text{constant}.\{1-(\alpha-\tfrac{1}{2}\beta).v\}.$$

Secular inequalities of the perigee and excentricity, insensible.

The ratio of the excentricity to the semi-major axis is, therefore, subjected to a secular equation, arising from the resistance of the ether; but it is insensible,
[5713] *in comparison with the corresponding acceleration of the moon's mean motion;* because this last acceleration is, as we shall soon show [5714], multiplied by the square of v. *This resistance does not produce any secular equation in*
[5713'] *the motion of the perigee* [5710c].*

[5711c] $\frac{x}{f} = 1+\frac{1}{2}\beta v$; consequently, $x = \frac{e}{a} = f.(1+\frac{1}{2}\beta v)$, as in [5711].

[5711d] Moreover, we have in [5709], $a = \text{constant} \times \{1-\alpha\,.v\}$; substituting this in [5711], after multiplying both members by a, we get [5712]. If we represent the increments of
[5711e] a, e, arising from this cause, by δa, δe, respectively, we shall have, as in [5708c,5712], the following expressions;

[5711f] $$\delta a = -\alpha a\,.\,v; \qquad \delta e = -c\,.\,(\alpha-\tfrac{1}{2}\beta)\,.\,v;$$

c being the constant factor of [5712]. These values will be of use in the next note.

* (3117.) Neglecting terms of the order e^2, γ^2, we have, in [5081p],

[5714a] $$dt = a^{\frac{3}{2}}.\{1-2e.\cos.cv\}.dv,$$

in which cv is used for $cv-\varpi$, for brevity. Supposing this quantity to vary, by augmenting a by δa, and e by δe [5711f], it will be increased by

[5714b] $$\tfrac{3}{2}\,a^{\frac{1}{2}}.\,\delta a\,.\{1-2e.\cos.cv\}\,.\,dv - 2\,a^{\frac{3}{2}}.\,\delta e\,.\cos.\,cv\,.\,dv.$$

Substituting the values [5711f], it becomes, successively,

[5714c] $$\begin{aligned}&-\tfrac{3}{2}\alpha.a^{\frac{3}{2}}.\{1-2e.\cos.cv\}.vdv.+a^{\frac{3}{2}}.c.(2\alpha-\beta).vdv.\cos.cv\\ &=-\tfrac{3}{2}a^{\frac{3}{2}}.\alpha v.dv+a^{\frac{3}{2}}.(5\alpha-\beta).e.vdv.\cos.cv.\end{aligned}$$

The integral of this last expression gives the corresponding increment of $t+\varepsilon$, which will, therefore, be represented by

[5714d] $$-\tfrac{3}{4}\,a^{\frac{3}{2}}.\,\alpha v^2+a^{\frac{3}{2}}.(5\alpha-\beta)\,.\,e\,.\{v.\sin.cv+\cos.cv.\};$$

as is easily proved by differentiation, and putting $c=1$. We can neglect the part depending on $e.\cos.cv$, which may be considered as included in the elliptical motion; then

The expression of dt [5081p] gives, in the value of $t+\varepsilon$, the terms [5714d, &c.],

$$-\tfrac{3}{4}.\alpha v^2+(5\alpha-\beta).ve.\sin.(cv-\varpi). \quad \text{[Increment of } t+\varepsilon\text{]}$$ [5714]

Substituting $t+\varepsilon+2e.\sin.(ct-\varpi)$ for v, we shall obtain, in the expression of v, the secular equation,* [5714′] Secular inequality of the moon's motion.

$$\delta v=\tfrac{3}{4}.\alpha t^2-(2\alpha-\beta).t.e.\sin.(ct-\varpi); \quad \text{[Secular equation in } v\text{]}$$ [5715]

therefore, the resistance of the ether produces in the moon's mean motion, a secular equation, which accelerates that mean motion, without producing any secular variation in the motion of the perigee. [5716]

We may prove, in the same manner, that the resistance of the ether does not produce any sensible secular equation, either in the motion of the nodes, or in the inclination of the lunar orbit to the ecliptic.† [5717] Secular inequalities of the node and inclination are insensible.

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putting $a^{\frac{3}{2}}=n^{-1}=1$ [4827, 5701a], it becomes as in [5714]. This contains the *square* of [5714e] v; but δa, δe [5711f] depend chiefly on its *first* power; hence it is evident, that the secular variation of the mean motion [5714], must be much more sensible than those of [5714f] a and e, as in [5713, &c.].

* (3118). Putting $c=1$, and $\varepsilon=0$, in [5701b], we get $t=v-2e.\sin.(cv-\varpi)$. [5715a] Transposing the last term, and substituting in it t for v, which may be done, if we neglect terms of the order e^2, we get $v=t+2e.\sin.(ct-\varpi)$ [5714′]. Substituting this in the first members of [5715b,c], and neglecting e^2, they become as in the second [5715a'] members of these expressions; their sum gives the value of the function [5714], as in [5715d];

$$-\tfrac{3}{4}.\alpha v^2=-\tfrac{3}{4}\alpha t^2-3\alpha.t.e.\sin.(ct-\varpi); \quad [5715b]$$

$$(5\alpha-\beta).v\,e.\sin.(cv-\varpi)=\quad +(5\alpha-\beta).t.e.\sin.(ct-\varpi); \quad [5715c]$$

$$\text{Sum}=-\tfrac{3}{4}\alpha t^2+(2\alpha-\beta).t.e.\sin.(ct-\varpi). \quad [5715d]$$

This last expression represents the correction of t [5714, 5715a]; and it is evident, that we must change its sign, to get the corresponding correction of v, as in [5715]. [5715e]

† (3119.) In finding the secular motions of γ, θ, depending upon the resistance of the ether, we must proceed with the equation [4755, or 5051b], as we have done with [4754, or 4973c], in finding the secular motions of e, ϖ, [5692—5715]; making the [5717a] necessary changes, to correspond to this case. In examining the reductions of the equation

Hence it follows, that the resistance of the ether can become sensible, in the moon's mean motion only. Ancient and modern observations evidently prove,
[5718] *that the mean motions of the moon's perigee and nodes, are subjected to very sensible secular inequalities. The secular motion of the perigee, deduced from the comparison of ancient and modern observations, is less by eight*
[5719] *or nine sexagesimal minutes, than that which results from the comparison of the observations made in the last century. This phenomenon, of which no doubt can remain, must, therefore, depend upon some other cause than the resistance of the ether. We have seen, in* [4983, &c.], *that it depends on the*
[5720] *variation of the excentricity of the earth's orbit; and, as the secular equations resulting from that variation satisfy, completely, all the ancient and modern observations, we may conclude, that the acceleration, produced by the resistance of an ethereal fluid, on the moon's mean motion, is yet insensible.*

Effect of an ethereal fluid on the moon's mean motion is insensible.

30. *The acceleration, produced by that resistance in the mean motion*
[5721] *of the earth, is much less than the corresponding acceleration in the moon's mean motion.* To prove this, we shall resume the formula [931]; and, if we apply it to the earth, we shall get, in the expression of $\delta v'$, the term,*

[5722]
$$\delta v' = -\frac{3a}{S} \cdot \int\int dv' . \mathrm{d}'Q' ;$$

[4755], we find, that the integral expression, in the first member of [5702], is multiplied, in
[5717b] [4755], by the factor $\frac{dds}{dv^2} + s$, which is of the *third* order in γ [5034a—b]; therefore it may be neglected. Now, it is on this term, that the value of α [5702, 5705]
[5717c] chiefly depends; and α produces also the part of the secular inequality of the mean motion corresponding to the square of the time, which is the most important part of the effect of the resistance of an ethereal fluid [5714f]. Hence it appears, that the remaining
[5717d] terms of Q produce, in like manner as in [5713], only insensible secular inequalities, in comparison with that of the mean motion.

* (3120). The chief term of [931] is,

[5722a]
$$\frac{3a}{\mu} \cdot \int\int n\, d\, t \, . \, \mathrm{d}\, R \, ;$$

which may be reduced to the form [5722], by changing ndt into dv [4828]; $\mathrm{d}R$
[5722b] into $-\mathrm{d}Q$ [5438]; μ into S [914′, 5722′], and then accenting the letters to conform to the present notation; the mass of the earth being neglected, in comparison with that of the sun, in estimating the value of μ.

S being the sun's mass; supposing the sum of the masses of the earth [5722']
and moon to be equal to unity; and, that the quantity Q', in the earth's
motion, corresponds to that which we have denoted by Q, in the moon's [5723]
theory. Moreover, the differential characteristic d' corresponds to the sun's co-ordinates. Then we have,*

$$\mathrm{d}'Q' = \left(\frac{dQ'}{dx'}\right) . dx' + \left(\frac{dQ'}{dy'}\right) . dy' + \left(\frac{dQ'}{dz'}\right) . dz' ; \qquad [5724]$$

$\left(\frac{dQ'}{dx'}\right)$, $\left(\frac{dQ'}{dy'}\right)$, $\left(\frac{dQ'}{dz'}\right)$ being the forces acting upon the earth, parallel [5725]
to the axes x', y', z', by means of the resistance of the ether. If we neglect the excentricity of the earth's orbit, and represent the element of [5726]
the time dt by the differential of the moon's mean motion, we shall have, as in [5672], for these forces, the following expressions;†

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* (3121). The equation [5724] is similar to that in [5674]; and, it is evident, from [5673c], that the quantities

$$\left(\frac{dQ'}{dx'}\right), \qquad \left(\frac{dQ'}{dy'}\right), \qquad \left(\frac{dQ'}{dz'}\right), \qquad [5724a]$$

represent the forces acting upon the earth, parallel to the axes of x', y', z', and arising from the resistance of the ether upon the earth.

† (3122). These forces are represented by the expressions [5672]; and we have, as in [5687b,c, 5681, 5683, &c.], by neglecting the excentricity of the orbit,

$$dx' = -a'dv'.\sin.v' \; ; \qquad dy' = a'dv'.\cos.v' \; ; \qquad dz' = a'ds' \; ;$$
$$\sqrt{dx'^2 + dy'^2 + dz'^2} = r'dq' = a'dv'. \qquad [5727a]$$

Substituting these values in the three expressions [5672], they become respectively,

$$K'.a'^2.\frac{dv'^2}{dt^2}.\sin.v' \; ; \qquad -K'.a'^2.\frac{dv'^2}{dt^2}.\cos.v' ; \qquad -K'.a'^2.\frac{dv'}{dt}.\frac{ds'}{dt}. \qquad [5727b]$$

Now we have, very nearly, $dv' = mdt$ [5687d]; substituting this in [5727b], we get the [5727c]
expressions [5727]; which represent the values of

$$\left(\frac{dQ'}{dx'}\right), \qquad \left(\frac{dQ'}{dy'}\right), \qquad \left(\frac{dQ'}{dz'}\right), \qquad [5727c']$$

respectively. Substituting these in the second member of [5724], and also the values

$$dx' = -a'.mdt.\sin.v' \; ; \qquad dy' = a'.mdt.\cos.v' \; ; \qquad dz' = a'ds' \; ; \qquad [5727d]$$

which are deduced from [5727a,c], we get,

$$\mathrm{d}'Q' = -K'.a'^3.m^3.dt.\left\{ \sin.^2v' + \cos.^2v' + \frac{ds'}{m^2dt}.\frac{ds'}{dt} \right\}. \qquad [5727e]$$

[5727] $$K'.a'^2.m^2.\sin.v' \;; \qquad -K'.a'^2.m^2.\cos.v' \;; \qquad -K'.a'^2.m.\frac{ds'}{dt} \;;$$

[5727'] therefore, by neglecting the square of $\frac{ds'}{dt}$, we shall have,

[5728] $$\mathrm{d}'Q' = -K'.a'^3.m^3.dt \;;$$

which gives,*

[5729] $$\delta v' = \frac{-3a'}{S}.\textstyle\int\int dv'.\mathrm{d}'Q' = \frac{3}{2}.\frac{K'.a'^4.m^4.t^2}{S}.$$

[5730] We must put $K' = H'.\varphi(u')$ [5699]; H' being a constant quantity, depending on the surface, and on the mass of the earth. Hence, *the secular equation, produced by the resistance of the ether, in the mean motion of the earth*, is,

[5731] $$\delta v' = \frac{\frac{3}{2}.H'.a'^4.m^4.t^2.\varphi(u')}{S}.$$ [Secular equation of the earth]

The corresponding acceleration of the moon's mean motion is, by what precedes [5730b],

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Neglecting the term depending on the square of ds' [5727'], and putting $\sin.^2v'+\cos.^2v'=1$, it becomes as in [5728].

* (3123). Substituting, in [5722], the value of $\mathrm{d}'Q'$ [5728], and $dv' = mdt$ [5727c], it becomes, by noticing only the part depending on t^2,

[5730a] $$\frac{3}{S}.K'.a'^4.m^4.\textstyle\int\int dt^2 = \frac{3}{2S}.K'.a'^4.m'^4.t^2, \quad \text{as in [5729]};$$

substituting K' [5730], we get [5731]. The acceleration of the moon's mean motion, depending on t^2, is $\frac{3}{4}\alpha t^2$ [5715]; and, by substituting α [5705], it becomes,

[5730b] $$\tfrac{3}{4}.H.a^3.mt^2.\left\{\frac{3\varphi(u')}{u'} - m.\varphi'(u')\right\};$$

which is easily reduced to the form [5732], by using $\frac{1}{u'} = a'$ [4937n]. Again, if we change the sun's mass m' [4757''] into S [5722'], also $\overline{m}^2$ into m^2, nearly [5094], we shall find that the expression [4865] becomes,

[5730c] $$\frac{Sa^3}{a'^3} = m^2, \quad \text{or} \quad S = \frac{a'^3.m^2}{a^3}, \quad \text{as in [5733]}.$$

$$\delta v = \tfrac{3}{4}.H.a^3.a'.mt^2.\left\{3\varphi(u') - \frac{m}{a'}.\varphi'(u')\right\}.$$ [Secular equation of the moon] [5732]

Moreover, we have $\frac{S.a^3}{a'^3} = m^2$ [5730c]; therefore, the acceleration of [5733]
the moon's mean motion, is to the corresponding acceleration of the earth's mean motion, as unity is to,*

$$\frac{2H'.m.\varphi(u')}{H.\left\{3\varphi(u') - \frac{m}{a'}.\varphi'(u')\right\}} = \frac{\text{secular motion of the earth}}{\text{secular motion of the moon}};$$ [5734]

consequently, as unity is to $\tfrac{2}{3}.\frac{H'.m}{H}$; neglecting the term $-\frac{m}{a'}.\varphi'(u')$. It [5735]
is evident, that,†

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* (3124). Dividing the expression of $\delta v'$ [5731], by that of δv [5732], and substituting S [5730c], we get [5734]. If we neglect the term of the denominator of [5734a]
[5734], which is multiplied by the small quantity $\frac{m}{a'}$; we find that the numerator and denominator become divisible by $\varphi(u')$, and the expression changes into $\tfrac{2}{3}.\frac{H'm}{H}$. [5734b]

† (3125) The resistance, which the moon suffers, must evidently be proportional to

$$\frac{\text{square of the moon's semi-diameter}}{\text{mass of the moon}};$$ [5736a]

and that of the earth is proportional to

$$\frac{\text{square of the earth's semi-diameter}}{\text{mass of the earth}}.$$ [5736a']

Now, these quantities are to each other as H to H' [5699, 5730]; hence we get,

$$\frac{H'}{H} = \frac{\text{mass of the moon}}{\text{mass of the earth}} \times \frac{\text{square of the earth's semi-diameter}}{\text{square of the moon's semi-diameter}}.$$ [5736b]

If we take, for the moon's semi-diameter, the angle under which it appears when viewed from the earth, at its mean distance; and, for the earth's semi-diameter, the angle under [5736c]
which it appears when viewed from the moon, or the moon's horizontal parallax; we shall find, that the expression [5736b] becomes as in [5736]. Substituting the values [5737—5738'], we get [5739]. Substituting this, and m [5117], in $\tfrac{2}{3}.\frac{H'.m}{H}$ [5735], [5736d]
it becomes as in [5740].

[5736] $$\frac{H'}{H} = \frac{\text{mass of the moon}}{\text{mass of the earth}} \times \frac{\text{square of the moon's parallax}}{\text{square of the app. semi-diameter of the moon}}.$$

From observation, we get,

[5737] The moon's apparent semi-diameter $= 943^s$;

[5738] The moon's parallax $= 3454^s$;

[5738′] and, in [4631], the moon's mass is $\frac{1}{68,5}$ of that of the earth; therefore, we have,

[5739] $$\frac{H'}{H} = 0,195804.$$

[5740] *Hence it follows, that the acceleration of the earth's mean motion, produced by the resistance of the ether, is equal to the corresponding acceleration of the moon's mean motion, multiplied by* 0,0097642 [5736*d*]; or, *about one hundredth part of the moon's acceleration.*

APPENDIX.

Presented, by the Author, to the Board of Longitude of France : August 17, 1808.*

THE object of this appendix is to render more complete the theory of the perturbations of the planets, which is given in the second and sixth books. In striving to give to the expressions of the elements of the orbits, the most simple forms which they can attain, we have been able to make them depend wholly [5741]

* (3126) This paper was given by the author, as an appendix to the third volume of this work; it was not, however, published, till after the appearance of the fourth volume; which is referred to in several places, as in [5764′, 5975]. The improvement, made by Mr. Poisson, in the demonstration of the permanency of the mean motions, which is treated of in [5794′—5846], was made known to the National Institute of France, in a paper, presented June 20, 1808, and printed in the eighth volume of the *Journal de l'Ecole Polytechnique*. This first demonstration was followed by a much more simple one, given by La Place, in this appendix; and some improvements were afterwards made by him, and published in the fifth volume of the present work [12508, &c.]. La Grange also gave an elegant demonstration, founded upon the principle of the variation of the constant quantities, in the *Mémoires de l'Institut de France*, for 1808, &c.; and in the second edition of his *Mécanique Analytique*. Subsequently, the subject was resumed by Mr. Poisson, in the same volume of the *Journal*, and in the *Mémoires* for 1816, with important improvements; in which he extended the demonstration of his theorem on the mean motions, so as to include terms of the *third* order of the disturbing masses, arising from those of the *second* order in the disturbed planet; and then, by induction, he supposes this will hold good for all powers of the masses, so far as they depend on the elements of the disturbed planet. He also demonstrated this remarkable theorem, 'That the perturbations of the rotatory motion of *a solid body*, of any form, arising from forces of attraction, depend upon

[5741a] [5741b] [5741c] [5741d] [5741e] [5741f] [5741g]

on the partial differentials of a single function [913, 1195, 1258, &c.],*
[5741′] taken relatively to these elements ; and, it is remarkable, that the coefficients of these differentials are functions of the elements themselves. These elements are the six arbitrary quantities of the three differential equations of the second order [915] ; by means of which, the motion of each planet is determined. *Supposing the orbit to be an ellipsis, which is*
[5741″] *variable at every instant, the elements will be represented in the following manner :*

Elements.

First. The semi-major axis, on which the mean motion of the planet depends, a ;

Second. The epoch of the mean longitude, ε ;

Third. The excentricity of the orbit, e ;

[5742] *Fourth.* The longitude of the perihelion, ϖ ;

Fifth. The inclination of the orbit to a fixed plane, φ ;

Sixth. The longitude of its node, θ.

La Grange gave, a long time ago, the above-mentioned form to the differential expression of the greater axis [5786] ; and proved, by means of
[5742′] it, in a very elegant manner, the invariableness of the mean motion, noticing only the first power of the disturbing masses. This invariableness was first discovered by me ; neglecting, however, the terms of the fourth and higher

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[5741h] the same equations as the perturbations of *a single particle* of matter, attracted towards a fixed centre ;' so that, the precession of the equinoxes, and the nutation of the earth's
[5741i] axis, can be expressed by the same formulas as the variations of the elliptical elements of the planets. We had intended to give a particular account of these improvements of
[5741k] La Grange and Poisson, together with some notice of the papers which Mr. Lubbock has published, on the secular and periodical inequalities of the planets, in the Transactions of the Royal Society of London, in 1830, 1831 ; but, we have been induced to postpone this
[5741l] notice, by reason of the great length of the appendix to this volume. We shall, however, resume the subject in the commentary on the fifteenth book.

* (3127) The function here spoken of is R [913]. The differentials of the
[5741m] elements a, ε, e, &c., are given in [1177, 1345, 1258, 1337b, &c.] ; and they are collected together, with improvements, in [5786—5791].

powers of the excentricities and inclinations of the orbits, which is sufficiently accurate for the purposes of astronomy. I have given, in the [5743]
second book [1258,1337, &c.], the same forms to the differential expressions of the excentricity of the orbit, of the inclination, and of the longitude of its node; nothing more is required, than to give the same form to the differential expressions of the longitudes of the epoch and of the perihelion; this I have now done in the present appendix.

The principal advantage of this form of the differential expressions of the elements is, to give their finite variations, by the development of the function, which is denoted by R, in the second book [913, &c]. If we reduce this function into a series of cosines of angles, increasing in proportion to the time [1011, &c.], we shall obtain, by taking the differential of each term, the corresponding terms of the variations of the elements. We have endeavored to satisfy this condition in the second book; but, we can do it in a more
simple and general manner, by means of some new expressions of these [5744]
variations. These last expressions have also the advantage of proving clearly, the beautiful theorem discovered by Mr. Poisson, on the invariableness of the mean motions, noticing the square of the disturbing force. We have proved, in the sixth book, by means of similar expressions, that this uniformity is not altered by the great inequalities of Jupiter and Saturn [3906″], which is the more important, as we have shown in the same book [3910—3912], that these great inequalities have a considerable influence upon the secular variations of the orbits of these two planets. The substitution of the new
formulas which we have just mentioned, shows, that the uniformity of the mean [5745]
motions of the planets is not troubled by any other periodical or secular equation. These expressions give also, the most general and simple solution of the secular variations of the elements of the planetary orbits. Lastly, they give, in a very simple manner, the two inequalities of the moon's motion in longitude, and in latitude [5967, 5971], depending on the oblateness of the earth, which have been determined in the second chapter of the seventh book [5357, 5389]. This confirmation of the results, which have been obtained relative to these inequalities, is interesting, because we can get, by comparing them with observations, the ellipticity of the earth, in as accurate a manner,
to say the least, as by the direct measures; with which they also agree, as [5746]
well as can be expected, considering the irregularities of the earth's surface.

In the theory of the great inequalities of Jupiter and Saturn, which is given in book VII, we have noticed the fifth power of the excentricities and inclinations of the orbits. Mr. Burckhardt has calculated the terms depending on these powers. But, it has been since found, that the inequality resulting from these terms, is taken with a wrong sign. Therefore, we shall correct, at the end of this appendix, the formulas of the motions of Jupiter and
[5747] Saturn, which are given in the eighth chapter of the tenth book. This produces a small alteration in the mean motions, as well as in the epochs of these two planets; and this change satisfies the observation of the conjunction of these two planets, made by Ibn Junis, at Cairo, in the year 1007. This observation varies from the formulas, by a quantity which is much less than the error to which the observation is liable. The ancient observations, quoted by Ptolemy, are equally well represented by these formulas. This agreement proves, that the mean motions of the two greatest planets in the system are now well known, and, that they have not suffered any sensible variation since
[5748] the time of Hipparchus; it guarantees, for a long time, the accuracy of the tables which Mr. Bouvard has constructed, by the theory, and which the Board of Longitude has just published.

In the same meeting at which I presented these investigations to the Board of Longitude, La Grange also communicated his learned researches on the same subject. He has, by a very elegant analysis, expressed the partial differential of R, taken relatively to each element, by a linear function of
[5749] the infinitely small differences of these elements; in which the coefficients of these differences are functions only of the elements themselves. If we determine, by means of these expressions, the differences of each element, we may, by proper reductions, obtain the very simple expressions which we have given; and, as they can thus be deduced from such different methods, their accuracy will thereby be confirmed.

1. We shall resume the expression of ede, given in [1262]; *putting,*
[5750] *for greater simplicity,* $\mu = 1$, we obtain,*

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* (3128) In the equations [1262, 5751], terms of the order of the square of the
[5751a] disturbing forces are neglected [1253a, &c.]; but it is correct in terms of the *first order of the disturbing forces*, for all powers of the excentricities and inclinations. The value
[5751b] $\mu = 1$, being substituted in [541'], gives $n = a^{-\frac{3}{2}}$, which is used in [5785, &c.].

$$ede = a.ndt.\sqrt{1-ee}\,.\left(\frac{dR}{dv}\right) - a.(1-e^2).\mathrm{d}R,$$ [5751]

In this equation, t *is the time*; nt *the mean motion of the planet* m; [5752]
a *the semi-major axis of its orbit*; e *the excentricity*; v *the true longitude* [5753]
of the planet; R *a function of the co-ordinates of the two planets* m, m'; [5754]
so that, by naming these co-ordinates x, y, z, x', y', z', respectively,
we shall have, as in [949, 949′],

$$R = m'.\frac{(xx'+yy'+zz')}{r'^3} - \frac{m'}{\rho};$$ [5755]

ρ *being the distance of the two planets from each other*; so that we shall have,

$$\rho = \sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}};$$ [5756]

r' *is the radius vector of the planet* m'; r *that of the planet* m; *lastly*, [5757]
the characteristic d *refers only to the co-ordinates of the planet* m [916′].

We may observe, that to obtain $\left(\frac{dR}{dv}\right)$, we must develop R in a series
of angles proportional to the time t; then take its differential relative to nt, [5758]
and divide it by ndt, adding to the quotient the partial differential $\left(\frac{dR}{d\varpi}\right)$;
ϖ being the longitude of the perihelion of the orbit of m. For, we must [5759]
not notice, in finding the partial differential of R, relative to v, the angle
nt, introduced into R, by the radius vector r of the planet m, or
by the periodical part of the elliptical expression of v, developed in a [5760]
series of sines of angles, proportional to the time. Now, in these functions
[669], the angle nt is always connected with the angle $-\varpi$, which is [5761]
introduced into R, by this means only; therefore by adding to the partial
differential $\frac{\mathrm{d}R}{ndt}$, the partial differential $\left(\frac{dR}{d\varpi}\right)$, we shall have the value* [5762]

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* (3129) The two first terms of $nt+\varepsilon$ [669], are not connected with $-\varpi$, but,
it is found in all the remaining terms; so that we have $v = nt+\varepsilon+\varphi(nt+\varepsilon-\varpi)$, [5763a]
φ being the characteristic of a function. If, for a moment, we consider R to be a function
of v, as in [3742], and represent it by $R=f(v)$, we shall have, by the usual [5763b]
notation, $\left(\frac{dR}{dv}\right)=f'(v)$. Substituting v [5763a], in R [5763b], we get, [5763c]

$$R = f\{nt+\varepsilon+\varphi(nt+\varepsilon-\varpi)\}.$$

[5763] of $\left(\frac{dR}{dv}\right)$. Hence, the preceding expression of ede, will give,*

[5764] $$de = \frac{a.\sqrt{1-e^2}}{e}.(1-\sqrt{1-e^2}).\mathrm{d}R + \frac{a.\sqrt{1-e^2}}{e}.ndt.\left(\frac{dR}{d\varpi}\right).$$

[5764′] Then we have, as in [7886],†

[5763c′] Its differentials, considering successively, nt and ϖ, as the variable quantities, also putting, for brevity, $nt+\varepsilon-\varpi=\mathrm{w}$, $\left(\frac{d.\varphi(\mathrm{w})}{d\mathrm{w}}\right)=\varphi'(\mathrm{w})$, give,

[5763d] $$\frac{\mathrm{d}R}{ndt}=\{1+\varphi'(nt+\varepsilon-\varpi)\}.f'\{nt+\varepsilon+\varphi(nt+\varepsilon-\varpi)\}.$$

[5763e] $$\left(\frac{dR}{d\varpi}\right)=-\varphi'(nt+\varepsilon-\varpi).f'\{nt+\varepsilon+\varphi(nt+\varepsilon-\varpi)\}.$$

The sum of the two expressions [5763d,e], being successively reduced, by using [5763a,c], becomes as in [5762]; namely,

[5763f] $$\frac{\mathrm{d}R}{ndt}+\left(\frac{dR}{d\varpi}\right)=f'\{nt+\varepsilon+\varphi(nt+\varepsilon-\varpi)\}=f'(v)=\left(\frac{dR}{dv}\right).$$

* (3130) Substituting the value of $\left(\frac{dR}{dv}\right)$ [5763f], in [5751], it becomes,

[5764a] $$ede=a.ndt.\sqrt{1-e^2}.\left\{\frac{\mathrm{d}R}{ndt}+\left(\frac{dR}{d\varpi}\right)\right\}-a.(1-e^2).\mathrm{d}R.$$

Dividing this by e, and making a slight reduction, we obtain [5764].

† (3131) The formula [5765] is the same as that which the author has demonstrated in [7886], in nearly the following manner. The first of the equations [606] becomes, by

[5765a] changing the origin of the time t, as in [668v]; $nt+\varepsilon-\varpi=u-e.\sin.u$; and, if we also change nt into $\int ndt$, as in [5793], we shall get,

[5765b] $$\int n\,dt+\varepsilon-\varpi=u-e.\sin.u.$$

In which $\int ndt+\varepsilon$ is the mean longitude of the planet m; $\int ndt+\varepsilon-\varpi$ its mean

[5765c] anomaly; $v-\varpi$ its true anomaly; and u its excentrical anomaly [603″,&c.,668v,669]. The differential of [5765b], supposing the ellipsis to be invariable, is,

[5765d] $$n\,dt=du.(1-e.\cos.u);$$

[5765e] and, as this must also hold good for the variable ellipsis [1168‴], we may take the general differential of [5765b], supposing all the elements to be variable; subtracting from this, the expression [5765d], we get,

[5765f] $$d\varepsilon-d\varpi=du.(1-e.\cos.u)-de.\sin.u;$$

$$d\varepsilon - d\varpi = -\frac{d\varpi.(1-e.\cos.u)^2}{\sqrt{1-ee}} - \frac{de.\sin.u.(2-e^2-e.\cos.u)}{1-e^2}. \qquad [5765]$$

[5766] In this formula, u *is the excentrical anomaly* [603″—604], and ε *the longitude of the epoch* [669′]. We may put the second member of [5765] under the form,*

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supposing du, *in the second member, to be restricted to the variations arising from* ε, ϖ; instead of referring to the time t, as in [5765*d*]. The third of the equations [606] becomes, by changing the origin of t, as in [5765*a*]; [5765*g*]

$$\text{tang.}\tfrac{1}{2}.(v-\varpi) = \sqrt{\frac{1+e}{1-e}}.\text{tang.}\tfrac{1}{2}u. \qquad [5765h]$$

If we take its differential, supposing ε, ϖ, u, to be the variable quantities; and u, to vary as in [5765*g*]; we shall get, by multiplying by 2,

$$-\frac{d\varpi}{\cos.^2\frac{1}{2}(v-\varpi)} = \frac{du}{\cos.^2\frac{1}{2}u}.\sqrt{\frac{1+e}{1-e}} + \frac{2de.\text{tang.}\frac{1}{2}u}{(1-e).\sqrt{1-e^2}}. \qquad [5765i]$$

Now we have, by using [5765*h*],

$$\frac{1}{\cos.^2\frac{1}{2}.(v-\varpi)} = 1+\text{tang.}^2\tfrac{1}{2}.(v-\varpi) = 1+\frac{1+e}{1-e}.\text{tang.}^2\tfrac{1}{2}u = 1+\text{tang.}^2\tfrac{1}{2}u+\frac{2e}{1-e}.\text{tang.}^2\tfrac{1}{2}u$$
$$= \frac{1}{\cos.^2\frac{1}{2}u} + \frac{2e}{1-e}.\text{tang.}^2\tfrac{1}{2}u. \qquad [5765k]$$

Substituting this in [5765*i*]; then, multiplying by $\cos.^2\frac{1}{2}u$; and reducing, by putting,

$$\cos.^2\tfrac{1}{2}u.\text{tang.}\tfrac{1}{2}u = \cos.\tfrac{1}{2}u.\sin\tfrac{1}{2}u = \tfrac{1}{2}.\sin.u; \quad (\cos.\tfrac{1}{2}u.\text{tang.}\tfrac{1}{2}u)^2 = \sin.^2\tfrac{1}{2}u = \tfrac{1}{2}-\tfrac{1}{2}.\cos.u; \qquad [5765l]$$

we get, $$-d\varpi.\left\{1+\frac{2e}{1-e}.(\tfrac{1}{2}-\tfrac{1}{2}.\cos.u)\right\} = du.\sqrt{\frac{1+e}{1-e}} + \frac{de.\sin.u}{(1-e).\sqrt{1-e^2}}. \qquad [5765m]$$

Multiplying this by $\sqrt{\frac{1-e}{1+e}}$, and reducing, we get,

$$du = -\frac{d\varpi.(1-e.\cos.u)}{\sqrt{1-e^2}} - \frac{de.\sin.u}{1-e^2}. \qquad [5765n]$$

Substituting this value of du, in [5765*f*], we get the expression [5765]; in which nothing is neglected. [5765*o*]

* (3132) We have $-(1-e.\cos.u)^2 = -(1-e^2)+e.(2.\cos.u-e-e.\cos.^2u)$, as is easily proved, by developing its first member. Substituting this in the numerator of the first term of the second member of [5765], it becomes as in [5767]. [5767*a*]

[5767] $$-d\varpi.\sqrt{1-e^2}+\frac{ed\varpi}{\sqrt{1-ee}}.(2.\cos.u-e-e.\cos.^2u)-\frac{de.\sin.u}{1-ee}.(2-e^2-e.\cos.u).$$

[5768] The excentrical anomaly u, is given in terms of the true anomaly $v-\varpi$, by means of the equations [603,606],

[5769] $$r=\frac{a.(1-e^2)}{1+e.\cos.(v-\varpi)}=a.(1-e.\cos.u.);$$

whence we deduce,*

[5770] $$\cos.u=\frac{e+\cos.(v-\varpi)}{1+e.\cos.(v-\varpi)};$$

[5771] $$\sin.u=\frac{\sqrt{1-ee}.\sin.(v-\varpi)}{1+e.\cos.(v-\varpi)};$$

consequently,†

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* (3133) Dividing the two values of r [5769] by a, we get, by successive reductions,

[5770a] $$e.\cos.u=1-\frac{(1-e^2)}{1+e.\cos.(v-\varpi)}=\frac{e.\cos.(v-\varpi)+e^2}{1+e.\cos.(v-\varpi)}.$$

[5770a′] Dividing by e, we obtain [5770]; and if we put for a moment, for brevity, $\cos.(v-\varpi)=\mathrm{w}$, it becomes,

[5770b] $$\cos.u=\frac{e+\mathrm{w}}{1+e\mathrm{w}};$$

whence we obtain,

[5770c] $$\sin.u=\sqrt{(1-\cos.^2u)}=\sqrt{1-\frac{(e+\mathrm{w})^2}{(1+e\mathrm{w})^2}}=\frac{\sqrt{\{(1+e\mathrm{w})^2-(e+\mathrm{w})^2\}}}{1+e\mathrm{w}}=\frac{\sqrt{(1-e^2-\mathrm{w}^2+e^2\mathrm{w}^2)}}{1+e\mathrm{w}}$$
$$=\frac{\sqrt{(1-e^2)}.\sqrt{(1-\mathrm{w}^2)}}{1+e\mathrm{w}}.$$

[5770d] Re-substituting the values of $\mathrm{w}=\cos.(v-\varpi)$, and $\sqrt{(1-\mathrm{w}^2)}=\sin.(v-\varpi)$, it becomes as in [5771].

† (3134) The value of $\cos.u$ [5770b], being substituted in the first member of [5772a], we get, by successive reductions, the expression in its last member. In like manner, from $\sin.u$ [5770c], we get [5772b],

[5772a] $$2.\cos.u-2e=\frac{2e+2\mathrm{w}}{1+e\mathrm{w}}-2e=\frac{2(1-e^2).\mathrm{w}}{1+e\mathrm{w}}=\frac{2(1-e^2).\mathrm{w}.(1+e\mathrm{w})}{(1+e\mathrm{w})^2}$$
$$=\frac{(1-e^2)}{(1+e\mathrm{w})^2}.\{2\mathrm{w}+2e\mathrm{w}^2\};$$

$$\frac{ed\varpi}{\sqrt{1-ee}}.(2.\cos.u-e-e.\cos.^2u)-\frac{de.\sin.u}{1-e^2}.(2-e^2-e.\cos.u) \qquad 1$$

$$=\sqrt{1-e^2}.\frac{\{2.\cos.(v-\varpi)+e+e.\cos.^2(v-\varpi)\}}{\{1+e.\cos.(v-\varpi)\}^2}.ed\varpi \qquad 2 \qquad [5772]$$

$$-\sqrt{1-e^2}.\frac{\{2+e.\cos.(v-\varpi)\}}{\{1+e.\cos.(v-\varpi)\}^2}.de.\sin.(v-\varpi). \qquad 3$$

Substituting the values of $ed\varpi$, de [1258], we find, that the second member of the equation [5772] can be reduced to the following form;*

$$e-e.\cos.^2u=e.\sin.^2u=\frac{(1-e^2)}{(1+e\mathrm{w})^2}.\{e-e\mathrm{w}^2\}. \qquad [5772b]$$

The sum of these two expressions, gives,

$$2.\cos.u-e-e.\cos.^2u=\frac{(1-e^2)}{(1+e\mathrm{w})^2}.\{2\mathrm{w}+e+e\mathrm{w}^2\}. \qquad [5772c]$$

Substituting this in the term which is connected with $d\varpi$, in the first member of [5772], get the term depending on $d\varpi$, in its second member [5772, line 2]. In a similar manner, we get, from the value of $\cos.u$ [5770b],

$$e^2-e.\cos.u=e.(e-\cos.u)=e.\left(e-\frac{(e+\mathrm{w})}{1+e\mathrm{w}}\right)=e.\left(\frac{ee\mathrm{w}-\mathrm{w}}{1+e\mathrm{w}}\right)$$

$$=-\frac{e\mathrm{w}}{1+e\mathrm{w}}.(1-e^2). \qquad [5772d]$$

Adding to the first, and to the last members of this expression, the quantity $2.(1-e^2)$; we obtain,

$$2-e^2-e.\cos.u=2.(1-e^2)-\frac{e\mathrm{w}}{1+e\mathrm{w}}.(1-e^2)=\frac{(1-e^2)}{1+e\mathrm{w}}.\{2.(1+e\mathrm{w})-e\mathrm{w}\}$$

$$=\frac{(1-e^2)}{1+e\mathrm{w}}.\{2+e\mathrm{w}\}. \qquad [5772e]$$

Hence $\frac{2-e^2-e.\cos.u}{1-e^2}=\frac{2+e\mathrm{w}}{1+e\mathrm{w}}$; multiplying these by $\sin.u=\frac{\sqrt{1-e^2}.\sin.(v-\varpi)}{1+e\mathrm{w}}$ [5772f] [5771]; and substituting the result in the term depending on de [5772 line 1], we get the corresponding term of the second member [5772 line 3].

* (3135) If we substitute $\mu=1$ [5750], in [1258], we shall obtain the following expressions of $ed\varpi$, de, in which terms of the order of the square of the disturbing forces are neglected [1253']; [5773a]

[5773] $$2a.ndt.r.\left(\frac{dR}{dr}\right);$$ [Value of the function 5772]

[5774] and, as we have $r.\left(\frac{dR}{dr}\right)=a.\left(\frac{dR}{da}\right)$ [962], it becomes,

[5775] $$2a^2.ndt..\left(\frac{dR}{da}\right).$$ [Value of the function 5772]

Hence, the expression of $d\varepsilon-d\varpi$ [5765] gives the following very simple

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[5773b] $$ed\varpi=-\frac{a.ndt}{\sqrt{1-e^2}}.\sin.(v-\varpi).\{2+e.\cos.(v-\varpi)\}.\left(\frac{dR}{dv}\right)+a^2.ndt.\sqrt{1-e^2}.\cos.(v-\varpi).\left(\frac{dR}{dr}\right);$$

[5773c] $$de=-\frac{a.ndt}{\sqrt{1-e^2}}.\{2.\cos.(v-\varpi)+e+e.\cos^2(v-\varpi)\}.\left(\frac{dR}{dv}\right)-a^2.ndt.\sqrt{1-e^2}.\sin.(v-\varpi).\left(\frac{dR}{dr}\right).$$

These are to be substituted in [5772 lines 2, 3]; and, in performing the operation, we may neglect the part depending on $\left(\frac{dR}{dv}\right)$; because, the terms depending on $ed\varpi$
[5773d] [5772 line 2, 5773b], are equal to those depending on de [5772 line 3, 5773c]; and, they have different signs; so that they mutually destroy each other; as is easily seen by the mere inspection of the formulas. The remaining part of the second member of
[5773e] [5772], arising from the substitution of the parts of [5773b, c], depending on $\left(\frac{dR}{dr}\right)$, becomes, without any reduction, as in [5573f]; omitting, for the sake of brevity, the symbol ϖ, which is connected with the angle $v-\varpi$, as in [4821f];

[5773f] $$\frac{(1-e^2).a^2.ndt}{(1+e.\cos.v)^2}.\left(\frac{dR}{dr}\right).\{(2.\cos.v+e+e.\cos.^2v).\cos.v+(2+e.\cos.v).\sin.^2v\}.$$

The terms of the factor, between the braces, being arranged according to the powers of e, and then successively reduced, become,

[5773g] $$2.(\cos.^2v+\sin.^2v)+e.\cos.v.\{1+(\cos.^2v+\sin.^2v)\}=2+e.\cos.v.\{1+1\}=2.\{1+e.\cos.v\}.$$

Substituting this last expression in [5773f], it becomes,

[5773h] $$\frac{2.(1-e^2)}{1+e.\cos.v}.a^2.ndt.\left(\frac{dR}{dr}\right);$$

which is easily reduced to the form [5773], by the substitution of,

[5773i] $$r=\frac{a.(1-e^2)}{1+e.\cos.v}\quad[603].$$

Lastly, the substitution of [5774], in [5773], gives [5775], for the value of the second member of the equation [5772].

equation, which was first discovered by Mr. Poisson ;*

$$d\varepsilon = d\varpi\,.\,(1-\sqrt{1-ee})+2a^2.\left(\frac{dR}{da}\right).ndt.$$ [5775′]

Poisson's expression of $d\varepsilon$.

If we refer, as in [1030′, &c.], *the motion of the planet* m, *to that of its primitive orbit, and put, as in* [1032],

$$p = \text{tang.}\varphi\,.\,\sin.\theta\ ; \qquad q = \text{tang.}\varphi\,.\,\cos.\theta\ ;$$ [5776]

φ *being the inclination of the orbit* [1030′], *and* θ *the longitude of its ascending node,* we shall have, as in [1337b,5751b],† [5777]

$$dp=-\frac{dt}{\sqrt{a.(1-ee)}}\cdot\left(\frac{dR}{dq}\right)\ ;$$ [5778]

$$dq = \frac{dt}{\sqrt{a.(1-ee)}}\cdot\left(\frac{dR}{dp}\right)\ .$$ [5779]

Now we have, by § 44, of the second book,‡

$$0=\left(\frac{dR}{da}\right).da+\left(\frac{dR}{de}\right).de+\left(\frac{dR}{d\varpi}\right).d\varpi+\left(\frac{dR}{d\varepsilon}\right).d\varepsilon+\left(\frac{dR}{dp}\right).dp+\left(\frac{dR}{dq}\right).dq;$$ [5780]

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* (3136) The expression of $d\varepsilon-d\varpi$ [5765] is reduced, in [5767], to *three* separate terms ; of which the *first* is $-d\varpi.\sqrt{1-e^2}$. The *second* and *third* terms constitute the [5775a] first member of [5772], which is successively reduced to the form $2a^2.ndt\,.\left(\frac{dR}{da}\right)$, in [5775] ; hence we get,

$$d\varepsilon-d\varpi = -d\varpi.\sqrt{1-e^2}+2a^2.\,ndt\,.\left(\frac{dR}{da}\right)\ ;$$ [5775b]

and, by transposing $-d\varpi$, we obtain [5775′] ; which is correct in terms of the order m', [5775c] as in [5773a].

† (3137) We have $an = a^{-\frac{1}{2}}$ [5751b] ; substituting this in [1337b], we get [5778, 5779] ; which are exact in terms of the order m' [1337b line 3]. [5778a]

‡ (3138) R is a function of $\int ndt$ [5793], and of the elements a, e, ϖ, ε. [5780a] p, q. Now, we may take its differential, relative to t, considering the elements as constant, and the ellipsis invariable. We may also take it, supposing all the quantities to be variable, as in [1168′, &c.]. The first of these differentials, being subtracted from the [5780b] second, gives [5780].

moreover, we obtain, from [1177, 5750],

[5781] $$da = -2a^2.\mathrm{d}R\ ;$$

[5782] and $\left(\frac{dR}{d\varepsilon}\right) = \frac{\mathrm{d}R}{ndt}$; because the angle nt is always connected with $+\varepsilon$;* therefore, by substituting the preceding values of da, de, $d\varepsilon$, dp, and dq; we shall have this very simple equation,†

[5783] $$d\varpi = -\frac{a.ndt.\sqrt{1-ee}}{e}\cdot\left(\frac{dR}{de}\right);$$

which gives,

[5784] $$d\varepsilon = -\frac{a.ndt.\sqrt{1-ee}}{e}\cdot(1-\sqrt{1-ee})\cdot\left(\frac{dR}{de}\right) + 2a^2.\,ndt\cdot\left(\frac{dR}{da}\right).$$

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* (3139) We see, in [953, 954, &c.], that nt is always connected with ε, in the form of $nt+\varepsilon$, or rather $fndt+\varepsilon$ [5793]; so that if we suppose R to be a function of $fndt+\varepsilon$, we may represent it by $R = f(fndt+\varepsilon)$; and, by using a notation similar to that in [5763c], we have $\left(\frac{\mathrm{d}R}{ndt}\right) = f'(fndt+\varepsilon)$, and $\left(\frac{dR}{d\varepsilon}\right) = f'\,(fndt+\varepsilon)$; whence we get $\left(\frac{dR}{d\varepsilon}\right) = \left(\frac{\mathrm{d}R}{ndt}\right)$, which is equivalent to that in [5782].

[5782a]

[5782b]

† (3140) Of the *six* terms of which the function [5780] is composed, the *fifth* and *sixth* destroy each other, by the substitution of the values of dp, dq [5778, 5779], as

[5783a] is evident by inspection. Again, the second term of $d\varepsilon$ [5775'], namely $2a^2.\left(\frac{dR}{da}\right).ndt$, being substituted in [5780], produces,

[5783b] $$2a^2.\left(\frac{dR}{da}\right).\,ndt\,.\left(\frac{dR}{d\varepsilon}\right) = 2a^2.\left(\frac{dR}{da}\right).\,\mathrm{d}R \quad [5782]\,;$$

and this is destroyed by means of the first term of [5780], namely $\left(\frac{dR}{da}\right).\,da$, as is evident, by the substitution of da [5781]. Hence the function [5780], is reduced to the *three* terms depending on de, $d\varpi$, $d\varepsilon$; taking for $d\varepsilon$, the first term of [5775'] only;

[5783c] namely, $d\varepsilon = d\varpi.\{1-\sqrt{1-e^2}\}$; hence the function [5780] becomes, by the substitution of this value of $d\varepsilon$, and that of $\left(\frac{dR}{d\varepsilon}\right)$ [5782],

[5783d] $$0 = \left(\frac{dR}{de}\right).\,de + \left\{\left(\frac{dR}{d\varpi}\right) + \frac{\mathrm{d}R}{ndt}\cdot(1-\sqrt{1-e^2})\right\}.\,d\varpi.$$

Connecting together, in one table, these different equations, we shall have, by observing, that $n=a^{-\frac{3}{2}}$ [5751b], and, that the sign d, affects only the co-ordinates of the body m;* [5785]

Now, the value of de [5764), can be separated into two factors, so that we may put it under the following form,

$$de=\left\{\left(\frac{dR}{d\varpi}\right)+\frac{\mathrm{d}R}{ndt}\cdot(1-\sqrt{1-e^2})\right\}\cdot\frac{a.ndt.\sqrt{1-e^2}}{e}; \qquad [5783e]$$

as is easily proved, by multiplying the terms. Substituting this in [5783d], and then dividing by the common factor $\left(\frac{dR}{d\varpi}\right)+\frac{\mathrm{d}R}{ndt}\cdot(1-\sqrt{1-e^2})$, we get,

$$0=\frac{a.ndt.\sqrt{1-e^2}}{e}\cdot\left(\frac{dR}{de}\right)+d\varpi; \qquad [5783f]$$

whence we obtain $d\varpi$ [5783]. Substituting this value of $d\varpi$, in that of $d\varepsilon$ [5775'], we get [5784]. The expressions of $d\varpi$, $d\varepsilon$ [5783, 5784], are exact in terms of the order m', for all powers of the excentricities and inclinations; but some terms of the order m'^2 are neglected. [5783g]

* (3141) The equations [5786—5791], are the same as those which are given in [5781, 5784, 5764, 5783, 1337b], respectively. *The equations* [5787—5791] *are correct, in terms of the first order of the disturbing masses, for all powers of the excentricities and inclinations; but, some terms of the order of the square of the disturbing masses are neglected.* [5786a] [5786b]

We may observe, that, in estimating the values of dp, dq [5790, 5791], we have taken the primitive orbit of the disturbed planet, for the fixed plane; so that p, q, are considered as very small quantities, of the order of the disturbing masses; whose squares are neglected. To avoid this restriction, the author has given other forms to these expressions in [12528, 12529]; by taking another fixed plane independent of the primitive orbit. Then, if γ' *be the inclination of the orbit of the disturbed planet to this new plane*, and θ' *the distance of its node from a fixed point in the same plane*, we shall have, instead of p, q, dp, dq [5776,5790,5791], the system of equations [5786e—g], representing the values of p', q', dp', dq'; corresponding to this plane. From these we easily deduce the values of $d\gamma'$, $d\theta'$ [5786h, i]. The investigation of these equations is given by the author in [12513—12537]; and it is unnecessary to repeat it here. [5786c] [5786d]

$$p'=\sin.\gamma'.\sin.\theta'; \qquad q'=\sin.\gamma'.\cos.\theta'; \qquad [12520] \qquad [5786e]$$

[5786] $$da = -2a^2.\mathrm{d}R\,; \qquad (1)$$

[5787] $$d\varepsilon = -\frac{a.ndt.\sqrt{1-e^2}}{e}.(1-\sqrt{1-e^2}).\left(\frac{dR}{de}\right)+2a^2.ndt.\left(\frac{dR}{da}\right)\,; \quad (2)$$

Differentials of the elements, exact in terms of the order m'.

[5788] $$de = \frac{a.\sqrt{1-e^2}}{e}.(1-\sqrt{1-e^2}).\mathrm{d}R + \frac{a.\sqrt{1-e^2}}{e}.ndt.\left(\frac{dR}{d\varpi}\right)\,; \quad (3)$$

[5789] $$d\varpi = -\frac{a.ndt.\sqrt{1-e^2}}{e}.\left(\frac{dR}{de}\right)\,; \qquad (4)$$

[5790] $$dp = -\frac{a.ndt}{\sqrt{1-e^2}}.\left(\frac{dR}{dq}\right)\,; \qquad (5)$$

[5791] $$dq = \frac{a.ndt}{\sqrt{1-e^2}}.\left(\frac{dR}{dp}\right). \qquad (6)$$

[5792] We may substitute, in these equations, $ndt.\left(\frac{dR}{d\varepsilon}\right)$ for $\mathrm{d}R$ [5782], and by this means, reduce the preceding expressions, so as to contain only the partial differentials of the elements; but, it is as simple, to retain the differential $\mathrm{d}R$.

[5793] *In the motion, considered as elliptical, we must substitute* $\int ndt$ *for* nt,* *if we wish to be rigorously correct*; now, $n=a^{-\frac{3}{2}}$ [5785]; therefore, by
[5793'] *putting* ζ *equal to the mean motion of the planet* m, we shall have [1183, 5750],

Mean motion.

[5794] $$\zeta = \int ndt = 3\int\int a.ndt.\mathrm{d}R. \qquad (7)$$

2. *From these equations, we easily deduce the same result, as that which*
[5794'] *was discovered by Mr. Poisson, relative to the invariableness af the mean*

[5786f] $$dp' = -\frac{a.ndt}{\sqrt{1-e^2}}.\cos.\gamma'.\left(\frac{dR}{dq'}\right)\,; \qquad [12528]$$

[5786g] $$dq' = \frac{a.ndt}{\sqrt{1-e^2}}.\cos.\gamma'.\left(\frac{dR}{dp'}\right)\,; \qquad [12529]$$

[5786h] $$d\gamma' = \frac{a.ndt}{\sqrt{1-e^2}.\sin.\gamma'}.\left(\frac{dR}{d\theta'}\right)\,; \qquad [12536]$$

[5786i] $$d\theta' = -\frac{a.ndt}{\sqrt{1-e^2}.\sin.\gamma'}.\left(\frac{dR}{d\gamma'}\right). \qquad [12537]$$

* (3142) We have the differential of the *first* order $d\zeta = ndt$ [1180'', or 5794],
[5794a] which corresponds to the *variable* ellipsis, and, therefore, also, to the *invariable* ellipsis [1168']. In the invariable ellipsis, we have n constant, and its integral is $\zeta = nt+\varepsilon$;

motions of the planets; noticing the square of the disturbing force. If we denote any finite variation by the characteristic δ, and *vary, in* R, *only* [5794''] Symbol δ.
what relates to the planet m; observing, that $\left(\frac{dR}{d\varepsilon}\right)=\frac{\mathrm{d}R}{ndt}$ [5782]; we [5794''']
shall have,*

$$\delta R=\frac{\mathrm{d}R}{ndt}.\{\delta(\textstyle\int ndt)+\delta\varepsilon\}+\left(\frac{dR}{da}\right).\delta a+\left(\frac{dR}{de}\right).\delta e+\left(\frac{dR}{d\varpi}\right).\delta\varpi+\left(\frac{dR}{dp}\right).\delta p+\left(\frac{dR}{dq}\right).\delta q. \qquad [5795]$$

Substituting, for δa, δe, $\delta\varpi$, &c., the integrals of the preceding values of da, de, $d\varpi$, &c. [5786, &c.], we shall have,†

Terms of δR, of the order m^2, arising from the variation of the elements of the planet m.

$$\delta R=\frac{\mathrm{d}R}{ndt}.\delta(\textstyle\int ndt) \qquad 1$$

$$+\ 2a^2.\left\{\frac{\mathrm{d}R}{ndt}.\textstyle\int ndt.\left(\frac{dR}{da}\right)-\left(\frac{dR}{da}\right).\int \mathrm{d}R\right\} \qquad 2$$

$$+\ \frac{a.\sqrt{1-e^2}}{e}.(1-\sqrt{1-ee}).\left\{\left(\frac{dR}{de}\right).\textstyle\int \mathrm{d}R-\frac{\mathrm{d}R}{ndt}.\int ndt.\left(\frac{dR}{de}\right)\right\} \qquad 3 \qquad [5796]$$

$$+\ \frac{a.\sqrt{1-e^2}}{e}.\left\{\left(\frac{dR}{de}\right).\textstyle\int ndt.\left(\frac{dR}{d\varpi}\right)-\left(\frac{dR}{d\varpi}\right).\int ndt.\left(\frac{dR}{de}\right)\right\} \qquad 4$$

$$+\ \frac{a}{\sqrt{1-e^2}}.\left\{\left(\frac{dR}{dq}\right).\textstyle\int ndt.\left(\frac{dR}{dp}\right)-\left(\frac{dR}{dp}\right).\int ndt.\left(\frac{dR}{dq}\right)\right\}. \qquad 5$$

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but, in the variable ellipsis, n is variable, and we have $\zeta=\int ndt+\varepsilon$. Hence it is evident, that the mean motion nt, corresponding to the invariable ellipsis, must be changed into $\int ndt$, in the variable ellipsis, as in [5793]. [5794b]

* (3143) R is a function of $\int ndt$, and of the elements ε, a, e, ϖ, p, q; now, if these quantities vary by the increments $\delta.\int ndt$, $\delta\varepsilon$, δa, δe, $\delta\varpi$, δp, δq, [5795a]
respectively, we may obtain the development of R, in a series, proceeding according to the powers and products of these increments, by means of the formulas [610—612, &c.].
If we retain only the first power of these quantities, and put, for $\left(\frac{dR}{d\varepsilon}\right)$, its value, [5795b]
deduced from [5782]; namely, $\frac{\mathrm{d}R}{ndt}$; the increment of R will become as in [5795]. [5795c]
This equation is *correct in terms of the order* m'^2; because, R [5755] is of the order m'; and the variations $\delta\varepsilon$, δe, &c., which depend on R, are also of the order m'; therefore, the terms of the second member of [5795] are of the order m'^2; and the neglected terms of the order $R\delta\varepsilon^2$, $R\delta e^2$, &c., must be of the order m'^3. [5795d]

† (3144) The integral of the equation [5786] is $a=\text{constant}-2.\int a^2.\mathrm{d}R$; the [5796a]

[5797] To obtain the value of $\mathrm{d}\,.\left\{\delta R-\frac{\mathrm{d}R}{ndt}.\delta(\int ndt)\right\}$, given by the equation [5796],*

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constant quantity being equal to the value of a, at the *commencement of the integral.*
[5796b] Hence the increment of a, is represented by $\delta a=-2\int a^2.\mathrm{d}R$; so, that if we put $\int\mathrm{d}R=R'$, and integrate by parts, we shall have, successively,

[5796c] $$\delta a=-2\int a^2.\mathrm{d}R=-2a^2.R'+\int R'.4ada;$$

as is easily proved, by taking the differentials of these expressions of δa, and re-substituting
[5796c'] $R'=\int\mathrm{d}R$. Now, R' and da [5786], are both of the order m'; therefore, $\int R'.4ada$, is of the order m'^2; and if we neglect terms of this order, in δa, which will only produce terms of the order m'^3, in [5795], we shall have,

[5796d] $$\delta a=-2\int a^2.\mathrm{d}R=-2a^2.\int\mathrm{d}R.$$

Hence it appears, that in finding the integral of $a^2.\mathrm{d}R$, we may bring the factor a^2 from under the sign of integration; neglecting terms of the order m'^2. For similar reasons we may bring a, e, from under the sign $\int$, in the integrals of the other
[5796e] expressions [5786—5791], leaving for symmetry, n under that sign, connected with dt, as in [5794, 5795, &c.]; hence we get the following expressions, which represent, respectively, the integrals of the five equations [5787—5791];

[5796f] $$\delta\varepsilon=-\frac{a.\sqrt{1-ee}}{e}.(1-\sqrt{1-ee}).\int ndt.\left(\frac{dR}{de}\right)+2a^2.\int ndt.\left(\frac{dR}{da}\right);$$

[5796g] $$\delta e=\frac{a.\sqrt{1-ee}}{e}.(1-\sqrt{1-ee}).\int\mathrm{d}R+\frac{a.\sqrt{1-ee}}{e}.\int ndt.\left(\frac{dR}{d\varpi}\right);$$

[5796h] $$\delta\varpi=-\frac{a.\sqrt{1-ee}}{e}.\int ndt.\left(\frac{dR}{de}\right);$$

[5796i] $$\delta p=-\frac{a}{\sqrt{1-ee}}.\int ndt.\left(\frac{dR}{dq}\right);$$

[5796k] $$\delta q=\frac{a}{\sqrt{1-ee}}.\int ndt.\left(\frac{dR}{dp}\right).$$

[5796l] Substituting these, and also δa [5796d] in [5795], we get [5796], *which is exact in*
[5796m] *terms of the order* m'^2. If, for brevity, we represent by $R_{,}$ the four lower lines of the second member of [5796], we shall obtain, by substitution and reduction,

[5796n] $$\delta R-\frac{\mathrm{d}R}{ndt}.\delta(\int ndt)=R_{,};$$

[5796o] so that $\mathrm{d}R_{,}$ represents the value of the function which is mentioned in [5797].

* (3145) If we vary in R, what relates to the planet m, as in [5794″, &c.], we

we must take its differential, relative to the quantities corresponding to the planet m only [5785]. *To obtain the differential relative to the elements of that planet, it will be sufficient to suppress the sign* f, *which has been* [5798]

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shall get the expression of δR [5796]; or the equivalent expression [5796n]; and the object of the author, in [5797—5812], is to prove, that $d.\delta R$ contains nothing but *periodical* quantities. The value of $d.\delta R$, deduced from [5796n], is of the following form; [5797a]

$$d.\delta R = d.\left\{\frac{dR}{ndt}.\,\delta\,(fndt)\right\} + dR_{,}. \qquad [5797b]$$

The calculation in [5798—5806] is to prove, in the first place, that the second term of this expression $dR_{,}$, produces periodical quantities only; the process in [5807—5812] serves the same purpose, relative to the other term; namely,

$$d.\left\{\frac{dR}{ndt}.\,\delta\,(fndt)\right\} \quad [5797b]. \qquad [5797c]$$

In these calculations, the terms of $R_{,}$ are supposed to be represented by,

$$M.fNdt - N.fMdt \quad [5800]; \qquad [5797d]$$

and, it is very easy to reduce them to this form. For, if we change fdR into $fndt.\frac{dR}{ndt}$ in [5796 lines 2, 3], for the sake of symmetry, we shall find, that any one of the lines of the function [5796 lines 2—5], is composed of two terms of the form,

$$A.\{R_2.fndt.R_3 - R_3.fndt.R_2\}; \qquad [5797e]$$

A being the factor without the braces; and R_2, R_3, the differential coefficients, depending on the partial differentials of R, which occur in that line. Now, if we put $AR_2 = M$; $nR_3 = N$; the preceding expression becomes, [5797f]

$$M.fNdt - \frac{AN}{n}.fndt.\frac{M}{A}; \qquad [5797f']$$

M and N being each of the order m'; therefore, MN is of the order m'^2; and, if we neglect terms of the order m'^3, we may introduce the factor $\frac{A}{n}$, of the second term of [5797f'], under the sign f; and then, by reduction, the expression becomes, [5797f'']

$$M.fNdt - N.fMdt, \text{ as in } [5800]. \qquad [5797g]$$

Similar processes and reductions are used, in calculating the part of $d.\delta R$, arising from the variation in δR, relative to the planet m', in [5813, &c.]; and those relative to the planet m'', in [5832, &c.]. [5797h]

introduced only by the integrals of the differential expressions of these
[5798'] *elements* [5786—5791]; and then, that expression becomes identically nothing;* so that, if we wish to obtain the differential d of the
[5799] function $\delta R - \frac{\mathrm{d}R}{ndt}.\delta(\int ndt)$, it will suffice to take its differential relative to nt, noticing only the quantities without the sign $\int$ [5798e]. The expression of this function is composed of terms of the form,

[5800] $$M.\int Ndt - N.\int Mdt \quad [5797g]. \qquad \text{[Function } R_{,}]$$

M and N may be developed in terms, depending on cosines of angles of the following forms;†

[5801] $$M = k.\cos.(i'n't - int + A) \; ; \qquad N = k'.\cos.(i'n't - int + A') \; ;$$

[5802] i' and i being any integral numbers, positive or negative. We must

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* (3146) The integrals of the expressions [5786—5791], introduce the sign $\int$ in the values of the variations of the elements of the planet m [5796f—k]; and by this
[5798a] means they occur also in [5796]; and, as these integrals have reference to the elements of m, their differentials relative to the characteristic d [5785], must be equivalent to the complete differentials; so that we shall have,

[5798b] $$\mathrm{d}.\int Ndt = Ndt \; ; \qquad \mathrm{d}.\int Mdt = Mdt \, .$$

Hence the differential of the function $R_{,}$ [5800], relative to d, is,

[5798c] $$\mathrm{d}R_{,} = \mathrm{d}M.\int Ndt - \mathrm{d}N.\int Mdt + MNdt - MNdt \, .$$

The two last terms of this expression destroy each other, as in [5798']; and we finally get,

[5798d] $$\mathrm{d}R_{,} = \mathrm{d}M.\int Ndt - \mathrm{d}N.\int Mdt \, .$$

Hence we obtain the same rule as in [5799], for finding the differential of

[5798e] $$R_{,} = M.\int Ndt - N.\int Mdt \quad [5800] \; ;$$

namely, by taking the differential of $R_{,}$, supposing the terms without the sign $\int$, to be the only variable quantities.

† (3147) The functions M, N [5797f], depend on R, which is of the same
[5801a] form as the assumed values of M, N [5801]; as appears in [957''']. Substituting these in [5803], we get [5804].

combine these two terms together, to obtain the non-periodical terms in $\mathrm{d}.\{M.\int Ndt - N.\int Mdt\}$; then this function becomes, [5803]

$$\begin{array}{ll} k.indt.\sin.(i'n't-int+A).\int k'dt.\cos.(i'n't-int+A') & 1 \\ -k'.indt.\sin.(i'n't-int+A').\int kdt.\cos.(i'n't-int+A). & 2 \end{array}$$ [Function $\mathrm{d}R$.] [5804]

The integrations which are indicated in this function, being made, we find, that the terms destroy each other, and the whole expression vanishes.* This agrees with what we have demonstrated, in [3906'], relative to the great inequalities of Jupiter and Saturn. The expression of [5805]

$$\mathrm{d}.\left\{\delta R - \frac{\mathrm{d}R}{ndt}.\delta.\int ndt\right\},$$ [5806]

is, therefore, *a periodical function.*

The expression of $\mathrm{d}.\left\{\frac{\mathrm{d}R}{ndt}.\delta.\int ndt\right\}$, contains only periodical quantities; for, we have,†

$$\mathrm{d}.\left\{\frac{\mathrm{d}R}{ndt}.\delta.\int ndt\right\} = \frac{\mathrm{dd}R}{ndt}.\delta.\int ndt + \frac{\mathrm{d}R}{ndt}.dt.\delta n.$$ [5807]

Substituting for δn, its value‡ $\delta n = 3\int an.\mathrm{d}R$, we shall have, [5808]

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* (3148) The integrations, which occur in [5804], are made in the usual manner, by changing *cos.* into *sin.*, and dividing by $i'n'-in$; and when this is done, the terms mutually destroy each other. We may remark, that the coefficient of t, in the values of M, N [5801], are *equal* to each other, being represented by $i'n'-in$. It is useless to notice other terms, in which these coefficients are *unequal;* because they produce nothing, except periodical terms, in the function [5804]; as is evident from [17] Int. [5805a] [5805b] [5805c]

† (3149) The complete differential of the first member of [5807] $\left\{\frac{\mathrm{d}R}{ndt}.\delta.\int ndt\right\}$, taken relatively to the characteristic d, contains the two terms in the second member of [5807]; and also the additional term $-\frac{\mathrm{d}R.dn}{n^2.dt}.\delta.\int ndt$; but this term contains the three factors $\mathrm{d}R$, dn, $\delta.\int ndt$; each of which is of the order m'; hence it is of the order m'^3, and may be neglected; and the expression becomes as in [5807]. [5807a] [5807b]

‡ (3150) Taking the differentials of the two expressions of ζ [5794], and dividing

[5809] $$\mathrm{d}\,.\left\{\frac{\mathrm{d}R}{ndt}\,.\,\delta\,.\,\textstyle\int ndt\right\}=3\,an\,.\,\frac{\mathrm{dd}R}{ndt}\,.\,\textstyle\int\int \mathrm{d}R\,.\,dt+3\,an\,.\,\frac{\mathrm{d}R}{ndt}\,.\,dt\,.\,\textstyle\int \mathrm{d}R\,.$$

We may unite, in one expression, all the terms of the development of R, which depend on the same angle $i'n't-int$, and it becomes of the form,

[5810] $$R=k\,.\,\cos.(i'n't-int+A)\quad[957''''].$$

Substituting it for R, in the functions $\frac{\mathrm{dd}R}{ndt}\,.\int\int\mathrm{d}R\,.\,dt$, and $\frac{\mathrm{d}R}{ndt}\,.\int\mathrm{d}R$,

[5811] we find, that they are reduced to sines of double the angle* $i'n't-int+A$;

[5808a] them by dt, we get $n=3\int an\,.\mathrm{d}R$; or, as it may be written, $\delta n=3\int an\,.\mathrm{d}R$, as in [5808]. Substituting this in the development of $\delta\,.\int ndt$, we easily obtain,

[5808b] $$\delta\,.\textstyle\int ndt=\int\delta n\,.\,dt=3\int\int an\,.\mathrm{d}R\,.dt\,.$$

These values being introduced into the second member of [5807], we get,

[5808c] $$\mathrm{d}\,.\left\{\frac{\mathrm{d}R}{ndt}\,.\,\delta\,.\,\textstyle\int ndt\right\}=\frac{\mathrm{dd}R}{ndt}\,.\,3\textstyle\int\int an\,.\,\mathrm{d}R\,.\,dt+\frac{\mathrm{d}R}{ndt}\,.\,dt\,.\,3\textstyle\int an\,.\,\mathrm{d}R\,.$$

Each of the two terms of the second member of this expression, is of the order m'^2;
[5808d] and, if we neglect terms of the order m'^3, we may bring the factor an, from under the sign $\int$, as in [5796d—e]; and then the equation [5808c], becomes as in [5809].

* (3151) From $R=k.\cos.(i'n't-int+A)$ [5810], we easily deduce the following
[5810a] expressions,

[5810b] $$\frac{\mathrm{d}R}{ndt}=ki.\sin.(i'n't-int+A)\,;\qquad \textstyle\int\mathrm{d}R=-\frac{k.in}{i'n'-in}\,.\,\cos.(i'n't-int+A)\,;$$

[5810c] $$\textstyle\int\int\mathrm{d}R\,.\,dt=-\frac{k.in}{(i'n'-in)^2}\,.\,\sin.(i'n't-int+A)\,;\quad \frac{\mathrm{dd}R}{ndt}=-\,k.i^2n.\cos.(i'n't-int+A).dt$$

In finding these expressions, we have neglected the variations of the elements n, n', &c., because they produce only terms of the order m'^3, in [5809]. The product of the two expressions [5810c], being substituted in the first term of the second member of [5809], produces a term depending on,

[5810d] $$\sin.(i'n't-int+A)\times\cos.(i'n't-int+A)=\tfrac{1}{2}.\sin.2.(i'n't-int+A).$$

In like manner, the product of the two expressions [5810b], being substituted in the second term of [5809], produces another term depending on,

[5810e] $$\tfrac{1}{2}\,.\,\sin.2.(i'n't-int+A)\,,\quad\text{as in [5811].}$$

thus, the differential $\mathrm{d}\,.\left(\frac{\mathrm{d}R}{ndt}\,.\,\delta\,.\int ndt\right)$, contains only *periodical quantities.* [5811′]
Hence it follows, that $\mathrm{d}.\delta R$, *contains only periodical quantities, when we vary in* δR, *the quantities relative to the planet* m *only.* [5812]

To obtain the complete value of $\mathrm{d}.\delta R$, *we must also vary in* δR, *what relates to the planet* m'. For this purpose, we shall put R', for what R becomes, relative to the planet m', disturbed by the action of m. We shall then have,* [5813]

$$R' = \frac{m.(xx'+yy'+zz')}{r^3} - \frac{m}{\rho}\,; \qquad [5814]$$

hence,

$$R = \frac{m'}{m}\,.R' + m'.(xx'+yy'+zz').\left(\frac{1}{r'^3} - \frac{1}{r^3}\right). \qquad [5815]$$

The variation of R, so far as it depends upon the variations of what relates to the planet m', is, therefore, equal to the variation of the second member of the equation [5815], *arising from the variations of the co-ordinates of* m'. *We shall denote, by* δ', *the variations which correspond* [5816] [5817] Symbol δ'.

From what has been proved, it appears, that the two functions [5806, 5811′], which compose the value of $\mathrm{d}.\delta R$ [5796n], produce nothing except periodical terms, noticing quantities relative to m, as in [5812]. [5810f]

* (3152) Changing, reciprocally, the elements of m into those of m', we shall get, from R [5755], the expression of R' [5814]. Multiplying this by $\frac{m'}{m}$, we obtain,

$$\frac{m'}{m}\,.\,R' = \frac{m'.(xx'+yy'+zz')}{r^3} - \frac{m'}{\rho}\,; \qquad [5815a]$$

subtracting this from [5755], we get,

$$R - \frac{m'}{m}\;R' = m'.(xx'+yy'+zz')\,.\left(\frac{1}{r'^3} - \frac{1}{r^3}\right); \qquad [5815b]$$

from which we easily obtain R [5815]. This expression of R does not contain ρ, and, on that account, it is more convenient than the expression [5755], in making the calculations relative to m', in [5813—5831′]. [5815c]

to these co-ordinates. We evidently see, by the preceding analysis, that,*

[5818] $$\frac{m'}{m}\cdot\left(\delta' R'-\frac{\mathrm{d}'R'}{n'dt}\cdot\delta'.\int n'dt\right)$$

is composed of terms of the form

[5819] $$M.\int Ndt - N.\int Mdt.$$

To obtain their differentials, relative to the characteristic d, we must

[5819'] vary only the quantities without the sign of integration; because the quantities under that sign correspond to the elements of the planet m'.†

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* (3153) If we change, in [5795, 5796], the elements n, ε, a, e, ϖ, p, q,

[5818a] into n', ε', a', e', ϖ', p', q', respectively, we shall get, by using the characteristic

[5818b] δ', as in [5817], and supposing d' to affect the co-ordinates of m' only;

[5818c] $$\delta' R'=\frac{\mathrm{d}'R'}{n'dt}\cdot\{\delta'(\int n'dt)+\delta'\varepsilon'\}+\left(\frac{dR'}{da'}\right).\delta' a'+\left(\frac{dR'}{de'}\right).\delta' e'+\left(\frac{dR'}{d\varpi'}\right).\delta'\varpi'+\left(\frac{dR'}{dp'}\right).\delta' p'+\left(\frac{dR'}{dq'}\right).\delta' q';$$

$$\delta' R' = \frac{\mathrm{d}'R'}{n'dt}\cdot\delta'(\int n'dt) \qquad 1$$

$$+\ 2a'^2.\left\{\frac{\mathrm{d}'R'}{n'dt}.\int n'dt.\left(\frac{dR'}{da'}\right)-\left(\frac{dR'}{da'}\right).\int \mathrm{d}'R'\right\} \qquad 2$$

[5818d] $$+\ \frac{a'.\sqrt{1-e'^2}}{e'}.(1-\sqrt{1-e'^2}).\left\{\left(\frac{dR'}{de'}\right).\int \mathrm{d}'R'-\frac{\mathrm{d}'R'}{n'dt}.\int n'dt.\left(\frac{dR'}{de'}\right)\right\} \qquad 3$$

$$+\ \frac{a'.\sqrt{1-e'^2}}{e'}.\left\{\left(\frac{dR'}{de'}\right).\int n'dt.\left(\frac{dR'}{d\varpi'}\right)-\left(\frac{dR'}{d\varpi'}\right).\int n'dt.\left(\frac{dR'}{de'}\right)\right\} \qquad 4$$

$$+\ \frac{a'.\sqrt{1-e'^2}}{1-e'^2}.\left\{\left(\frac{dR'}{dq'}\right).\int n'dt.\left(\frac{dR'}{dp'}\right)-\left(\frac{dR'}{dp'}\right).\int n'dt.\left(\frac{dR'}{dq'}\right)\right\}. \qquad 5$$

If we represent by $R_{\prime}'$ the four lower lines of the second member of this last equation,

[5818e] we shall get, by substitution and transposition, the following expression, which is similar to [5796n];

[5818f] $$\delta' R'-\frac{\mathrm{d}'R'}{n'dt}\cdot\delta'(\int n'dt) = R_{\prime}';$$

[5818g] and we may prove, as in [5797g], that $R_{\prime}'$ is composed of terms of the forms mentioned in [5819].

[5819a] † (3154) The terms under the sign of integration, in the second member of [5818d],

We shall suppose these two corresponding terms of M, N, to be represented by,*

$$M = k.\cos.(i'n't-int+A) \;; \qquad N = k'.\cos.(i'n't-int+A'). \qquad [5820]$$

Then, we must combine these terms together, to obtain the non-periodical quantities in

$$\mathrm{d}.(M.\textstyle\int Ndt-N.\int Mdt) \;; \qquad [5821]$$

and, it is evident, that this differential function does not contain such quantities. We may easily prove, that

$$\mathrm{d}\,.\left(\frac{\mathrm{d}'R'}{n'dt}\,.\,\delta'.\textstyle\int n'd\,t\right) \qquad [5821']$$

does not contain any ; by the same manner of reasoning as that which we have used in proving that

$$\mathrm{d}\,.\left(\frac{\mathrm{d}R}{ndt}\,.\,\delta\,.\textstyle\int n\,d\,t\right) \quad [5811']$$

contains only periodical quantities ;† therefore, $\mathrm{d}.\delta'R'$ contains only similar quantities. [5821'']

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arise from the quantities $\delta'\varepsilon'$, $\delta'a'$, $\delta'e'$, $\delta'\varpi'$, $\delta'p'$ $\delta'q'$, $\delta'.\int n'dt$; which contain terms with the sign $\int$, like the similar expressions of $\delta\varepsilon$, δa, &c. [5796d—k]. Now, these quantities $\delta'\varepsilon'$, $\delta'a'$, &c., *depend on the co-ordinates of the planet* m'; therefore, their differentials relative to d [5785] must vanish. On the contrary, the factors $\frac{\mathrm{d}'R'}{n'dt}$, $\left(\frac{dR'}{da'}\right)$, $\left(\frac{dR'}{de'}\right)$, &c., in the function [5818c], may produce, in [5818d], some terms without the sign $\int$, containing the elements of the planet m, which will be affected by the differential d. [5819b] [5819c]

* (3155) The calculation in [5819—5821''], is similar to that in [5800—5812] ; and the functions [5800, 5801, 5803, &c.], correspond, respectively, to [5819,5820,5821, &c.]; hence we obtain a result, similar to that in [5806]; namely, that the differential d, of the function [5818f], does not contain non-periodical quantities. [5820a] [5820b]

† (3156) If we develop the function [5821'], we shall get, by observing, that n', and $\int n'dt$, are not affected by d ;

$$\mathrm{d}\,.\left\{\frac{\mathrm{d}'R'}{n'dt}.\delta'.(\textstyle\int n'dt)\right\} = \frac{\mathrm{dd}'R'}{n'dt}.\delta'.(\textstyle\int n'dt). \qquad [5821a]$$

It now remains to consider the second term of R [5815], which we shall denote by,

[5822] $$P = m'.(xx'+yy'+zz').\left(\frac{1}{r'^3}-\frac{1}{r^3}\right).$$

We have, as in [915],*

We may substitute, in the second member of this equation, for $\delta'.(fn'dt)$, its value
[5821b] $f\delta'n'.dt$; and, for $\delta'n'$, its value $\delta'n'=3fa'n'.\mathrm{d}'R'$, which is similar to [5808]; hence we shall have,

[5821c] $$\mathrm{d}.\left\{\frac{\mathrm{d}'R'}{n'dt}.\delta'.(fn'dt)\right\}=\frac{\mathrm{dd}'R'}{n'dt}.3ffa'n'.\mathrm{d}'R'.dt=\frac{\mathrm{dd}'R'}{n'dt}.3\,a'n'.ff\mathrm{d}'R'.dt;$$

observing, that we may bring $a'n'$ from under the sign ff, by neglecting terms of the order m'^3, as in [5796e, &c.]. Now, if we put

[5821d] $$R'=k.\cos.(i'n't-int+A),$$

we shall get, in like manner as in [5810c];

[5821e] $$ff\mathrm{d}'R'.dt=\frac{k.i'n'}{(i'n'-in)^2}.\sin.(i'n't-int+A);\quad \frac{\mathrm{dd}'R'}{n'dt}=k.i'in.\cos.(i'n't-int+A).\,at$$

The product of these two quantities being substituted in the last expression of [5821c],
[5821f] produces only a periodical term depending on $\sin.2.(i'n't-int+A)$; in like manner as in [5810d, &c.]. Having found, that the differential relative to d, of the function [5818f], and that of [5821'], produce only periodical quantities; their sum representing the value
[5821g] of $\mathrm{d}.\delta'R'$, deduced from $\delta'R'$ [5818f], must also consist of such periodical quantities;
[5821h] which may be neglected: therefore, we may reject the term $\frac{m'}{m}.R'$, in the value of R [5815], and it will be reduced to $R=P$; using the abridged symbol P [5822].

[5823a] * (3157) Substituting, in the first of the equations [915], the value of $M+m=\mu$ [914'], we get,

[5823b] $$0=\frac{ddx}{dt^2}+\frac{(M+m).x}{r^3}+\left(\frac{dR}{dx}\right);\quad\text{or,}\quad \frac{Mx}{r^3}=-\frac{ddx}{dt^2}-\frac{mx}{r^3}-\left(\frac{dR}{dx}\right);$$

multiplying this by $\frac{m'}{M}$, we obtain [5823]. The second and third of the equations
[5823c] [915], give similar expressions of $\frac{m'y}{r^3}$, $\frac{m'z}{r^3}$. If, in these equations, we change m, x, y, z, r, into m', x', y', z', r', respectively, and the contrary; and, then multiply
[5823d] by $\frac{m'}{m}$, we shall get the corresponding values of $\frac{m'x'}{r'^3}$, $\frac{m'y'}{r'^3}$, $\frac{m'z'}{r'^3}$. The first of these expressions is explicitly given, in [5825].

$$\frac{m'x}{r^3}=-\frac{m'}{M}\cdot\frac{ddx}{dt^2}-\frac{m\,m'}{M}\cdot\frac{x}{r^3}-\frac{m'}{M}\cdot\left(\frac{dR}{dx}\right)\;; \qquad [5823]$$

M being the sun's mass. We have also, [5824]

$$\frac{m'x'}{r'^3}=-\frac{m'}{M}\cdot\frac{ddx'}{dt^2}-\frac{m'^2}{M}\cdot\frac{x'}{r'^3}-\frac{m'}{M}\cdot\left(\frac{dR'}{dx'}\right). \qquad [5825]$$

The co-ordinates y, z, y', z', produce similar equations; hence we easily deduce,*

$$P=\frac{m'}{M}\cdot\frac{d.(x'dx-xdx'+y'dy-ydy'+z'dz-zdz')}{dt^2}+Q\;; \qquad [5826]$$

Q being a function of x, y, z, x', y', z', of the order of the [5827]
square of the masses m, m' [5825c, c'']. The variation of the part of P [5826], which is independent of Q, may be expressed by,

$$\delta'P=\frac{m'}{M}\cdot\frac{d.\delta'.(x'dx-xdx'+y'dy-ydy'+z'dz-zdz')}{dt^2}\;; \qquad [5828]$$

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* (3158) The terms depending on x, x', in [5822], are $\frac{m'xx'}{r'^3}-\frac{m'xx'}{r^3}$. The [5825a]
value of this function is found, by multiplying [5823] by $-x'$, also [5825] by $+x$, and then taking the sum of the products. Hence we have,

$$\frac{m'.xx'}{r'^3}-\frac{m'.xx'}{r^3}=\frac{m'}{M}\cdot\frac{(x'ddx-xddx')}{dt^2}+\frac{m'.xx'}{M}\cdot\left\{\frac{m}{r^3}-\frac{m'}{r'^3}\right\}$$
$$+\frac{m'}{M}\cdot\left\{x'\cdot\left(\frac{dR}{dx}\right)-x\cdot\left(\frac{dR'}{dx'}\right)\right\}. \qquad [5825b]$$

If we substitute, in the first term of the second member, for $x'ddx-xddx'$, its value
$d.(x'dx-xdx')$; which is easily proved to be identical, by development; and put Q_1 for [5825c]
the remaining terms of the second member, which are of the order m^2; we get the
expression [5825d]. The similar expressions in y, y', z, z', give [5825e, f]; [5825c']
Q_2, Q_3, being quantities similar to Q_1, and of the order m^2. The sum of the
equations [5825d, e, f], being substituted in [5822], putting $Q=Q_1+Q_2+Q_3$, becomes [5825c'']
as in [5826];

$$m'.xx'.\left(\frac{1}{r'^3}-\frac{1}{r^3}\right)=\frac{m'}{M}\cdot\frac{d.(x'dx-xdx')}{dt^2}+Q_1, \qquad [5825d]$$

$$m'.yy'.\left(\frac{1}{r'^3}-\frac{1}{r^3}\right)=\frac{m'}{M}\cdot\frac{d.(y'dy-ydy')}{dt^2}+Q_2\;; \qquad [5825e]$$

$$m'.zz'.\left(\frac{1}{r'^3}-\frac{1}{r^3}\right)=\frac{m'}{M}\cdot\frac{d.(z'dz-zdz')}{dt^2}+Q_3\;; \qquad [5825f]$$

[5829] and, as this is an exact differential, we shall obtain the part of $\int \mathrm{d}.\delta' P$, which depends on the function [5828], by changing, in this function, d into
[5829′] d [5829e];* and then, it is evident, that it contains, in terms of the order m^2, none but periodical quantities [5829i].

[5830] The term Q will give, in $\int dP$, the quantity $\int dQ$. If we notice

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[5829a] * (3159) If we neglect, for a moment, the consideration of the quantity Q, the remaining part of the second member of [5826] will be an exact differential, which we
[5829b] shall represent by $dP_{,}$; so that we shall have $P = dP_{,}$. Its variation, relative to δ',
[5829c] gives $\delta' P = d.\delta' P_{,}$, which corresponds to [5828]. Integrating this, we get $\int \delta' P = \delta' P_{,}$; and its differential, relative to the characteristic d, gives $\int \mathrm{d}.\delta' P = \mathrm{d}.\delta' P_{,}$. Comparing
[5829d] this with $\delta' P$ [5829c], we easily perceive, that $\int \mathrm{d}.\delta' P$ can be deduced from this
[5829e] expression of $\delta' P$ [5829c, or 5828], by changing, in its second member, d into d, as in [5829′]; hence we shall have,

[5829f]
$$\int \mathrm{d}.\delta' P = \frac{m'}{M} \cdot \frac{\mathrm{d}.\delta'.(x'dx - xdx' + y'dy - ydy' + z'dz - zdz')}{dt^2}.$$

If the function $\frac{1}{dt}.\{x'dx - xdx' + y'dy - ydy' + z'dz - zdz'\}$, by the substitution of the
[5829g] elliptical values of x, y, z, x', y', z', produces a term represented by $A_{,}$; its
[5829h] variation, relative to δ', will become $\delta' A_{,}$; which is of the *first* order relative to the masses, as is evident from the import of δ' [5817]; and, when $\delta' A_{,}$ is multiplied,
[5829i] as in [5829f], by $\frac{m'}{M}$, it becomes of the *second* order. This is finally reduced to the *third* order, in the second member of [5829f], by taking the differential relative to d, of the non-periodical terms; because it produces the differentials of the elements [5786, &c.],
[5829k] which are of the *first* order. Hence it appears, that, if we neglect the non-periodical terms of the *third* order, relative to the masses, we may put the part of $\int \mathrm{d}.\delta' P$, which depends on the function [5828], equal to nothing. Then, there will remain to be noticed
[5829l] only the part of the function $\int \mathrm{d}.\delta' P$, depending on Q [5829a]; which may be
[5829m] represented by $\int \mathrm{d}.\delta' P = \int \mathrm{d}.\delta' Q$. But, Q is of the order m^2 [5827], and, if we represent it by $m^2 A_2$, we shall have $\delta' Q$, of the order $m^2.\delta' A_2$; which is of the
[5829n] *third* order relative to the masses, as is evident from [5829h]: therefore, it may be neglected. What is proved in [5814—5831], relative to the planet m', may also be applied to the other planets m'', m''', &c.; but, it will still be necessary to notice the effect
[5829o] of m'' on m'; m''' on m', &c.; m''' on m'', &c.; in the value of R. This is done in [5833, &c.].

only terms of the order m^2 in dQ, it will suffice to substitute in Q, the elliptical values of the co-ordinates, and then $\int dQ$ will contain only periodical inequalities. Thus, $\int d.\delta P$, will contain only similar quantities. [5831]
Hence it follows, that $\int d.\delta R$ will contain, in terms of the order m^2, only periodical quantities, when we vary in R, the co-ordinates of the [5831']
two planets m and m'.

If there be a third planet m'', *it adds to* R *the function*,*

$$R = \frac{m''.(xx''+yy''+zz'')}{r''^3} - \frac{m''}{\rho'}\,; \qquad \text{[Action of } m'' \text{ on } m] \qquad [5832]$$

ρ' *being the distance from* m'' *to* m. The part of R, relative to the action of m' upon m, then acquires a variation depending on the action of m'' upon m'. This part of R is,

$$R = \frac{m'.(xx'+yy'+zz')}{r'^3} - \frac{m'}{\rho} \quad [5755]\,; \qquad \text{[Action of } m' \text{ on } m] \qquad [5833]$$

the variation of the co-ordinates x', y', z', by the action of m'', produces in [5833], some terms, multiplied by $m'm''$, which are functions of the [5834]
elliptical co-ordinates x, y, z, and of the angles $n't$, $n''t$.† But

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* (3160) The expression [5832] is the same as [5755]; changing the elements of m' into those of m''. It corresponds to the second terms in the expressions of R, λ [5832a]
[913,914].

† (3161) The co-ordinates x', y', z'. contain the elliptical values of the orbit of m', augmented by the terms arising from the action of the bodies m, m'', m''', [5834a]
&c. When these are substituted in [5833], they produce terms of the second order of the masses, which we shall represent by $m'm''.A_3$; A_3 containing among other terms, the quantities x, y, z. The co-ordinates of the planets m', m'', which occur in A_3, [5834b]
introduce the angles $n't$, $n''t$ [5834, 950, 952, 953], and the co-ordinates x, y, z, contain the angle nt. The products of the sines and cosines of such angles, produce, in dR, some terms, which depend on the angles $int+i'n't+i''n''t$ [1214''']; and, as n, n', [5834c]
n'' are incommensurable [1197''], these terms will be periodical. Therefore, by noticing only the non-periodical terms, in dR, we must consider i', i'' as equal to nothing; [5834d]
or, in other words, we must notice in R, only those terms which are independent of $n't$, $n''t$, as in [5835], and then it becomes of the form $m'm''X$ [5836]; X being a function of the co-ordinates of m, as in [5836]. Putting this equal to R, it gives [5834e]

[5835] these angles must vanish from the non-periodical part of dR, and as they cannot be destroyed by the angle nt, which is introduced by means of the values of x, y, z; we need only notice, in the development of the variation R, the terms which are independent of $n't$ and $n''t$. These
[5836] terms will be of the form $m'm''X$; X being a function of the co-ordinates of the planet m, they introduce into $\int dR$ some terms of the form, $m'm''.\int dX$, or $m'm''X$; which produce only non-periodical quantities of the order
[5837] $m'm''$; and such quantities we have neglected in $\int dR$.

In like manner, the variation of the co-ordinates x, y, z, by the
[5838] action of m'', can introduce in the preceding part of R [5833], only the angles nt and $n''t$; therefore, we need only consider, in this part,
[5839] the terms independent of $n't$, consequently of the form $m'm''X$; X being a function of the co-ordinates x, y, z, only; which, as we have
[5840] just seen, can only produce quantities that may be neglected. Thus, by noticing only the non-periodical quantities, of the order m^2, in $\int dR$, we may suppose, that m'' is nothing, when we consider the part of R, relative to the action of m' upon m; and we may suppose m' nothing,
[5841] when we consider the part of R relative to the action of m'' upon m: we have just seen, that in these two cases [5837, 5840], the secular variation of $\int dR$ is nothing. This variation is, therefore, generally nothing,
[5842] when we consider the reciprocal action of three, or of any number of planets, if we only notice as far as the squares and products of the disturbing forces, inclusively, in the value of dR.

We shall now resume the equation [5794],

[5843] $$\zeta = 3\int\int a.ndt.dR.$$

Its variation is,*

[5834f] $dR = m'm''.dX$; whence $\int dR = m'm''.\int dX = m'm''X$; this last expression being deduced from that which precedes it, by omitting the double sign $\int d$, taking into consideration, that X is a function of the co-ordinates of m only, as in [5836];
[5834g] consequently the signs $\int d$ represent *inverse* operations which mutually cancel each other. The variation of the expression $\int dR = m'm''X$, produces in $\int d.\delta R$, non-periodical quantities, of the *third* order of the masses m, m', &c., which are neglected.

[5844a] * (3162) Multiplying $an = a^{-\frac{1}{2}}$ [5751b] by dR, we get $an.dR = a^{-\frac{1}{2}}.dR$

$$\delta\zeta = 3an\,.\textstyle\iint dt.\mathrm{d}.\delta R + 3a^2.\iint(ndt.\mathrm{d}R\,.\int\mathrm{d}R). \qquad [5844]$$

We have just seen [5812, 5821'', &c.], that $\mathrm{d}.\delta R$ is nothing, noticing only the secular quantities, as far as the order of the square of the masses of the planets, inclusively. We have seen, likewise, that $\mathrm{d}R\,.\int\mathrm{d}R$ is nothing,* noticing only the same quantities. Therefore, *if we take into consideration only the secular quantities, which acquire, by the double integration, a denominator of the order of the square of the masses of the planets; we shall find, that the variation* $\delta\zeta$ *vanishes. Hence it is manifest, that, if we notice the secular, as well as the periodical quantities, this variation cannot exceed a term of the the order of the disturbing masses.*† *This important result was first obtained by Mr. Poisson.* [5845] [5846] [5846']

Important result, by M. Poisson.

Taking its variation, and then substituting the value of δa [5796d], also $a^{-\frac{1}{2}} = an$, $a^{\frac{1}{2}} = a^2n$, we get, successively, by neglecting terms of the order m^3, [5844a']

$$\delta.(an.\mathrm{d}R) = \delta.(a^{-\frac{1}{2}}.\mathrm{d}R) = a^{-\frac{1}{2}}.\mathrm{d}.\delta R - \tfrac{1}{2}.a^{-\frac{3}{2}}.\mathrm{d}R.\delta a = an.\mathrm{d}.\delta R + a^2n.\mathrm{d}R\,.\textstyle\int\mathrm{d}R. \qquad [5844b]$$

Multiplying this by $3dt$, and prefixing to the double sign $\iint$, we obtain,

$$3\,\delta\,.\textstyle\iint(an.\mathrm{d}R) = 3an\,.\iint dt.\mathrm{d}.\delta R + 3a^2.\iint(ndt.\mathrm{d}R\,.\int\mathrm{d}R)\,; \qquad [5844c]$$

the terms an, a^2, being placed without the signs $\iint$; which can be done, by neglecting terms of the order m^3, as in [5796c, &c.]. Now, taking the variation δ of ζ [5843], and substituting [5844c], we get [5844]. [5844d]

* (3163) We have seen, in [5810e], that the product of the two equations [5810b], which represents the value of $\frac{1}{ndt}\,.\mathrm{d}R.\int\mathrm{d}R$, produces only periodical inequalities, as in [5845]. [5845a]

† (3164) The elements of the orbit of a planet are represented in [1102, 1133], by systems of terms of the forms,

$$N\,.\,\mathrm{sin.}\,(gt+\beta)\;; \qquad N\,.\,\mathrm{cos.}\,(gt+\beta)\;; \qquad [5846a]$$

in which g is of the same order as that of the disturbing masses m', m'', &c. [1097c]. The double integration of a quantity, depending on an angle of this kind, in [5843 &c.], introduces, into ζ or $\delta\zeta$, the divisor g^2, of the *second* order of the disturbing masses. But, the terms of the *first* and *second* orders, vanish from the expressions in [5845]; therefore, this divisor can operate only upon those of the *third* or higher orders; and, when those of the *third* order m'^3 are divided by g^2, they produce terms of the *first* order only; [5846b] [5846c]

3. *We shall now consider two planets* m and m', *in motion about the*
[5847] *sun, whose mass we shall take for unity.* We shall put v *for the angular*
[5848] *distance of the planet* m, *from the line of intersection of the two orbits;* v' *for the angular distance of the planet* m', *from the same right line; also*
[5849] γ *for the mutual inclination of the two orbits; taking the orbit of* m *for the*
[5850] *plane of the co-ordinates,* and *the line of the nodes of the orbit, for the origin of* x; *we shall have,**

[5851]
$$x = r.\cos.v\ ; \qquad y = r.\sin.v\ ; \qquad z = 0\ ; \qquad 1$$
$$x' = r'.\cos.v'\ ; \qquad y' = r'.\cos.\gamma.\sin.v'\ ; \quad z' = r'.\sin.\gamma.\sin.v'\ ; \qquad 2$$

which gives, by putting,†

[5846d] or, *of the same order as the periodical inequalities.* This agrees with [5846']; and, the importance of this result of Mr. Poisson, is manifest from the consideration, that, if the
[5846e] terms of the second order, relative to the masses, instead of vanishing, as in [5846], were, on the contrary, of their usual magnitude, or of the order m'^2, they would produce, in
[5846f] ζ [5843], by the double integrations, terms of the order $\frac{m'^2}{g^2}$, or of a *finite* order;
[5846g] which might become sensible, in the theory of the planetary motions.

* (3165) The formulas [5851], are as in [3740, 3740'], changing the value of γ,
[5851a] which represents, in [3739, &c.], the tangent of the inclination of the two orbits; but, γ
[5851b] is used for the inclination itself in [5849]; so that, we must change $\frac{1}{\sqrt{1+\gamma^2}}$, $\frac{\gamma}{\sqrt{1+\gamma^2}}$ [3740c], into $\cos.\gamma$, $\sin.\gamma$, respectively; by this means, y', z' [3740'], become as in [5851, line 2].

† (3166) Substituting the values of x, x', &c. [5851], in the first member of
[5853a] [5853b], it becomes as in its second member. Putting, in this, $\cos.\gamma = 1-\beta$ [5852] and successively reducing, we obtain [5853d]. In like manner, by developing the first
[5853a'] member of [5853e], and substituting $x^2+y^2+z^2=r^2$; $x'^2+y'^2+z'^2=r'^2$ [3742d]; we get [5853e, f];

[5853b] $$xx'+yy'+zz' = rr'.\{\cos.v.\cos.v'+\cos.\gamma.\sin.v.\sin.v'\}$$
[5853c] $$= rr'.\{\cos.v.\cos.v'+\sin.v.\sin.v'-\beta.\sin.v.\sin.v'\}$$
[5853d] $$= rr'.\{\cos.(v'-v)-\beta.\sin.v.\sin.v'\}\ ;$$
[5853e] $$(x'-x)^2+(y'-y)^2+(z'-z)^2 = (x^2+y^2+z^2)+(x'^2+y'^2+z'^2)-2.(xx'+yy'+zz')$$
[5853f] $$= r^2+r'^2-2rr'.\cos.(v'-v)+2\beta.rr'.\sin.v.\sin.v'.$$

$$\beta = 1-\cos.\gamma = 2.\sin.^2\tfrac{1}{2}\gamma\,;$$ [5852]

$$R=\frac{m'.(xx'+yy'+zz')}{r'^3}-\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} \qquad 1$$

[5853]

$$=\frac{m'.r}{r'^2}.\{\cos.(v'-v)-\beta.\sin.v.\sin v'\}-\frac{m'}{\sqrt{r^2+r'^2-2rr'.\cos(v'-v)+2\beta.rr'.\sin v.\sin v'}};2$$ R.

under this form, R *becomes independent of the plane to which the co-ordinates are referred* [5853g]. Developing it, in terms of sines and cosines of angles, increasing in proportion to the time t, by the substitution of the elliptical values of r, r', v, v' [952, 953], it becomes a function of the mean angular distances $nt+\varepsilon$, $n't+\varepsilon'$, of the planets, from the line of nodes; of the distances of the perihelia from the same line; of the semi-axes a, a'; of the excentricities e, e'; and of β, or the mutual inclination of the orbits: β being a very small quantity, of the order of the square of that inclination [5852]. Under this form, R does not contain explicitly the variable quantities p and q [1032]; but, we may introduce them in the following manner. [5854] [5855] [5856]

Instead of referring the motions of the planets to their orbits, we may refer them to the fixed plane of the primitive orbit of m*;* then z will not vanish, and it will be represented by $z=rs$;* s being the sine of the latitude of m, above that plane. If we neglect the square of the disturbing forces, we may reject the square of s; then we shall have, instead of R, the following function, which we shall denote by,† [5857] [5858] [5859]

Substituting [5853d,f] in [5755, or 5853 line 1], we get the expression in [5853 line 2]; which is a function of v, v', β [5852]; and these quantities depend entirely on the relative position of the two orbits, and are wholly independent of any arbitrary plane, to which the co-ordinates can be referred; as in [5854]. [5853g]

* (3167) This is similar to [3787], z being the perpendicular, and r the hypothenuse of a right-angled plane triangle, of which the sine of the angle at the base is s. [5858a]

† (3168) If we use $z=rs$ [5858], and z' [5851 line 2], we get,

$$zz'=rr'.s.\sin.\gamma.\sin.v', \quad \text{instead of} \quad zz'=0 \quad [5851];$$ [5859a]

[5860] $\bar{R}$.

$$\bar{R} = \frac{m'.r}{r'^2}.\{\cos.(v'-v)-\beta.\sin.v.\sin.v'+s.\sin.\gamma.\sin.v'\} \qquad 1$$

$$-\frac{m'}{\sqrt{r^2+r'^2-2rr'.\cos.(v'-v)+2\beta.rr'.\sin.v.\sin.v'-2rr'.s.\sin.\gamma.\sin.v'}}. \qquad 2$$

[5861] In these values of R and $\bar{R}$, we shall subtract, from v, v', the longitude θ' of the node of the orbit of m' upon m; this longitude being counted in the orbit of m. This is equivalent to a change in *the*
[5862] *origin of* v *and* v'; and we shall suppose,*

[5863] $$s = q.\sin.(v-\theta') - p.\cos.(v-\theta').$$

Then we shall have,†

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hence the value of $xx'+yy'+zz'$ [5853d] must be increased by that quantity; moreover, the value of $(x'-x)^2+(y'-y)^2+(z'-z)^2$ [5853f], which contains
[5859b] $-2.(xx'+yy'+zz')$, must be, for the same reason, augmented by the term $-2rr'.s.\sin.\gamma.\sin.v'$; and these corrections being applied to the corresponding terms of [5853 line 2], it becomes as in [5860]; observing, that the value of $y=r.\sin.v$ [5851 line 1], may be retained in this hypothesis, if we neglect quantities of the order s^2. For, the correct value of y, being similar to that of y' [5851]; namely,

[5859c] $$y=r.\cos.\gamma_{,}.\sin.v=r.\sin.v-r.\sin.v.(2.\sin.\tfrac{1}{2}\gamma_{,})^2;$$

$\gamma_{,}$ being of the order s, we may, by neglecting s^2, suppose

[5859d] $$y=r.\sin.v, \quad \text{as in [5851 line 1]}.$$

[5862a] * 3169) The expression [5863], is like that in [1335'], altering the origin of the angles v, by writing $v-\theta'$ for v, as in [5861]. We may remark, that the angle $v-\theta'$ is counted, as in [3739], from the line of nodes, or mutual intersection of the
[5862b] orbits, *on the orbit of m*; and the angle $v'-\theta'$ is counted from the same line of nodes, *on the orbit of m'*; so that we may consider the origin of the angle v to be *on the orbit of* m, at a point, which is distant, by the angle θ', from the node, and
[5862c] *counted upon the orbit of m*. In like manner, the origin of the angle v' is taken upon the *plane of the orbit of m'*, and at the same distance θ' from the node, but counted
[5862d] *on the orbit of m'*. This is evident from the investigation, in [3737—3740'], of the formulas [3740, 3740'], which are similar to those in [5851].

† (3170) If we decrease the angles v, v' by θ', as in [5861], we shall find, that the angle $v'-v$ is not altered; but the expression $\sin.v.\sin.v'$ becomes,

[5864a] $$\sin.(v-\theta').\sin.(v'-\theta');$$

$$R=\frac{m'.r}{r'^2}.\{(1-\tfrac{1}{2}\beta).\cos.(v'-v)+\tfrac{1}{2}\beta.\cos.(v'+v-2\theta')\}$$ 1

$$-\frac{m'}{\sqrt{r^2+r'^2-2rr'.\{(1-\frac{1}{2}\beta).\cos.(v'-v)+\frac{1}{2}\beta.\cos.(v'+v-2\theta')\}}};$$ 2 [5864]

$$\bar{R}=\frac{m'.r}{r'^2}.\left\{\begin{array}{c}(1-\frac{1}{2}\beta+\frac{1}{2}q.\sin.\gamma).\cos.(v'-v)+(\frac{1}{2}\beta-\frac{1}{2}q.\sin.\gamma).\cos.(v'+v-2\theta')\\ -\frac{1}{2}p.\sin.\gamma.\sin.(v'-v)-\frac{1}{2}p.\sin.\gamma.\sin.(v'+v-2\theta')\end{array}\right\}$$ 1, 2

$$-\frac{m'}{\sqrt{r^2+r'^2-2rr'.\left\{\begin{array}{c}(1-\frac{1}{2}\beta+\frac{1}{2}q.\sin.\gamma).\cos.(v'-v)+(\frac{1}{2}\beta-\frac{1}{2}q.\sin.\gamma).\cos.(v'+v-2\theta')\\ -\frac{1}{2}p.\sin.\gamma.\sin.(v'-v)-\frac{1}{2}p.\sin.\gamma.\sin.(v'+v-2\theta')\end{array}\right\}}};$$ 3, 4 [5865]

now, it is evident, that we may change R into $\bar{R}$, if we vary in R, β by $\delta\beta$; v by δv; and θ' by $\delta\theta'$ [5867i]; so that we may have,* [5866]

$$\delta\beta=-q.\sin.\gamma;$$ 1

$$(1-\tfrac{1}{2}\beta).\delta v=\cos.^2\tfrac{1}{2}\gamma.\delta v=-\tfrac{1}{2}p.\sin.\gamma;$$ 2 [5867]

$$\beta.\delta\theta'-\tfrac{1}{2}\beta.\delta v=-\tfrac{1}{2}p.\sin.\gamma.$$ 3

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which may be reduced to,

$$\tfrac{1}{2}\cos.(v'-v)-\tfrac{1}{2}\cos.(v'+v-2\theta')\quad[17]\text{ Int.}$$ [5864b]

Substituting these in [5853 line 2], we get [5864]. Now if we multiply the expression [5863], by $\sin.\gamma.\sin.(v'-\theta')$, and reduce the product, by means of [17, 18] Int., we get,

$$s.\sin.\gamma.\sin.(v'-\theta')=\sin.\gamma.\left\{\begin{array}{c}\frac{1}{2}q.\cos.(v'-v)-\frac{1}{2}q.\cos.(v'+v-2\theta')-\frac{1}{2}p.\sin.(v'-v)\\ -\frac{1}{2}p.\sin.(v'+v-2\theta')\end{array}\right\}.$$ [5864c]

Substituting, in [5860], for $\sin.v.\sin.v'$, its value [5864b]; and for $s.\sin.\gamma.\sin.v'$, its value [5864c], we obtain [5865].

* (3171) If we put the factor of $\frac{m'.r}{r'^2}$ in [5864 line 1], equal to w; and that of [5865] equal to $w+\delta w$; we shall have, [5867a]

$$w=(1-\tfrac{1}{2}\beta).\cos.(v'-v)+\tfrac{1}{2}\beta.\cos.(v'+v-2\theta');$$ [5867b]

$$\delta w=\tfrac{1}{2}q.\sin.\gamma.\cos.(v'-v)-\tfrac{1}{2}p.\sin.\gamma.\sin.(v'-v)-\tfrac{1}{2}q.\sin.\gamma.\cos.(v'+v-2\theta')-\tfrac{1}{2}p.\sin.\gamma.\sin.(v'+v-2\theta').$$ [5867c]

Then R [5864], being considered as a function of w; that of $\bar{R}$ [5865], will be a similar function of $w+\delta w$. Now, if we take the variations of w [5867b], considering β, v, θ', as variable, and neglect the second and higher powers and products of $\delta\beta$, [5867c']

Thus we get,

[5868] $$\bar{R} = R - q.\sin.\gamma \cdot \left(\frac{dR}{d\beta}\right) - p.\text{tang}.\tfrac{1}{2}\gamma \cdot \left(\frac{dR}{dv}\right) - \frac{p}{\sin.\gamma} \cdot \left(\frac{dR}{d\theta'}\right);$$

and we have, as in [5763f],

[5869] $$\left(\frac{dR}{dv}\right) = \frac{\mathrm{d}R}{ndt} + \left(\frac{dR}{d\varpi}\right).$$

This being premised, the equations [5790, 5791] give* the two following

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δv, $\delta\theta'$, we get,

[5867d] $$\delta \text{w} = -\tfrac{1}{2}\delta\beta.\cos.(v'-v) + (1-\tfrac{1}{2}\beta).\delta v.\sin.(v'-v) + \tfrac{1}{2}\delta\beta.\cos.(v'+v-2\theta') + (\beta.\delta\theta' - \tfrac{1}{2}\beta.\delta v).\sin.(v'+v-2\theta').$$

Comparing the coefficients of $\cos.(v'-v)$, $\sin.(v'-v)$ &c., in the expressions of δw [5867c, d], we obtain,

[5867e] $$-\tfrac{1}{2}\delta\beta = \tfrac{1}{2}q.\sin.\gamma; \quad (1-\tfrac{1}{2}\beta).\delta v = -\tfrac{1}{2}p.\sin.\gamma; \quad \beta.\delta\theta' - \tfrac{1}{2}\beta.\delta v = -\tfrac{1}{2}p.\sin.\gamma.$$

These equations agree with those in [5867]; observing, in [5867 line 2, 5852], that we
[5867f] have $1-\tfrac{1}{2}\beta = 1-\sin.^2\tfrac{1}{2}\gamma = \cos.^2\tfrac{1}{2}\gamma$. If we substitute $\sin.\gamma = 2.\sin.\tfrac{1}{2}\gamma.\cos.\tfrac{1}{2}\gamma$ [31]
[5867g] Int., in [5867 line 2], and divide the result by $\cos.^2\tfrac{1}{2}\gamma$, we get $\delta v = -p.\text{tang}.\tfrac{1}{2}\gamma$.
[5867g′] Subtracting the equation [5867 line 2], from that in [5867 line 3], we get, $\beta.\delta\theta' - \delta v = 0$, hence,

[5867h] $$\delta\theta' = \frac{\delta v}{\beta} = -\frac{p.\text{tang}.\tfrac{1}{2}\gamma}{2.\sin.^2\tfrac{1}{2}\gamma} = -\frac{p}{2.\sin.\tfrac{1}{2}\gamma.\cos.\tfrac{1}{2}\gamma} = -\frac{p}{\sin.\gamma} \quad [5867g, 5852].$$

It is evident, by inspection, that the symbols β, v, θ', occur in R [5864], by means of the quantity w only [5867b]; hence it is plain, from [5867c′], that we may
[5867i] consider R as a function of β, v, θ'; and $\bar{R}$ as a similar function of $\beta+\delta\beta$, $v+\delta v$, $\theta'+\delta\theta'$, as in [5866]. If we develop $\bar{R}$, according to the powers and products of $\delta\beta$,
[5867k] δv, $\delta\theta'$, by formulas [610—612], and retain only the first power of these quantities, which are of the order m', we shall have,

[5867l] $$\bar{R} = R + \left(\frac{dR}{d\beta}\right).\delta\beta + \left(\frac{dR}{dv}\right).\delta v + \left(\frac{dR}{d\theta'}\right).\delta\theta'.$$

Substituting in this, the expressions of $\delta\beta$, δv, $\delta\theta'$ [5867 line 1, 5867g, h], we get [5868].

* (3172) The function $\bar{R}$ [5860], is equivalent to R, in the formulas [5790, 5791],
[5869a] where the fixed plane is supposed to be the primitive orbit of m [5775′ line 2]. Therefore we must substitute $\bar{R}$ [5868], for R, in the values of dp, dq [5790, 5791],

expressions,

$$dp = \frac{andt}{\sqrt{1-e^2}}\cdot \sin.\gamma\cdot\left(\frac{dR}{d\beta}\right); \qquad (8) \qquad [5870]$$

$$dq = -\frac{andt}{\sqrt{1-e^2}.\sin.\gamma}\cdot\left\{\left(\frac{dR}{d\theta'}\right)+\beta\cdot\left[\frac{\mathrm{d}R}{ndt}+\left(\frac{dR}{d\varpi}\right)\right]\right\}. \qquad (9) \qquad [5871]$$

Connecting these equations with those in [5786—5789, 5794], we shall have, by taking the differential of the terms of the development of R, the corresponding terms of each of the elements of the motion of m. This facilitates very much the computation of these different terms. We shall put, [5871']

$$R = m'.k.\cos.(i'n't - int + i'\varepsilon' - i\varepsilon - g\varpi - g'\varpi' - 2g''\theta'), \qquad [5872]$$

for one of the terms of the development of R. Then, the corresponding terms *of the semi-major axis* a; *of the mean motion* $\int ndt$; *of the epoch* ε; *of the excentricity* e; *of the longitude of the perihelion* ϖ; [5872'] *and of the quantities* p, q; *will be represented by the following expressions, respectively;**

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observing, that the partial differentials of [5868], relative to p, q, give the expressions [5869b—d], by using [5869];

$$\left(\frac{d\overline{R}}{dq}\right) = -\sin.\gamma\cdot\left(\frac{dR}{d\beta}\right); \qquad [5869b]$$

$$\left(\frac{d\overline{R}}{dp}\right) = -\tan g.\tfrac{1}{2}\gamma\cdot\left(\frac{dR}{dv}\right) - \frac{1}{\sin.\gamma}\cdot\left(\frac{dR}{d\theta'}\right) \qquad [5869c]$$

$$= -\tan g.\tfrac{1}{2}\gamma\cdot\left\{\frac{\mathrm{d}R}{ndt}+\left(\frac{dR}{d\varpi}\right)\right\} - \frac{1}{\sin.\gamma}\cdot\left(\frac{dR}{d\theta'}\right). \qquad [5869d]$$

Now we have,

$$\tan g.\tfrac{1}{2}\gamma = \frac{\sin.\frac{1}{2}\gamma}{\cos.\frac{1}{2}\gamma} = \frac{2.\sin.^2\frac{1}{2}\gamma}{2.\sin.\frac{1}{2}\gamma.\cos.\frac{1}{2}\gamma} = \frac{1-\cos.\gamma}{\sin.\gamma} = \frac{\beta}{\sin.\gamma} \quad [5852]. \qquad [5869e]$$

Substituting this in [5869d], and then using the resulting value for $\left(\frac{dR}{dp}\right)$, in [5791], we get [5871]. In like manner, the substitution of [5869b], for $\left(\frac{dR}{dq}\right)$, in [5790], gives [5870].

* (3173) Substituting the expression R [5872], in the first member of the

[5873] $$\delta a = \frac{2m'.a^2.in}{i'n'-in} \cdot k.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta');$$

[5874] $$\delta\zeta = \delta.\int ndt = -\frac{3m'.in^2}{(i'n'-in)^2}.ak.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta');$$

[5875] $$\delta\varepsilon = -\frac{m'.an}{i'n'-in}.\left\{(1-\sqrt{1-e^2}).\frac{\sqrt{1-e^2}}{e}.\left(\frac{dk}{de}\right)-2a.\left(\frac{dk}{da}\right)\right\}\sin(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta');$$

Periodical inequalities in the elements.

[5876] $$\delta e = -\frac{m'.an.\sqrt{1-e^2}}{e}.k.\frac{\{g+i.(1-\sqrt{1-e^2})\}}{i'n'-in}.\cos(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta');$$

[5877] $$\delta\varpi = -\frac{m'.an.\sqrt{1-e^2}}{e.(i'n'-in)} \cdot \left(\frac{dk}{de}\right) \cdot \sin.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta');$$

[5878] $$\delta p = \frac{m'.an.\sin.\gamma}{\sqrt{1-e^2}.(i'n'-in)} \cdot \left(\frac{dk}{d\beta}\right) \cdot \sin.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta');^*$$

[5879] $$\delta q = \frac{2m'.an.k}{(i'n'-in).\sqrt{1-e^2}.\sin.\gamma}.\{g''+(i+g).\sin.^2\tfrac{1}{2}\gamma\}.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta');$$

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following integral, we get,

[5869f] $$\int dR = \frac{-in}{(i'n'-in)} \cdot m'k.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta').$$

Substituting this in [5796d], we obtain δa [5873]. The same value of R, being used in ζ [5794], gives [5874]. In like manner, from $\delta\varepsilon$, δe, $\delta\varpi$ [5796f—h], we deduce [5875, 5876, 5877].

* (3174) Taking the partial differentials of R [5872], relative to β, θ', $\int ndt$,
[5878a] ϖ, and using for brevity, $T = i'n't - int + i'\varepsilon' - i\varepsilon - g\varpi - g'\varpi' - 2g''\theta'$; we get,

[5878b] $$\left(\frac{dR}{d\beta}\right) = m'.\left(\frac{dk}{d\beta}\right).\cos.T; \qquad \left(\frac{dR}{d\theta'}\right) = 2m'.k.g''.\sin.T;$$

[5878c] $$\frac{dR}{ndt} = m'.k.i.\sin.T; \qquad \left(\frac{dR}{d\varpi}\right) = m'.k.g.\sin.T.$$

Substituting the first of the values [5878b], in [5870], we get, by integration, p or δp [5878]; and by using the remaining three equations, we obtain from [5871], the expression
[5878d] of δq [5879]; observing that we have, $\beta = 2.\sin.^2\frac{1}{2}\gamma$. [5852].

These results are conformable to those in chap. viii, of the second book; but these new expressions have the great advantage of including all the powers of the excentricities and inclinations.* [5880]

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* (3175) We may show, that the expressions of δa, $\delta.\mathit{f}ndt$, δe, $\delta\varpi$ [5873, 5874, 5876, 5877], are similar to those in [1197, 1286, 1294, &c.], in the following manner. The assumed value of R in [1195'], is $R=m'.k.\cos.(i'\zeta'-i\zeta+A)$; so that, if we [5880a] substitute the mean values $\zeta'=n't$, $\zeta=nt$, and $A=i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-2g''\theta'$, it becomes, by using T [5878a], $R=m'.k.\cos.T$, as in [5872]. Substituting these [5880b] values of A, T, and $\mu=1$ [5750], in [1197]; prefixing also the sign δ before [5880c] the terms, in the first members of these equations, to conform to the present notation; we get,

$$\delta.\left(\frac{1}{a}\right)=-\frac{2m'.in}{i'n'-in}.k.\cos.T;\qquad \delta\zeta=-\frac{3m'.in^2}{(i'n'-in)^2}.ak.\sin.T. \qquad [5880d]$$

Now, by neglecting the square and higher powers of δa, we have $\delta.\left(\frac{1}{a}\right)=-\frac{\delta a}{a^2}$; [5880e] substituting this in the first of the equations [5880d], and then multiplying by $-a^2$, we get δa [5873]. The expression of $\delta\zeta$ [5880d], is the same as that in [5874]. Again, if we neglect e^2, as in [1283'], we may change the factor $\sqrt{1-e^2}$ into 1, in [5876], and then it will become,

$$\delta e=-\frac{m'.an}{e}.k.\frac{g}{i'n'-in}.\cos.T, \qquad [5880f]$$

as in [1286, 1285]. In like manner, if we change the factor $\sqrt{1-e^2}$ into 1, in [5877], and multiply the expression by e, we get,

$$e\delta\varpi=-\frac{m'.an}{i'n'-in}.\left(\frac{dk}{de}\right).\sin.T; \qquad [5880g]$$

which is the same as the integral of $ed\varpi$ [1294].

The expression of $\delta\varepsilon$ [5875], may be derived from that in [1345], neglecting terms of the order e^3. For, if we multiply $ed\varpi$ [1258], by $-\frac{1}{2}e$, and add the product to [1345], we get, by reduction, an expression of $d\varepsilon-\frac{1}{2}e^2d\varpi$, which is equivalent to [5880h] that in [5775'], using the value of r [5769]; and, from this we easily obtain [5875]. We have thought it unnecessary to go through the details of this calculation, as it is evident [5880i] that the result must correspond with [5775']. For similar reasons, we shall omit the reduction of δp, δq, [5878, 5879], to the forms [1341, &c.]

[5880′] *We shall have the secular variations of the elements of the orbit of* m, *by reducing* R *to its non-periodical part, which we shall denote by,*

[5881] $$R = m'F.$$ [Non-periodical part of R]

[5881′] Then $\mathrm{d}R$ vanishes,* as well as da, and we shall have,

[5882] $$d\varepsilon = -\frac{m'.andt.\sqrt{1-e^2}}{e}.(1-\sqrt{1-e^2}).\left(\frac{dF}{de}\right)+2a^2.\left(\frac{dF}{da}\right).m'.ndt;$$

[5883] $$de = \frac{m'.a.\sqrt{1-e^2}}{e}.ndt.\left(\frac{dF}{d\varpi}\right);$$

[5884] $$d\varpi = -\frac{m'.andt.\sqrt{1-e^2}}{e}.\left(\frac{dF}{de}\right);$$

Secular inequalities.

[5885] $$dp = -\frac{m'.andt}{\sqrt{1-e^2}}.\left(\frac{dF}{dq}\right);$$

[5886] $$dq = \frac{m'.andt}{\sqrt{1-e^2}}.\left(\frac{dF}{dp}\right);$$

or,

[5887] $$dp = \frac{m'.andt}{\sqrt{1-e^2}}.\sin.\gamma.\left(\frac{dF}{d\beta}\right);$$

[5888] $$dq = -\frac{m'.andt}{\sqrt{1-e^2}.\sin.\gamma}.\left\{\left(\frac{dF}{d\theta'}\right)+\beta.\left(\frac{dF}{d\varpi}\right)\right\}.$$

We may here observe, that we have, as in [5755],

[5889] $$R = \frac{m'.(xx'+yy'+zz')}{r'^3} - \frac{m'}{\rho};$$

[5882a] * (3176) Taking for R its non-periodical part $m'F$, we shall have $\mathrm{d}R = 0$ [5812, 5821′, 5831,&c.]. Subtituting this in [5786], we get $da=0$ [5881′]. With this

[5882b] value of $\mathrm{d}R$, and $\left(\frac{dR}{d\varpi}\right) = m'.\left(\frac{dF}{d\varpi}\right)$ [5881], we obtain, from [5788], the expression of de [5883]. In like manner, from [5789], we get [5884]; from [5787], we obtain [5882]; from [5790, 5791], we deduce [5885, 5886] respectively; lastly, from [5870, 5871],

[5882c] we get [5887, 5888], respectively. In all the equations [5882—5888], quantities of the

[5882d] order m'^2 are neglected; but they are exact in terms of the order m', for all powers and products of the excentricities and inclinations.

and by neglecting quantities of the order m'^2 it becomes,* [5889']

$$R = - m' \cdot \frac{(xddx'+yddy'+zddz')}{dt^2} - \frac{m'}{\rho};$$ [5890]

Therefore, the non-periodical terms of R depend on the non-periodical part of $-\frac{m'}{\rho}$; hence we have,†

$$F = \text{non-periodical part of } \frac{R}{m'} = \text{non-periodical part of} - \frac{1}{\rho};$$ [5890']

this part being developed in a series of cosines of angles, increasing in proportion to the time t; *and* F *is the same, for both planets* [5756]. If we [5891] vary in F, the elements e, ϖ, p, q, of the orbit of m, and substitute for δe, $\delta\varpi$, δp, δq, their values, which are given by the integrals of the preceding

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* (3177) If we neglect terms of the order m'^2 in [5825] we get,

$$\frac{m'x'}{r'^3} = - \frac{m'}{M} \cdot \frac{ddx'}{dt^2}; \quad \text{or, simply,} \quad \frac{m'x'}{r'^3} = - m' \cdot \frac{ddx'}{dt^2};$$ [5890a]

because, by neglecting quantities of the order m'^2, we may put $M=1$ [3709a]. [5890b] In like manner, we have,

$$\frac{m'y'}{r'^3} = - m' \cdot \frac{ddy'}{dt^2}; \qquad \frac{m'z'}{r'^3} = - m' \cdot \frac{ddz'}{dt^2}.$$ [5890c]

Multiplying these three equations by x, y, z, respectively, and taking the sum of the products, we get,

$$\frac{m' \cdot (xx'+yy'+zz')}{r'^3} = - m' \cdot \frac{(xddx'+yddy'+zddz')}{dt^2}.$$ [5890d]

Substituting this in [5889], we obtain [5890].

† (3178) If we neglect terms of the order m'^2, we may substitute the elliptical values of x, y, z, x', y', z' [950, 952, 953, &c.], in the terms of the second member of [5890], which are divided by dt^2; and then we shall see, that it contains no terms of the proposed order, except such as are periodical. For, if x' contain a non-periodical term, its second differential ddx' will depend on the differentials of the elements a', e', &c., which are of the order R, or m [5786, &c.]; and, [5891a] [5891b] [5891c]

[5892] differential equations [5883—5886], we shall find, that δF vanishes,* and the same result is obtained with the variations of the elements of the orbit of m'. This is demonstrated, in [3767], supposing the terms of fourth and higher orders of the excentricities and inclinations to be neglected.

We have, as in [5867 line 1, 5867h],

[5893] $$\delta\beta = -q.\sin.\gamma \,; \qquad \delta\theta' = -\frac{p}{\sin.\gamma}\,.$$

[5894] If we suppose, that $\delta\beta$ and $\delta\theta'$ are increased by the quantities $d\beta$, $d\theta'$, respectively, we shall have,†

[5891d] when ddx' is multiplied by $m'x$, as in [5890], it becomes of the same order as the neglected terms [5889']. It is unnecessary to notice the periodical terms of ddx', because they produce no non-periodical terms of the first order in $m'.xddx'$; therefore, this term may be neglected; and, for similar reasons, we may reject $m'.yddy'$, $m'.zddz'$.

[5891e] Hence we have, by noticing only the non-periodical terms, $R = -\frac{m'}{\rho}$ [5890]. Substituting this in [5881], and dividing by m', we get $F = -\frac{1}{\rho}$, as in [5890']. Finally, as the value of ρ [5756] is symmetrical, in the co-ordinates of the two planets

[5891f] x, y, z, x', y', z', respectively; it is plain, that the non-periodical part of R, or F, must be the same for both planets, as in [5891].

* (3179) If we vary in F, the elements e, ϖ, p, q, of the orbit of m, we shall get, in like manner as in [5795a—b, 5795], by noticing only the secular variations of these elements;

[5892a] $$\delta F = \left\{\left(\frac{dF}{de}\right).\delta e + \left(\frac{dF}{d\varpi}\right).\delta\varpi\right\} + \left\{\left(\frac{dF}{dp}\right).\delta p + \left(\frac{dF}{dq}\right).\delta q\right\}.$$

[5892b] The integrals of the values of de, $d\varpi$, dp, dq [5883, 5884, 5885, 5886], are found, by changing, in these functions, dt into t, neglecting terms of the order m'^2; by this means, we get δe, $\delta\varpi$, δp, δq, respectively. Substituting these values of

[5892c] δe, $\delta\varpi$, in [5892a], we find, that the terms depending on these quantities mutually destroy each other. In like manner, the terms which depend on δp, δq, mutually destroy each other in [5892a]; therefore, the whole of the second member of [5892a]

[5892d] vanishes, and we have, as in [5892], $\delta F = 0$. In a similar manner, we find, that δF vanishes, by the substitution of the variations of the elements e', ϖ', p', q', of the planet m'.

[5893a] † (3180) Taking the differentials of [5893], and writing, as in [5894], $d\beta$, $d\theta'$, for $d.\delta\beta$, $d.\delta\theta'$, we get,

$$d\beta = -\, dq.\sin.\gamma\,; \qquad d\delta' = -\,\frac{dp}{\sin.\gamma}. \qquad [5894']$$

Substituting the values of dp, dq, we shall get,*

$$d\delta' = -\,\frac{m'.andt}{\sqrt{1-e^2}}\cdot\left(\frac{dF}{d\beta}\right); \qquad [5895]$$

$$d\gamma = \frac{m'.andt}{\sqrt{1-e^2}.\sin.\gamma}\cdot\left\{\left(\frac{dF}{d\delta'}\right)+\beta\cdot\left(\frac{dF}{d\varpi}\right)\right\}. \qquad [5896]$$

We have,†

$$d\beta = -\,dq.\sin.\gamma - qd\gamma.\cos.\gamma\,; \qquad d\delta' = -\,\frac{dp}{\sin.\gamma}+\frac{pd\gamma.\cos.\gamma}{\sin.^2\gamma}. \qquad [5893b]$$

Now, γ [5849] is of the same order as the greatest latitude of the planet m', above the orbit of m; and this varies, in consequence of the perturbations of the latitude, by quantities of the order m. Moreover, p, q [5863], are of the same order as s, which is of the order m [5858]; therefore, $pd\gamma$, $qd\gamma$, are of the *second* order in m, m', and may be neglected; hence the formulas [5893b], become as in [5894']. [5893c]

* (3181) Substituting dp [5887], in the expression of $d\delta'$ [5894'], we get [5895]; moreover, the differential of $\beta = 1-\cos.\gamma$ [5852], gives $d\beta = d\gamma.\sin.\gamma$. [5895a] Now, it is evident, that we may put this value of $d\beta$ equal to that in [5894']; because β would be constant, if it were not for the mutual action of the planets; so that the whole of this variation of β, arises from that of $\delta\beta$; hence we get, [5895b]

$$-\,dq.\sin.\gamma = d\gamma.\sin.\gamma\,; \quad \text{consequently,} \quad d\gamma = -\,dq. \qquad [5895c]$$

Substituting the value of dq [5888], we get [5896].

† (3182) If we put $g'''=g''$, $\theta=\theta'$, in the term of R [958], it becomes of the same form as in [5872]. Making these substitutions in [959], we get,

$$0 = i'-i-g-g'-2g''\,; \qquad [5897a]$$

which must be satisfied for all the terms of R [5872]. Now, F [5881] comprises the non-periodical terms of R, or those which do not contain $i'n't-int$ [5872]; and, as n, n' are incommensurable [1197'], we must necessarily have, in this case, $i'=0$, $i=0$. Substituting these values of i', i, in [5897a], we get, [5897b]

$$0 = g+g'+2g'', \quad \text{as in [5899]}\,; \qquad [5897c]$$

and the value of R [5872] becomes,

[5897] $$\left(\frac{dF}{d\theta'}\right) = -\left(\frac{dF}{d\varpi}\right) - \left(\frac{dF}{d\varpi'}\right);$$

because, if F be developed in cosines of the form,

[5898] $$F = H.\cos.(g\varpi + g'\varpi' + 2g''\theta');$$

the sum $g+g'+2g''$ of the coefficients of the angles ϖ, ϖ', θ', must
[5899] be equal to nothing, to render this term independent of the arbitrary origin of those angles [5897c]. Therefore, we have,*

[5900] $$d\gamma = -\frac{m'.andt}{\sqrt{1-ee}.\sin.\gamma}.\left\{(1-\beta).\left(\frac{dF}{d\varpi}\right) + \left(\frac{dF}{d\varpi'}\right)\right\}.$$

Hence we obtain, by means of the preceding expressions of de, de',†

[5897c'] $$R = m'.k\cos.(-g\varpi - g'\varpi' - 2g''\theta') = m'.k.\cos.(g\varpi + g'\varpi' + 2g''\theta').$$

Hence we get, by means of [5890'],

[5897d] $$F = k.\cos.(g\varpi + g'\varpi' + 2g''\theta'), \quad \text{as in [5898]};$$

H being used for k. The partial differentials of F, relative to ϖ, ϖ', θ' give,
[5897e] by putting, for abridgement, $\text{w} = g\varpi + g'\varpi' + 2g''\theta'$,

[5897f] $$\left(\frac{dF}{d\varpi}\right) = -gk.\sin.\text{w}; \qquad \left(\frac{dF}{d\varpi'}\right) = -g'k.\sin.\text{w}; \qquad \left(\frac{dF}{d\theta'}\right) = -2g''k.\sin.\text{w};$$

hence,

[5897g] $$\left(\frac{dF}{d\varpi}\right) + \left(\frac{dF}{d\varpi'}\right) + \left(\frac{dF}{d\theta'}\right) = -k.(g+g'+2g'').\sin.\text{w} = 0 \quad [5897c].$$

This last expression is equivalent to that in [5897].

[5900a] * (3183) Substituting the value of $\left(\frac{dF}{d\theta'}\right)$ [5897], in [5896], we get [5900].

[5901a] † (3184) The expression of $d\gamma$ [5900] depends upon the disturbing force of m'; and, if we call this part $d\gamma_1$, and put the other part, depending upon the disturbing
[5901b] force of m upon m', equal to $d\gamma_2$, we shall have the whole value $d\gamma = d\gamma_1 + d\gamma_2$. Substituting $d\gamma_1$ for $d\gamma$, in [5900], also $1-\beta = \cos.\gamma$ [5852], then multiplying
[5901c] by $\frac{\sin.\gamma}{\cos.\gamma}$, we get [5901e]. Multiplying [5883] by $\frac{e}{1-ee}$, we obtain [5901f]; adding this to [5901e], we get the first of the formulas [5901g]; and, by substituting the value of $\left(\frac{dF}{d\varpi'}\right) = \frac{e'de'}{m.a'n'dt.\sqrt{(1-e'^2)}}$, which is easily deduced from [5883], by changing reciprocally
[5901c'] the elements of m into those of m', which does not change F [5891], we get the last

$$\frac{d\gamma.\sin.\gamma}{\cos.\gamma}+\frac{ede}{1-e^2}+\frac{e'de'}{1-e'^2}=-\frac{m.a'n'.ede}{m'.an.\sqrt{1-e^2}.\sqrt{1-e'^2}.\cos.\gamma}$$
$$-\frac{m'.an.e'de'}{m.a'n'.\sqrt{1-e^2}.\sqrt{1-e'^2}.\cos.\gamma}. \qquad [5901]$$

Multiplying this equation by $-2.\sqrt{1-e^2}.\sqrt{1-e'^2}.\cos.\gamma$, and taking its integral, we get,*

$$2.\sqrt{1-e^2}.\sqrt{1-e'^2}.\cos.\gamma=\text{constant}-\frac{m.\sqrt{a}}{m'.\sqrt{a'}}.(1-e^2)-\frac{m'.\sqrt{a'}}{m.\sqrt{a}}.(1-e'^2). \qquad [5902]$$

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expression in [5901g]. The similar formula, corresponding to the action of m on m', is found, by changing the elements of m into those of m', and the contrary; by this means, [5901d]
we get [5901h]. Adding together the expressions [5901g,h], and substituting $d\gamma$ [5901b], we get [5901];

$$\frac{d\gamma_1.\sin.\gamma}{\cos.\gamma}=-\frac{m'.andt}{\sqrt{1-e^2}}.\left\{\left(\frac{dF}{d\varpi}\right)+\frac{1}{\cos.\gamma}.\left(\frac{dF}{d\varpi'}\right)\right\}; \qquad [5901e]$$

$$\frac{ede}{1-e^2}=-\frac{m'.andt}{\sqrt{1-e^2}}.\left\{-\left(\frac{dF}{d\varpi}\right)\right\}; \qquad [5901f]$$

$$\frac{d\gamma_1.\sin.\gamma}{\cos.\gamma}+\frac{ede}{1-e^2}=-\frac{m'.andt}{\sqrt{1-e^2}}.\frac{1}{\cos.\gamma}.\left(\frac{dF}{d\varpi'}\right)=-\frac{m'.an.e'de'}{m.a'n'.\sqrt{1-e^2}.\sqrt{1-e'^2}.\cos.\gamma}; \qquad [5901g]$$

$$\frac{d\gamma_2.\sin.\gamma}{\cos.\gamma}+\frac{e'de'}{1-e'^2}=-\frac{m.a'n'.ede}{m'.an.\sqrt{1-e^2}.\sqrt{1-e'^2}.\cos.\gamma}. \qquad [5901h]$$

* (3185) Multiplying the equation [5901] by $-2.\sqrt{1-e^2}.\sqrt{1-e'^2}.\cos.\gamma$, we obtain,

$$-2.\sqrt{1-e^2}.\sqrt{1-e'^2}.d\gamma.\sin.\gamma-\frac{2ede.\sqrt{(1-e'^2)}.\cos.\gamma}{\sqrt{(1-e^2)}}-\frac{2e'de'.\sqrt{(1-e^2)}.\cos.\gamma}{\sqrt{(1-e'^2)}}$$
$$=\frac{m.a'n'}{m'.an}.2ede+\frac{m'.an}{m.a'n'}.2e'de'. \qquad [5902a]$$

The integral of the first member of this equation, is the same as that in [5902]. In the second member, we must substitute $an=a^{-\frac{1}{2}}$, $a'n'=a'^{-\frac{1}{2}}$ [5778a], and it becomes, [5902b]

$$\frac{m.\sqrt{a}}{m'.\sqrt{a'}}.2ede+\frac{m'.\sqrt{a'}}{m.\sqrt{a}}.2e'de'; \qquad [5902c]$$

which, by integration, gives the second member of [5902]. Finally, we may observe, that, in all the differential equations [5882—5902], we have neglected terms of the second order in m, m'. [5902d]

If we put, for brevity,

[5903] $$\sqrt{a.(1-e^2)}=f\,; \qquad \sqrt{a'.(1-e'^2)}=f'\,;$$

we shall have,*

[5904] $$\beta=\frac{(mf+m'f')^2-c^2}{2mm'.ff'}\,;$$

[5904′] c^2 being an arbitrary constant quantity, independent of the elements.

The preceding value of $d\theta'$ [5895], expresses the motion of the
[5905] intersection of the two orbits, produced by the action of m', and referred to the orbit of m' [5862d]. We shall suppose *an intermediate plane, between*
[5905′] *these two orbits, and passing through their mutual intersection; and shall put* φ *for the inclination of the orbit of* m *to this plane.* To obtain the differential
[5906] motion of the node of the orbit of m, upon this plane, arising from the action
[5907] of m' upon m, we must multiply the preceding value of $d\theta'$ by† $\dfrac{\sin.\gamma}{\sin.\varphi}$.

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* (3186) From [5903], we obtain,

[5904a] $$\sqrt{1-e^2}=\frac{f}{\sqrt{a}}\,; \qquad \sqrt{1-e'^2}=\frac{f'}{\sqrt{a'}}\,; \quad \text{also} \quad \cos.\gamma=1-\beta \quad [5852].$$

Substituting these in [5902], we get,

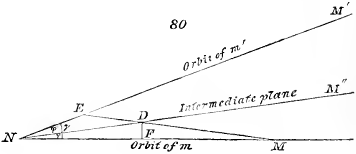
[5904b] $$2.(1-\beta).\frac{ff'}{\sqrt{aa'}}=\text{constant}-\frac{m.f^2}{m'.\sqrt{aa'}}-\frac{m'.f'^2}{m.\sqrt{aa'}}\,;$$

multiplying this by $mm'.\sqrt{aa'}$, and putting,

[5904c] $$mm'.\sqrt{aa'}\times\text{constant}=c^2, \quad \text{we get,} \quad 2.(1-\beta).mm'.ff'=c^2-m^2.f^2-m'^2.f'^2\,;$$

whence we easily deduce β [5904].

[5907a] † (3187) In the annexed figure, NM, NM', represent the orbits of the planets m, m', respectively, supposing them to be viewed from the sun, and referred to the concave
[5907b] surface of the starry heavens; NDM'' is the intermediate plane, or orbit; and N the common intersection, or node, at the commencement of the time dt.



Putting this motion equal to $d\theta$, we shall have, [5907']

$$d\theta = -\frac{m'.dt}{f}\cdot\frac{\sin.\gamma}{\sin.\varphi}\cdot\left(\frac{dF}{d\beta}\right).$$ [5908]

If we put φ' for the inclination of the orbit of m, upon the same plane, [5909]
we shall have $\varphi+\varphi'=\gamma$; and, [5909']

Then we shall have, as in [5905', 5909],

the angle $MNM''=\varphi$; the angle $M'NM''=\varphi'$; the angle $MNM'=\varphi+\varphi'=\gamma$; [5907c]
the arc $ND=d\theta$; the arc $NE=d\theta'$ [5907', 5905]. [5907d]

We shall now suppose, that the action of the body m' upon m, changes the orbit of m, from MN to the infinitely near orbit MDE, in the time dt; by this means, the node N moves through the space $NE=d\theta'$ [5905, 5862, c, d] upon the orbit of m'; [5907e]
or, through the space $ND=d\theta$, upon the intermediate orbit. Then, in the infinitely small triangle NDE, we have,

$$\text{sine}\,NDE : \text{sine}\,NED :: NE : ND;$$ [5907f]

and, if we neglect infinitely small quantities, we have,

$$\text{angle}\,NDE=\varphi; \qquad \text{angle}\,NED=180^d-\gamma;$$ [5907f']

hence we have, in symbols,

$$\sin.\varphi : \sin.\gamma :: d\theta' : d\theta; \text{ consequently, } d\theta = d\theta'.\frac{\sin.\gamma}{\sin.\varphi} \quad [5907].$$ [5907f'']

Substituting in this, the value of $d\theta'$ [5895], we get,

$$d\theta = -\frac{m'.andt}{\sqrt{1-e^2}}\cdot\frac{\sin.\gamma}{\sin.\varphi}\cdot\left(\frac{dF}{d\beta}\right);$$ [5907g]

and, since $an=a^{-\frac{1}{2}}$ [5902b], we have,

$$\frac{an}{\sqrt{1-e^2}} = \frac{1}{\sqrt{a.(1-e^2)}} = \frac{1}{f} \quad [5903];$$ [5907h]

hence the preceding expression of $d\theta$ becomes as in [5908]. In like manner, we obtain the value of $d'\theta$ [5910], which represents the motion of the node of the planet m', by the action of m; and, we can easily deduce this value of $d'\theta$ [5910], from that of $d\theta$ [5907i]
[5908], by changing reciprocally the elements and mass of m into those of m'; by which means, f changes into f', in [5903]; and $d\theta$ [5908], changes into $d'\theta$ [5910]; F remaining unaltered [5891].

[5910] $$d'\theta = -\frac{m\,.\,dt}{f'}\,.\,\frac{\sin.\gamma}{\sin.\varphi'}\,.\,\left(\frac{dF}{d\beta}\right);$$

[5911] $d'\theta$ being the motion of the orbit of m', upon this plane, produced, by the
[5912] action of m upon m'. *The motions $d\theta$ and $d'\theta$ will be equal, and the intersection of the two orbits will remain upon the plane we have just*
[5912'] *considered, if it divides the angle of the mutual inclination of the orbits γ, so that we may have,**

[5913] $$mf\,.\sin.\varphi = m'f'.\sin.\varphi'.$$

[5913'] This result is the same as is found in [1164]; *where we see, that the plane in question, is that of the maximum of areas*; and, that we have,

[5914] $$c = mf\,.\cos.\varphi + m'f'.\cos.\varphi'.$$

This equation [5914], being combined with [5913], gives the integral corresponding to [5904]; namely,†

[5915] $$\beta = \frac{(mf+m'f')^2 - c^2}{2mm'.ff'}.$$

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* (3188) Putting the two expressions [5908, 5910] equal to each other, and dividing
[5912a] by the common factor $-dt.\sin.\gamma.\left(\frac{dF}{d\beta}\right)$, we get, $\frac{m'}{f.\sin.\varphi} = \frac{m}{f'.\sin.\varphi'}$; which is easily reduced to the form [5913]. This equation, by the substitution of the values of f, f'
[5912b] [5903], becomes as in [1164 line 1], corresponding to the equation of the maximum of the areas; and, by a similar reduction, we may prove the identity of the expressions of c in [1165, 5914].

† (3189) The equation [5913] may be put under the form,

[5915a] $$0 = -mf.\sin.\varphi + m'f'.\sin.\varphi'.$$

Adding the square of this equation to the square of c [5914], we get successively, by using γ, β [5909', 5852];

[5915b] $$c^2 = m^2.f^2.(\cos.^2\varphi+\sin.^2\varphi) + 2mm'.ff'.(\cos.\varphi'.\cos.\varphi - \sin.\varphi'.\sin.\varphi) + m'^2.f'^2.(\cos.^2\varphi'+\sin.^2\varphi')$$

[5915c] $$= m^2.f^2 + 2mm'.ff'.\cos.(\varphi'+\varphi) + m'^2.f'^2 = m^2.f^2 + 2mm'.ff'.\cos.\gamma + m'^2.f'^2$$

[5915d] $$= m^2.f^2 + 2mm'.ff'.(1-\beta) + m'^2.f'^2 = (m.f+m'.f')^2 - 2mm'.ff'.\beta.$$

[5915e] From this last expression, we easily deduce the value of β [5915]; and, by an inverse operation, we might deduce [5914] from [5904, 5913].

These two equations, give also the following expressions;*

$$\sin.\varphi = \frac{m'f'.\sin.\gamma}{c}\,; \qquad \sin.\varphi' = \frac{mf.\sin.\gamma}{c}\,;$$ [5916]

$$\cos.\varphi = \frac{c^2+m^2f^2-m'^2f'^2}{2mf.c}\,; \qquad \cos.\varphi' = \frac{c^2+m'^2f'^2-m^2f^2}{2m'f'.c}\,;$$ [5917]

$$d\theta = -\frac{cdt}{ff'}\cdot\left(\frac{dF}{d\beta}\right) = \frac{dt.\sqrt{(mf+m'f')^2-2mm'.ff'.\beta}}{ff'}\cdot\left(\frac{dF}{d\beta}\right).$$ [5918]

We shall denote by $\varpi_{,}$ *and* $\varpi_{,}'$, *the perihelion distances of* m *and* m', *from the line of mutual intersection of the orbits.* Then we shall obtain $d\varpi_{,}$, [5919]
by subtracting from the differential $d\varpi$, the motion of that intersection $d\theta$, referred to the orbit of m;† and, it is evident, that, for this purpose, it [5920]

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* (3190) From [5913] we get $mf = m'f'.\frac{\sin.\varphi'}{\sin.\varphi}$; substituting this in [5914], we [5916a]
obtain successively, by using γ [5909'];

$$c = \frac{m'f'}{\sin.\varphi}.\{\cos.\varphi.\sin.\varphi'+\sin.\varphi.\cos.\varphi'\} = \frac{m'f'}{\sin.\varphi}.\sin.(\varphi+\varphi') = \frac{m'f'}{\sin.\varphi}.\sin.\gamma.$$ [5916b]

From this last value of c, we easily obtain $\sin.\varphi$ [5916]. Substituting this expression of $\sin.\varphi$ in [5913], and dividing by $m'f'$, we get $\sin.\varphi'$ [5916]. Again, we have,

$$c-mf.\cos.\varphi = m'f'.\cos.\varphi' \quad [5914]\,;$$ [5916c]

adding the square of this to the square of [5913], and reducing, we obtain,

$$c^2-2mfc.\cos.\varphi+m^2f^2 = m'^2f'^2\,;$$ [5916d]

whence we easily deduce the value of $\cos.\varphi$ [5917]; substituting this in [5914], we get $\cos.\varphi'$ [5917]. Using the value of $\sin.\varphi$ [5916], we get, from [5908].

$$d\theta = -\frac{cdt}{ff'}\cdot\left(\frac{dF}{d\beta}\right) \quad [5918]\,;$$ [5916e]

and, by substituting the value of c [5915d], we get the second form of $d\theta$ [5918].

† (3191) Drawing DF perpendicular to NM, in fig. 80, page 742, we have,

$$NF = ND.\cos.FND = d\theta.\cos.\varphi \quad [5907c, d]\,;$$ [5921a]

and, if we substitute the first value of $d\theta$ [5918], and that of $\cos.\varphi$ [5917], we get,

is only necessary to multiply it by $\cos.\varphi$; now, we have,

[5921] $$d\theta.\cos.\varphi = -\frac{(mf+m'f'-m'f'.\beta)}{ff'}.dt.\left(\frac{dF}{d\beta}\right).$$

therefore, we shall have,*

[5922] $$ed\varpi_{,} = -m'.andt.\sqrt{1-e^2}.\left(\frac{dF}{de}\right) + \frac{(mf+m'f'-m'f'.\beta)}{ff'}.edt.\left(\frac{dF}{d\beta}\right);$$

[5923] $$ede = m'.andt.\sqrt{1-e^2}.\left(\frac{dF}{d\varpi_{,}}\right).$$

[5921b] $$d\theta.\cos.\varphi = -\frac{(c^2+m^2f^2-m'^2f'^2)}{2m.f^2f'}.dt.\left(\frac{dF}{d\beta}\right).$$

Substituting c^2 [5915d], and dividing the numerator and denominator by $2mf$, we
[5921c] get [5921]. Subtracting this quantity from the whole motion of the perihelion of the planet m; namely, $d\varpi$, we get $d\varpi_{,}$ [5921e]; which represents the increment of the distance of the perigee of the planet m from the moveable node. In the same
[5921d] manner, we get $d\varpi_{,}'$ [5921f]; or, it may be more easily derived from $d\varpi_{,}$ [5921e], by interchanging the elements of m, m', in the usual manner;

[5921e] $$d\varpi_{,} = d\varpi + \frac{(mf+m'f'-m'f'.\beta)}{ff'}.dt.\left(\frac{dF}{d\beta}\right);$$

[5921f] $$d\varpi_{,}' = d\varpi' + \frac{(mf+m'f'-mf.\beta)}{ff'}.dt.\left(\frac{dF}{d\beta}\right).$$

* (3192) Multiplying the expression of $d\varpi_{,}$ [5921e], by e, and substituting $d\varpi$, [5884], we get [5922]. In like manner, multiplying the expression of $d\varpi_{,}'$ [5921f], by e', and substituting,

[5922a] $$d\varpi' = -\frac{m.a'n'dt.\sqrt{1-e'^2}}{e'}.\left(\frac{dF}{de'}\right);$$

which is deduced from [5884], by changing reciprocally, the elements of m into those of m'; we get [5924]. Now, we may suppose, as in [5926], that $\varpi_{,}$, $\varpi_{,}'$, take the places of ϖ, ϖ', respectively, in the function F; and then we shall have,

[5922b] $$\left(\frac{dF}{d\varpi}\right) = \left(\frac{dF}{d\varpi_{,}}\right).\left(\frac{d\varpi_{,}}{d\varpi}\right); \qquad \left(\frac{dF}{d\varpi'}\right) = \left(\frac{dF}{d\varpi_{,}'}\right).\left(\frac{d\varpi_{,}'}{d\varpi'}\right).$$

If we neglect quantities of the order m, we shall get from [5921e, f],

[5922c] $$\left(\frac{d\varpi_{,}}{d\varpi}\right) = 1; \qquad \left(\frac{d\varpi_{,}'}{d\varpi'}\right) = 1;$$

In like manner, we have,

$$e'd\varpi_,' = -\; m.a'n'dt.\sqrt{1-e'^2}.\left(\frac{dF}{de'}\right) + \frac{(mf+m'f'-mf\,.\,\beta)}{ff'}\,.\,e'dt\,.\left(\frac{dF}{d\beta}\right); \qquad [5924]$$

$$e'de' = \; m.a'n'dt.\sqrt{1-e'^2}.\left(\frac{dF}{d\varpi_,'}\right). \qquad [5925]$$

F is a function of a, a', e, e', $\varpi_,$, $\varpi_,'$, and β. If we eliminate β from the second members of these equations, by means of its value, [5926]

$$\beta = \frac{(mf+m'f')^2-c^2}{2mm'.ff'} \quad [5915], \qquad [5927]$$

we shall obtain four differential equations between the four variable quantities e, e', $\varpi_,$, $\varpi_,'$. We may give them a still more simple form,* by putting,

$$h = e.\sin.\varpi_, ; \qquad l = e.\cos.\varpi_, ; \qquad [5928]$$

$$h' = e'.\sin.\varpi_,' ; \qquad l' = e'.\cos.\varpi_,'. \qquad [5929]$$

This renders them linear, when we neglect the higher powers of the excentricities, and facilitates the farther integrations, by approximation, to any powers of the excentricities.† Thus we shall have the position [5929']

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so that by rejecting quantities of the order m, we shall have,

$$\left(\frac{dF}{d\varpi}\right) = \left(\frac{dF}{d\varpi_,}\right); \qquad \left(\frac{dF}{d\varpi'}\right) = \left(\frac{dF}{d\varpi_,'}\right). \qquad [5922d]$$

Substituting the first of these expressions in [5883], and multiplying by e, we get [5923], in which terms of the order m^2 are neglected. The second of the expressions [5922d], being substituted in the value of de', deduced from de [5883], by interchanging the elements of m, m', gives [5925]. [5922e]

* (3193) We have already seen the effect of similar substitutions, in simplifying such results, in [1022, 1046, 1089, &c]. [5928a]

† (3194) After we have obtained the values of h, h', l, l', by methods analogous to those in [1097, &c.], we may determine e, e', $\varpi_,$, $\varpi_,'$, from [5928, 5929]. [5929a]
Then a, a', being constant [5881'], we shall have f, f', from [5903]. The constant quantity c^2 is known, from the values of f, f', β, at the epoch [5929b]
when $t=0$, by means of [5915d]; and at any other time t, the value of β will be known, by substituting the corresponding values of f, f', in [5927], then from β, [5929c]

of the orbits, relatively to the variable position of the line of their
[5930] mutual intersection. We shall then have the inclinations of their orbits to each other, by means of the preceding value of β; and we may thence obtain their inclinations upon the plane of the *maximum* of the
[5931] areas, by means of the preceding values of φ and φ'. Lastly, we shall have the motion of the intersection of the two orbits, upon this *maximum* plane, by integrating the preceding expression of $d\theta$ [5908]. *This*
[5931'] *seems to be the most general and simple solution of the problem of the secular variations of the planetary orbits.*

We shall now resume the equation [5915*d*],

[5932] $$c^2 = (mf+m'f')^2 - 2mm'.ff'.\beta.$$

[5932'] If we neglect quantities of the fourth power of the excentricities and inclinations, it will give,*

[5933] $$\text{constant} = m.\sqrt{a}.\,e^2 + m'.\sqrt{a'}.\,e'^2 + \frac{2mm'.\sqrt{aa'}.\beta}{m.\sqrt{a}+m'.\sqrt{a'}};$$

[5929*d*] we obtain γ [5852]. With these values of m, m', c, f, f', γ, we deduce φ, φ', from [5916 or 5917], and $d\theta$ from [5918], whose integral gives θ. Thus we shall obtain all the elements, in the same manner as in [5930, 5931].

* (3195) The quantity β is of the second order in γ [5852], and by neglecting terms of the *fourth* order, we may put,

[5933*a*] $$-2mm'.ff'.\beta = -2mm'.\sqrt{aa'}.\beta \quad [5903];$$

also,

[5933*a'*] $$ff' = \sqrt{aa'}.\sqrt{1-e^2}\sqrt{1-e'^2} = \sqrt{aa'}.(1-\tfrac{1}{2}e^2-\tfrac{1}{2}e'^2).$$

Hence the expression [5932], becomes, without reduction,

[5933*b*] $$c^2 = m^2.a.(1-e^2) + 2mm'.\sqrt{aa'}.(1-\tfrac{1}{2}e^2-\tfrac{1}{2}e'^2) + m'^2.a'.(1-e'^2) - 2mm'.\sqrt{aa'}.\beta.$$

Then, by transposition, we get [5933*c*], and its second member is easily reduced to the form [5933*d*];

[5933*c*] $$-c^2+m^2.a+m'^2.a'+2mm'.\sqrt{aa'} = m^2.a.e^2+mm'.\sqrt{aa'}.(e^2+e'^2)+m'^2.a'.e'^2+2mm'.\sqrt{aa'}.\beta$$

[5933*d*] $$=(m.\sqrt{a}+m'.\sqrt{a'}).(m.\sqrt{a}.e^2+m'.\sqrt{a'}.e'^2)+2mm'.\sqrt{aa'}.\beta.$$

If we divide this by $m.\sqrt{a}+m'.\sqrt{a'}$, we shall find, that the first member is a constant quantity, and the second member becomes as in [5933].

and by what has been said in [5786, 5842, 5881′, &c.], a and a', are constant, noticing the square of the disturbing force; therefore, we shall have,* [5934]

$$0 = m.\sqrt{a}\,.\,e\delta e + m'.\sqrt{a'}\,.\,e'\delta e' + \frac{mm'.\sqrt{aa'}.\gamma\delta\gamma}{m.\sqrt{a}+m'.\sqrt{a'}}\,. \qquad [5935]$$

This equation is of the same form as that which is found in [3964], noticing the terms depending upon the great inequalities of Jupiter and Saturn. Hence it appears, that the invariable plane, determined in [1162′, &c., 5913], remains invariable, even when we notice some terms of the order of the square of the disturbing force [5935e]. [5936]

4. We may, by means of the differential expressions of the elements, determine, in a very simple manner, the influence of the figure of the earth upon the moon's motion. We have seen, in [5340, 5438], that this action produces in R, the following term;

$$(\alpha\rho - \tfrac{1}{2}\alpha\varphi).\frac{D^2}{r^3}.(\mu^2 - \tfrac{1}{3})\;; \qquad \text{[Term of } R\text{]} \qquad [5937]$$

$\alpha\rho$ *is the oblateness of the earth* [5333]; $\alpha\varphi$ *is the ratio of the centrifugal force to gravity, at the equator* [5333′]; D *is the mean radius of the terrestrial spheroid* [5334]; and μ *the sine of the moon's declination* [5334′]; which is represented as in [5344], by, [5938] [5939]

$$\mu = \sqrt{1-s^2}.\sin.\lambda.\sin.fv + s.\cos.\lambda\;; \qquad [5940]$$

or, more accurately, as in [5344e],

$$\mu = \frac{\sin.\lambda.\sin.fv + s.\cos.\lambda}{\sqrt{1+ss}}\;; \qquad [5941]$$

fv *being the true longitude of the moon, counted from the vernal equinox* [5345]; λ *the obliquity of the ecliptic* [5341]; and s *the tangent of the moon's latitude* [4759″]. [5942]

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* (3196) We have $2\beta = 4.\sin.^2\tfrac{1}{2}\gamma$ [5852], and, if we neglect terms of the order γ^4, we get $2\beta = \gamma^2$. Substituting this in [5933]; taking its variation, dividing by 2, and neglecting terms of the second order in δe, $\delta e'$, $\delta\gamma$, we obtain [5935], which is similar to that in [3964]. The equation [5935] is correct in some of the terms of the [5935a] [5935b]

[5943] *The part of R, depending on the sun's action, is of the form* r^2Q', neglecting terms depending on the sun's parallax, which are very small* [5944c]. Then we shall have, very nearly,

[5944] $$R=r^2Q'+(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{r^3}.(\sin.^2\lambda.\sin.^2fv+2s.\sin.\lambda.\cos.\lambda.\sin.fv-\tfrac{1}{3}) \quad [5944e,\&c.];$$

which gives,

[5945] $$2r.\left(\frac{dR}{dr}\right)=2a.\left(\frac{dR}{da}\right)=4r^2Q'-6.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{r^3}.(\sin^2\lambda.\sin^2fv+2s.\sin\lambda.\cos\lambda.\sin fv-\tfrac{1}{3}).$$

We shall here notice only the inequalities depending on the angle $gv-fv$;
[5946] gv being what is called the *argument of latitude;* then we shall have,

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[5935c] order m^2 [3964', &c.], but others of the order m^3, $m^2.e^3.\delta e$, δe^2, &c., are neglected, as in [1150', 5932', 5935b, &c.].

* (3197) Substituting the values of u, u', [4776, 4777e] in Q [4780], and developing it in a series ascending according to the powers of r, we get,

[5944a] $$Q=\frac{1}{r}+\frac{m'}{r'}.\left\{1+A.\frac{r^2}{r'^2}+B.\frac{r^3}{r'^3}+\&c.\right\};$$

[5944a'] A, B, &c., being quantities which contain v, s, v', s'. Substituting this in [5438], we get,

[5944b] $$R=-\frac{m'}{r'}.\left\{1+A.\frac{r^2}{r'^2}+B.\frac{r^3}{r'^3}+\&c.\right\}.$$

The first term of this expression of R, produces nothing, in its partial differentials, taken relatively to the elements of the moon's orbit; we may, therefore, neglect it; and
[5944c] also the terms depending on r^3, r^4, &c., on account of their smallness [5943]. By this means, the expression of R, is reduced to its *greatest term*, depending upon r^2, which is represented by r^2Q' in [5943], and is of the same order as that of the disturbing force of the sun upon the moon; Q' being a function of v, s, r', v', s'
[5944d] [5944a, a']. Finally, we may remark, that the symbol Q' is denoted by Q, in the original work, but we have placed an accent upon it, in order to distinguish it from the value of Q [5944a]. Adding this chief term of R to that in [5937], we get,

[5944e] $$R=r^2Q'+(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{r^3}.(\mu^2-\tfrac{1}{3}).$$

Substituting the value of μ [5940], and neglecting s^2, it becomes as in [5944]. Its
[5944f] partial differential, relative to r, being multiplied by $2r$, and then substituting [5774], gives [5945].

very nearly, $s=\gamma.\sin.gv$ [4818]; γ *being the inclination of the moon's orbit to the ecliptic* [4813]. Thus, we shall obtain,* [5946′]

$$R=r^2Q'+(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^3}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv);$$ [5947]

$$2a^2.\left(\frac{dR}{da}\right)=4a.r^2Q'-6.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv).$$ [5948]

We have seen, in [5842], that the variation of dR is nothing,† even when we notice the square of the disturbing force; therefore, the coefficient of $\cos.(gv-fv)$ must vanish from R. *We shall denote by the characteristic* δ, *placed before any function, the part of that function, which depends on the oblateness of the earth*; and, we shall then have, [5948′] [5949] Symbol δ.

$$0=\delta.(r^2Q')+(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^3}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv);$$ [5950]

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* (3198) The value of s [5946′], is the same as in [4818], *supposing the origin of* gv *to correspond to* $\theta=0$. From this expression, we get, in $2s.\sin.fv$, the term $\gamma.\cos.(gv-fv)$. Substituting this in [5944], it becomes as in [5947]; and, from [5945], multiplied by a, we get [5948]; observing, that in the terms which are connected with $\alpha\rho-\frac{1}{2}\alpha\varphi$, we may put $r=a$. Moreover, we have, as in [5347q], $f=1+\frac{1}{3\,4\,0\,0\,0\,0}$, $g=1+\frac{1}{2\,5\,0}$, nearly; so that the angle $gv-fv$ is very small in comparison with v; the mean increment of $gv-fv$ in a given time, being the same as that of the longitude of the moon's node [5388c], and $g-f$ is of the order m^2 [4828e], *or of the same order as the disturbing force* of the sun upon the moon; consequently the factor $m'.(g-f)$ which occurs in dR [5949e], must be considered as of the *second* order, relative to the powers and products of the disturbing forces. [5947a] [5947b] [5947c] [5947d] [5947e] [5947f]

† (3199) The secular variation of $d.\delta R$, or of dR vanishes, as is shown in [5844 line 2, 5794″, &c.], *noticing the terms of the order of the square of the disturbing forces*. Now the *secular* inequalities are those which are *independent of the configuration* of the heavenly bodies; that is to say, they depend on the variations of the elements, or on the motions of the nodes, perihelia, inclinations, &c., as in [4242—4251, &c.]; and as the angle $gv-fv$ represents the longitude of moon's node [5947d], it partakes of the nature of the secular quantities, being similar to those in [5846a], which are represented by the angle $gt+\beta$, applied to the moon's orbit. If we notice only the terms of R [5881] which depend on the angle $gv-fv$, we may put it under the form, [5949a] [5949b] [5949c]

$$R=m'.F'.\cos.(gv-fv);$$ [5949d]

hence we deduce,*

[5951] $$\delta\,.\left\{2a^2.\left(\frac{dR}{da}\right)\right\}=-10.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv).$$

We shall now resume the expression of $d\varepsilon$ [5784],

[5952] $$d\varepsilon=-\frac{andt.\sqrt{1-e^2}}{e}.(1-\sqrt{1-e^2}).\left(\frac{dR}{de}\right)+2a^2.\left(\frac{dR}{da}\right).ndt.$$

It is evident, that, if we neglect the excentricity of the orbit, we shall have,†

[5953] $$d\varepsilon=2a^2.\left(\frac{dR}{da}\right).ndt;$$

therefore, by noticing only the cosine of the angle $gv-fv$, and substituting

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whose differential, relative to d, is,

[5949e] $$\mathrm{d}R=-m'.(g-f).F'.\sin.(gv-fv).dv;$$

[5949f] and, as the factor $m'.(g-f).F'$, is of the *second* order relative to the disturbing forces
[5949f'] [5947f], it must vanish from $\mathrm{d}R$ [5949a]; therefore we must put $F'=0$; and then the expression of R [5949d], becomes $R=0$. Substituting this in [5947],
[5949g] and retaining in r^2Q', the part $\delta.(r^2Q')$ [5949], corresponding to the angle $(gv-fv)$, we get [5950]; observing, that the co-ordinates of the moon produce in r^2Q',
[5949h] terms depending on the angle $gv-fv$, in the same manner as arguments of similar forms appear in the expressions of the moon's mean motion and parallax in [5220, 5331, &c.].

* (3200) If we retain, in [5948], only those terms which depend on the angle $(gv-fv)$, and use the sign δ, as in [5949], we shall get,

[5951a] $$\delta\,.\left\{2a^2.\left(\frac{dR}{da}\right)\right\}=4a.\delta.(r^2Q')-6.(\alpha\rho-\tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv);$$

Adding this to the product of [5950], by $-4a$, we obtain [5951].

[5953a] † (3201) We have, by development, $1-\sqrt{1-e^2}=\frac{1}{2}e^2+$ &c.; substituting this in the first term of [5952], we find, that it becomes of the order e; and by neglecting terms of this order, we get [5953]. If we retain, in the second member of this last
[5953b] expression, the term depending on the angle $gv-fv$, which is given in [5951], and change ndt into dv, as in [5378'], we shall get [5954].

dv for ndt [5378′], we shall get, as in [5379],

$$d\varepsilon = -10.(\alpha_{\gamma}-\tfrac{1}{2}\alpha\rho)\,.\frac{D^2}{a^2}.\gamma.dv.\sin.\lambda.\cos.\lambda.\cos.(gv-fv) \quad [5953b]. \qquad [5954]$$

This value of $d\varepsilon$ [5952, or 5954], is measured in the plane of the moon's orbit ;* *to refer it to the ecliptic, we must add to it the quantity* $\tfrac{1}{2}.(qdp-pdq)$ [5955] [5955c]. We shall now determine p and q.

The equation,

$$s = \gamma\,.\sin.gv \quad [5946'], \qquad [5956]$$

may be put under the form,†

$$s = \gamma.\cos.(gv-fv).\sin.fv+\gamma.\sin.(gv-fv).\cos.fv. \qquad [5957]$$

If we compare it with the following expression,‡

$$s = q.\sin.fv-p.\cos.fv, \qquad [5958]$$

we shall obtain,

$$p = -\gamma.\sin.(gv-fv)\;; \qquad q = \gamma.\cos.(gv-fv)\,. \qquad [5959]$$

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* (3202) In computing the value of $d\varepsilon$ [5784 or 5952], from the expression [5775′], we have taken, in [5775′line 2], the primitive orbit of m, for the plane of the projection; [5955a] so that the angle $nt+\varepsilon$, or $\int ndt+\varepsilon$ [5782, 5793], is counted *on this primitive orbit.* If we represent the differential of this expression by $dv = ndt + d\varepsilon$, and put $dv_{\prime}$ for [5955b] its projection upon the *fixed plane of the ecliptic* [3778, &c.], we shall have, as in [3782], $dv_{\prime} = dv + \tfrac{1}{2}.(qdp-pdq)$; so that, to obtain $dv_{\prime}$ from dv, we must add to $d\varepsilon$ the correction $\tfrac{1}{2}.(qdp-pdq)$, as in [5955]. [5955c]

† (3203) We have $gv = fv+(gv-fv)$; hence,

$$\sin.gv = \cos.(gv-fv).\sin.fv+\sin.(gv-fv).\cos.fv \quad [21]\text{ Int.} \qquad [5957a]$$

Multiplying this by γ, we get the second member of [5956], and this value of s becomes as in [5957].

‡ (3204) The expression [5958] may be deduced from [1335′], by changing v into fv; which is the same as to count the longitudes from the moveable equinox, instead of the fixed equinox [5315] Comparing the coefficients of $\sin.fv$, $\cos.fv$, in [5957, 5958], [5958a] we get [5959].

From these, we get,*

[5960] $$dp = -(g-f).qdv\ ;$$

[5961] $$dq = \quad (g-f).pdv\ .$$

The value of R contains the term,†

[5962] $$(\alpha\rho - \tfrac{1}{2}\alpha\varphi)\ .\ \frac{D^2}{a^3}\ .\ \sin.\lambda.\ \cos.\lambda.\ q\ :$$

by the equations [5790, 5791], it adds to the value of dp the term,

[5963] $$-(\alpha\rho - \tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda.\,dv\ ;$$ [Term of dp]

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* (3505) The differentials of [5959] give,

[5960a] $$dp = -(g-f).\gamma.\cos.(gv-fv).dv\ ; \qquad dq = -(g-f).\gamma.\sin.(gv-fv).\,dv.$$

Substituting, in the second members of these equations, the values of p, q [5959], we get [5960,5961].

† (3206) Substituting for $\gamma.\cos.(gv-fv)$, its value q [5959], in the last term of R [5947], and retaining only this part of R, we get,

[5963a] $$R = (\alpha\rho - \tfrac{1}{2}\alpha\varphi)\ .\frac{D^2}{a^3}.\sin.\lambda.\cos.\lambda.q \quad [5962].$$

This is to be substituted in [5790, 5791], as the most important part of R corresponding to the values of dp, dq, now under discussion; the other parts having the small factor $\frac{m'}{r'^3}$, which is contained in r^2Q' [5944d, b]. Its partial differentials relative to p, q, give,

[5963b] $$\left(\frac{dR}{dp}\right) = 0 \qquad \left(\frac{dR}{dq}\right) = (\alpha\rho - \tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^3}.\sin.\lambda.\cos.\lambda.$$

Substituting these, in [5790, 5791], we get,

[5963c] $$dp = -\frac{andt}{\sqrt{1-e^2}}.(\alpha\rho - \tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^3}.\sin.\lambda.\cos\lambda\ ; \qquad dq = 0\ .$$

Neglecting terms of the order e^2, and changing ndt into dv [5953b], we find, that this term of dp becomes as in [5963]. Adding this part of dp to that in
[5963d] [5960], we get [5964]; dq [5961] is the same as in [5965], not being altered by the term $dq = 0$ [5963c].

then we have the two equations,

$$dp = -(g-f).qdv - (\alpha\rho - \tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda.dv\ ; \qquad [5964]$$

$$dq = \quad (g-f).pdv. \qquad [5965]$$

These equations give, in the expression of q, *the constant term*,*

$$-\frac{(\alpha\rho - \frac{1}{2}\alpha\varphi)}{g-f}.\frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda \quad [5965d]. \qquad [5966]$$

[Constant part of q]

From this we obtain, in the latitude s, the inequality,

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* (3207) Taking the differential of [5965], supposing dv to be constant, we get,

$$ddq = (g-f).dp.dv. \qquad [5965a]$$

Substituting the value of dp [5964], dividing by dv^2, and reducing, we obtain,

$$0 = \frac{ddq}{dv^2} + (g-f)^2.q + (g-f).(\alpha\rho - \tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda. \qquad [5965b]$$

This equation is of the same form as in [865a, 870′], changing y, t, a, b, φ, &c., into q, v, $g-f$, γ, 0, &c. respectively; by this means, we obtain from the integral [865b, 871] the following expression [5965d], which satisfies [5965b]; as is easily proved by substitution and reduction, by mere inspection, if we take separately into consideration the two terms of q; [5965c]

$$q = \gamma.\cos.(gv-fv) - \frac{(\alpha\rho - \frac{1}{2}\alpha\varphi)}{g-f}.\frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda. \qquad [5965d]$$

The differential of this value of q, being substituted in the first member of [5965], and then dividing by $(g-f).dv$, gives,

$$p = -\gamma.\sin.(gv-fv), \quad \text{as in [5959].} \qquad [5965e]$$

Multiplying [5965d] by $\sin.fv$, and [5965e] by $-\cos.fv$; then taking the sum of the products, and reducing the factor of γ, by means of [5957a], we obtain the value of the second member of [5958], or the expression of s; namely,

$$s = \gamma.\sin.gv - \frac{(\alpha\rho - \frac{1}{2}\alpha\varphi)}{g-f}.\frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda.\sin.fv. \qquad [5965f]$$

The term depending on $\alpha\rho - \frac{1}{2}\alpha\varphi$, being represented by δs, is as in [5967]; and if we change the divisor $g-f$ into $g-1$; f being nearly equal to unity [5947c]; it becomes as in [5351]. [5965g]

[5967] $$\delta s = -\frac{(\alpha\rho - \frac{1}{2}\alpha\varphi)}{g-f} \cdot \frac{D^2}{a^2} \cdot \sin.\lambda.\cos.\lambda.\sin.fv \quad [5965f];$$

which agrees with the result in [5351].

[5968] The constant part of q [5966] produces, in the function $\frac{1}{2}.(qdp-pdq)$, the following term, as in [5385];*

[5969] $$\tfrac{1}{2}.(\alpha\rho - \tfrac{1}{2}\alpha\varphi) \cdot \frac{D^2}{a^2} \cdot \gamma \cdot \sin.\lambda.\cos.\lambda.\cos.(gv-fv).dv.$$

[5969′] Putting, therefore, $d\varepsilon_{,}$ equal to the preceding value of $d\varepsilon$, referred to the ecliptic, we shall have,

[5970] $$d\varepsilon_{,} = -\tfrac{19}{2}.(\alpha\rho - \tfrac{1}{2}\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv).dv;$$

which gives, in $\varepsilon_{,}$, and, therefore, in the moon's motion in longitude, the inequality,

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* (3208) Multiplying the expressions [5964, 5965] by $\frac{1}{2}q$ and $-\frac{1}{2}p$, respectively, and adding the products, we get,

[5968a] $$\tfrac{1}{2}.(qdp-pdq) = -\tfrac{1}{2}.(g-f).(p^2+q^2).dv - \tfrac{1}{2}.(\alpha\rho-\alpha\varphi).\frac{D^2}{a^2}.\sin.\lambda.\cos.\lambda.qdv.$$

Taking the sum of the squares of q, p [5965d, e], and neglecting terms of the order $(\alpha_{,}-\alpha\varphi)^2$, we get,

[5968b] $$p^2+q^2 = \gamma^2 - \frac{2.(\alpha\rho-\alpha\varphi)}{g-f} \cdot \frac{D^2}{a^2} \cdot \gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv).$$

Substituting this in [5968a], and retaining only the terms depending on $(\alpha\rho-\alpha\varphi)$, we get,

[5968c] $$\tfrac{1}{2}.(qdp-pdq) = (\alpha\rho-\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv).dv - \tfrac{1}{2}.(\alpha\rho-\alpha\varphi).\frac{D^2}{a^2}.\sin\lambda.\cos\lambda.qdv.$$

[5968d] We may put $q=\gamma.\cos.(gv-fv)$ [5965d], in the last term of [5968c], and then we shall have, as in [5969],

[5968e] $$\tfrac{1}{2}.(qdp-pdq) = \tfrac{1}{2}.(\alpha\rho-\alpha\varphi).\frac{D^2}{a^2}.\gamma.\sin.\lambda.\cos.\lambda.\cos.(gv-fv).dv.$$

[5968f] This value of $\frac{1}{2}.(qdp-pdq)$ is to be added to $d\varepsilon$ [5954], as in [5955], to obtain the quantity which is called $d\varepsilon_{,}$ [5969′]; and the sum evidently becomes as in [5970]. Its integral gives the term of $\varepsilon_{,}$, or δv [5971]; which agrees with that in [5387].

$$\delta v = -\tfrac{19}{2} \cdot \frac{(\alpha\rho - \frac{1}{2}\alpha\varphi)}{g-f} \cdot \frac{D^2}{a^2} \cdot \gamma \cdot \sin.\lambda \, . \cos.\lambda \, . \sin.(gv - fv). \qquad [5971]$$

This result is wholly conformable to that in [5387].

Lastly, the function R being indeterminate, *the preceding differential expressions of the elements of the orbits, can also be used to determine the* [5972] *variations they suffer, either by the resistance of an ethereal medium, by the impulsion of the sun's light, or, by the change which the course of time may* [5973] *produce in the masses of the sun and planets. It is only necessary, for this purpose, to determine the function R, which results from it,* by the considerations explained in chap. vii, of the tenth book* [8884—9036].

ON THE TWO GREAT INEQUALITIES OF JUPITER AND SATURN.

5. In the theory of these inequalities, given in the sixth book, we have noticed the fifth powers of the excentricities and inclinations of the orbits. But it has been discovered, that the values of $N^{(0)}$, $N^{(1)}$, &c. [3860–3860^{ix}] [5974] are taken with a wrong sign [3860a, &c.]. To correct this mistake, we must change the signs of this part of the inequalities. This can be done, by adding to the expression of the mean longitude, which is given in the eighth [5975] chapter of the tenth book, the double of this part, taken with a contrary sign. This part, for Jupiter, is as in [4431, 4430a];

$$\begin{aligned} \delta v^{iv} = \; & (12^s{,}536393 - t.0^s{,}001755).\sin.(5n^{v}t - 2n^{iv}t + 5\epsilon^{v} - 2\epsilon^{iv}) && 1 \\ & - (8^s{,}120963 + t.0^s{,}004885).\cos.(5n^{v}t - 2n^{iv}t + 5\epsilon^{v} - 2\epsilon^{iv}); && 2 \end{aligned} \qquad [5976]$$

and, for Saturn, as in [4487, 4483e line 4];

$$\begin{aligned} \delta v^{v} = & -(29^s{,}144591 - t.0^s{,}004081).\sin.(5n^{v}t - 2n^{iv}t + 5\epsilon^{v} - 2\epsilon^{iv}) && 1 \\ & + (18^s{,}879594 + t.0^s{,}011356).\cos.(5n^{v}t - 2n^{iv}t + 5\epsilon^{v} - 2\epsilon^{iv}). && 2 \end{aligned} \qquad [5977]$$

The addition, to the mean longitudes of Jupiter and Saturn, *of the double*

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* (3209) This method of finding Q, or R [5438], has already been used in estimating the resistance of the earth and moon, from an ethereal fluid [5672, 5673]. Similar methods are used in ascertaining the values of R, in other cases, like those [5973a] which are mentioned in [5973].

[5978] *of these inequalities taken with a contrary sign*, can affect only the mean motions and the epochs of these two planets. It cannot alter, except by insensible quantities, the other elliptical elements, deduced from the observations made between the years 1750 and 1800; because, during that
[5978′] interval, the variations of these inequalities are very nearly proportional to the time. We may, therefore, determine the corrections of the mean motions, so as to make *the double of these inequalities, affected with a*
[5979] *contrary sign, vanish, in* 1750, *when* $t=0$, *and, in* 1800, *when* $t=50$. Thus we find, by noticing the correction of Saturn's mass, given in chap. viii, of the tenth book [9121], that we must add to the mean longitude q^{iv} of Jupiter, given in [9137], the function,*

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* (3210) We have, in [9128, 9129],

[5980a] $$n^{\text{iv}}t+\varepsilon^{\text{iv}} = 3^d\,45^m\,47^s{,}5+t\,.\,30^d\,20^m\,56^s{,}4\;;$$

[5980b] $$n^{\text{v}}t+\varepsilon^{\text{v}} = 231^d\,21^m\,53^s{,}9+t\,.\,12^d\,13^m\,17^s{,}1\;.$$

Multiplying the second of these expressions by 5, and the first by -2; and then putting the sum of these products equal to T, for brevity, we shall have,

[5980c] $$T = 5n^{\text{v}}t-2n^{\text{iv}}t+5\varepsilon^{\text{v}}-2\varepsilon^{\text{iv}} = 69^d\,17^m\,54^s{,}5+t\,.\,24^m\,32^s{,}7.$$

Now, if we double the expression of δv^{iv} [5976], and change its sign, as in [5978];
[5980d] then decrease the result, in the ratio of 19,232 to 20,232, on account of the change in the estimated value of the mass of Saturn [9121], it becomes,

[5980e] $$\begin{aligned}&A^{\text{iv}}+B^{\text{iv}}t\\ &-2\times\frac{19{,}232}{20{,}232}.(12^s{,}536393-t.0^s{,}001755).\sin.T\\ &+2\times\frac{19{,}232}{20{,}232}.(8^s{,}120963+t.0^s{,}004885).\cos.T\;;\end{aligned}$$

the terms $A^{\text{iv}}+B^{\text{iv}}t$, being added so as to make the expression vanish in 1750, and in 1800, when $t=0$, and $t=50$, as in [5979]. To obtain the values of A^{iv}, B^{iv},
[5980f] we must first put $t=0$ in [5980c], and we shall get the value of T corresponding to this time. Substituting this, and $t=0$, in [5980e], then putting the result equal to nothing, as in [5979], we get the value of A^{iv}. Again, with $t=50$, we get a new value of T [5980c]; substituting these expressions of t, T, A^{iv}, in [5980e], we
[5980g] obtain $50B^{\text{iv}}$, from which B^{iv} may be determined. The result of this calculation agrees very nearly with that in [5980].

In like manner, if we multiply the expression [5977] by 2, and change its signs, adding
[5980h] also the terms $A^{\text{v}}+B^{\text{v}}t$, we shall obtain the formula [5981]. Having computed the

$$\begin{aligned}\delta q^{\text{iv}} = \;& 16^s{,}84 + t.0^s{,}1347 & 1\\ & -(23^s{,}84 - t.0^s{,}0033).\sin.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}) & 2\\ & +(15^s{,}44 + t.0^s{,}0093).\cos.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}})\,; & 3\end{aligned} \qquad [5980]$$

and, to the mean longitude q^{v} of Saturn [9138], the function,

$$\begin{aligned}\delta q^{\text{v}} = & -41^s{,}19 - t.0^s{,}3309 & 1\\ & +(58^s{,}304 - t.0^s{,}008162).\sin.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}) & 2\\ & -(37^s{,}759 + t.0^s{,}022744).\cos.(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\varepsilon^{\text{v}} - 2\varepsilon^{\text{iv}}). & 3\end{aligned} \qquad [5981]$$

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expressions [5980, 5981], it will be easy to complete the calculations relative to the observations of Ebn Junis [5982, &c.].

It is probable, that the coefficients of the function [5981], as well as those of the other inequalities of the motions of Saturn, arising from the action of Jupiter, must be increased in consequence of an augmentation of the estimated value of the mass of Jupiter by Gauss, Nicolai, Encke, and Airy. The first estimate, made by La Place, in [4065], is founded on the observed elongations of the satellites, by Pound, and is $\frac{1}{1067{,}09}$. But these elongations have been lately observed with much greater accuracy, by Professor Airy, and the result of his measures, given in vol. 10, page 404, of the *Astronomische Nachrichten* makes the mass $\frac{1}{1048{,}69}$. Nicolai, by the observations of the perturbations of Juno, gives $\frac{1}{1053{,}924}$. Encke, by those of Vesta, $\frac{1}{1050{,}117}$; and by the perturbations of the comet which bears his name, $\frac{1}{1051{,}4}$. All these observations indicate, that the mass, assumed by La Place, is too small by about $\frac{1}{70}$ part; and that the perturbations of Saturn, and several of the other planets, require some correction on this account. On the contrary, the calculations of Bouvard, from numerous observations of the perturbations of Saturn and Uranus, make the mass equal to $\frac{1}{1070{,}5}$. The cause of this difference must be ascertained by future observations and investigations. Some have supposed this discrepancy to arise from a difference between the action of Jupiter upon Saturn, and upon the other planets; but we have nothing, analogous to this, in any known experiments or observations on the effect of universal gravitation. [5980i] [5980k] [5980l] [5980m] [5980n] [5980o] [5980p]

In closing this volume, we may remark, that the sequel of the work of Hansen, upon the inequalities of the motions of Jupiter and Saturn; which is mentioned in [4458c], and also the work on the lunar theory, by Plana and Carlini, [4752a], have not been received in this country at the time of writing this article. We must therefore defer any notice of these works in the present volume. [5980q] [5980r]

[5982] These corrections have the advantage of making the formulas of the motion of Jupiter and Saturn, given in the above-mentioned chapter, agree better with a very important observation of Ebn Junis. This observation, reduced to the meridian of Paris, took place the 31st of October, 1007, at $3^h\,50^m$. These formulas give 729^s for the excess of the geocentric longitude of
[5983] Saturn over that of Jupiter, at that time; and the Arabian astronomer found it, by observation, to be 1440^s: the difference being 711^s. The preceding corrections increase, by 388^s, the excess of the longitude of Jupiter over that of Saturn; consequently, the new computation corresponds more accurately with the observation, by that quantity; and the difference is reduced to nearly five sexagesimal minutes; which is much less than the error to which this observation is liable.

APPENDIX, BY THE TRANSLATOR.

We shall, in this appendix, point out some of the important improvements made by
Gauss, Olbers, and others, in the calculation of the orbit of a planet or comet, moving in
an ellipsis, parabola or hyperbola; with the methods of computing the place of the moving
body, at any time, by means of several auxiliary tables. For the sake of convenient [5984]
reference, we shall insert in the tables [5985, 5986, 5988], the most important theorems,
relative to this subject, which have been already introduced in the preceding part of the
work; together with several new formulas, given by Gauss, in his *Theoria Motus Corporum* (1)
Cœlestium, conforming, however, to the notation generally used by La Place, in this work.

In the demonstrations of the formulas included in the table [5985 lines 1—19], we shall
refer to any particular line of it, by including the number of the line in a parenthesis; (2)
thus, in referring to the value of e [5985 line 1], we shall use the abridged notation (1).

From the assumed value of $e = \sin.\varphi$ (1), we easily deduce the expressions (2, 3, 4); (3)
observing, in the formulas (3), that the development of $\{\sqrt{1+e} \mp \sqrt{1-e}\}^2$ becomes, by (4)
reduction, equal to $2 \mp 2.\sqrt{1-e^2} = 2 \mp 2.\cos.\varphi$; and, that, (5)

$$2 - 2.\cos.\varphi = 4.\sin.^2\tfrac{1}{2}\varphi, \qquad 2 + 2.\cos.\varphi = 4.\cos.^2\tfrac{1}{2}\varphi \quad [1, 6] \text{ Int.} \qquad (6)$$

The expression of p [378s], is the same as in (5); those of D, α (6), are as in (7)
[681″]. The second and third values of p (5), are easily deduced from the first, by
using φ, D (1, 6). The formulas (7, 8, 9), are as in [606], using the second of the (8)
expressions (4). The first of the formulas (10), is the same as in [603]; the second and
third values are obtained by means of (5). The expression of $\cos.u$ (11), is the same (9)
as in [603b]; and, from this, we easily obtain the value of $\cos.v$, in the same line. The
first expressions of $\sin.\frac{1}{2}u$, $\cos.\frac{1}{2}u$ (12, 13), are the same as in [1, 6] Int. The second (10)
values in these lines, are deduced from the first, by the substitution of the formulas,

$$1 \mp \cos.u = \frac{(1 \mp e).(1 \mp \cos.v)}{1 + e.\cos.v} \quad [603b \text{ line } 5], \qquad (11)$$

and putting $\frac{1}{2}.(1 - \cos.v) = \sin.^2\frac{1}{2}v$, $\frac{1}{2}.(1 + \cos.v) = \cos.^2\frac{1}{2}v$ [1, 6] Int. The third (12)

[5981]

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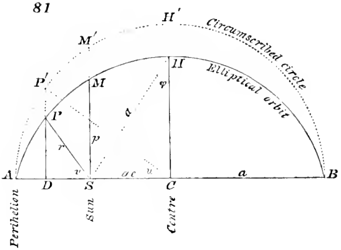

expressions are deduced from the second, by the substitution of

(13) $$\frac{1}{1+e.\cos.v}=\frac{r}{p}\quad(10);$$

the fourth, or last of these values, is deduced from third, by the substitution of

(14) $$p=a.(1-e^2)\quad(5).$$

The last of the formulas (14), is the same as in (8); and the second is deduced from this by

(15) using the value of $\text{tang.}(45^d-\frac{1}{2}\varphi)$ (4). Multiplying together the last values of
(16) $\sin.\frac{1}{2}u$, $\cos.\frac{1}{2}u$ (12, 13), reducing by means of [31] Int., and using $\sqrt{1-e^2}=\cos.\varphi$ (1), we get the last expression (15); the second expression (15), is deduced from this, by
(17) using $\frac{1}{a.\cos.\varphi}=\frac{\cos.\varphi}{p}$ (5). The first of the formulas (16), is deduced from the first of (15); then substituting $p=a.\cos.^2\varphi$, and $\sqrt{pa}=a.\cos.\varphi$, we get the third and
(18) fourth expressions in that line. Multiplying together the first values of r and $\cos.v$
(19) (9, 11), we get the first expression of $r.\cos.v$ (17); substituting $e=\sin.\varphi$, or rather
(20) $-e=\cos.(90^d+\varphi)$, we get,

(21) $$\begin{aligned}\cos.u-e&=\cos.u+\cos.(90^d+\varphi)\\&=2.\cos.(\tfrac{1}{2}u+\tfrac{1}{2}\varphi+45^d).\cos.(\tfrac{1}{2}u-\tfrac{1}{2}\varphi-45^d)\quad[27]\text{ Int.};\end{aligned}$$

whence we easily obtain the last expression (17). Multiplying the third value of $\cos.\frac{1}{2}u$
(22) (13), by $\sin.\frac{1}{2}v$, and the third value of $\sin.\frac{1}{2}u$ (12), by $-\cos.\frac{1}{2}v$; then taking the sum of the products, and reducing, by means of [22, 31] Int., we obtain,

(23) $$\sin.\tfrac{1}{2}.(v-u)=\tfrac{1}{2}\sin.v.\sqrt{\frac{r}{p}}.\{\sqrt{1+e}-\sqrt{1-e}\};$$

substituting the first of the formulas (3), we get the first of the expressions (18); and, by
(24) using the value of $\sin.v=\frac{\sqrt{pa}.\sin.u}{r}$ (16), we obtain the second of the formulas (8). If we repeat this last calculation, changing the factor $-\cos.\frac{1}{2}v$, into $+\cos.\frac{1}{2}v$, we get,

(25) $$\sin.\tfrac{1}{2}.(v+u)=\tfrac{1}{2}\sin.v.\sqrt{\frac{r}{p}}.\{\sqrt{1+e}+\sqrt{1-e}\},$$

and by using the second expression (3), we get the first formula (19); then, substituting the preceding value of $\sin.v$, we get the second of the formulas (19).

FORMULAS IN AN ELLIPTICAL ORBIT. [5985]

$e = \sin.\varphi$; $\sqrt{(1-e^2)} = \cos.\varphi$; [Excentricity e] (1)

$1-e = 2.\sin.^2(45^d-\frac{1}{2}\varphi) = 2.\cos.^2(45^d+\frac{1}{2}\varphi)$; $1+e = 2.\cos.^2(45^d-\frac{1}{2}\varphi) = 2.\sin.^2(45^d+\frac{1}{2}\varphi)$; (2)

$\sqrt{(1+e)}-\sqrt{(1-e)} = 2.\sin.\frac{1}{2}\varphi$; $\sqrt{(1+e)}+\sqrt{(1-e)} = 2.\cos.\frac{1}{2}\varphi$; (3)

$\frac{1-e}{1+e} = \text{tang.}^2(45^d-\frac{1}{2}\varphi)$; $\frac{1+e}{1-e} = \text{tang.}^2(45^d+\frac{1}{2}\varphi)$; (4)

$p = a.(1-e^2) = a.\cos.^2\varphi = (1+e).D$; [Parameter $2p$] (5)

$D = a.(1-e) = a\alpha$; $\alpha = 1-e$; [Perihelion distance D] (6)

$nt = u - e.\sin.u$; [Mean anomaly nt] (7)

$\text{tang.}\frac{1}{2}v = \sqrt{\left(\frac{1+e}{1-e}\right)}.\text{tang.}\frac{1}{2}u = \text{tang.}(45^d+\frac{1}{2}\varphi).\text{tang.}\frac{1}{2}u$; [Time from Perihelion t expressed in days] (8)

$r = a.(1-e.\cos.u)$; [Excentric anomaly u] (9)

$r = \frac{a.(1-e^2)}{1+e.\cos.v} = \frac{a.\cos.^2\varphi}{1+e.\cos.v} = \frac{p}{1+e.\cos.v}$; [Radius vector r] (10)

Elliptical formulas.

$\cos.v = \frac{\cos.u-e}{1-e.\cos.u}$; $\cos.u = \frac{e+\cos.v}{1+e.\cos.v}$; [True anomaly v] (11)

$\sin.\frac{1}{2}u = \sqrt{\frac{1}{2}.(1-\cos.u)} = \sin.\frac{1}{2}v.\left(\frac{1-e}{1+e.\cos.v}\right)^{\frac{1}{2}} = \sin.\frac{1}{2}v.\left(\frac{r.(1-e)}{p}\right)^{\frac{1}{2}} = \sin.\frac{1}{2}v.\left(\frac{r}{a.(1+e)}\right)^{\frac{1}{2}}$; (12)

$\cos.\frac{1}{2}u = \sqrt{\frac{1}{2}.(1+\cos.u)} = \cos.\frac{1}{2}v.\left(\frac{1+e}{1+e.\cos.v}\right)^{\frac{1}{2}} = \cos.\frac{1}{2}v.\left(\frac{r.(1+e)}{p}\right)^{\frac{1}{2}} = \cos.\frac{1}{2}v.\left(\frac{r}{a.(1-e)}\right)^{\frac{1}{2}}$; (13)

$\text{tang.}\frac{1}{2}u = \text{tang.}\frac{1}{2}v.\text{tang.}(45^d-\frac{1}{2}\varphi) = \sqrt{\left(\frac{1-e}{1+e}\right)}.\text{tang.}\frac{1}{2}v$; (14)

$\sin.u = \frac{r.\sin.v.\cos.\varphi}{p} = \frac{r.\sin.v}{a.\cos\varphi}$; (15)

$r.\sin.v = \frac{p.\sin.u}{\cos.\varphi} = a.\cos.\varphi.\sin.u = \sqrt{pa}.\sin.u$; (16)

$r.\cos.v = a.(\cos.u-e) = 2a.\cos.(\frac{1}{2}u+\frac{1}{2}\varphi+45^d).\cos.(\frac{1}{2}u-\frac{1}{2}\varphi-45^d)$; (17)

$\sin.\frac{1}{2}.(v-u) = \sqrt{\frac{r}{p}}.\sin.\frac{1}{2}\varphi.\sin.v = \sqrt{\frac{a}{r}}.\sin.\frac{1}{2}\varphi.\sin.u$; (18)

$\sin.\frac{1}{2}.(v+u) = \sqrt{\frac{r}{p}}.\cos.\frac{1}{2}\varphi.\sin.v = \sqrt{\frac{a}{r}}.\cos.\frac{1}{2}\varphi.\sin.u.$ (19)

[5986] FORMULAS IN A PARABOLIC ORBIT.

The equations of the motion in a *parabola* [5986 lines 2—10], are the same as in [691, 693, &c.]; in which 2π represents the circumference of a circle, whose radius is
(1) unity [691‴ line 4], and $T = 365^{\text{days}}.25638$, is the length of a sideral year [691‴, 750′].

(2) $$p = 2D\,;$$ [Parameter $2p$]

(3) $$D = \tfrac{1}{2}p\,;$$ [Perihelion distance D]

(4) $$r = \frac{D}{\cos.^2\frac{1}{2}v} = \frac{p}{1+\cos.v}\,;$$ [Radius vector r]

(5) $$t = \frac{D^{\frac{3}{2}}T}{\pi.\sqrt{2}}.\{\text{tang}.\tfrac{1}{2}v + \tfrac{1}{3}.\text{tang}.^3\tfrac{1}{2}v\}$$ [True anomaly v]

(6) $$= \frac{D^{\frac{3}{2}}.\sqrt{2}}{k}.\{\text{tang}.\tfrac{1}{2}v + \tfrac{1}{3}.\text{tang}.^3\tfrac{1}{2}v\}$$ [Time from the Perihelion t, expressed in days]

Parabolic formulas.

(7) $$= D^{\frac{3}{2}}t'\,;$$

(8) $$t' = \frac{T}{\pi.\sqrt{2}}.\{\text{tang}.\tfrac{1}{2}v + \tfrac{1}{3}.\text{tang}.^3\tfrac{1}{2}v\}$$ [Time from the perihelion t' days, when $D=1$.]

(9) $$= \frac{\sqrt{2}}{k}.\{\text{tang}.\tfrac{1}{2}v + \tfrac{1}{3}.\text{tang}.^3\tfrac{1}{2}v\}$$

(10) $$= \frac{t}{D^{\frac{3}{2}}}.$$

[5987] In the expressions of t [5986 lines 5, 8], we
(1) ought, in strictness, to change T into $T.\sqrt{1+m''}$; m'' being the mass of the earth, and 1 the mass of the sun; this is evident from [692′, &c.],
(2) where $\mu = 1 + m''$. It is common, however, to neglect the mass m'', as we have already observed in [692′ line 4]. Instead of T, or
(3) rather $T.\sqrt{1+m''}$, the symbol $k = \dfrac{2\pi}{T.\sqrt{1+m''}}$ is used by Gauss, and by most of the
(4) German astronomers. We have already found, in [750′],

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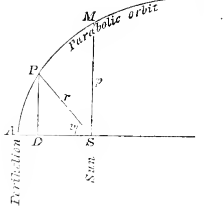

(5) $$\frac{T}{12\pi} = 9^{\text{days}},688724\ldots, \quad \text{or} \quad \log.\frac{T}{12\pi} = 0{,}98626669\ldots\,;$$

and, by neglecting m'', we have, [5987]

$$k = \frac{2\pi}{T}, \quad \text{or} \quad \log.k = \log.\frac{2\pi}{T} = 8{,}2355820\ldots; \qquad (6)$$

but, if we notice m'', we shall get,

$$\log.k.\sqrt{1+m''} = \log.\frac{2\pi}{T} = 8{,}2355820\ldots; \qquad (7)$$

and, since $\log.\sqrt{1+m''} = 0{,}0000006\ldots$, we shall obtain the corrected value of,

$$\log.k = 8{,}2355814\ldots; \qquad (8)$$

being nearly as it is given by Gauss, in his *Theoria Motus Corporum Cœlestium*; differing from the former expression, by the very small fraction 0,0000006... We may remark, that the mean angular motion of any planet, in the time t, is represented in [605'', 605'], by $nt = \frac{t.\sqrt{1+m}}{a^{\frac{3}{2}}}$; m being the mass of the planet; a its mean distance from the (9) sun; that of the earth from the sun being taken for unity. *The second member of this expression must be multiplied by a constant quantity, which is the same for all the planets, to reduce it to the unit of the measures of these angles.* (10) To ascertain this quantity, we shall observe, that the mean angular motion of the earth in a sideral year T, is represented by the whole circumference 2π [691$^{\text{iv}}$]; and, if we change, in the second member of (11) [5987 (9)], t, m, a into T, m'', 1 respectively, it becomes $T.\sqrt{1+m''}$. To reduce this to 2π [5987 (11)], we must evidently multiply it by $\frac{2\pi}{T.\sqrt{1+m''}}$, or by the quantity k [5987 (3)]; which therefore represents the constant quantity [5987 (10)]; hence the mean motion [5987 (9)] becomes $nt = \frac{t.k\sqrt{1+m}}{a^{\frac{3}{2}}}$; consequently $n = \frac{k\sqrt{1+m}}{a^{\frac{3}{2}}}$. (12) This value of n must be used in [5985 (7)]. If we wish to express the mean motion in seconds, we must multiply the expression of nt [5987 (12)] by the radius in seconds $206264^s{,}67$; (13) or, to avoid this labor, we may use the value of k in seconds; namely, $k = 3548^s{,}18761$, or $\log.k = 3{,}55000657$. In estimating the motion of a comet, we may neglect its mass (14) m, on account of its smallness; and then the expression of the mean motion [5987 (12)] becomes $\frac{kt}{a^{\frac{3}{2}}}$. This is expressed in [702'] by $\frac{t.\sqrt{\mu}}{a^{\frac{3}{2}}}$; the accent on a' being (15) omitted, to conform to the present notation. Hence it appears, that we must put $\sqrt{\mu} = k$, (16) to reduce the formulas of the author, in [702', &c.], to the notation of this article.

The expressions in [5986 lines 2, 3] are the same as in [807', 807'']. The first formula in [5986 (4)], is the same as in [691 line 1]; the second expression is easily deduced (17) from the first, by the substitution of

$$\cos.^2\tfrac{1}{2}v = \tfrac{1}{2} + \tfrac{1}{2}.\cos.v,$$

[5987] and the value of D [5986 (3)]. The expression of t [5986 (5)], is the same as in [693]. Substituting in this, the value,

(18) $$T = \frac{2\pi}{k} \quad [5987\ (6)],$$

we get [5986 (6)]. This expression of t may be put equal to $D^{\frac{3}{2}} t'$ [693a],
(19) as in [5986 (7)]; t' being the time from the perihelion, corresponding to the anomaly v, in a parabolic orbit, whose perihelion distance D is equal to unity. This parabola
(20) is usually called *the parabola of* 109 *days*: because, it requires about 109 days to
(21) describe an arc of 90^d from the perihelion, in a parabola whose perihelion distance is unity. Dividing the three expressions in [5986 (5, 6, 7)], by $D^{\frac{3}{2}}$, we get the formulas
(22) [5986 (8, 9, 10)]. From that in line 8 or 9, Burckhardt has computed Table III, of this appendix, changing v into U; and putting,

(23) $$\frac{T}{3\pi.\sqrt{2}} = 27^{\text{days}},4038...$$

Then, by means of this table, we can find, by inspection, the anomaly U, or v, from $\log. t'$, or the contrary.

FORMULAS IN A HYPERBOLIC ORBIT.

[5988] The formulas for computing the motion of a body, in a hyperbolic orbit, are given in [702]; but, it will be convenient to alter the forms of these expressions, by writing
(1) a for a', and introducing the auxiliary quantities ψ, u, proposed by Gauss; so that

(2) $$e.\cos.\psi = 1, \qquad \text{and} \qquad u = \text{tang.}(45^d + \tfrac{1}{2}\varpi);$$

by this means, we obtain the following system of equations, corresponding to the motion in a hyperbolic orbit.

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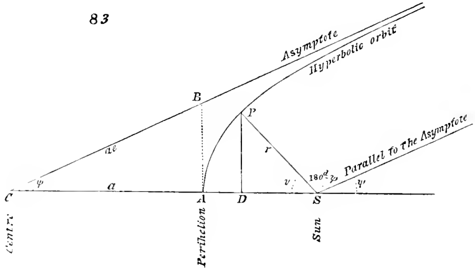

[5988]

$e = \frac{1}{\cos.\psi} = \text{secant}\,\psi = \sqrt{1+\text{tang.}^2\psi}$; $\sqrt{e^2-1} = \text{tang.}\psi$; $\frac{e-1}{e+1} = \text{tang.}^2\tfrac{1}{2}\psi$; [Excentricity e] (3)

$p = a.(e^2-1) = a.\text{tang.}^2\psi = (e+1).D$; [Parameter $2p$] (4)

$D = a.(e-1) = -a\alpha$; $\alpha = -(e-1)$; [Perihelion distance D] (5)

$\frac{k}{a^{\frac{3}{2}}}\cdot t = e.\text{tang.}\varpi - \text{hyp.log.tang.}(45^d+\tfrac{1}{2}\varpi)$; [Semi-transverse axis a] (6)

$\frac{\lambda k}{a^{\frac{3}{2}}}\cdot t = \tfrac{1}{2}\lambda e.\frac{(u^2-1)}{u} - \text{common log. tang.}(45^d+\tfrac{1}{2}\varpi)$; [Time from the perihelion t, expressed in days] (7)

$\lambda = 0{,}43429448\ldots$; $\log.\lambda = 9{,}6377843113\ldots$; $\log.\lambda k = 7{,}8733657527\ldots$; (8)

$k = \frac{2\pi}{T.\sqrt{1+m''}} = 0{,}01720209895$ parts of radius; $\log.k = 8{,}2355814414\ldots$; (9)

$\text{tang.}\tfrac{1}{2}\varpi = \sqrt{\left(\frac{e-1}{e+1}\right)}\cdot\text{tang.}\tfrac{1}{2}v = \text{tang.}\tfrac{1}{2}\psi.\text{tang.}\tfrac{1}{2}v$; [Auxiliary angle ϖ] (10)

$r = \frac{p}{1+e.\cos.v} = \frac{a.(e^2-1)}{1+e.\cos.v} = \frac{p.\cos.\psi}{2.\cos.\frac{1}{2}(v-\psi).\cos.\frac{1}{2}(v+\psi)}$; [Radius vector r] (11)

$r = a.\left(\frac{e}{\cos.\varpi}-1\right) = \tfrac{1}{2}a.\left\{e.\left(u+\frac{1}{u}\right)-2\right\}$; (12)

Hyperbolic formulas.

$u = \text{tang.}(45^d+\tfrac{1}{2}\varpi) = \frac{1+\text{tang.}\frac{1}{2}\varpi}{1-\text{tang.}\frac{1}{2}\varpi} = \frac{\cos.\frac{1}{2}(v-\psi)}{\cos.\frac{1}{2}(v+\psi)}$; [Auxiliary quantity u] (13)

$\frac{1}{\cos.\varpi} = \tfrac{1}{2}\cdot\left(u+\frac{1}{u}\right) = \frac{1+\cos.\psi.\cos.v}{2.\cos.\frac{1}{2}(v-\psi).\cos.\frac{1}{2}(v+\psi)} = \frac{e+\cos.v}{1+e.\cos.v}$; (14)

$\sin.\varpi = \frac{u^2-1}{u^2+1}$; $\cos.\varpi = \frac{2u}{u^2+1}$; $\text{tang.}\varpi = \frac{u^2-1}{2u}$; (15)

$\sin.\tfrac{1}{2}\varpi = \frac{u-1}{\sqrt{2(u^2+1)}}$; $\cos.\tfrac{1}{2}\varpi = \frac{u+1}{\sqrt{2(u^2+1)}}$; $\text{tang.}\tfrac{1}{2}\varpi = \frac{u-1}{u+1}$; (16)

$\sin.\tfrac{1}{2}v.\sqrt{r} = \sin.\tfrac{1}{2}\varpi.\sqrt{\left(\frac{p}{(e-1).\cos.\varpi}\right)} = \sin.\tfrac{1}{2}\varpi.\sqrt{\left(\frac{a.(e+1)}{\cos.\varpi}\right)}$ [True anomaly v, from the perihelion] (17)

$= \tfrac{1}{2}\cdot(u-1).\sqrt{\left(\frac{p}{(e-1).u}\right)} = \tfrac{1}{2}\cdot(u-1).\sqrt{\left(\frac{a.(e+1)}{u}\right)}$; (18)

$\cos.\tfrac{1}{2}v.\sqrt{r} = \cos.\tfrac{1}{2}\varpi.\sqrt{\left(\frac{p}{(e+1).\cos.\varpi}\right)} = \cos.\tfrac{1}{2}\varpi.\sqrt{\left(\frac{a.(e-1)}{\cos.\varpi}\right)}$ (19)

$= \tfrac{1}{2}\cdot(u+1).\sqrt{\left(\frac{p}{(e+1).u}\right)} = \tfrac{1}{2}\cdot(u+1).\sqrt{\left(\frac{a.(e-1)}{u}\right)}$; (20)

$r.\sin.v = p.\cot.\psi.\text{tang.}\varpi = a.\text{tang.}\psi.\text{tang.}\varpi$ (21)

$= \tfrac{1}{2}p.\cot.\psi.\left(u-\frac{1}{u}\right) = \tfrac{1}{2}a.\text{tang.}\psi.\left(u-\frac{1}{u}\right)$; (22)

$r.\cos.v = a.\left(e-\frac{1}{\cos.\varpi}\right) = \tfrac{1}{2}a.\left(2e-u-\frac{1}{u}\right)$. (23)

[5989] In the demonstrations of these formulas [5988], we shall refer to any one of them, by
(1) placing the number of the line in which it is situated, in a parenthesis, in the same manner as in the elliptical formulas [5984 (2), &c.]. From the assumed value

$e=\frac{1}{\cos.\psi}$ (3), we get, by means of [1, 6] Int.,

(2) $$e-1=\frac{1-\cos.\psi}{\cos.\psi}=\frac{2.\sin.^2\frac{1}{2}\psi}{\cos.\psi}\,; \qquad e+1=\frac{1+\cos.\psi}{\cos.\psi}=\frac{2.\cos.^2\frac{1}{2}\psi}{\cos.\psi}\,;$$

dividing the first of these expressions by the second, we get the third of the formulas (3).
(3) In a hyperbola, the semi-axis a becomes negative, and is represented by $-a'$ [698″]; hence the values of p, D [5985 (5, 6)] become, in the hyperbola,

(4) $$p=a'.(e^2-1)\;; \qquad D=a'.(e-1)\;;$$

(5) and, if we neglect the accent upon a', for the sake of simplicity in the notation, we shall obtain the first expressions of p, D (4, 5); the others are deduced from these, by the substitution of ψ, D (3, 5). If we change, in the first equation [702], the
(6) symbol $\sqrt{\mu}$ into k, as in [5987 (16)], and omit the accent on a', as above, it becomes as in (6); using hyperbolic logarithms. We must multiply this by

(7) $$\lambda=0{,}43429448\ldots \quad (8),$$

when common logarithms are used; the quantity λ being the ratio of a common logarithm to a hyperbolic logarithm; and then, (6) changes into (7), by the substitution of tang.ϖ (15); this value of tang.ϖ being deduced from the assumed value of u (13),
(8) as in [5989 (14)]. The first formula (10) is the same as that in [702 line 3]; from this we easily deduce the second form, by the substitution of tang.$\frac{1}{2}\psi$ (3). The first value of r (11) is the same as in [379b], using p (4); and this form is common to
(9) the other conic sections [5985 (10), 5986 (4)]. Substituting in this, the first value of p (4), we get the second form of r (11). Multiplying the numerator and denominator
(10) of the first form of r (11), by $\cos.\psi$, and substituting $e.\cos.\psi=1$ (2), in the denominator, we find, that this denominator becomes,

(11) $$\cos.\psi+\cos.v=2.\cos.\tfrac{1}{2}(v-\psi).\cos.\tfrac{1}{2}(v+\psi) \quad [20]\text{ Int.},$$

and we obtain the third expression of r (11). The first expression of r (12) is the same as in [702 line 2], omitting the accent upon a', as above. To obtain the second form, we must use the auxiliary quantity u (13); namely,

(12) $$u=\text{tang.}(45^d+\tfrac{1}{2}\varpi)=\frac{1+\text{tang.}\frac{1}{2}\varpi}{1-\text{tang.}\frac{1}{2}\varpi} \quad [29]\text{ Int.};$$

from which we get,

(13) $$\text{tang.}\,\tfrac{1}{2}\varpi=\frac{u-1}{u+1} \quad (16)\;;$$

and then, from [30′] Int., we have, as in (15),

[5989]

$$\text{tang.}\varpi = \frac{2.\text{tang.}\tfrac{1}{2}\varpi}{1-\text{tang.}^2\tfrac{1}{2}\varpi} = \frac{2.\left(\frac{u-1}{u+1}\right)}{1-\left(\frac{u-1}{u+1}\right)^2} = \frac{2.(u^2-1)}{(u+1)^2-(u-1)^2} = \frac{u^2-1}{2u}\,. \qquad (14)$$

From this, we get,

$$\text{sec.}\varpi = \frac{1}{\text{cos.}\,\varpi} = (1+\text{tang.}^2\varpi)^{\frac{1}{2}} = \frac{u^2+1}{2u} = \tfrac{1}{2}.\left(u+\frac{1}{u}\right), \quad \text{as in } (14, 15). \qquad (15)$$

Multiplying together the expressions of $\text{cos.}\varpi$, and $\text{tang.}\varpi$ (15), we get $\text{sin.}\varpi$ (15). In like manner, if we substitute the value of $\text{tang.}\frac{1}{2}\varpi$ (16), in the expression,

$$\text{cos.}\tfrac{1}{2}\varpi = (1+\text{tang.}^2\tfrac{1}{2}\varpi)^{-\frac{1}{2}} \quad [34'''] \text{ Int.}, \qquad (16)$$

we get its value (16) ; multiplying together these two expressions of $\text{cos.}\frac{1}{2}\varpi$, $\text{tang.}\frac{1}{2}\varpi$, we get $\text{sin.}\frac{1}{2}\varpi$ (16). Substituting the first value of $\frac{1}{\text{cos.}\varpi}$ (14) in the first expression of r (12), we obtain its second form. If we substitute, in the second expression of u (13), the value, (17)

$$\text{tang.}\tfrac{1}{2}\varpi = \text{tang.}\tfrac{1}{2}\psi.\text{tang.}\tfrac{1}{2}v \quad (10)\ ; \qquad (18)$$

then, multiply the numerator and denominator by $\text{cos.}\frac{1}{2}\psi.\text{cos.}\frac{1}{2}v$, we shall find, that the numerator becomes,

$$\text{cos.}\tfrac{1}{2}\psi.\text{cos}\tfrac{1}{2}v+\text{sin.}\tfrac{1}{2}\psi.\text{sin.}\tfrac{1}{2}v = \text{cos.}\tfrac{1}{2}(v-\psi)\ ;$$

and, the denominator, (19)

$$\text{cos.}\tfrac{1}{2}\psi.\text{cos.}\tfrac{1}{2}v-\text{sin.}\tfrac{1}{2}\psi.\text{sin.}\tfrac{1}{2}v = \text{cos.}\tfrac{1}{2}(v+\psi)\ ;$$

as in the last of the formulas (13). If we now substitute the last value of u (13), in the first expression of (14), it becomes,

$$\frac{1}{\text{cos.}\varpi} = \tfrac{1}{2}.\left\{\frac{\text{cos.}\tfrac{1}{2}(v-\psi)}{\text{cos.}\tfrac{1}{2}(v+\psi)} + \frac{\text{cos.}\tfrac{1}{2}(v+\psi)}{\text{cos.}\tfrac{1}{2}(v-\psi)}\right\}; \qquad (20)$$

reducing these to a common denominator,

$$2.\text{cos.}\tfrac{1}{2}(v-\psi).\,\text{cos.}\tfrac{1}{2}(v+\psi), \qquad (21)$$

we find, that the numerator becomes, by using [6, 20] Int.,

$$\begin{aligned}\text{cos.}^2\tfrac{1}{2}(v-\psi)+\text{cos.}^2\tfrac{1}{2}(v+\psi) &= \{\tfrac{1}{2}+\tfrac{1}{2}.\text{cos.}(v-\psi)\}+\{\tfrac{1}{2}+\tfrac{1}{2}.\text{cos.}(v+\psi)\} \\ &= 1+\tfrac{1}{2}.\text{cos.}(v-\psi)+\tfrac{1}{2}.\text{cos.}(v+\psi) = 1+\text{cos.}\psi.\text{cos.}v\ ;\end{aligned} \qquad (22)$$

as in the second formula (14). Multiplying the numerator and denominator of the second formula (14) by e, and substituting the values [5989 (11)] and (2), we get (23)

[5989] the third formula (14). If we add ∓ 1 to the last of the values $\frac{1}{\cos.\varpi}$ (14), and

(24) substitute $\frac{1}{1+e.\cos.v}=\frac{r}{p}$ (11), we get,

(25) $$\frac{1\mp\cos.\varpi}{\cos.\varpi}=\frac{(e\mp 1).(1\mp\cos.v)}{1+e.\cos.v}=\frac{(e\mp 1).(1\mp\cos.v).r}{p}.$$

If we use the upper sign, and put

(26) $$1-\cos.\varpi=2.\sin.^2\tfrac{1}{2}\varpi\,; \qquad 1-\cos.v=2.\sin.^2\tfrac{1}{2}v \quad [1]\,\text{Int.};$$

(27) we get, by extracting the square root, the first of the formulas (17). If we use the lower sign, and put,

(28) $$1+\cos.\varpi=2.\cos^2\tfrac{1}{2}\varpi\,; \qquad 1+\cos.v=2.\cos.^2\tfrac{1}{2}v\,;$$

we get the first of the formulas (19). The second of the formulas (17, or 19), is deduced

(29) from the first, by the substitution of $p=a.(e^2-1)$ (4). Substituting, in the first of the formulas (17, or 19), the values of $\sin.\frac{1}{2}\varpi$, $\cos.\frac{1}{2}\varpi$, $\cos.\varpi$ (16, 15), which give,

(30) $$\frac{\sin.\frac{1}{2}\varpi}{\sqrt{\cos.\varpi}}=\frac{u-1}{2.\sqrt{u}}\,; \qquad \frac{\cos.\frac{1}{2}\varpi}{\sqrt{\cos.\varpi}}=\frac{u+1}{2.\sqrt{u}}\,;$$

we get the first of the formulas (18, 20); finally, substituting in these, the value of $p=a.(e^2-1)$, we get the last of the formulas (18, 20). Multiplying by two the product of the first of the formulas (17, 19), we get,

(31) $$2r.\sin.\tfrac{1}{2}v.\cos.\tfrac{1}{2}v=\frac{2\sin.\frac{1}{2}\varpi.\cos.\frac{1}{2}\varpi}{\cos.\varpi}\cdot\frac{p}{\sqrt{(e^2-1)}}\,;$$

and by substituting,

(32) $$2.\sin.\tfrac{1}{2}v.\cos.\tfrac{1}{2}v=\sin.v\,; \qquad 2.\sin.\tfrac{1}{2}\varpi.\cos.\tfrac{1}{2}\varpi=\sin.\varpi=\cos.\varpi.\text{tang}.\varpi \quad [31, 34']\ \text{Int.},$$

also $\sqrt{(e^2-1)}=\text{tang}.\psi=(\text{cotang}.\psi)^{-1}$ (3), we get the first equation (21). The

(33) second formula (21), is easily deduced from the first, by the substitution of $p=a.\text{tang}.^2\psi$ (4).

(34) Substituting in these two expressions, the value of $\text{tang}.\varpi=\frac{1}{2}.\left(u-\frac{1}{u}\right)$ (15), we get the first and second formulas (22). Multiplying the second value of r (11), by $\cos.v$, and reducing, we get, by using the last formula (14),

(35) $$r.\cos.v=\frac{a.(e^2-1).\cos.v}{1+e.\cos.v}=ae-a.\frac{(e+\cos.v)}{1+e.\cos.v}=ae-a.\frac{1}{\cos.\varpi}\,;$$

as in the first expression (23). Substituting in this, the first value of $\frac{1}{\cos.\varpi}$ (14), it

(36) becomes as in the second formula (23).

From the first of the formulas (11), it appears, that r increases with v, and becomes infinite, when [5989]

$$1+e.\cos.v=0, \quad \text{or} \quad \cos.v=-\frac{1}{e}=-\cos.\psi \quad (3); \qquad (37)$$

which gives $v=180^d-\psi$. Now the radius r, corresponding to a point of the hyperbola, at an infinite distance from the focus, *must evidently be parallel to the asymptote;* (38)
therefore, the angle ψ represents the angle of inclination of the asymptote to the axis. Hence it is evident, that the *maximum* value of v is represented by $180^d-\psi$; and (39)
the greatest *minimum* value is $-(180^d-\psi)$; moreover, it follows, from the last of the formulas (13), that when $v=0$, $u=\frac{\cos.\frac{1}{2}.(-\psi)}{\cos.\frac{1}{2}.(+\psi)}=1$; and that u increases with v, (40)
and becomes infinite, when $v=180^d-\psi$, or $\frac{1}{2}.(v+\psi)=90^d$. It decreases when v is negative, and becomes nothing at the other limit, where $v=-(180^d-\psi)$; or $\frac{1}{2}.(v-\psi)=-90^d$. (41)

TO COMPUTE THE TRUE ANOMALY FROM THE TIME, OR THE CONTRARY, IN AN ELLIPTICAL ORBIT.

The true anomaly v, in an elliptical orbit, can be easily obtained from the mean anomaly nt, by means of the formula [668], in cases where the excentricity e is so small, that it is only necessary to notice two or three terms of the series; but as the value of e augments, the number of terms must be increased, so that the method finally becomes very laborious, and it is much better to use the indirect method of solution, first given by Kepler, who was the original proposer of the problem. This method is very simple, and has the decided advantage of being applicable to all the varieties of the ellipsis; but when the excentricity is nearly equal to unity, it requires the use of a table of logarithms, to more than seven places of decimals; this difficulty is obviated partially in the method of Simpson, and wholly in the method of Gauss, which we shall give hereafter. [5990] (1) Kepler's problem. (2)

To illustrate this indirect method of solution, we shall apply it, according to the precepts of Gauss, to the determination of the true anomaly in an elliptical orbit. We shall suppose $u_{\prime}$ to be an approximate value of u, and x its correction; so that $u=u_{\prime}+x$, satisfies the equation [5985 (7)]. We must compute the value of $e.\sin.u_{\prime}$ in seconds, by logarithms; and, while performing the operation, we must take from the tables, the variation λ of the $\log.\sin.u_{\prime}$, corresponding to $1''$ in the value of $u_{\prime}$; also the variation μ of the logarithm $e.\sin.u_{\prime}$, corresponding to the variation of *one unit* in the number $e.\sin.u_{\prime}$; the signs of λ, μ being neglected, and *both the logarithms being taken to the same number of decimals.* Now when $u_{\prime}$ is nearly equal to u, or $u_{\prime}+x$, the variations of the log. sines of the arcs from $u_{\prime}$ to $u_{\prime}+x$, will, in general, be nearly uniform; hence we shall have, with a considerable degree of accuracy, (3) (4) (5) (6) (7)

$$e.\sin.(u_{\prime}+x)=e.\sin.u_{\prime}\pm\frac{\lambda x}{\mu}; \qquad (8)$$

[5990] the *upper sign* being used in the *first* and *fourth* quadrants; the *lower sign* in the *second*
(9) and *third* quadrants; these signs being evidently the same as those of $e.\cos.u_{,}$ [5990(13)]. Substituting this, and $u=u_{,}+x$, in [5985(7)], we get, by reduction,

(10)
$$x=\frac{\mu}{\mu\mp\lambda}\cdot(nt-u_{,}+e\,.\sin.u_{,})\,;$$

Indirect solution of Kepler's problem.

or,

(11)
$$u=u_{,}+x=nt+e\,.\sin.u_{,}\pm\frac{\lambda}{\mu\mp\lambda}\cdot(nt-u_{,}+e\,.\sin.u_{,})\,;$$

(12) in which we must notice the sign of the factor $\pm\frac{\lambda}{\mu\mp\lambda}$, according to the above directions; and we must also have regard to the sign of the other factor $(nt-u_{,}+e.\sin.u_{,})$.
(13) We may remark, that the factor $\pm\frac{\lambda}{\mu}=e.\cos.u_{,}$, as is easily proved by the substitution of $\sin.(u_{,}+x)=\sin.u_{,}+x\cos.u_{,}$ [60] Int., in the first member of [5990(8)]; and, as
(14) $e<1$, $\cos.u_{,}<1$, we shall have $\mu>\lambda$; therefore, $\frac{\lambda}{\mu\mp\lambda}$ has the same sign as $\frac{\lambda}{\mu}$.
(15) If the assumed value of $u_{,}$ should differ considerably from $u_{,}+x$, we must repeat the operation; using this computed value of $u_{,}+x$ for a new value of $u_{,}$; and this process must be repeated, until the correct value of u is found. In most cases which occur in practical astronomy, it will be easy to assume, in the first instance, a value of $u_{,}$ which
(16) does not differ much from u. This is particularly the case, when forming a table of the values of u, corresponding to the regular intervals of nt, from 0^d to 360^d. If we have no means of ascertaining this first value of $u_{,}$, we may make the first computation in a rough manner, using small tables of logarithms, to five places of decimals, and to minutes of a degree. *It will tend to simplify the operation, to take for* $u_{,}$ *a quantity*
(17) *whose sine can be obtained from the tables by inspection, without any interpolation*; as, for example, by taking the value of $u_{,}$ to minutes, when the table of sines is given for every
(18) minute; or for tens of seconds, when the tables are arranged for tens of seconds; &c.

Use of the letter n, affixed to the figures in a numerical calculation.

In making these calculations, and others of a similar nature, *it has been found convenient to annex the small letter* n *to the last figure of the logarithm of any factor*
(19) *which has a negative value: since, by this means, we can very easily ascertain the sign of a quantity, which depends on the product of a number of factors, of different signs,*
(20) whose logarithms are to be added together, to obtain the logarithm of the required number. It being evident, that the sign of this number must be *positive*, if the number of the letters
(21) n be *even*, but *negative* if the number be *odd*. Thus, in finding the logarithm corresponding to the quantity $-3.\sin.192^d$, composed of the two factors -3 and $\sin.192^d$, we may put for their logarithms the quantities $0{,}4771213_n$ and $9{,}3178789_n$, whose
(22) sum $9{,}7950002$ corresponds to a positive quantity. We must also carefully notice the signs of any quantities, depending on the sine, cosine or tangent of an arc; observing that, according to the usual rules, we have,

[5990]

sin. or *cosec.* is $+$ in the *first* and *second* quadrants; $-$ in the *third* and *fourth*. (23)

cos. or *sec.* is $+$ in the *first* and *fourth* quadrants; $-$ in the *second* and *third*. (24)

tang. or *cot.* is $+$ in the *first* and *third* quadrants; $-$ in the *second* and *fourth*. (25)

To show by an example the use of the formula [5990 (11)], we shall suppose the mean motion to be $nt = 332^d\,28^m\,54^s{,}77$, log. e in seconds $= 4{,}7011513$, or $e = 50600^s$ (26) nearly. Then, for a first operation, we shall take $u_{,} = 326^d$, from which we find, as below, $u_{,}+x = 324^d\,16^m\,20^s$. Taking this for $u_{,}$, in a second operation, we finally obtain $u = 324^d\,16^m\,29^s{,}5$; which is its true value, as will appear by the following calculations.

FIRST OPERATION $u_{,} = 326^d$.

$u_{,} = 326^d$ log.sin. $9{,}7475617_n$ $\lambda = 31$

e log. $4{,}7041513$

$e.\sin.u_{,}$ log. $4{,}4517130_n$ $\mu = 153$

$e.\sin.u_{,} = -28295^s = -7^d\,51^m\,35^s$ $\mu - \lambda = 122$

$nt = 332^d\,28^m\,55^s$

$nt + e.\sin.u_{,} = 324^d\,37^m\,20^s = A$

$u_{,} = 326^d\,00^m\,00^s$

$(nt - u_{,} + e.\sin.u_{,}) = -1^d\,22^m\,40^s = -4960^s$

multiply this by

$\pm\dfrac{\lambda}{\mu\mp\lambda} = +\frac{31}{122}$ gives $-21^m\,00^s = B$ nearly

$A + B = u_{,}+x = 324^d\,16^m\,20^s$.

SECOND OPERATION $u_{,} = 324^d\,16^m\,20^s$.

$u_{,} = 324^d\,16^m\,20^s$ log.sin. $9{,}7663644_n$ $\lambda = 29$

e log. $4{,}7041513$

$e.\sin.u_{,}$ log. $4{,}4705157_n$ $\mu = 147$

$e.\sin.u_{,} = -29547^s{,}16 = -8^d\,12^m\,27^s{,}16$ $\mu - \lambda = 118$

$nt = 332^d\,28^m\,54^s{,}77$ (27)

$nt + e.\sin.u_{,} = 324^d\,16^m\,27^s{,}6 = A$

$u_{,} = 324^d\,16^m\,20^s$

$(nt - u_{,} + e.\sin.u_{,}) = +7^s{,}6$

multiply this by

$\pm\dfrac{\lambda}{\mu\mp\lambda} = +\frac{29}{118}$ gives $+1^s{,}9 = B$

$A + B = u = 324^d\,16^m\,29^s{,}5$.

Having obtained the value of u, we may compute r, v from [5985 (9, 11)]: but as the method of making this calculation is sufficiently obvious, we shall not give an example.

When the excentricity e is very nearly equal to unity, this indirect method requires the use of tables of logarithms to more than seven places of decimals. For, if the logarithms were correct, to the nearest unit, in the seventh decimal place, there might be an error of 46^s, in computing the anomaly, in an orbit, where $1-e = 0.001$; and, the error would exceed this, by decreasing $1-e$. *In this case, we may use the method of Simpson, given by La Place in* [694—698], neglecting all the powers of $1-e = \alpha$, above the first. This degree of accuracy is not, however, sufficient, in Halley's comet, where $1-e = 0{,}03$, nearly; for, it is found to be necessary to notice the terms depending on the second power of $1-e$; which exceed 30^s, when the anomaly is 100^d. If we use the same notation as in [691', &c.], we easily perceive, that the true anomaly $v = U + t$, in the ellipsis, may be derived from the value of U, corresponding to the parabola, by an expression of the following form, in which the third and higher powers of $1-e = \alpha$, are neglected;

[5991] (1) (2) (3) (4) (5) (6)

[5991]

(7) $$v = U + S\,.\,(1-e) + B\,.\,(1-e)^2 = U + S.\alpha + B.\alpha^2\ ;$$

(8) *S being the value of the function [698], corresponding to Simpson's method, and B the function [5991(30)], introduced by Bessel, in his tables, published in vol. 12, p. 207, of the Monatliche Correspondenz.* (9) *The same formula may be applied, without any modification, to a hyperbolic orbit, which approaches very near to a parabolic form, by merely noticing the sign of $1-e$, which then becomes negative.*

(10) In the computation of S and B, we may put, for brevity, $\text{tang.}\tfrac{1}{2}U = \theta$, or,

(11) $$\cos.^2\tfrac{1}{2}U = \frac{1}{1+\text{tang.}^2\tfrac{1}{2}U} = \frac{1}{1+\theta^2}.$$

Substituting these, in the expression of S [698], it becomes,

(12) $$S = \tfrac{1}{10}\,.\,\theta\,.\left\{4 - \frac{3}{1+\theta^2} - \frac{6}{(1+\theta^2)^2}\right\} = \frac{(-\tfrac{1}{2}\theta + \tfrac{1}{2}\theta^3 + \tfrac{2}{5}\theta^5)}{(1+\theta^2)^2}.$$

Method of Simpson, improved by Bessel.

To obtain B, we shall develop the expression [690], according to the powers of α, neglecting terms of the order α^3; hence we get the first of the following expressions; (13) the second form is deduced from the first, by multiplying the terms, between the braces, by the external factor $1+\tfrac{1}{4}\alpha+\tfrac{3}{32}\alpha^2$; the third form is obtained, by arranging the terms according to the powers of α;

$$t = \frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.(1+\tfrac{1}{4}\alpha+\tfrac{3}{32}\alpha^2).\text{tang.}\tfrac{1}{2}v\,.\left\{\begin{array}{l} 1+(\tfrac{1}{3}-\tfrac{1}{3}\alpha-\tfrac{1}{6}\alpha^2).\text{tang.}^2\tfrac{1}{2}v \\ \quad +(-\tfrac{1}{5}\alpha+\tfrac{1}{20}\alpha^2).\text{tang.}^4\tfrac{1}{2}v+\tfrac{3}{28}\alpha^2.\text{tang.}^6\tfrac{1}{2}v \end{array}\right\}$$

(14) $$= \frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.\left\{(1+\tfrac{1}{4}\alpha+\tfrac{3}{32}\alpha^2).\text{tang.}\tfrac{1}{2}v+(\tfrac{1}{3}-\tfrac{1}{4}\alpha-\tfrac{7}{32}\alpha^2).\text{tang.}^3\tfrac{1}{2}v-\tfrac{1}{5}\alpha.\text{tang}^5\tfrac{1}{2}v+\tfrac{3}{28}\alpha^2.\text{tang.}^7\tfrac{1}{2}v\right\}$$

(15) $$= \frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.\left\{\begin{array}{l} \text{tang.}\tfrac{1}{2}v+\tfrac{1}{3}.\text{tang}^3\tfrac{1}{2}v+\alpha\,.\,(\tfrac{1}{4}.\text{tang.}\tfrac{1}{2}v-\tfrac{1}{4}.\text{tang.}^3\tfrac{1}{2}v-\tfrac{1}{5}.\text{tang.}^5\tfrac{1}{2}v) \\ \qquad\qquad +\alpha^2.(\tfrac{3}{32}.\text{tang.}\tfrac{1}{2}v-\tfrac{7}{32}.\text{tang.}^3\tfrac{1}{2}v+\tfrac{3}{28}.\text{tang.}^7\tfrac{1}{2}v) \end{array}\right\}.$$

If $\alpha = 0$, v changes into U, and the expression of t becomes as in [691]. Putting these two values of t equal to each other, and dividing by the common factor $\frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}$, we get,

(16) $$\begin{array}{l} \text{tang.}\tfrac{1}{2}U+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}U = \text{tang.}\tfrac{1}{2}v+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}v+\alpha\,.\,\{\tfrac{1}{4}.\text{tang.}\tfrac{1}{2}v-\tfrac{1}{4}.\text{tang.}^3\tfrac{1}{2}v-\tfrac{1}{5}.\text{tang.}^5\tfrac{1}{2}v\} \\ \qquad\qquad\qquad +\alpha^2.\{\tfrac{3}{32}.\text{tang.}\tfrac{1}{2}v-\tfrac{7}{32}.\text{tang.}^3\tfrac{1}{2}v+\tfrac{3}{28}.\text{tang.}^7\tfrac{1}{2}v\}. \end{array}$$

(17) If we put, for brevity, $x = S\alpha + B\alpha^2$, we shall have $v = U + x$ [5991(7)]; and, by neglecting x^3, which is of the order α^3, we shall get, by means of [29, 45] Int.,

(18) $$\text{tang.}\tfrac{1}{2}v = \text{tang.}\tfrac{1}{2}.(U+x) = \frac{\text{tang.}\tfrac{1}{2}U+\text{tang.}\tfrac{1}{2}x}{1-\text{tang}\tfrac{1}{2}U.\text{tang.}\tfrac{1}{2}x} = \frac{\theta+\tfrac{1}{2}x}{1-\tfrac{1}{2}\theta.x} = \theta+\tfrac{1}{2}x.(1+\theta^2)+\tfrac{1}{4}x^2\theta.(1+\theta^2).$$

[5991]

Re-substituting the value of x [5991 (17)], and putting, for brevity, $1+\theta^2=\theta_1$, (19)
we get the following expression of $\tang.\frac{1}{2}v$; from which we easily deduce its powers $\tang.^3\frac{1}{2}v$, &c.;

$$\tang.\tfrac{1}{2}v = \theta+\tfrac{1}{2}\alpha.S\theta_1+\alpha^2.\theta_1.\{\tfrac{1}{2}B+\tfrac{1}{4}S^2\theta\}; \qquad (20)$$

$$\tang.^3\tfrac{1}{2}v = \theta^3+\tfrac{3}{2}\alpha.S\theta^2\theta_1+\alpha^2.\theta_1.\{\tfrac{3}{2}B\theta^2+\tfrac{3}{4}S^2(\theta\theta_1+\theta^3)\}; \qquad (21)$$

$$\tang.^5\tfrac{1}{2}v = \theta^5+\tfrac{5}{2}\alpha.S\theta^4\theta_1+\&c.; \qquad \tang.^7\tfrac{1}{2}v = \theta^7+\&c. \qquad (22)$$

If we substitute these in [5991 (16)], the terms independent of α will mutually destroy each other; also those depending on the first power of α: and, if we notice, in the second members of the following expressions, only the terms multiplied by α^2, we shall have, by using the values [5991 (20—22)], and $S\theta'^2=(-\frac{1}{2}\theta+\frac{1}{2}\theta^3+\frac{2}{5}\theta^5)$ [5991 (12)]; (23)

$$\begin{aligned}\tang.\tfrac{1}{2}v+\tfrac{1}{3}.\tang.^3\tfrac{1}{2}v &= \alpha^2\theta_1.\{\tfrac{1}{2}B.(1+\theta^2)+\tfrac{1}{4}S^2\theta.(1+\theta_1+\theta^2)\}=\alpha^2.\{\tfrac{1}{2}B\theta_1^{\,2}+\tfrac{1}{2}S^2\theta\theta_1^{\,2}\}\\ &= \tfrac{1}{2}\alpha^2.B.\theta_1^{\,2}+\tfrac{1}{2}\alpha^2.S.\{-\tfrac{1}{2}\theta^2+\tfrac{1}{2}\theta^4+\tfrac{2}{5}\theta^6\};\end{aligned} \qquad (24)$$

$$\alpha.\{\tfrac{1}{4}.\tang.\tfrac{1}{2}v-\tfrac{1}{4}.\tang.^3\tfrac{1}{2}v-\tfrac{1}{5}.\tang.^5\tfrac{1}{2}v\} = \alpha^2.S\theta_1.\{\tfrac{1}{8}-\tfrac{3}{8}\theta^2-\tfrac{1}{2}\theta^4\} = \tfrac{1}{2}\alpha^2.S.\{\tfrac{1}{4}-\tfrac{1}{2}\theta^2-\tfrac{7}{4}\theta^4-\theta^6\}; \qquad (25)$$

$$\alpha^2.\{\tfrac{3}{32}.\tang.\tfrac{1}{2}v-\tfrac{7}{32}.\tang.^3\tfrac{1}{2}v+\tfrac{3}{28}.\tang.^7\tfrac{1}{2}v\} = \alpha^2.\{\tfrac{3}{32}\theta-\tfrac{7}{32}\theta^3+\tfrac{3}{28}\theta^7\}. \qquad (26)$$

The sum of these three formulas represents the terms depending on α^2, in the second member of [5991 (16)]; and, as this sum is to be put equal to nothing, we shall get, by (27)
dividing by $\frac{1}{2}\alpha^2\theta_1^{-2}$, the first of the following expressions. Substituting in this, the value of $S\theta_1^{\,2}$ [5991 (23)]; also $\theta_1^{\,2}=1+2\theta^2+\theta^4$; and then reducing, we obtain the (28)
second value of $B\theta_1^{\,4}$; dividing this by $\theta_1^{\,4}$, we get the value of B;

$$\begin{aligned}B\theta_1^{\,4} &= S\theta_1^{\,2}.\{-\tfrac{1}{4}+\theta^2+\tfrac{5}{4}\theta^4+\tfrac{3}{5}\theta^6\}+\theta_1^{\,2}.\{-\tfrac{3}{16}\theta+\tfrac{7}{16}\theta^3-\tfrac{3}{14}\theta^7\}\\ &= -\tfrac{1}{16}\theta-\tfrac{9}{16}\theta^3+\tfrac{37}{80}\theta^5+\tfrac{531}{560}\theta^7+\tfrac{13}{35}\theta^9+\tfrac{9}{350}\theta^{11};\end{aligned} \qquad (29)$$

$$B = \frac{-\frac{1}{16}\theta-\frac{9}{16}\theta^3+\frac{37}{80}\theta^5+\frac{531}{560}\theta^7+\frac{13}{35}\theta^9+\frac{9}{350}\theta^{11}}{(1+\theta^2)^4}. \qquad (30)$$

The values of the logarithms of S, B, in seconds, computed by Bessel, by means of the formulas [5991 (12, 30)], with their first and second differences, are given in Table IV, (31)
of this collection.

To show, by an example, the use of Table IV, we have here inserted the computation of the true anomaly v, in an orbit which does not differ much from that of Halley's comet; supposing the time from the perihelion to be 60 days;

$$e=0{,}9675212; \qquad \log.(1-e)=8{,}5115999; \qquad \log.\text{peri. dist.}=9{,}7665598. \qquad (32)$$

With these data, we find,

[5991]

(33)

D	log. co.	0,2334402		
Its half		0,1167201		
$t=60$ days	log.	1,7781513		
Table III. $U=97^d\ 29^m\ 58^s,6$	log.	2,1283116		
Table IV. S	log.	4,56506		
$1-e$	log.	8,51160		
Simpson's corr. $1193^s,1$	log.	3,07666		
Table IV. B	log.	4,4333	From Table III, for the parabola,	$U=97^d\ 29^m\ 58^s,6$
$1-e$	log.	8,5116	Simpson's correction, Table IV	$+\ 19^m\ 53^s,1$
$1-e$	log.	8,5116	Bessel's correction, Table IV	$+\ 28^s,6$
Bessel's corr. $28^s,6$	log.	1,4565	Sum is true anom. in the ellipsis	$v=97^d\ 50^m\ 20^s,3$

In a hyperbolic orbit, in which $e=1,0324788$, we shall have,

(34) $$\log(e-1)=8,5115999\ ;$$

and, if we suppose $t=60$ days, the numerical calculation will be the same as before; but, $1-e$ being negative, the value of Simpson's correction will be negative; and, we shall have, in this hyperbolic orbit,

(35)

From Table III, for the parabola	$U=$	$97^d\ 29^m\ 58^s,6$
Simpson's correction	$-$	$19^m\ 53^s,1$
Bessel's correction	$+$	$28^s,6$
True anomaly in the hyperbolic orbit	$v=$	$97^d\ 10^m\ 34^s,1$

The inverse problem, of finding the time t, *from the perihelion, when* v *is given, is*
(36) *easily solved*, if $1-e$ be so small, that Bessel's correction, depending on B, may be
(37) neglected. For, in this case, the expression [5991 (7)] becomes $U=v-S.(1-e)$; and S may be obtained from Table IV, with the argument v instead of U. Having found U, we easily deduce from it, the value of t, by means of Table III. Hence
(38) it appears, *that this inverse problem, in Simpson's method, merely requires a change in the sign of the quantity* S. If $1-e$ should be so great, that it is necessary to notice the term B, it will be necessary to repeat the operation, by an indirect method; or, more
(39) conveniently, by forming a table, similar to that used in finding B, by which the correction of Bessel may be directly obtained. But, in this case, it is better to use the
(40) method of Gauss, which is not restricted to the first and second powers of $1-e$, but includes also the higher powers of this quantity.

[5992]
(1) *We shall now proceed to the investigation of this method of Gauss, for the direct solution of Kepler's problem, for computing the true anomaly* v, *from the time* t, *in*

an ellipsis or a hyperbola, which approaches nearly to a parabolic form; and, in the demonstrations, we shall refer to any line of [5992], by merely putting the number of the line in a parenthesis, as we have done in [5981 (2)], omitting, for brevity, the number [5992]. In this solution we do not, as in the preceding method, deduce the anomaly in the ellipsis, from that in a parabola having the same perihelion distance D; but we obtain it from a parabola, whose perihelion distance is increased to [5992] (2)

$$D_{\prime} = D \cdot \left(\frac{B^2}{1-0{,}9 \cdot \alpha}\right)^{\frac{1}{3}} \quad (41)\ ; \qquad (3)$$

B being a quantity which exceeds unity, by terms of the fourth order in u (18). By this means, the interpolations in Table V become very easy, on account of the smallness of $\log. B$, and $C-1$, as well as the smallness of their variations; so that we are enabled to notice all the powers of α, with but very little additional labor. The same remarks may be applied to the use of Table VI, relative to a hyperbolic orbit. (4)

We shall first treat of an elliptical orbit, using the same elements as before; namely, *a the semi-transverse axis; e the excentricity;* $2p = 2a.(1-e^2)$ *the parameter; D the perihelion distance; nt the mean anomaly; u the excentric anomaly; v the true anomaly.* We shall also use the following abridged symbols, in which α, α', C, differ from those used by Gauss; this change is made in order to conform to the notation generally used in this work, and to render some of the formulas more simple. (5) (6) (7) (8)

$$\alpha' = \sqrt{0{,}1+0{,}9.e}\ ; \qquad \alpha = 1-e\ ; \qquad (9)$$

$$\beta = \frac{5-5e}{1+9e} = \frac{\alpha}{2\alpha'^2}\ ; \qquad \gamma = \sqrt{\left(\frac{5+5e}{1+9e}\right)} = \sqrt{\left(\frac{1+e}{2\alpha'^2}\right)}\ ; \qquad (10)$$

$$\mathrm{T} = \mathrm{tang.}^2\tfrac{1}{2}u = \left(\frac{1-e}{1+e}\right).\mathrm{tang.}^2\tfrac{1}{2}v\ ; \qquad (11)$$

$$A = \frac{15.(u-\sin.u)}{9u+\sin.u}\ ; \qquad (12)$$

$$B = \frac{9u+\sin.u}{20A^{\frac{1}{2}}}\ ; \qquad (13)$$

$$C = \frac{A}{\mathrm{T}} + \tfrac{4}{5}A\ ; \qquad \text{or,} \qquad \mathrm{T} = \frac{A}{C-\frac{4}{5}A}\,. \qquad (14)$$

The quantities A, B, C, may be expressed in series, by the substitution of

$$\sin.u = u-\tfrac{1}{6}u^3+\tfrac{1}{120}u^5-\&c. \quad [43]\ \text{Int.}\ ; \qquad (15)$$

which gives,

$$u-\sin.u = \tfrac{1}{6}u^3-\tfrac{1}{120}u^5+\tfrac{1}{5040}u^7-\&c.\ ;$$
$$9u+\sin.u = 10u-\tfrac{1}{6}u^3+\tfrac{1}{120}u^5-\&c. \qquad (16)$$

[5992] Substituting these in A (12), it becomes,

$$A = \tfrac{1}{4}u^2 - \tfrac{1}{120}u^4 - \tfrac{1}{20160}u^6 - \&c. ;$$

(17) or,

$$\sqrt{A} = \tfrac{1}{2}u - \tfrac{1}{120}u^3 - \tfrac{1}{8400}u^5 - \&c. ;$$

and, in like manner, we get,

(18)
$$B = 1 + \tfrac{3}{2800}u^4 - \&c. ;$$

(19) so that *A is of the second order, relative to u,* and *B differs from unity, by a quantity of the fourth order only.* We may obtain the value of A, in terms of T, by the following process. From [48] Int., putting $z=\frac{1}{2}u$, we get u (20), and from [30″] Int., we have $\sin.u$ (20), by using T (11);

$$u = 2.(\mathrm{tang}.\tfrac{1}{2}u - \tfrac{1}{3}\mathrm{tang}.^3\tfrac{1}{2}u + \tfrac{1}{5}\mathrm{tang}.^5\tfrac{1}{2}u - \&c.) = 2\mathrm{T}^{\frac{1}{2}}.(1 - \tfrac{1}{3}\mathrm{T} + \tfrac{1}{5}\mathrm{T}^2 - \&c.) ;$$

(20)
$$\sin.u = \frac{2.\mathrm{tang}.\frac{1}{2}u}{1+\mathrm{tang}.^2\frac{1}{2}u} = \frac{2.\mathrm{T}^{\frac{1}{2}}}{1+\mathrm{T}} = 2.\mathrm{T}^{\frac{1}{2}}.(1 - \mathrm{T} + \mathrm{T}^2 - \&c.) ;$$

hence we get, by substitution,

(21)
$$15.(u - \sin.u) = 30.\mathrm{T}^{\frac{1}{2}}.\{\tfrac{2}{3}\mathrm{T} - \tfrac{4}{5}\mathrm{T}^2 + \&c.\} = 20.\mathrm{T}^{\frac{1}{2}}.\{\mathrm{T} - \tfrac{6}{5}\mathrm{T}^2 + \tfrac{9}{7}\mathrm{T}^3 - \tfrac{12}{9}\mathrm{T}^4 + \&c.\} ;$$

(22)
$$(9u + \sin.u) = 2\mathrm{T}^{\frac{1}{2}}.\{\tfrac{10}{1} - \tfrac{12}{3}\mathrm{T} + {}^{14}\mathrm{T}^2 - \&c.\} = 20.\mathrm{T}^{\frac{1}{2}}.\{1 - \tfrac{6}{15}\mathrm{T} + \tfrac{7}{25}\mathrm{T}^2 - \tfrac{8}{35}\mathrm{T}^3 + \&c.\}.$$

Substituting these in A (12), it becomes,

(23)
$$A = \frac{\mathrm{T} - \frac{6}{5}\mathrm{T}^2 + \frac{9}{7}\mathrm{T}^3 - \frac{12}{9}\mathrm{T}^4 + \frac{15}{11}\mathrm{T}^5 - \&c.}{1 - \frac{6}{15}\mathrm{T} + \frac{7}{25}\mathrm{T}^2 - \frac{8}{35}\mathrm{T}^3 + \frac{9}{45}\mathrm{T}^4 - \&c.}$$

(24)
$$= \mathrm{T} - \tfrac{4}{5}\mathrm{T}^2 + \tfrac{24}{35}\mathrm{T}^3 - \tfrac{1592}{2625}\mathrm{T}^4 + \tfrac{236563}{433125}\mathrm{T}^5 - \tfrac{98097192}{197071875}\mathrm{T}^6 + \&c.$$

Inverting this series, we get,

(25)
$$\frac{A}{\mathrm{T}} = 1 - \tfrac{4}{5}A + \tfrac{8}{175}A^2 + \tfrac{8}{525}A^3 + \tfrac{1896}{336875}A^4 + \tfrac{28744}{13138125}A^5 + \&c.;$$

as we may easily prove, by substituting in it the value of A (24), and reducing, by which means we shall find, that the terms mutually destroy each other.

If we substitute this value of $\frac{A}{\mathrm{T}}$ in C (14), it becomes,

(26)
$$C = 1 + \tfrac{8}{175}A^2 + \tfrac{8}{525}A^3 + \tfrac{1896}{336875}A^4 + \tfrac{28744}{13138125}A^5 + \&c.$$

Hence it appears, *that C differs from unity by terms of the second order in A, or*
(27) *of the fourth order in u* (17). The quantities A, B, C, are functions of T,
(28) which have been computed by the preceding formulas, and inserted in Table V. By
(29) means of this table we can easily find, by inspection, the values of A, C, $\log.B$, for

any given value of T, or the contrary; and, as the quantities C, $\log.B$, vary so slowly, in the most useful part of the table, it is very easy to take out the corresponding numbers, which we shall hereafter find to be one of the great advantages of the method of Gauss. After this digression on the method of computing Table V, we shall proceed to the investigation of its uses in *the direct solution of Kepler's problem, of finding* r, v *from* t, *in a very excentric ellipsis.* [5992] (30) (31)

Substituting the value of nt [5987 (12)], in [5985 (7)], neglecting m on account of its smallness, and then putting $a=\frac{D}{1-e}$ [5985 (6)], we get (34). From this we easily deduce (35), since by multiplying together the two factors of (35), and reducing, it becomes identical with the second member of (34). Now, the value of B (13), gives, (32)

$$9u+\sin.u=20A^{\frac{1}{2}}.B\,; \tag{33}$$

substituting this in (35), in the factor without the braces, also the value of A (12), we get (36); whence we easily deduce the expression (37);

$$k.t.\left(\frac{1-e}{D}\right)^{\frac{3}{2}}=u-e.\sin.u \tag{34}$$

$$=(9u+\sin.u)\,.\left\{\frac{1-e}{10}+\frac{1+9e}{10}\cdot\frac{u-\sin.u}{9u+\sin.u}\right\} \tag{35}$$

$$=20A^{\frac{1}{2}}.\,B\,.\left\{\frac{1-e}{10}+\frac{1+9e}{10}\cdot\frac{A}{15}\right\} \tag{36}$$

$$=2B\,.\left\{(1-e)\,.\,A^{\frac{1}{2}}+\frac{1}{15}\cdot(1+9e)\,.\,A^{\frac{3}{2}}\right\}; \tag{37}$$

in which we must substitute the value of $\log.k=\log.\sqrt{\mu}=8{,}2355814...$ [5987 (8,16)]. If we now suppose,

$$A^{\frac{1}{2}}=\left(\frac{5.(1-e)}{1+9e}\right)^{\frac{1}{2}}.\,\text{tang}.\tfrac{1}{2}w\,,\quad\text{or}\quad A=\beta.\text{tang}.^2\tfrac{1}{2}w\quad(10), \tag{38}$$

and substitute it in the preceding expression, every term will have the factor $(1-e)^{\frac{3}{2}}$; then dividing by this quantity, we get,

$$t\,.\,\frac{k}{D^{\frac{3}{2}}}=2B\,.\left(\frac{5}{1+9e}\right)^{\frac{1}{2}}.\,\{\text{tang}.\tfrac{1}{2}w+\tfrac{1}{3}.\text{tang}.^3\tfrac{1}{2}w\}. \tag{39}$$

Multiplying this by,

$$\frac{\alpha'}{Bk}=\left(\frac{1+9e}{5}\right)^{\frac{1}{2}}\cdot\frac{1}{Bk.\sqrt{2}}\quad(9)\,; \tag{39}$$

we finally obtain,

[5992]

(40)
$$\frac{\alpha'}{BD^{\frac{3}{2}}}\cdot t = \frac{\sqrt{2}}{k}\cdot\{\text{tang.}\tfrac{1}{2}w+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}w\}.$$

Now, from the construction of Table III [5987 (22)], it appears, that the tabular number, corresponding to the anomaly w, represents the logarithm of the second member of this expression; so that, if we put,

(41)
$$D_{,}=D.\left(\frac{B^2}{0{,}1+0{,}9.e}\right)^{\frac{1}{3}}=D.\left(\frac{B^2}{1-0{,}9.\alpha}\right)^{\frac{1}{3}};$$

and then substitute $D_{,}$ and α' (9), in (40), we shall get, by making successive reductions in its first member, the following expressions;

(42)
$$\frac{\alpha'}{BD^{\frac{3}{2}}}.t=\frac{(0{,}1+0{,}9.e)^{\frac{1}{2}}t}{BD^{\frac{3}{2}}}=\frac{t}{D_{,}^{\frac{3}{2}}}=\text{number of the log., in Table III, corresponding to the anomaly } w\ ;$$

so that, if B, and, therefore, $D_{,}$, be known, we can determine the relation of w and t, by means of Table III. Hence it appears, that, *in the direct solution of Kepler's problem*, in a very excentrical orbit; where t is given, to find r, v; we can obtain w from t, by means of (42); and then, from w, we get A, by means of formula (38); namely,

(43)
$$A=\beta.\text{tang.}^2\tfrac{1}{2}w=\frac{5.(1-e)}{1+9.e}\cdot\text{tang.}^2\tfrac{1}{2}w=\frac{\alpha}{2\alpha'^2}.\text{tang.}^2\tfrac{1}{2}w.$$

Now, B differs so little from unity (18), that we may, in a first rough calculation,
(44) suppose $B=1$; and, upon this supposition, we can compute the approximate values
(45) of w and A (42, 43). With this value of A, we find, from Table V, the expression of $\log.B$; and, by repeating the calculation, with this value, we get the corrected expressions of w, A. In general, this second operation will be sufficiently accurate, except u be very great. It frequently happens, when several observations
(46) are computed, for successive days, that the value of $\log.B$ is very nearly known at the commencement of the operation; in this case, we must use this approximate value of B,
(47) in the first operation; and, it will generally happen, that one operation, in such cases, will be sufficient to obtain the correct value of w.

(48) Having obtained the value of A, we find, from Table V, the corresponding value of C; from which we get,

(49)
$$\mathrm{T}=\text{tang.}^2\tfrac{1}{2}u=\frac{A}{C-\frac{1}{5}A}\quad(11, 14),$$

with more accuracy and less labor, than it could be directly obtained from Table V. Substituting this value of $\text{tang.}^2\frac{1}{2}u$, in (11), we get the first expression of $\text{tang.}\frac{1}{2}v$ (51).
(50) Substituting in this, the second value of A (43), rejecting the factor $(1-e)^{\frac{1}{2}}$, which occurs in the numerator and denominator, then introducing the first value of γ (10), we get the second expression (51);

[5992]

$$\text{tang.}\tfrac{1}{2}v = \sqrt{\frac{1+e}{1-e}} \cdot \sqrt{\frac{A}{C-\frac{4}{5}A}} = \frac{\gamma.\text{tang.}\frac{1}{2}w}{\sqrt{C-\frac{4}{5}A}}. \qquad (51)$$

Having found u, v, we may compute r from either of the formulas [5985 (9,10)], or from the following ;

$$r = \frac{D.\cos.^2\frac{1}{2}u}{\cos.^2\frac{1}{2}v} = \frac{D}{(1+\text{T}).\cos.^2\frac{1}{2}v} = \frac{(C-\frac{4}{5}A).D}{(C+\frac{1}{5}A).\cos.^2\frac{1}{2}v}. \qquad (52)$$

The first of these expressions is easily deduced from the last formula [5985 (13)], by substituting $a.(1-e)=D$ [5985 (6)], then squaring and reducing. The second is (53) obtained from the first, by putting,

$$\cos.^2\tfrac{1}{2}u = \frac{1}{1+\text{tang.}^2\frac{1}{2}u} = \frac{1}{1+\text{T}} \quad (11) ; \qquad (54)$$

and the third is deduced from the second, by the substitution of the value of T (14).

The inverse problem of finding the time t, from the true anomaly v, is also solved by means of Table V. In this case, we must first compute T, from v, by the formula (11) ; (55)

$$\text{T} = \frac{1-e}{1+e}.\text{tang.}^2\tfrac{1}{2}v. \qquad (56)$$

With the argument T, we must enter Table V, and take out the number A, and the log.B; or, what is more convenient, and, at the same time, more accurate, the (57) number C, and the log.B; then compute A, by the formula (14),

$$A = \frac{C\text{T}}{1+\frac{4}{5}\text{T}}; \qquad (58)$$

lastly, we must find t, by means of the formula (37). This expression, being divided by the factor of t, gives,

$$t = \frac{2}{k}.D^{\frac{3}{2}}.A^{\frac{1}{2}}.B.(1-e)^{-\frac{1}{2}}.\left\{1+\tfrac{1}{15}.A.(1+9e).(1-e)^{-1}\right\}; \qquad (59)$$

and, if we put,

$$t_1 = \frac{2}{k}.D^{\frac{3}{2}}.A^{\frac{1}{2}}.B.(1-e)^{-\frac{1}{2}}; \qquad t_2 = t_1.\tfrac{1}{15}.A.(1+9e).(1-e)^{-1}; \qquad (60)$$

we shall have,

$$t = t_1 + t_2 ; \qquad (61)$$

and, it is under this form, that the value of t is computed in the introduction to Table V, observing, that we have,

$$\log.\frac{2}{k} = 2{,}0654486 \quad [5987\ (8)], \quad \text{and} \quad \log.\tfrac{1}{15} = 8{,}8239087. \qquad (62)$$

[5992] We may also compute t, from v, by means of Table III; but, this table does
(63) not facilitate the operation, as it does when finding v from t. In using Table III, for this purpose, it will not be necessary to compute A. For, we have, in (43, 56),

(64) $$\text{tang.}\tfrac{1}{2}w = A^{\frac{1}{2}}.\left(\frac{1+9e}{5-5e}\right)^{\frac{1}{2}}; \qquad \text{tang.}\tfrac{1}{2}v = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}.\mathrm{T}^{\frac{1}{2}}.$$

Dividing the first of these expressions by the second, we get the first of the equations (66); substituting γ^2 (10), we get the second expression (66); and, by using A (58),
(65) we get the last of the formulas (66); from which we easily obtain the first value of $\text{tang.}\tfrac{1}{2}w$ (67). The second formula (67) is derived from the first, by the substitution of the second value of γ (10).

(66) $$\frac{\text{tang.}\frac{1}{2}w}{\text{tang.}\frac{1}{2}v} = \left(\frac{A}{\mathrm{T}}\right)^{\frac{1}{2}}.\left(\frac{1+9e}{5+5e}\right)^{\frac{1}{2}} = \left(\frac{A}{\mathrm{T}\gamma^2}\right)^{\frac{1}{2}} = \sqrt{\frac{C}{\gamma^2.(1+\frac{4}{5}\mathrm{T})}};$$

or,

(67) $$\text{tang.}\tfrac{1}{2}w = \sqrt{\frac{C}{\gamma^2.(1+\frac{4}{5}\mathrm{T})}}.\,\text{tang.}\tfrac{1}{2}v = \sqrt{\frac{2\,\alpha'^2.\,C}{(1+e).(1+\frac{4}{5}\mathrm{T})}}.\,\text{tang.}\tfrac{1}{2}v.$$

Having found, in Table III, the time corresponding to this anomaly w, we must
(68) multiply it by $\frac{BD^{\frac{3}{2}}}{\alpha'}$, to obtain the time t from the perihelion; as is evident from the first of the formulas (42).

Table V is given for every thousandth part of a unit, from $A = 0{,}000$ to
(69) $A = 0{,}300$. It was thought to be unnecessary to extend it any farther; because $A = 0{,}3$ corresponds to $T = 0{,}392374 = \text{tang.}^2\tfrac{1}{2}u$ (11), or $u = 64^d\,7^m$; and,
(70) with such large values of u, the indirect method of solution is the shortest, as we have already observed. This table is arranged so as to make it most convenient for use in
(71) finding B, C, with the argument A, in the first problem, where t is given to find v, which is by far the most frequently required. In this case, the number T is not used. In the second problem, the argument T is used to find B and C, which
(72) are small and easily computed; and then A is found directly, by means of the formula (58).

We shall apply this method to the computation of the same example, as in [5991 (33)].

EXAMPLE 1.

Given, $e = 0{,}9675212$, $t = 60^{\text{days}}$,

(73) log. perih. dist. $D = 9{,}7665598$, $\alpha = 1 - e = 0{,}0324788$.

$\alpha'^2 = 0{,}1 + 0{,}9.e = 0{,}9707691$; to find t, v, in an elliptical orbit.

[5992]

FIRST OPERATION TO FIND v.

α'^2	log.	9,9871159	
α'	log.	9,9935579	
D	log. co.	2334402	(74)
	Its half	1167201	
$t = 60$	log.	1,7781513	
$U = 96^d\,58^m$ Table III.	log. t'	2,1218695	
α	log.	8,5115999	
α'^2	log. co.	128841	
$\frac{1}{2}$	log.	9,6989700	(75)
β	log.	8,2234540	
$\frac{1}{2}U = 48^d\,29^m$	tang.	0,05294	
	same	0,05294	
Approx. $A = 0,021347$	log.	8,32933	

TO FIND THE RADIUS r.

$C + 0,2.A = 1,0042912$	log. co.	9,9981403	
$C - 0,8.A = 0,9829401$	log.	9,9925270	
D	log.	9,7665598	(76)
$\frac{1}{2}v = 48^d\,55^m\,10^s,47$	sec.	0,1823567	
	same	0,1823567	
r	log.	0,1219405	(77)

SECOND OPERATION TO FIND v.

$-B$, or 34, found by the first operation 2,1218661.

Hence $w = 96^d\,58^m\,21^s,6$, by means of Table III.

. . . .	same	8,2234540
$\frac{1}{2}w = 48^d\,29^m\,10^s,8$	tang.	0,0529828
	same	0,0529828
$A = 0,0213511$	log.	8,3294196
$C = 1,0000210$		
$0,8.A = 0,0170809$		
$C - 0,8.A = 0,9829401$	log. co.	0,0074730
$1 + e = 1,9675212$	log.	0,2939194
$1 - e = \alpha$	log. co.	1,4884001
tang.$^2\frac{1}{2}v$	log.	2)0,1192121
$\frac{1}{2}v = 48^d\,55^m\,10^s,47$	tang.	0,0596060

$v = 97^d\,50^m\,20^s,94$ (51).

This value of v differs $0^s,64$ from that found in [5991 (33)], by noticing only the corrections of Simpson and Bessel.

EXAMPLE II.

In the inverse problem, with the same elements, we have given,

$$\text{the anomaly } v = 97^d\,50^m\,20^s,94\,, \qquad (78)$$

to find t, in the following manner, by means of the formula (61).

$1 - e = 0,0324788$	log.	8,5115999
$1 + e = 1,9675212$	log. co.	9,7060806
$\frac{1}{2}v = 48^d\,55^m\,10^s,47$	tang.	0,0596060
	same	0,0596060
$T = 0,02172163$	log.	8,3368925
Hence $C = 1,0000210$	log.	91
$1 + 0,8.T = 1,0173773$	log. co.	9,9925180
$A = 0,0213511$	log.	8,3294196
Corresponding log. B in Table V		0,0000034

	Constant log.	2,0654486	
	$\frac{3}{2}$ log. D	9,6498397	
	$\frac{1}{2}$ log. A	9,1647098	(79)
	log. B	34	
	$\frac{1}{2}$ log. $(1-e)$ arith. co.	0,7442000	
$t_1 = 42^{\text{days}},092$	log.	1,6242015	
. .	Constant	8,8239087	
. .	A log.	8,3294196	
. . $1 + 9e = 9,707691$	log.	0,9871159	(80)
. .	$(1-e)$ log. co.	1,4884001	
$t_2 = 17^{\text{days}},908$	log.	1,2530458	

$t = 60^{\text{days}} = t_1 + t_2$.

[5992] We shall now compute the same example by means of Table III; by which means it
(81) will evidently appear, that the preceding form is the shortest and most simple.

	$1-e$	log.	8,5115999	$\alpha'^2 = 0{,}9707691$	log.	9,9871159
	$1+e$	log. co,	9,7060806	. . . .	. .	9,7060806
	$\frac{1}{2}v = 48^d\ 55^m\ 40^s{,}47$	tang.	0,0596060		2 log.	0,3010300
		same	0,0596060			.
(82)	$T = 0{,}02172163$	log.	8,3368925			:
	$C = 1{,}0000210$	log.	91	. . . .		91
	$1+0{,}8.T = 1{,}0173773$	log. co.	9,9925180	. . . .		9,9925180
	$A = 0{,}0213511$	log.	8,3294196		sum	2)9,9867536
(83)	Corresponding log. B, Table V, is		0,0000034		half	9,9933768
				$\frac{1}{2}v = 48^d\ 55^m\ 10^s{,}47$	tang.	0,0596060
				$\frac{1}{2}w = 48^d\ 29^m\ 10^s{,}8$	tang.	0,0529828
				$w = 96^d\ 58^m\ 21^s{,}6$ Table III	log. t'	2,1218662
				Table V	log. B	34
				D	log.	9,7665598
				$D^{\frac{1}{2}}$	log.	9,8832799
				α'	log. co.	64421
(84)				$t = 60^{\text{days}}$	log.	1,7781514

[5993] *We shall now proceed to the explanation of the method of computation in a hyperbolic*
(1) *orbit;* in which the elements are; a the semi-transverse axis; e the excentricity;
(2) $2p = 2a.(e^2-1)$ the parameter; D the perihelion distance. We shall also use the
(3) following abridged symbols, which are similar to those in [5992 (9—14)], corresponding
(4) to an elliptical orbit. In the demonstrations in this article, we shall refer to any line of
(5) [5993], by merely putting the number of the line in a parenthesis, as in [5984 (2), &c.].

(6) $$\alpha' = \sqrt{0{,}1+0{,}9.e}\ ;$$

(7) $$\beta = \frac{5e-5}{1+9e} = \frac{e-1}{2\alpha'^2}\ ; \qquad \gamma = \sqrt{\left(\frac{5(e+1)}{1+9e}\right)} = \sqrt{\left(\frac{e+1}{2\alpha'^2}\right)}\ ;$$

(8) $$T = \text{tang.}^2\tfrac{1}{2}\varpi = \frac{e-1}{e+1}.\text{tang.}^2\tfrac{1}{2}v = \left(\frac{u-1}{u+1}\right)^2\ ;$$

(9) $$u = \text{tang.}(45^d+\tfrac{1}{2}\varpi)\ ;$$

(10) $$A = \frac{\frac{3}{2}.\left\{\frac{1}{2}.\left(u-\frac{1}{u}\right)-\log.u\right\}}{\frac{1}{20}.\left(u-\frac{1}{u}\right)+\frac{9}{10}.\log.u}\ ;$$

(11) $$B = \frac{\frac{1}{20}.\left(u-\frac{1}{u}\right)+\frac{9}{10}.\log.u}{2\sqrt{A}}\ ;$$

(12) $$C = \frac{A}{T} - \tfrac{4}{5}A\ ; \qquad \text{or,} \qquad T = \frac{A}{C+\frac{4}{5}A}\,.$$

We may observe, that the expression of u (9) is the same as in [5989 (12)]; and the last expression of $\tan^2.\frac{1}{2}w$, or T (8), is deduced from it, in the same manner as in [5989 (13)]. This last value of T (8) gives, [5993]

$$u=\frac{1+T^{\frac{1}{2}}}{1-T^{\frac{1}{2}}}\,; \qquad \text{hence,} \tag{13}$$

$$u-\frac{1}{u}=\frac{(1+T^{\frac{1}{2}})}{1-T^{\frac{1}{2}}}-\frac{(1-T^{\frac{1}{2}})}{1+T^{\frac{1}{2}}}=\frac{4T^{\frac{1}{2}}}{1-T}=4T^{\frac{1}{2}}.(1+T+T^2+T^3+\&c.)\,; \tag{13'}$$

and, from [58] Int., we have,

$$\log.u=\log.\left(\frac{1+T^{\frac{1}{2}}}{1-T^{\frac{1}{2}}}\right)=\log.\left(1+T^{\frac{1}{2}}\right)-\log.\left(1-T^{\frac{1}{2}}\right) \tag{14}$$

$$=\left(T^{\frac{1}{2}}+\tfrac{1}{2}T^{\frac{2}{2}}+\tfrac{1}{3}T^{\frac{3}{2}}+\&c.\right)-\left(-T^{\frac{1}{2}}+\tfrac{1}{2}T^{\frac{2}{2}}-\tfrac{1}{3}T^{\frac{3}{2}}+\&c.\right)$$

$$=2T^{\frac{1}{2}}.(1+\tfrac{1}{3}T+\tfrac{1}{5}T^2+\tfrac{1}{7}T^3+\&c.)\,; \qquad \text{hence,} \tag{15}$$

$$\tfrac{3}{2}.\left\{\tfrac{1}{2}.\left(u-\frac{1}{u}\right)-\log.u\right\}=3T^{\frac{1}{2}}.\{\tfrac{2}{3}T+\tfrac{4}{5}T^2+\tfrac{6}{7}T^3+\&c\}=2T^{\frac{1}{2}}.\{T+\tfrac{6}{5}T^2+\tfrac{9}{7}T^3+\tfrac{12}{9}T^4+\&c\}; \tag{16}$$

$$\tfrac{1}{20}.\left(u-\frac{1}{u}\right)+\tfrac{9}{10}.\log.u=2T^{\frac{1}{2}}\{(\tfrac{1}{10}+\tfrac{1}{10}T+\tfrac{1}{10}T^2+\tfrac{1}{10}T^3+\&c)+\tfrac{9}{10}(1+\tfrac{1}{3}T+\tfrac{1}{5}T^2+\tfrac{1}{7}T^3+\&c)\} \tag{17}$$

$$=2T^{\frac{1}{2}}.\{1+\tfrac{6}{15}T+\tfrac{7}{25}T^2+\tfrac{8}{35}T^3+\&c.\}. \tag{18}$$

Substituting the expression (16, 18) in the value of A (10), and rejecting $2T^{\frac{1}{2}}$ from the numerator and denominator, we get,

$$A=\frac{T+\frac{6}{5}T^2+\frac{9}{7}T^3+\frac{12}{9}T^4+\&c.}{1+\frac{6}{15}T+\frac{7}{25}T^2+\frac{8}{35}T^3+\&c.}=T+\tfrac{4}{5}T^2+\tfrac{24}{35}T^3+\tfrac{1592}{2625}T^4+\&c. \tag{19}$$

this may be derived from the expressions [5992 (23, 24)], by changing the signs of A, T; and, if we make these changes in [5992 (25, 26)], we shall get, for an hyperbolic orbit, (20)

$$\frac{A}{T}=1+\tfrac{4}{5}A+\tfrac{8}{175}A^2-\tfrac{8}{525}A^3+\tfrac{1896}{336875}A^4-\tfrac{28744}{13138125}A^5+\&c.; \tag{21}$$

$$C=1+\tfrac{8}{175}.A^2-\tfrac{8}{525}.A^3+\tfrac{1896}{336875}.A^4-\tfrac{28744}{13138125}A^5+\&c. \tag{22}$$

Extracting the square root of the expression of A (19), we get,

$$\sqrt{A}=T^{\frac{1}{2}}.\{1+\tfrac{6}{15}\,T+\tfrac{46}{175}\,T^2+\&c.\}\,; \tag{23}$$

substituting this, and (18), in B (11), we get,

$$B=\frac{1+\frac{6}{15}T+\frac{7}{25}\,T^2+\&c.}{1+\frac{6}{15}T+\frac{46}{175}T^2+\&c.}=1+\tfrac{3}{175}\,T^2+\&c. \tag{24}$$

[5993] Now, *if we consider* ϖ *as a small quantity of the first order, we shall have* T (8)
(25) *of the second order, and* A (19) *of the second order. Hence* C, B (22, 24) *differ*
(26) *from unity, by quantities of the fourth order.* These values of A, B, C, T have relations to each other, which are very similar to those in the ellipsis [5992 (15—31)].
(27) These quantities, for the hyperbola, are given in Table VI; which is arranged in the same way as Table V, for the ellipsis; and is used in the same manner as in [5992 (29), &c.].
(28) The numbers in Table VI are computed for values of A, from 0 to 0,300; which is sufficiently extensive for practical purposes.

We have in [5988 (6)],

$$\frac{k}{a^{\frac{3}{2}}} \cdot t = e \cdot \text{tang.}\,\varpi - \log.\,\text{tang.}(45^d + \tfrac{1}{2}\varpi). \tag{29}$$

Substituting in this the values of $\text{tang.}\varpi$, $\text{tang.}(45^d + \frac{1}{2}\varpi)$ [5988 (15, 13)];

$$\text{also,} \qquad a = \frac{D}{e-1} \qquad [5988\,(4)], \tag{30}$$

we get (34); and, as we have identically,

$$\tfrac{1}{2}e = \tfrac{1}{20} \cdot (e-1) + \tfrac{1}{2} \cdot (\tfrac{1}{10} + \tfrac{9}{10}\,e)\,; \qquad -1 = \tfrac{9}{10} \cdot (e-1) - (\tfrac{1}{10} + \tfrac{9}{10}\,e)\,; \tag{31}$$

as is easily proved by reduction; we may substitute these factors of $u - \frac{1}{u}$, and of $\log.u$, in the second member of (34), and it will become as in (35). This may be still farther reduced, by observing, that the product of the expression (11), by $2A^{\frac{1}{2}}$, gives,

$$\tfrac{1}{20} \cdot \left(u - \frac{1}{u}\right) + \tfrac{9}{10} \cdot \log.u = 2B.A^{\frac{1}{2}}\,; \tag{32}$$

substituting this, in the denominator of the value of A (10), and then multiplying it by $\frac{2}{3} \times 2B.\sqrt{A}$, we get,

$$\left\{ \tfrac{1}{2} \cdot \left(u - \frac{1}{u}\right) - \log.u \right\} = \tfrac{4}{3}\,B.A^{\frac{3}{2}}. \tag{33}$$

Using these two last expressions, we find, that the function (35), is reduced to the form (36), or the equivalent expression (37); which is very similar to that corresponding to the ellipsis in [5992 (37)].

$$k \cdot \left(\frac{e-1}{D}\right)^{\frac{3}{2}} \cdot t = \tfrac{1}{2}e \cdot \left(u - \frac{1}{u}\right) - \log.u \tag{34}$$

$$= (e-1) \cdot \left\{ \tfrac{1}{20} \cdot \left(u - \frac{1}{u}\right) + \tfrac{9}{10} \cdot \log.u \right\} + (\tfrac{1}{10} + \tfrac{9}{10}e) \cdot \left\{ \tfrac{1}{2} \cdot \left(u - \frac{1}{u}\right) - \log.u \right\} \tag{35}$$

$$= (e-1) \cdot 2B.A^{\frac{1}{2}} + (\tfrac{1}{10} + \tfrac{9}{10}e) \cdot \tfrac{4}{3} \cdot B.A^{\frac{3}{2}} \tag{36}$$

$$= 2B. \left\{ (e-1).A^{\frac{1}{2}} + \tfrac{1}{15} \cdot (1 + 9e).A^{\frac{3}{2}} \right\}. \tag{37}$$

If we suppose, [5993]

$$A = \left(\frac{5.(e-1)}{1+9e}\right).\text{tang.}^2\tfrac{1}{2}w = \beta.\text{tang.}^2\tfrac{1}{2}w \quad (7), \qquad (38)$$

and then substitute this first value of A in (37), we shall get, by dividing by $(e-1)^{\frac{3}{2}}$,

$$\frac{k}{D^{\frac{3}{2}}}.t = 2B.\left(\frac{5}{1+9e}\right)^{\frac{1}{2}}.\{\text{tang.}\tfrac{1}{2}w + \tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}w\}. \qquad (39)$$

Multiplying this by,

$$\frac{\alpha'}{Bk} \quad \text{or} \quad \left(\frac{1+9e}{5}\right)^{\frac{1}{2}}.\frac{1}{B.k.\sqrt{2}} \quad (6);$$

we finally obtain,

$$\frac{\alpha'}{BD^{\frac{3}{2}}}.t = \frac{\sqrt{2}}{k}.\{\text{tang.}\tfrac{1}{2}w + \tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}w\}. \qquad (40)$$

This is of exactly the same form as [5992 (40)], in an ellipsis; and, if we put, as in [5992 (41)],

$$D_{,} = D.\left(\frac{B^2}{0{,}1+0{,}9.e}\right)^{\frac{1}{3}}; \qquad (41)$$

we shall get, as in [5992 (42)],

$$\frac{\alpha'}{BD^{\frac{3}{2}}}.t = \frac{(0{,}1+0{,}9.e)}{BD^{\frac{3}{2}}}.t = \frac{t}{D_{,}^{\frac{3}{2}}} = \text{number of the log., in Table III, corresponding to the anomaly } w; \qquad (42)$$

so that, if B be known, we can determine the value of w, by means of Table III. Therefore, in the direct solution of Kepler's problem, in a hyperbolic orbit; where t is given, to find r and v; we can find w from t, by means of (42); and then, from w, we get A, by the following expression, which is the same as in (38);

$$A = \beta.\text{tang.}^2\tfrac{1}{2}w = \frac{5.(e-1)}{1+9.e}.\text{tang.}^2\tfrac{1}{2}w = \frac{e-1}{2\alpha'^2}.\text{tang.}^2\tfrac{1}{2}w. \qquad (43)$$

Now, B *differs so little from unity* (24), *that we may, at first, suppose* $B=1$; (44)
and, with this assumed value, we can find the approximate values of w, A (42, 43).
With this value of A, we obtain, from Table VI, the expression of $\log.B$; and, (45)
by repeating the calculation, with this value, we get the corrected expressions of
w, A. In general, this second operation will be sufficiently accurate, as we have (46)
observed in the similar calculation for an elliptical orbit [5992 (46)].

Having obtained the value of A, we find, from Table VI, the corresponding value of C; from which we get,

[5993]

(47) $$\mathrm{T} = \text{tang.}^2\tfrac{1}{2}\varpi = \frac{A}{C+\frac{4}{5}A} \quad (8, 12),$$

with greater accuracy and with less labor, than it could be directly obtained from Table VI.
(48) Substituting this value of $\text{tang.}^2\frac{1}{2}\varpi$, in (8), we get the first expression of $\text{tang.}\frac{1}{2}v$ (49). Substituting in this, the second value of A (43), rejecting the factor $(e-1)^{\frac{1}{2}}$, which occurs in the numerator and denominator, then introducing the first value of γ (7), we get the second expression (49);

(49) $$\text{tang.}\tfrac{1}{2}v = \sqrt{\frac{e+1}{e-1}} \cdot \sqrt{\frac{A}{C+\frac{4}{5}A}} = \frac{\gamma.\text{tang.}\frac{1}{2}w}{\sqrt{C+\frac{4}{5}A}}.$$

Having found ϖ, v, we may compute r, from either of the formulas [5988(11,12)], or from the following expressions; which are similar to those in an ellipsis [5992 (52)];

(50) $$r = \frac{D}{(1-\mathrm{T}).\cos.^2\frac{1}{2}v} = \frac{(C+\frac{4}{5}A).D}{(C-\frac{1}{5}A).\cos.^2\frac{1}{2}v}.$$

The first of these formulas is deduced from the last expression in [5988 (20)], which gives, by squaring and reducing,

(51) $$r = \frac{(u+1)^2}{4u} \cdot \frac{a.(e-1)}{\cos.^2\frac{1}{2}v}.$$

Now, from the value of T (8), we have,

(52) $$1-\mathrm{T} = 1-\left(\frac{u-1}{u+1}\right)^2 = \frac{4u}{(u+1)^2}, \quad \text{also} \quad a.(e-1) = D \quad (30).$$

Substituting these in the preceding value of r, it becomes like the first expression (50); and the second expression is deduced from the first, by using the second value of T (47).

(53) *The inverse problem of finding the time t, from the true anomaly v,* may be solved by means of Table VI. To effect this, we must first compute T, from v, by the formula (8);

(54) $$\mathrm{T} = \frac{e-1}{e+1} \cdot \text{tang.}^2\tfrac{1}{2}v.$$

With the argument T, we must enter Table VI, and take out the number A, and the $\log.B$; or, what is more convenient, and, at the same time, more accurate, the
(55) number C, and the $\log.B$; then compute A, by the following formula; which is easily deduced from (47);

(56) $$A = \frac{C\mathrm{T}}{1-\frac{4}{5}\mathrm{T}};$$

lastly, we must find t, by the formula (37). This expression, being divided by the factor of t, gives,

[5993]

$$t=\frac{2}{k}\cdot D^{\frac{3}{2}}.A^{\frac{1}{2}}.B.(e-1)^{-\frac{1}{2}}.\left\{1+\tfrac{1}{15}.A.(1+9e).(e-1)^{-1}\right\};$$ (57)

observing, that we have, as in [5992 (62)],

$$\log.\frac{2}{k}=2{,}0654486\ ; \qquad \log.\tfrac{1}{15}=8{,}8239087.$$ (58)

Then, if we put,

$$t_1=\frac{2}{k}\cdot D^{\frac{3}{2}}.A^{\frac{1}{2}}.B.(e-1)^{-\frac{1}{2}}\ ; \qquad t_2=t_1.\tfrac{1}{15}.A.(1+9e).(e-1)^{-1}\ ;$$ (59)

we shall have the following expression of t, which is exactly similar to that for an ellipsis [5992 (61)];

$$t=t_1+t_2\ ;$$ (60)

and, it is under this form, that the value of t is computed in the introduction to Table VI.

If we wish to use Table III, which does not, however, facilitate the operation, it will not be necessary to compute A. Then, we shall have,

$$\text{tang.}\tfrac{1}{2}w=A^{\frac{1}{2}}.\left(\frac{1+9e}{5e-5}\right)^{\frac{1}{2}}\ \ (38); \qquad \text{tang.}\tfrac{1}{2}v=\text{T}^{\frac{1}{2}}.\left(\frac{e+1}{e-1}\right)^{\frac{1}{2}}\ \ (51).$$ (61)

If we divide the first of these expressions by the second, then substitute the values of γ (7), also that of A (56), we shall get, by successive reductions,

$$\frac{\text{tang.}\frac{1}{2}w}{\text{tang.}\frac{1}{2}v}=\left(\frac{A}{\text{T}}\right)^{\frac{1}{2}}\cdot\left(\frac{1+9e}{5e+5}\right)^{\frac{1}{2}}=\left(\frac{A}{\text{T}\gamma^2}\right)^{\frac{1}{2}}=\sqrt{\frac{C}{\gamma^2.(1-\frac{4}{5}\text{T})}}\ ;$$ (62)

$$\text{tang.}\tfrac{1}{2}w=\sqrt{\frac{C}{\gamma^2.(1-\frac{4}{5}\text{T})}}\cdot\text{tang.}\tfrac{1}{2}v=\sqrt{\frac{2\,a'^2.\,C}{(e+1).(1-\frac{4}{5}\text{T})}}\cdot\text{tang.}\tfrac{1}{2}v.$$ (63)

Having computed the value of w, from (63), we may then find, in Table III, the time corresponding to the anomaly w. We must multiply this time by $\frac{BD^{\frac{3}{2}}}{a'}$ (42), to obtain the time t from the perihelion. The remarks made in [5992 (69—72)], relative to the construction of Table V, will apply, with the proper modifications, to Table VI. (64)

To illustrate this method of computation we shall give the following examples.

EXAMPLE I.

Given. $e=1{,}2500000$; log. perih. dist. 0,0200000 ; $t=60^{\text{days}}$, to find t, v. (65)

[5993]

APPROXIMATE OPERATION TO FIND v.

$\alpha'^2 = 0{,}1 + 0{,}9.e = 1{,}225$	log.	0,0881361
α'	log.	0,0440680
D	log. co.	9,9800000
	its half	9,9900000
$t = 60^{\text{days}}$	log.	1,7781513
(66) Approx.	log. t'	1,7922193
Hence $U = 66^d\, 44^m$, nearly, in Table III		
$e - 1 = 0{,}25$	log.	9,3979400
α'^2	log. co.	9,9118639
Constant	log.	9,6989700
Sum given β	log.	9,0087739
$\frac{1}{2}\, U = 33^d\, 22^m$	tang.	9,8185849
	same	9,8185849
(67) Approx. $A = 0{,}044253$	log.	8,6459437
Corresponding log. $B = 0{,}0000145$ Table VI.		

TO FIND THE RADIUS r.

$C - 0{,}2.A = 0{,}9912360$	log. co.	0,0038230
$C + 0{,}8.A = 1{,}0354969$	log.	0,0151488
(68) D	log.	0,0200000
$\frac{1}{2}\, v = 31^d\, 48^m\, 31^s{,}3$	sec.	0,0706768
	same	0,0706768
r	log.	0,1803254

CORRECTED OPERATION TO FIND v.

Subtract $\log. B = 0{,}0000145$ gives correc. $\log. t'$ 1,7922048

Hence U or $w = 66^d\, 44^m\, 16^s{,}9$ in Table III.

	same	9,0087739
$\frac{1}{2}\, w = 33^d\, 22^m\, 08^s{,}4$	tang.	9,8186233
	same	9,8186233
Constant $A = 0{,}0442609$	log.	8,6460205
$0{,}8.A = 0{,}0354087$		
$C = 1{,}0000882$		
$C + 0{,}8.A = 1{,}0354969$	log. co.	9,9848512
$e + 1 = 2{,}25$	log.	0,3521825
$e - 1 = 0{,}25$	log. co.	0,6020600
Sum is 2 log. tang $\frac{1}{2}\, v$		9,5851142
$\frac{1}{2}\, v = 31^d\, 48^m\, 31^s{,}3$	tang.	9,7925571
$v = 63^d\, 37^m\, 02^s{,}6$		

EXAMPLE II.

In the inverse problem, with the same elements, we have given,

(69) $e = 1{,}2500000$; log. perih. dist. 0,0200000 ; and $v = 63^d\, 37^m\, 02^s{,}6$;

to find t, in the following manner, by means of formula (60).

$e - 1 = 0{,}25$	log.	9,3979400
$e + 1 = 2{,}25$	log. co.	9,6478175
$\frac{1}{2} v = 31^d\, 48^m\, 31^s{,}3$	tang.	9,7925572
	same	9,7925572
$T = 0{,}0427437$	log.	8,6308719
Hence $C = 1{,}0000882$	Tab. VI log.	383
(70) $1 - 0{,}8.T = 0{,}9658050$	log. co.	0,0151106
$A = 0{,}0442609$	log.	8,6460208

	Constant log.	2,0654486
	$\frac{3}{2}$ log. D	0,0300000
	$\frac{1}{2}$ log. A	9,3230104
	log. B	145
	$\frac{1}{2}$ log. $(e - 1)$ arith. co.	3010300
$t_1 = 52^{\text{days}}{,}421$	log.	1,7195035
	Constant	8,8239087
	A log.	8,6460208
$1 + 9e = 12{,}25$	log.	1,0881361
$e - 1$	log. co.	6020600
$t_2 = 7^{\text{days}}{,}579$	log.	0,8796291
$t = 60^{\text{days}} = t_1 + t_2$.		

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We shall now compute the same example by means of Table III; by which means it
will appear, that the preceding method is the most simple. (71)

$e - 1 = 0{,}2500000$	log.	9,3979400
$e + 1 = 2{,}2500000$	log. co.	9,6478175
$\frac{1}{2}v = 31^d\,48^m\,31^s{,}3$	tang.	9,7925572
	same	9,7925572
$T = 0{,}0427437$	log.	8,6308719
$1 - 0{,}8.T = 0{,}9658050$	log. co.	0,0151106
$C = 1{,}0000882$	log.	383
$A = 0{,}0442609$	log.	8,6460208
Corresponding log. B, Table VI, is		0,0000145

$\alpha'^2 = 1{,}225$	log.	0,0881361
. . . .		9,6478175
	2 log.	0,3010300
. . . .		0,0151106
. . . .		383
	sum	2)0,0521325
	half	0,0260662
$\frac{1}{2}v = 31^d\,48^m\,31^s{,}3$	tang.	9,7925572
$\frac{1}{2}w = 33^d\,22^m\,08^s{,}4$	tang.	9,8186234
$w = 66^d\,44^m\,16^s{,}8$ Table III	log. t'	1,7922044
Table VI	log. B	145
D	log.	0,0200000
$D^{\frac{1}{2}}$	log.	0,0100000
α'	log. co.	9,9559320
$t = 60^{\text{days}}$ nearly.	log.	1,7781509

(72)

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ON THE METHOD OF COMPUTING THE ORBIT OF A COMET.

A short time before the publication of the first volume of the Mécanique Céleste,
containing La Place's method of computing the orbit of a comet [754—849], Dr. Olbers
gave a much shorter process for solving the same problem, in a work published at Gotha, (1)
in 1797, entitled *Abhandlung über die leichteste und bequemste Methode die Bahn eines
Cometen aus einigen Beobachtungen zu berechnen;* and as this method is but little
known in our country, we shall here give a full explanation of it, and shall simplify in some
respects, the calculation by means of Tables I, II, of this collection; which have been (2)
computed and examined with particular care, in order to render them correct, to the nearest
unit, in the last decimal place. We have used Table II, in an abridged form, for several
years, and have found it convenient and sufficiently accurate, as it regards the number of
decimal places. We shall first explain the method of Dr. Olbers, by the geometrical
process, which he used, and shall afterwards, in (262 &c.), show how his results can be (3)
obtained by an analytical process; noticing the small terms that he has neglected, and
which require attention in some particular cases. (4)

Olbers's method of computing the orbit of a Comet.

In finding the orbit of a comet, we have given, by observation, three geocentric longitudes
and latitudes, together with the times of observation; and from the solar tables we have (5)
the Sun's longitudes and the radii vectores. We shall use the symbols in the following table (6)
(9—29); most of them being like those which are given by La Place, [761^{iv}, 820^{vi} &c.].
*The unaccented letters being taken for the first observation; the same letters with one accent
for the middle observation; and with two accents for the third observation.* We have (7)
inserted in the same table (30—45), several theorems which are useful in these calculations
with the demonstrations in (46—130). In treating of this subject we shall refer to any (8)
line of [5994], by placing the number of the line in a parenthesis, as in [5984 (2), &c.].

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Symbols.

(9) t, t', t'', The times of observation;

(10) $\odot, \odot', \odot''$, Longitudes of the Sun, differing 180^d from those of the earth A, A', A'', respectively;

(11) $\alpha, \alpha', \alpha''$, Geocentric Longitudes of the comet;

(12) $\theta, \theta', \theta''$, Geocentric latitudes of the comet; *southern latitude being considered as negative;*

(13) R, R', R'', Distances of the earth from the sun;

(14) $\rho_1, \rho_1', \rho_1''$, Distances of the comet from the earth;

(15) ρ, ρ', ρ'', Curtate distances of the comet from the earth;

(16) r, r', r'', Radii vectores of the orbit of the comet;

(17) β, β', β'', Heliocentric longitudes of the comet;

(18) $\varpi, \varpi', \varpi''$, Heliocentric latitudes of the comet; *southern latitude being considered as negative;*

(19) $\varepsilon, \varepsilon', \varepsilon''$, The differences of the heliocentric longitudes of the comet and the earth;

(20) Ω, Longitude of the ascending node of the comet;

(21) φ, Inclination of the comet's orbit to the ecliptic;

(22) $\mathrm{u}, \mathrm{u}', \mathrm{u}''$, Arguments of latitude of the comet, or distances from the ascending node counted on the orbit;

(23) w, w', w'', Arguments of latitude of the comet reduced to the ecliptic and counted from the ascending node;

(24) $\chi = \mathrm{u}'' - \mathrm{u}$;

(25) v, v', v'', The true anomalies of the comet;

(26) D The perihelion distance of the comet;

(27) c The chord of the path of the comet between the first and third observations.

(28) $m = \dfrac{\text{tang.}\theta'}{\sin.(\odot' - \alpha')}$;

(29) $\rho'' = M.\rho$;

(30) $M = \dfrac{m\sin.(\odot' - \alpha) - \text{tang.}\theta}{\text{tang.}\theta'' - m.\sin.(\odot' - \alpha'')} \cdot \dfrac{t'' - t}{t' - t}$; [Approximate value of M.]

Fundamental equations.

(31) $r^2 = R^2 - 2R\,\rho.\cos.(\odot - \alpha) + \rho^2.\sec^2.\theta$; (A)

(32) $r''^2 = R''^2 - 2R''.M\rho.\cos.(\odot'' - \alpha'') + M^2.\rho^2.\sec^2.\theta''$; (B)

(33)
$$\begin{aligned} c^2 = {} & r^2 + r''^2 - 2RR''.\cos.(\odot'' - \odot) \\ & + \{2R''.\cos.(\odot'' - \alpha) + 2MR.\cos.(\odot - \alpha'')\}.\rho \\ & + \{-2M.\cos.(\alpha'' - \alpha) - 2M.\text{tang.}\theta.\text{tang.}\theta''\}.\rho^2. \end{aligned} \qquad (C)$$

(34) $\sin.\varpi = \dfrac{\rho}{r}.\text{tang.}\theta$;

(35) $\sin.\varpi'' = \dfrac{\rho''}{r''}.\text{tang.}\theta''$;

(36) $\sin.\varepsilon = \dfrac{\rho.\sin.(\odot - \alpha)}{r.\cos.\varpi}$;

(37) $\sin.\varepsilon'' = \dfrac{\rho''.\sin.(\odot'' - \alpha'')}{r''.\cos.\varpi''}$;

(38) $\cot.w = \text{tang.}\varpi''.\cot.\varpi.\text{cosec.}(\beta'' - \beta) - \cot.(\beta'' - \beta)$; or,

(38') $\text{tang.}\left(w + \dfrac{\beta'' - \beta}{2}\right) = \dfrac{\sin.(\varpi'' + \varpi)}{\sin.(\varpi'' - \varpi)} \cdot \text{tang.}\left(\dfrac{\beta'' - \beta}{2}\right)$;

(39) $\Omega = \beta - w$ = longitude of the ascending node; $\mho$ = longitude of the descending node $= 180^d + \Omega$;

(40) $\text{tang.}\varphi = \text{tang.}\varpi.\text{cosec.}w$. [$\varphi$ = Inclination]

(41) $\cos.\mathrm{u} = \cos.\varpi.\cos.w = \cos.\varpi.\cos.(\beta - \Omega)$;

(42) $\cos.\mathrm{u}'' = \cos.\varpi''.\cos.(\beta'' - \Omega)$;

(43) $\text{tang.}\xi = \left(\dfrac{r}{r''}\right)^{\frac{1}{2}}$.

(44) $\text{tang.}\tfrac{1}{2}v = \cot.\tfrac{1}{2}\chi - \left(\dfrac{r}{r''}\right)^{\frac{1}{2}}.\text{cosec.}\tfrac{1}{2}\chi$;

(44') $\text{tang.}\tfrac{1}{2}v'' = -\cot.\tfrac{1}{2}\chi + \left(\dfrac{r''}{r}\right)^{\frac{1}{2}}.\text{cosec.}\tfrac{1}{2}\chi$;

(44'') $\text{tang.}(\tfrac{1}{2}v + \tfrac{1}{4}\chi) = \text{tang.}(45^d - \xi).\cot.\tfrac{1}{4}\chi$;

(45) $D = r.\cos.^2\tfrac{1}{2}v$ = Perihelion distance.

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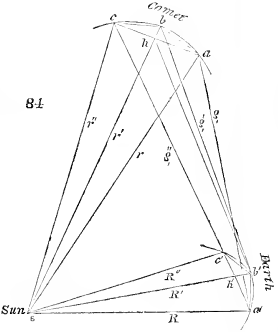
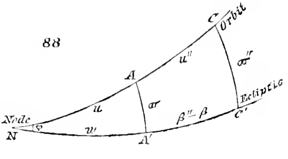


88



[5994]

We shall suppose, in the annexed figure 84, that s is the place of the sun; a, b, c, the places of the comet in the first, second, and third observations; a', b', c', the corresponding places of the earth. Draw the chords $a\,h\,c$, $a'\,h'\,c'$, intersecting the radii $s\,b$, $s\,b'$, in the points h, h', respectively; then by Kepler's first law [365'], we have, (46)

$$t'-t \;:\; t''-t' \;::\; \text{sector } s\,a\,b \;:\; \text{sector } s\,b\,c\,. \tag{47}$$

Now if we consider the chord $a\,c$ as a very small quantity of the *first* order, in comparison with the radius sb, the segment $h\,b$ will be of the *second* order. In this case, the triangle $s\,a\,c$ will be of the *first* order, and the elliptic, hyperbolic or parabolic segment $a\,b\,c\,h\,a$ of the *third* order; so, that the sector $s\,a\,b\,c$ will differ but very little from the triangle $s\,a\,h\,c$. In like manner if we suppose the chords ab, cb to be drawn, we shall find that the sectors $s\,a\,b$, $s\,c\,b$, differ respectively from the corresponding plane triangles $s\,a\,b$, $s\,c\,b$, by quantities of the *third* order; therefore the error will be but very small, if we substitute in (47), the ratios of the areas of these plane triangles, instead of the ratio of the areas of the sectors. Now these plane triangles have the same common base $s\,b$, and the perpendiculars let fall upon it from the points a, c, are evidently proportional to $a\,h$, $c\,h$, therefore their areas must be in the same proportion; hence, we shall have, very nearly,

(48) Geometrical investigation of Olbers's (49) method.

(50)

$$t'-t \;:\; t''-t' \;:\; \text{triangle } s\,a\,b \;:\; \text{triangle } s\,c\,b \;::\; a\,h \;:\; c\,h\,. \tag{51}$$

The same reasoning may be applied to the segments of the chord $a'c'$, described by the earth; therefore, we shall have, very nearly,

$$t'-t \;:\; t''-t' \;::\; a\,h \;:\; c\,h \;::\; a'h' \;:\; c'h'\;; \tag{52}$$

which is equivalent to the supposition, that if the two chords $a\,c$, $a'c'$, be described with uniform velocities, in the time $t''-t$, by a fictitious comet and planet, the fictitious bodies will be at the points h, h', when the real bodies are at b, b', respectively; and it is upon this hypothesis that the method of Dr Olbers essentially depends. (53)

We shall now take the point h', as a centre, and shall suppose the line $h's$, to be continued infinitely, till it meets the concave surface of the starry heavens, in the point S, figure 85, representing the geocentric place of the sun at the second observation. Moreover, we shall suppose *three* lines to be drawn through h' figure 84, page 797, parallel to the lines $a'a$, $b'b$, $c'c$, in the same directions, and continued infinitely to the heavens in the points A, B, C, figure 85, representing respectively the geocentric places of the comet, in the first, second, and third observations. Through the extreme points A, C, we shall draw the great circle $CHAN$, intersecting the ecliptic SN in the point N; also the great circle SB, intersecting the arc AC in H. To avoid the confusion of having many lines on the same figure, we have not actually drawn these three lines through the point h', but have merely marked, in figure 84, page 797, the point a'' of the line $h'a''A$, and the point c'' of the line $h'c''C$; supposing $h'a''=a'a$, $h'c''=c'c$. Then it is (54) (55) (56)

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evident from this construction, and from the proportions between the lines ah, ch, $a'h'$, $c'h'$
(52), that the right line, connecting the points a'', c'', will pass through the point h; and
(57) this line will be divided by the point h into the segments ha'', hc'', which have the same
ratio to each other, as ha, hc; as will be more particularly explained in a similar case in
(70, &c.). Then as the line shb, when viewed from h', is projected on the concave
(58) surface of the heavens, in the great circle SB; and the line $a''hc''$, when viewed
from the same point h', is projected in the great circle AC; it follows, that the point h,
(59) which is the intersection of these two lines sb, $a''c''$, must be projected in the heavens in
(60) the point H where the two great circles AHC, SHB intersect each other. Therefore,
H will be the geocentric place of the comet in the heavens, in the middle observation, if
(61) the bodies were to move uniformly in the chords ac, $a'c'$ and the comet be at the point h,
when the earth is at h'.

(62) Now if we suppose P, figure 85, to represent the pole of the ecliptic; ♈ the first point of Aries; PAA', PBB', PHH', PCC' circles of latitude; we shall have,

(63) $AA' = \theta$, $BB' = \theta'$, $CC' = \theta''$; ♈$A' = \alpha$, ♈$B' = \alpha'$, ♈$C'' = \alpha''$, ♈$S = \odot'$,

and we shall put for the geocentric longitude and latitude of the point H,

$$♈H' = \alpha_2 ; \qquad HH' = \theta_2 ;$$

(64) also, the angles, $ASA' = b$; $BSB' = b'$; $CSC' = b''$;

and the arcs, $SA' = \odot' - \alpha$; $SB' = \odot' - \alpha'$; $SC' = \odot' - \alpha''$.

Then, in the rectangular spherical triangle ASA', we have,

(65) $$\text{tang.}ASA' = \frac{\text{tang.}AA'}{\sin.SA'} \quad [1345^{31}];$$

which, in symbols, is the same as the first of the equations (66); the second and third of these equations, are found in the same manner, from the rectangular spherical triangles, BSB', CSC'; the second of these expressions, is evidently equal to the assumed value of m (28);

(66) $$\text{tang.}b = \frac{\text{tang.}\theta}{\sin.(\odot' - \alpha)}; \qquad \text{tang.}b' = \frac{\text{tang.}\theta'}{\sin.(\odot' - \alpha')} = m; \qquad \text{tang.}b'' = \frac{\text{tang.}\theta''}{\sin.(\odot' - \alpha'')}.$$

We shall suppose in figure 86, that the *paths of the earth and comet are projected*
(67) *orthographically upon the plane of that circle of latitude which is perpendicular to the
radius, drawn from the sun to the earth at the time of the middle observation*; or in other
words, that the plane of projection is perpendicular to the line $b'h's$ figure 84; so that
(68) the point h_1, of figure 86, is the projection of this line, or of the three points b', h', s,
upon the plane of this figure. We shall suppose, that the points a_1, c_1, A_1, C_1,
represent, respectively, the projections of the places of the earth a', c', and of the comet
a, c, at the times of the first and third observations; also H_2 the projection of the
(69) point h; so that the points S, A, H, C, in figure 85, correspond respectively to
the points s, A_1, H_2, C_1 in figure 86. Then it is evident from the principles of the

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orthographic projection, that the lines a_1c_1, A_1C_1, are divided in the points h_1, H_2, in the same proportion as the lines $a'c'$, ac, in figure 84, are by the points h', h. Therefore, if we draw the line A_1A_2, parallel and equal to a_1h_1; C_1C_2, parallel and equal to c_1h_1; then join H_2C_2, H_2A_2, we shall have, as in (52), very nearly, (70)

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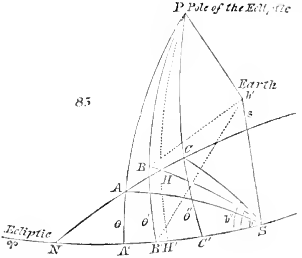
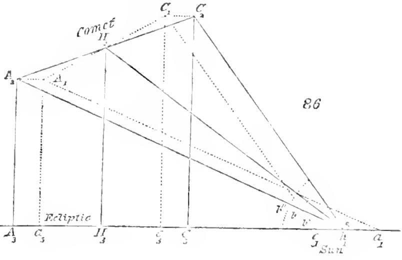


$$t'-t : t''-t' :: A_1H_2 : C_1H_2 :: A_1A_2 : C_1C_2 .$$ (71)

Now by construction the angles $C_2C_1H_2$, $A_2A_1H_2$, are equal, and as the sides about these angles are proportional, in the triangles $C_1C_2H_2$, $A_1A_2H_2$, therefore these triangles are similar, and the angle $A_1H_2A_2 =$ the angle $C_1H_2C_2$; consequently the three (72) points A_2, H_2, C_2, are situated in a straight line; which is divided by the point H_2, in the same proportion as the line A_1C_1 is divided by the same point; so that we shall (73) have, as in (52),

$$t'-t : t''-t' :: A_2H_2 : C_2H_2 .$$ (74)

From the above construction, it is also evident, that the line h_1A_2 is equal and parallel to a_1A_1; and h_1C_2, equal and parallel to c_1C_1; so that if the lines h_1A_2, h_1C_2 be (75) continued infinitely, they will represent the projections of the lines $h'A$, $h'C$ (55), drawn in figure 85, from the centre of the sphere h' to the geocentric places A, B, C, of the comet, (76) in the starry heavens, at the first and third observations. In like manner h_1H_2 represents the projection of the line $h'hH$ figure 84, drawn from the centre h', through the point h, towards the point H in the starry heavens. The line a_1c_1, figure 86, continued to A_3, (77) represents the projection of the ecliptic, upon which we shall let fall the perpendiculars A_2A_3, A_1a_3, H_2H_3, C_1c_3, C_2C_3. Then it is evident, from the construction, that $a_1a_3 = h_1A_3$ (78) is the projection of the curtate distance ρ at the time of the first observation; and as the geocentric longitude of the comet is then equal to α, and the longitude of the axis (79)



[5994] drawn through a_1, perpendicular to the plane of the figure is $\odot'$, the inclination of the
(79') line ρ to this axis is $\odot'-\alpha$; consequently this projection of ρ is represented by,

(80) $$a_1a_3 = h_1A_3 = \rho.\sin.(\odot'-\alpha).$$

In like manner the projection of ρ'' is,

(81) $$c_1c_3 = h_1C_3 = \rho''.\sin.(\odot'-\alpha'').$$

Again, as the line Sh' figure 85, is perpendicular to the plane of projection in figure 86, the angles formed about the point S figure 85, will be projected about the point h_1,
(82) figure 86, without any alterations in their magnitudes or relative positions; so that the angle AST' figure 85, will be projected into $A_2h_1A^3$ in figure 86, and so on for the other angles; hence we shall have, by using the same symbols as in (64);

(83) $$A_3h_1A_2 = b; \quad H_3h_1H_2 = b'; \quad C_3h_1C_2 = b''; \quad A_2h_1H_2 = b'-b; \quad C_2h_1H_2 = b''-b'.$$

Now in the rectangular plane triangles $A_3h_1A_2$, $C_3h_1C_2$, we have by using the values (80, 81, 83),

(84) $$h_1A_2 = \frac{h_1A_3}{\cos.A_3h_1A_2} = \frac{\rho.\sin.(\odot'-\alpha)}{\cos.b}; \qquad h_1C_2 = \frac{h_1C_3}{\cos.C_3h_1C_2} = \frac{\rho''.\sin.(\odot'-\alpha'')}{\cos.b''}.$$

In the plane triangle $A_2H_2h_1$ we have,

(85) $$\sin.A_2H_2h_1 : \sin.A_2h_1H_2 :: h_1A_2 : A_2H_2;$$

hence we get the first expression (86), using for brevity,

(85') $$\sin.H = \sin.A_2H_2h_1 = \sin.C_2H_2h_1;$$

and by substituting the symbols (83, 84) in its second member we get the second expression (86). In like manner from the triangle $C_2H_2h_1$ we get the expression (87);

(86) $$A_2H_2.\sin.H = h_1A_2.\sin.A_2h_1H_2 = \frac{\rho.\sin.(\odot'-\alpha)}{\cos.b}.\sin.(b'-b);$$

(87) $$C_2H_2.\sin.H = h_1C_2.\sin.C_2h_1H_2 = \frac{\rho''.\sin.(\odot'-\alpha'')}{\cos.b''}.\sin.(b''-b').$$

Dividing the equation (87) by (86), then substituting in the first member, the expression,

(88) $$\frac{C_2H_2}{A_2H_2} = \frac{t''-t'}{t'-t} \quad (74);$$

putting also $\rho'' = M.\rho$, as in (29), we get the approximate values of M (92). Developing the first members of (89, 90), by [22, 34'] Int., and substituting the values (66), we get successively,

[5994]

$$\sin.(b'-b) = \sin.b'.\cos.b - \cos.b'.\sin.b = \cos.b.\cos.b'.(\text{tang}.b' - \text{tang}.b)$$

$$= \cos.b.\cos.b'.\left(m - \frac{\text{tang}.\theta}{\sin.(\odot'-\alpha)}\right). \qquad (89)$$

$$\sin.(b''-b') = \sin.b''.\cos.b' - \cos.b''.\sin.b' = \cos.b'.\cos.b''.(\text{tang}.b'' - \text{tang}.b')$$

$$= \cos.b'.\cos.b''.\left(\frac{\text{tang}.\theta''}{\sin.(\odot'-\alpha'')} - m\right). \qquad (90)$$

Substituting the values (89,90) in M (92), and rejecting the factor $\cos.b.\cos.b'.\cos.b''$, which occurs in the numerator and denominator, we finally obtain the approximate value of M (93); which is of the same form as in (30); (91)

$$M = \frac{t''-t'}{t'-t}\cdot\frac{\cos.b''.\sin.(b'-b)}{\cos.b.\sin.(b''-b')}\cdot\frac{\sin.(\odot'-\alpha)}{\sin.(\odot'-\alpha'')} \qquad (92)$$

[Approximate value of M.]

$$= \frac{t''-t'}{t'-t}\cdot\frac{m.\sin.(\odot'-\alpha) - \text{tang}.\theta}{\text{tang}.\theta'' - m.\sin.(\odot'-\alpha'')}. \qquad (93)$$

We shall show hereafter, in (306, &c.), how this approximate value of M may be corrected for the error of the hypothesis (50), where the ratio of the areas of the triangles, is used (93') instead of that of the sectors. Again, we have, in the right angled spherical triangle $BB'S$, figure 85, page 795,

$$\cos.SB = \cos.SB'.\cos.BB' = \cos.(\odot'-\alpha').\cos.\theta' \quad [1345^{27}]; \qquad (94)$$

and this evidently represents the cosine of the angular distance of the sun and comet $sb'b = SB$ figures 84, 85, in the second observation. In like manner, by decreasing by unity, the accents of the symbols, so as to make them correspond to the first observation, we get, (95)

$$\cos.s\,a'a = \cos.(\odot - \alpha).\cos.\theta. \qquad (96)$$

Now in this plane triangle $sa'a$, we have, $sa = r$, $sa' = R$, $aa' = \rho.\sec.\theta$, and, by using [62] Int. we obtain the expression (97), which is easily reduced to the form (98), being the same as the first equation of La Place's method [806]; (96')

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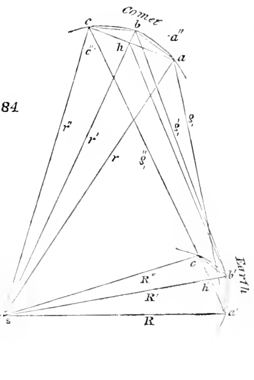


$$r^2 = R^2 - 2R.(\rho.\sec.\theta).\{\cos.(\odot-\alpha).\cos.\theta\} + (\rho.\sec.\theta)^2 \qquad (97)$$

$$= R^2 - 2R.\rho.\cos.(\odot-\alpha) + \rho^2.\sec^2.\theta. \qquad (98)$$

This last expression is the same as the value of r^2, (31), corresponding to the *first* observation. If we add two accents to the symbols of this expression we get,

$$r''^2 = R''^2 - 2R''.\rho''.\cos.(\odot''-\alpha'') + \rho''^2.\sec^2.\theta''; \qquad (99)$$

[5994] which, by substituting $\rho'' = M.\rho$ (29), becomes as in (32); corresponding to the third observation.

We shall now suppose, for a moment, that the place of the comet at the first observation, is determined by three rectangular co-ordinates x, y, z, whose origin is the centre of
(100) the sun. The axis of x is drawn in the plane of the ecliptic, towards the first point of
(101) Aries; the axis of y, is drawn in the same place, towards the first point of Cancer; the axis of z, is perpendicular to the ecliptic, and directed towards its northern pole. In like
(102) manner, we shall suppose, that x', y', z', represent the co-ordinates of the comet, at
(103) the second observation; also x'', y'' z'', those at the third observation; then it is
(104) evident, from the principles of the orthographic projection [118], that if c represent the line or chord, between the places of the comet at the first and third observations, we shall have the first of the expressions of c^2 (106) and by developing and substituting,

(105) $$r^2 = x^2 + y^2 + z^2; \qquad r''^2 = x''^2 + y''^2 + z''^2;$$

it becomes as in (107);

(106) $$c^2 = (x''-x)^2 + (y''-y)^2 + (z''-z)^2 = (x''^2 + y''^2 + z''^2) + (x^2 + y^2 + z^2) - 2.(xx'' + yy'' + zz'')$$

(107) $$= r''^2 + r^2 - 2.(xx'' + yy'' + zz'').$$

Now we have, as in [762,768],

(108) $$x = R.\cos.A + \rho.\cos.\alpha; \qquad y = R.\sin.A + \rho.\sin.\alpha; \qquad z = \rho.\mathrm{tang}.\theta.$$

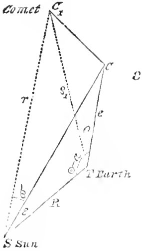
(109) $$x'' = R''.\cos.A'' + \rho''.\cos.\alpha''; \qquad y'' = R''.\sin.A'' + \rho''.\sin.\alpha''. \qquad z'' = \rho''.\mathrm{tang}.\theta''.$$

Substituting these values in the first member of (110), we get its second member, and by successive reductions, it becomes as in (112); using the values of A, A'' (10);

(110) $$xx'' + yy'' = R.R''.(\cos.A.\cos.A'' + \sin.A.\sin.A'') + R''.\rho.(\cos.A''.\cos.\alpha + \sin.A''.\sin.\alpha)$$
$$+ R.\rho''.(\cos.A.\cos.\alpha'' + \sin.A.\sin.\alpha'') + \rho.\rho''.(\cos.\alpha.\cos.\alpha'' + \sin.\alpha.\sin.\alpha'')$$

(111) $$= R.R''.\cos.(A''-A) + R''.\rho.\cos.(A''-\alpha.) + R.\rho''.\cos.(A-\alpha'') + \rho.\rho''.\cos.(\alpha''-\alpha)$$

(112) $$= R.R''.\cos.(\odot''-\odot) - R''.\rho.\cos.(\odot''-\alpha) - R.\rho''.\cos.(\odot-\alpha'') + \rho.\rho''.\cos.(\alpha''-\alpha).$$



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(113) Substituting this and $zz'' = \rho.\rho''.\mathrm{tang}.\theta.\mathrm{tang}.\theta''$ (108, 109), in (107); and then putting $\rho'' = M.\rho.$ (29), it becomes as in (33), the terms being arranged according to the powers of ρ. We have as in
(114) [5358] $z = r.\sin.\mathrm{lat.}$ or $z = r.\sin.\varpi$ (18); putting this equal to the value of z (108), we get $\sin.\varpi$ (34), corresponding to the first observation, and in like manner we obtain for the third observation $\sin.\varpi''$ (35). In fig. 87, if S be the place of the sun,
(115) C_1 that of the comet, and T that of the earth, at the first obervation; C_1C a line drawn from the comet C_1, perpendicular to the plane

of the ecliptic; we shall have in the plane triangle, STC, [5994]

$$TC = \rho, \quad SC = r.\cos.\varpi, \quad CST = \varepsilon, \quad STC = \odot - \alpha\,; \qquad (116)$$

and since,

$$SC : TC :: \sin.STC : \sin.CST\,; \qquad (117)$$

we shall have, in symbols,

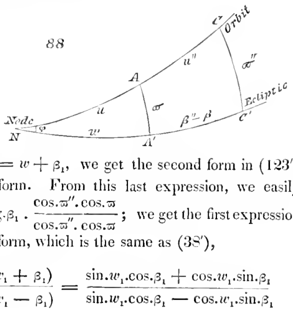
$$r.\cos.\varpi : \rho :: \sin.(\odot - \alpha) : \sin.\varepsilon\,; \qquad (118)$$

whence we obtain $\sin.\varepsilon$ (36), corresponding to the first observation; and by putting two accents upon the symbols, we obtain the similar expression of $\sin.\varepsilon''$ (37), corresponding to the third observation.

In figure 88, $NA'C'$ represents the ecliptic, A, C, the heliocentric places of the comet at the first and third observations; A', C', these places reduced to the ecliptic; N the ascending (119)
node of the comet's heliocentric orbit NAC. Then in the rectangular spherical triangle,
$NA'A$, we have $\cot.ANA' = \cot.AA'.\sin.NA'$ [1345[31]]; or, in symbols, (120)
$\cot.\varphi = \cot.\varpi.\sin.w$, which is easily reduced to the form (40). In like manner, in the (120′)
triangle $NC'C'$, we have, by putting for a moment, $\beta'' - \beta = 2\beta_1$, and $NC' = w + 2\beta_1$; (121)
$\cot.\varphi = \cot.\varpi''.\sin.(w + 2\beta_1) = \cot.\varpi''.\{\sin.w.\cos.2\beta_1 + \cos.w.\sin.2\beta_1.\}$. Putting these two (122)
expressions of $\cot.\varphi$ (120′, 122) equal to each other, and dividing by $\sin.w.\sin.2\beta_1.\cot.\varpi''$,
we get, $\tan g.\varpi''.\cot.\varpi.\mathrm{cosec}.2\beta_1 = \cot.2\beta_1 + \cot.w$, whence we get $\cot.w$ (38). This (122′)
expression may be reduced to the form (38′); which is rather more convenient, in using logarithms. For we have, in the triangles, NAA', $NC'C$;

$$\mathrm{tang}.\varphi = \frac{\mathrm{tang}.\varpi}{\sin.w}\,;$$

$$\mathrm{tang}.\varphi = \frac{\mathrm{tang}.\varpi''}{\sin.(w + \beta'' - \beta)}\,; \qquad (122'')$$



and by putting these two expressions equal to each other, we get the first equation (123′).
Putting for brevity, $w_1 = w + \frac{1}{2}.(\beta'' - \beta) = w + \beta_1$, we get the second form in (123′) (123)
and, by development we obtain the third form. From this last expression, we easily deduce (124), and by multiplying it by $\mathrm{tang}.\beta_1 . \frac{\cos.\varpi''.\cos.\varpi}{\cos.\varpi''.\cos.\varpi}$; we get the first expression (124′), which is easily reduced to the second form, which is the same as (38′),

$$\frac{\mathrm{tang}.\varpi''}{\mathrm{tang}.\varpi} = \frac{\sin.(w + \beta'' - \beta)}{\sin.w.} = \frac{\sin.(w_1 + \beta_1)}{\sin.(w_1 - \beta_1)} = \frac{\sin.w_1.\cos.\beta_1 + \cos.w_1.\sin.\beta_1}{\sin.w_1.\cos.\beta_1 - \cos.w_1.\sin.\beta_1} \qquad (123')$$

$$\frac{\mathrm{tang}.\varpi'' + \mathrm{tang}.\varpi}{\mathrm{tang}.\varpi'' - \mathrm{tang}.\varpi} = \frac{2\sin.w_1 . \cos.\beta_1}{2\cos.w_1 . \sin.\beta_1} = \frac{\mathrm{tang}.w_1}{\mathrm{tang}.\beta_1} \qquad (124)$$

$$\mathrm{tang}.w_1 = \frac{\sin.\varpi''.\cos.\varpi + \cos.\varpi''.\sin.\varpi}{\sin.\varpi''.\cos.\varpi - \cos.\varpi''.\sin.\varpi} . \mathrm{tang}.\beta_1 = \frac{\sin.(\varpi'' + \varpi)}{\sin.(\varpi'' - \varpi)} . \mathrm{tang}.\beta_1 . \qquad (124')$$

[5994] Again in the triangles NAA', NCC', fig. 88. we have

(125) $$\cos.NA = \cos.AA'.\cos.NA' ; \qquad \cos.NC = \cos.CC'.\cos.NC' \quad [1345^{27}],$$

which in symbols, becomes as in (41,42). These values of u, u'', give,

(126) $$\chi = u'' - u = \text{arc } AC;$$

adding this to v we get $v'' = v + \chi$. Then the formula (45), which is the same as [5986(4)], gives,

(127) $$D = r.\cos.^2\tfrac{1}{2}v = r''.\cos.^2\tfrac{1}{2}.(v+\chi) ;$$

hence,

(128) $$\left(\frac{r}{r''}\right)^{\frac{1}{2}} = \frac{\cos.\frac{1}{2}.(v+\chi)}{\cos.\frac{1}{2}v} = \frac{\cos.\frac{1}{2}v.\cos.\frac{1}{2}\chi - \sin.\frac{1}{2}v.\sin.\frac{1}{2}\chi}{\cos.\frac{1}{2}v} = \cos.\tfrac{1}{2}\chi - \text{tang}.\tfrac{1}{2}v.\sin.\tfrac{1}{2}\chi.$$

(128′) Dividing this by $\sin.\frac{1}{2}\chi$, we get the value of $\text{tang}.\frac{1}{2}v$ (44). In the same way, we may obtain the expression of $\text{tang}.\frac{1}{2}v''$ (44′); or more simply, by changing r, v, u, corresponding to the first observation, into r'', v'', u'', which corresponds to the third;
(129) by which means χ (24), changes into $-\chi$. The expression (44), may be reduced to the form (44″), by putting,

(129) $$\text{tang}.\xi = \sqrt{\frac{r}{r''}} \quad (43) ;$$

by which means it becomes,

(129′) $$\text{tang}.\tfrac{1}{2}v = \cot.\tfrac{1}{2}\chi - \frac{\text{tang}.\xi}{\sin.\frac{1}{2}\chi}, \qquad \text{or,} \qquad \text{tang}.\xi = \cos.\tfrac{1}{2}\chi - \sin.\tfrac{1}{2}\chi.\text{tang}.\tfrac{1}{2}v ;$$

hence we get, by successive reductions, and using [1,6,31,29] Int., the following expressions,

(130) $$\frac{1-\text{tang}.\xi}{1+\text{tang}.\xi} = \frac{1-\cos.\frac{1}{2}\chi + \sin.\frac{1}{2}\chi.\text{tang}.\frac{1}{2}v}{1+\cos.\frac{1}{2}\chi - \sin.\frac{1}{2}\chi.\text{tang}.\frac{1}{2}v} = \frac{2\sin.^2\frac{1}{4}\chi + 2\sin.\frac{1}{4}\chi.\cos.\frac{1}{4}\chi.\text{tang}.\frac{1}{2}v}{2\cos.^2\frac{1}{4}\chi - 2\sin.\frac{1}{4}\chi.\cos.\frac{1}{4}\chi.\text{tang}.\frac{1}{2}v}$$

(130′) $$= \text{tang}.\tfrac{1}{4}\chi.\ \frac{\text{tang}.\frac{1}{4}\chi + \text{tang}.\frac{1}{2}v}{1-\text{tang}.\frac{1}{4}\chi.\text{tang}.\frac{1}{2}v} = \text{tang}.\tfrac{1}{4}\chi.\text{tang}.(\tfrac{1}{2}v + \tfrac{1}{4}\chi).$$

Substituting $1 = \text{tang}.45^d$, in the first member of (130), and then reducing, by means of (30) Int. it becomes,

(130″) $$\frac{\text{tang}.45^d - \text{tang}.\xi}{1 + \text{tang}.45^d\,\text{tang}.\xi} = \text{tang}.(45^d - \xi) ;$$

hence the expression (130′) becomes as in (44″).

We shall now proceed to illustrate these formulas by an example in (173, &c.). The
(131) data being as in (174 — 175). With these, we can compute in (176 — 181), the coefficients of the fundamental equations (31,32,33), as in (182—186). From these equations,

[5994]

by the process, which is explained in (134—163), we may compute the values of ρ, r, r'', in successive approximations with the help of Tables I, II, as in (187—191); the arguments to be used in Table II, being the sum of the radii $r+r''$ at the top of the page, and the chord c at the side. Having obtained these approximate values of ρ, r, r'', we can deduce from them the approximate elements of the orbit, as in (193—205). The chief difficulty in this solution, is in finding the value of ρ, which will satisfy the equations (182, 183, 186), or, as they are called, (A), (B), (C); to which we may also annex the equation (D), or the sum of the equations (A), (B), which represents the value of $r^2+r''^2$. The method of operation, to find the value of ρ, is explained in the precepts, in the four first pages of Table II, to which we may refer, observing particularly the directions at the bottom of the fourth page, to vary ρ in the successive operations by some aliquot part of its last value, represented by $\frac{1}{p}\cdot\rho$; p being an integral number, positive or negative; by this means any term A, depending on the first power of ρ is augmented by $\frac{1}{p}\cdot A$, and if it depend on ρ^2, it is augmented by the quantity $\frac{2}{p}\cdot A+\frac{1}{2p}\cdot\left(\frac{2}{p}\cdot A\right)$. We may also observe, that in making the first rough estimate of the value of ρ, we can use with advantage the two equations (C), (D), or the values of $r^2+r''^2$, c^2; found to one or two places of decimals. In this process we must enter Table II with the argument $r^2+r''^2$ at the bottom, and c^2 at the right hand side column. In this case we have only two equations, (C), (D), to satisfy; instead of the *three* equations (A), (B), (C), required in the general and more accurate process. Most commonly, we may, for a first hypothesis, take $\rho=1$; and if the resulting time T, deduced from Table II, be too great, we must, in general, decrease proportionally the value of ρ; and in one or two trials, *without the trouble of taking any proportional parts, and with a very few minutes labor, we can get a pretty close approximation to the value of* ρ. When this is obtained, we can use it with the equations (A), (B), (C), in getting the correct value of ρ, by the process explained in page 2 of Table II, or by the similar calculation in (153—163). In the examples which we shall give in (207—242), for finding ρ, we have neglected the consideration of the equation (D), but it may not be amiss to show the advantage of using it, by applying it to these examples. Taking therefore the first example, and using the equations (D), (C), (184, 186), we find, that if we put $\rho=1$, and use two places of decimals we shall get

$$r^2+r''^2=2{,}02-1{,}50+2{,}01=2{,}53\,; \qquad c^2=0{,}02-0{,}11+0{,}50=0{,}41\,;$$

whence we find, by inspection, in Table II, $T=27^{\text{days}}$, instead of the real value by observation $T=8^{\text{days}}$; and as this is three times too great, we may decrease ρ in that ratio, and take for a second value $\rho=\frac{1}{3}$. This gives in (184, 186),

$$r^2+r''^2=1{,}747\,; \qquad c^2=0{,}0374,$$

[5994] whence $T = 7^{\text{days}},6$ nearly. This must be increased a little, because the time is too
(144) small, as we have done in (189). Again if we put $\rho = 1$, in the second example, (207, &c.) we shall get, from (210, 209),

$$r^2 + r''^2 = 4,71\ ; \qquad c^2 = 0,68\ ;$$

which gives, in Table II, $T = 42^{\text{days}}$, instead of $11^{\text{days}},9734$. We may, therefore,
(145) for a second supposition, put $\rho = \frac{1}{4}$, because these two values are nearly in that ratio. Substituting $\rho = \frac{1}{4}$ in (210, 209) we get,

$$r^2 + r''^2 = 2,42\ ; \qquad c^2 = 0,104\ ;$$

hence we get in Table II, $T = 14^{\text{days}}$ nearly; so that ρ must be still further
(146) decreased; and the value assumed in (212) is $\frac{1}{6}$. In Example III, (216, &c.), we have by putting $\rho = 1$, in (219, 218),

$$r^2 + r''^2 = 4,98\ ; \qquad c^2 = 0,91\ ;$$

whence $T = 49^{\text{days}}$ instead of 10^{days}; so that for a second value we may take $\rho = \frac{1}{5}$, which gives,

$$r^2 + r''^2 = 1,79\ ; \qquad c^2 = 0,009\ ;$$

(147) whence $T = 3^{\text{days}},7$. This is much too small, therefore we may take $\rho = \frac{1}{3}$; hence,

$$r^2 + r''^2 = 1,89\ ; \qquad c^2 = 0,050\ ;$$

whence $T = 9^{\text{days}}$, which must be increased a little as in (222). In Example IV, (226, &c) we have, by putting $\rho = 1$, in (229, 228),

$$r^2 + r''^2 = 1,98\ ; \qquad c^2 = 0,12\ ;$$

whence $T = 14^{\text{days}}$; which is nearly four times too great, therefore we may take for
(148) the next operation $\rho = \frac{1}{4}$, as in (231). In Example V, (235, &c.), we have, by putting $\rho = 1$, in (236, 235),

$$r^2 + r''^2 = 4,39\ ; \qquad c^2 = 0,16\ ;$$

whence $T = 20^{\text{days}}$, which is more than double the actual value, we may therefore
(149) assume $\rho = \frac{1}{2}$ as in (237) for the next operation. In Example VI, we have by putting $\rho = 1$, in (240, 239),

$$r^2 + r''^2 = 1,27\ ; \qquad c^2 = 0,45\ ;$$

whence $T = 24^{\text{days}}$; which is twice the actual value; we may therefore take for the
(150) next operation $\rho = \frac{1}{2}$, as in (241). What we have here stated will serve to show the
(151) method of using the equation (D). We shall now proceed to the explanation of the process with the equations (A), (B), (C); and it will suffice, for this purpose, to explain
(152) particularly, the calculations in the first example in (173—206).

In making the calculation of ρ, from the equations (A), (B), (C), or (182, 183, 186)
of the first example, we have placed, in the first column of the table (187—191), the (153)
successive values which are assumed for ρ. The second column contains the corresponding
terms of r^2, deduced from the equation (A), in the third, the value of r''^2, deduced (154)
from (B); the fourth the value of c^2, deduced from (C). In the fifth column are
the corresponding values of r, r'', c, deduced from r^2, r''^2, c^2, by means of Table I; and
in the sixth column is the resulting value of T, deduced from Table II. Thus by putting (155)
$\rho=1$ in the equation (A), we find that the terms become $r^2=1{,}014-0{,}288+1{,}103=1{,}829$,
as in column 2; and with this value of r^2, we get $r=1{,}35$, in Table I. In the (156)
same way, we get, from (B), (C), the expressions $r''=0{,}84$; $c=0{,}64$; then with
$r+r''=2{,}19$ and $c=0{,}64$, we obtain, by the mere inspection of Table II, $T=27^{\text{days}}$, (157)
nearly. This time being about three times as great as the actual value by observation,
$T=8^{\text{days}}$, we may take for a second hypothesis $\rho=\frac{1}{3}$; and by repeating the operation get
$T=7^{\text{days}},660$. The calculation of the coefficients of ρ, ρ^2 in these equations is made in (158)
columns 8, 9, 10, conformably to the precepts in pages 1, 2, 3, 4, of Table II; and the
results are transferred to columns 2, 3, 4. In going through these calculations, we have always (159)
varied ρ by an aliquot part $\frac{1}{p}\cdot\rho$ of its last value, according to the precepts in the table
and in (135). Thus we have, in the first instance, taken $\rho=1$ (187), then $\rho=\frac{1}{3}$ (188); to (160)
this second value $\frac{1}{20}$ part is added, for the next operation; and as this is found to be too (161)
great, it is decreased by $\frac{1}{200}$ part; finally this last value is increased by $\frac{1}{2500}$ part or
0,0004, multiplied by its last value; and then the resulting expression of T becomes (162)
8^{days}, agreeing with the observations. Similar processes are used in the other examples,
as may be seen by inspection of the calculations, without any particular explanation. (163)

In the first example (173—206), we have gone through the whole calculation (176—181)
for finding the coefficients of the equations (A), (B), (C), (182—186); and deducing from
them the values of ρ, r, r'' (187—192). From these last quantities we have finally (164)
deduced the elements of the orbit, as in (193—205). This one example will suffice for
the illustration of the method of calculating the coefficients (176—181), and the computation
of the elements (193—205); but for the sake of explaining more particularly the uses of (165)
Tables I, II, we shall insert several examples of the computation of ρ, r, r'', similar (166)
to (187—192), from the fundamental equations (A), (B), (C), corresponding to different
comets and shall select, for this purpose, some which have been already calculated by Olbers, (167)
Delambre, Ivory, &c. We may remark, *that if any one of the coefficients of the equations*
(A), (B), (C), *be negative, we may add its arithmetical complement to* 10,00000, *and* (168)
then reject this last quantity. Thus, in finding the first value of r^2, in the following table
(187); instead of using $1{,}014-0{,}288+1{,}103$ we may take, (169)

$$1{,}014+9{,}712+1{,}103-10{,}000\,;$$ (170)

and *as each figure of the arithmetical complement can be taken separately, while performing* (171)
the process of the addition of these quantities, without the trouble of actually writing down
the figures of the arithmetical complement, we can make this addition, by one operation,
notwithstanding the difference of the signs: by this means the calculation is somewhat (172)
abridged.

[5994]

EXAMPLE I.

This example is the same as that of Dr Olbers, in page 54 of his *Abhandlung*, &c., in which he gives the
(173) computation of the orbit of the comet of 1769, from the observations of September 4^{th}, 8^{th}, 12^{th}, 1769.

	$\odot$ long.	Comet's geo. long.	Com. geo. lat. South.	log. $\odot$ dist. earth.
1769 Sep. $4^d\,14^h$;	$\odot = 162^d\,42^m\,05^s$;	$\alpha = 80^d\,56^m\,11^s$;	$\theta = -17^d\,51^m\,39^s$;	$\log.R = 0{,}003132$;
(174) $8^d\,14^h$;	$\odot' = 166^d\,35^m\,31^s$;	$\alpha' = 101^d\,00^m\,54^s$;	$\theta' = -22^d\,05^m\,02^s$;	$\log.R' = 0{,}002665$;
$12^d\,14^h$;	$\odot'' = 170^d\,20^m\,20^s$;	$\alpha'' = 124^d\,19^m\,22^s$;	$\theta'' = -23^d\,43^m\,55^s$;	$\log.R'' = 0{,}002184$.

Hence we deduce,

$$\odot - \alpha = 81^d\,45^m\,54^s\,;\quad \odot - \alpha'' = 38^d\,22^m\,43^s\,;$$

(175)
$$\odot' - \alpha = 85^d\,39^m\,20^s\,;\quad \odot' - \alpha' = 65^d\,34^m\,37^s\,;\quad \odot' - \alpha'' = 42^d\,16^m\,09^s\,;$$

$$\odot'' - \alpha = 89^d\,33^m\,09^s\,;\quad \odot'' - \alpha'' = 46^d\,09^m\,58^s\,;\quad \odot'' - \odot = 7^d\,47^m\,15^s\,;\quad \alpha'' - \alpha = 43^d\,23^m\,11^s.$$

$$\log.R^2 = 0{,}006264\,;\quad \log.R''^2 = 0{,}004368\,;\quad \log.2R = 0{,}304162\,;\quad \log.2R'' = 0{,}303214\,.$$

In this example the comet's geocentric latitudes being *south* are considered as *negative*; *and the rules for the signs of the angles* [5990 (23, 24, 25)] *are to be observed in finding the coefficients of all the terms of the fundamental equations* (28—33).

I. CALCULATION OF THE THREE FUNDAMENTAL EQUATIONS (31, 32, 33).

(176) *To find* m, M (28, 30).

θ'	tang.	$9{,}608237_n$
$\odot' - \alpha'$	cosec.	$0{,}040712$
m	log.	$9{,}648949_n$
$-\sin.(\odot' - \alpha'')$	log.	$9{,}827766_n$
(177) $0{,}29972$	log.	$9{,}476715$
tang $\theta'' = -0{,}43963$		
$-0{,}13991$ = Denomin. of M		
m	log.	$9{,}648949_n$
$\odot' - \alpha$	sin.	$9{,}998750$
(178) $-0{,}44432$	log.	$9{,}647699_n$
$-\text{tang.}\theta = 0{,}32224$		
Num. of $M = -0{,}12208$	log.	$9{,}086644_n$
Den. of M	log. co. ar.	$0{,}854151^n$
$t'' - t'$	log.	$0{,}602060$
$t' - t$	log. co. ar.	$9{,}397940$
(179) M	log.	$9{,}940795$

To find r^2 (31).

R^2 1,01453	log.	$0{,}006264$
$-2R$	log.	$0{,}304162_n$
$\odot - \alpha$	cos.	$9{,}156045$
coeff. of $\rho = -0{,}28854$	log.	$9{,}460207_n$
θ	sec.	$0{,}021452$
$\sec^2.\theta = 1{,}10384$	log.	$0{,}042904$

To find r''^2 (32).

R''^2 1,01011	log.	$0{,}004368$
$-2R''$	log.	$0{,}303214_n$
$\odot'' - \alpha''$	cos.	$9{,}840464$
M	log.	$9{,}940795$
coeffi. of $\rho = -1{,}21471$	log.	$0{,}084473_n$
θ''	sec.	$0{,}038371$
M	log.	$9{,}940795$
$M.\sec.\theta''$	log.	$9{,}979166$
$M^2.\sec^2\theta'' = 0{,}90852$	log.	$9{,}958332$

To find c^2 (33).

$-2R$	log.	$0{,}304162_n$
R''	log.	$0{,}002184$
$\odot'' - \odot$	cos.	$9{,}995976$
$-2{,}00596$	log.	$0{,}302322_n$
$2R''$	log.	$0{,}303214$
$\odot'' - \alpha$	cos.	$7{,}892666$
1^{st} term $= 0{,}01570$		$8{,}195880$
$2R$	log.	$0{,}304162$
$\odot - \alpha''$	cos.	$9{,}894275$
M	log.	$9{,}940795$
2^{nd} term $= 1{,}37795$	log.	$0{,}139232$
coeff. of $\rho = 1{,}39365$		
-2	log.	$0{,}301030_n$
M	log.	$9{,}940795$
$\alpha'' - \alpha$	cos.	$9{,}861378$
1^{st} term $= -1{,}26825$		$0{,}103203_n$
-2	log.	$0{,}301030_n$
M	log.	$9{,}940795$
θ	tang.	$9{,}508175_n$
θ''	tang.	$9{,}643092_n$
(180) 2^{nd} term $= -0{,}24723$	log.	$9{,}393092_n$
(181) coeff. of $\rho^2 = -1{,}51548$		

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Three Fundamental Equations.

$$r^2 = 1{,}01453 - 0{,}28854.\rho + 1{,}10384\,\rho^2 \qquad (A)$$ (182)

$$r''^2 = 1{,}01011 - 1{,}21471.\rho + 0{,}90852.\rho^2 \qquad (B)$$ (183)

$$\text{Sum} \quad r^2 + r''^2 = 2{,}02464 - 1{,}50325.\rho + 2{,}01236\,\rho^2 \qquad (D)$$ (184)

$$\text{Add the other terms of } c^2, \quad - 2{,}00596 + 1{,}39365.\rho - 1{,}51548.\rho^2$$ (185)

$$\text{Sum is} \quad c^2 = 0{,}01868 - 0{,}10960.\rho + 0{,}49688.\rho^2 \qquad (C)$$ (186)

Computation of ρ *from the equations* (A), (B), (C).

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
Assumed values of ρ.	Corresponding terms of the equations (A), (B), (C). r^2	r''^2	c^2	Values of r, r'', c.	T
Hypothesis I. $\rho = 1{,}0$ as in (141).	1,014 —,288 1,103	1,010 -1,214 0,908	0,018 —0,109 0,496	$r = 1{,}35$ $r'' = 0{,}84$ $r + r'' = 2{,}19$	27^{days}
	1,829	0,704	0,405	$c = 0{,}64$	
Hypothesis II. $\rho = \frac{1}{3} = 0{,}33333$	1,01453 —,09618 ,12265	1,01011 —,40490 ,10095	0,01868 — ,03653 ,05521	$r = 1{,}02029$ $r'' = 0{,}84033$ $r + r'' = 1{,}86062$	7,529 1 ,130
	1,04100	0,70616	0,03736	$c = 0{,}19329$	7,660
Hypothesis III. Add $\frac{1}{20}$ or 0,01667 makes $\rho = 0{,}35$	1,01453 —,10099 ,13522	1,01011 —,42514 0,11129	0,01868 — ,03836 ,06087	$r = 1{,}02469$ $r'' = 0{,}83443$ $r + r'' = 1{,}85852$	7,903 18 ,117
	1,04876	0,69626	0,04119	$c = 0{,}20295$	8,038
Hypothesis IV. Less $\frac{1}{200}$ or 0,00175 makes $\rho = 0{,}34825$	1,01453 —,10048 ,13387	1,01011 —,42302 ,11018	0,01868 — ,03816 ,06025	$r = 1{,}02368$ $r'' = 0{,}83503$ $r + r'' = 1{,}85871$	7,903 18 75
	1,04792	0,69727	0,04077	$c = 0{,}20191$	7,996
Hypothesis V. Add 0,0004.ρ or 0,00014 makes $\rho = 0{,}34839$	1,01453 —,10052 ,13398	1,01011 -0,42319 0,11027	0,01868 — ,03818 ,06031	$r = 1{,}02372$ $r'' = 0{,}83499$ $r + r'' = 1{,}85871$	7,903 18 79
	1,04799	0,69719	0,04081	$c = 0{,}20201$	8,000

Col. 7.	Col. 8.	Col. 9.	Col. 10.
Coefficients of ρ.			
	0,28854	1,21471	0,10960
$\frac{1}{3}$ $\frac{1}{20}$	0,096180 4809	0,404903 20245	0,036533 1826
$\frac{1}{200}$	0,100989 —504	0,425148 —2125	0,038359 —191
,0004	0,100485 40	0,423023 169	0,038168 15
	0,100525	0,423192	0,038183

(187)

Coefficients of ρ^2.			
	1,10384	0,90852	0,49688
$\frac{1}{9}$ $\frac{1}{10}$ $\frac{1}{40}$	0,122649 12265 306	0,100947 10095 252	0,055208 5521 138
$-\frac{1}{100}$ $+\frac{1}{400}$	0,135220 —1352 3	0,111294 —1113 3	0,060867 —609 1
,0008	0,133871 107	0,110184 88	0,060256 48
	0,133978	0,110272	0,0603[illegible]

(188)

(189)

(190)

(191)

With these last found values of $\rho = 0{,}34839$, $r = 1{,}02372$, $r'' = 0{,}83499$, we shall now compute the elements of the orbit, by means of the formulas (34—45); observing, that as $r > r''$, the comet must be nearer the perihelion at the third observation than at the first. (192)

Computation of the elements of the orbit.

$\rho = 0{,}34839$		log.	9,542066
M		log.	9,940795
$\rho'' = M.\rho$		log.	9,482861
r''	0,83499	log. co.	,078319
θ''		tang.	$9{,}643092_n$
$\varpi'' = -$	$9^d\ 12^m\ 37^s$	sin.	$9{,}204272_n$
ρ		log.	9,542066
r	1,02372	log. co.	9,989819
θ		tang.	$9{,}508175_n$
$\varpi = -$	$6^d\ 17^m\ 45^s$	sin.	$9{,}040060_n$

ϖ		sec.	0,002627
r		log. co.	9,989819
ρ		log.	9,542066
$\odot - \alpha$		sin.	9,995499
$\epsilon =$	$19^d\ 48^m\ 25^s$	sin.	9,530011

(193)

$$A = 342^d\ 42^m\ 05^s = \odot + 180^d$$

$$\beta = 2^d\ 30^m\ 30^s = A + \epsilon$$ (194)

$$\beta'' = 5^d\ 55^m\ 06^s \quad \text{found in (196)}$$

$$\beta'' - \beta = 3^d\ 24^m\ 36^s$$

$$w = 7^d\ 11^m\ 25^s \quad \text{found in (197)}$$

$$\beta'' - \mho = 10^d\ 36^m\ 01^s = \beta'' - \beta + w$$ (195)

$$\mho = 355^d\ 19^m\ 05^s = \beta - w$$

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ϖ''		sec.	0,005636
r''		log. co.	0,078319
ρ''		log.	9,482861
$☉'' - \alpha''$		sin.	9,858147
$\varepsilon'' =$	$15^d\ 25^m\ 46^s$	sin.	9,424963
$A'' =$	$350^d\ 29^m\ 20^s = ☉'' + 180^d$		
(196) $\beta'' =$	$5^d\ 55^m\ 06^s = A'' + \varepsilon''$		
$\beta'' - \beta =$	$3^d\ 24^m\ 36^s$	cosec.	1,225625
ϖ		cot.	$0,957317_n$
ϖ''		tang.	$9,209914_n$
	24,7090	log.	1,392856
$-\cot.(\beta''-\beta)$	$-$ 16,7824		
(197) $\cot. w =$	7,9266	log.	0,899087
$w =$	$7^d\ 11^m\ 25^s$	cosec.	0,902518
ϖ		tang.	$9,042683_n$
(197') Inclination $\varphi =$	$-41^d\ 23^m\ 41^s$	tang.	$9,945201_n$

$-v'' = 135^d\ 54^m\ 00^s$ } found in the 2^d column
$u'' = 14^d\ 00^m\ 29^s$ }

Perihel. $-$ ☊ $= 149^d\ 54^m\ 29^s$
Long. ☊ $= 355^d\ 19^m\ 05^s$ found in (195)
Long. Perih. $= 145^d\ 13^m\ 34^s$ on the orbit.

ϖ		cos.	9,997373
w		cos.	9,996571
$u =$	$9^d\ 32^m\ 46^s$	cos.	9,993944
ϖ''		cos.	9,994364
(195) β'' $-$ ☊		cos.	9,992525
$u'' =$	$14^d\ 00^m\ 29^s$	cos.	9,986889
$\chi = u'' - u =$	$4^d\ 27^m\ 43^s$		
$\frac{1}{2}\chi =$	$2^d\ 13^m\ 52^s$	cosec.	1,409711
$\frac{1}{2}\log. r$		arith. co.	9,994909
$\frac{1}{2}\log. r''$			9,960840
Number	23,1985	log.	1,365460
$-\cot.\frac{1}{2}\chi = -$	25,6674		
$\tang.\frac{1}{2}v'' = -$	2,4689		
$\frac{1}{2}v'' = -$	$67^d\ 57^m\ 00$	cos.	9,574512
same		cos.	9,574512
r''		log.	9,921681
Per. Dist. $D =$	0,11768	log.	9,070705
half			9,535352
$v'' = -\ 135^d\ 54^m\ 00^s$	tab. III	log.	2,789133
Time from Per.	$24^{days},8422$	log.	1,395190
Third obs. Sept.	$12^{days},5833$		

Time of Perihelion, October $7^{days},4255$.

The value of v'' being negative, indicates, that the comet was approaching towards the perihelion at the time of the third observation. The heliocentric latitudes,

(198) $$\varpi = -\ 6^d\ 17^m\ 45^s; \qquad \varpi'' = -\ 9^d\ 12^m\ 37^s,$$

(199) being south and increasing, it is evident, that the comet had passed the descending node ☋, a short time before the first observation; and we have therefore calculated the longitude of that node $355^d\ 19^m\ 05^s$; to which corresponds $\varphi = -\ 41^d\ 23^m\ 41^s$, which is the
(200) same as to put ☊ $= 175^d\ 19^m\ 05^s$, and $\varphi = 41^d\ 23^m\ 41^s$. Hence the approximate elements of the orbit are,

(201) Longitude of the ascending node $175^d\ 19^m\ 05^s$;

(202) Inclination $41^d\ 23^m\ 41^s$;

(203) Longitude of the Perihelion $145^d\ 13^m\ 34^s$;

(204) Perihelion distance 0,11768;

(205) Time of passing the Perihelion 1769, Oct. $7^{days},4255$.

(206) To illustrate the process of finding ρ, r, r'', from the fundamental equations (31,32,33), we shall give the following additional examples.

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EXAMPLE II.

The equations in this example, correspond to those of the comet of 1805, as given by Mr Ivory in the Transactions of the Royal Society for 1814, page 170.

$$r^2 = 0{,}973662 + 1{,}408969.\rho + 1{,}000000.\rho^2 ; \quad (A) \qquad (207)$$

$$r''^2 = 0{,}969967 + 0{,}230047.\rho + 0{,}131450.\rho^2 ; \quad (B) \qquad (208)$$

$$c^2 = 0{,}043505 + 0{,}115200.\rho + 0{,}518768.\rho^2 ; \quad (C) \qquad (209)$$

$$r^2 + r''^2 = 1{,}943629 + 1{,}639016.\rho + 1{,}131450.\rho^2. \quad (D) \qquad (210)$$

Interval between the extreme observations $T = 11^{\text{days}},9734$. (211)

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
ρ	r^2	r''^2	c^2	$r,\ r'',\ c$	T
Hypothesis I. $\rho = \frac{1}{6}$ or 0,16667 as in (146).	0,97366 ,23483 2778	0,96997 3834 365	0,043505 19200 14410	$r = 1{,}11188$ $r'' = 1{,}00596$ $r + r'' = 2{,}11784$	11,392 21 ,324
	1,23627	1,01196	0,077115	$c = 0{,}27769$	11,737
Hypothesis II. Add $\frac{1}{20}$ or 0,00833 $\rho = 0{,}175$	0,97366 ,24657 3062	0,96997 4026 402	0,043505 20160 15887	$r = 1{,}11841$ $r'' = 1{,}00710$ $r + r'' = 2{,}12551$	11,841 15 87
	1,25085	1,01425	0,079552	$c = 0{,}28205$	11,943
Hypothesis III. Add $\frac{1}{140} = 0{,}00125$ $\rho = 0{,}17625$	0,97366 ,24833 3106	0,96997 4055 408	0,043505 20304 16115	$r = 1{,}11939$ $r'' = 1{,}00727$ $r + r'' = 2{,}12666$	11,841 19 ,114
	1,25305	1,01460	0,079924	$c = 0{,}28271$	11,974

Col. 7.	Col. 8.	Col. 9.	Col. 10.
	Coefficients of ρ.		
	1,408969	0,230047	0,115200
$\frac{1}{6}$	0,234828	0,038341	0,019200
$\frac{1}{20}$	11741	1917	960
	0,246569	0,040258	0,020160
$\frac{1}{140}$	1761	288	144
	0,248330	0,040546	0,020304

(212)

(213)

	Coefficients of ρ^2.		
	1,000000	0,131450	0,518768
$\frac{1}{6}$	0,166667	0,021908	0,086461
$\frac{1}{36}$	0,027778	0,003651	0,014410
$\frac{1}{10}$	2778	365	1441
$\frac{1}{40}$	69	9	36
	0,030625	0,004025	0,015887
$\frac{1}{70}$	437	57	227
$\frac{1}{280}$	2		1
	0,031064	0,004082	0,016115

(214)

Hence $\rho = 0{,}17625$, $r = 1{,}11939$, $r'' = 1{,}00727$.

Mr Ivory makes, $\rho = 0{,}17620$, $r = 1{,}11936$, $r'' = 1{,}00727$. (215)

EXAMPLE III.

These equations are similar to those given by Mr Ivory in the Transactions of the Royal Society for 1814, page 160; and correspond to the comet of 1781.

$$r^2 = 0{,}976625 - 0{,}303724.\rho + 1{,}000000.\rho^2 ; \quad (A) \qquad (216)$$

$$r''^2 = 0{,}972873 - 1{,}457243.\rho + 3{,}788166.\rho^2 ; \quad (B) \qquad (217)$$

$$c^2 = 0{,}030278 - 0{,}353719.\rho + 1{,}237818.\rho^2 ; \quad (C) \qquad (218)$$

$$r^2 + r''^2 = 1{,}949498 - 1{,}760967.\rho + 4{,}788166.\rho^2. \quad (D) \qquad (219)$$

Interval between the extreme observations $T = 10^{\text{days}}$. (220)

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	Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
	ρ	r^2	r''^2	c^2	$r,\ r'',\ c$	T
(221)	Hypothesis I. $\rho = 0,33333$ as in (147).	0,97663 —,10124 ,11111 0,98650	0,97287 —,48575 ,42091 0,90803	0,030278 —,117906 ,137535 0,049907	$r = 0,99323$ $r'' = 0,95291$ $r + r'' = 1,94614$ $c = 0,2234$	8,902 14 137 9,053
(222)	Hypothesis II. Add $\frac{1}{20}$ makes $\rho = 0,35$	0,97663 —,10630 ,12250 0,99283	0,97287 —,51003 ,46405 0,92689	0,030278 —,123801 ,151633 0,058110	$r = 0,99640$ $r'' = 0,96275$ $r + r'' = 1,95915$ $c = 0.24106$	9,735 23 43 9,801
(223)	Hypothesis III. Add $\frac{1}{100}$ makes $\rho = 0,3535$	0,97663 —,10737 ,12496 0,99422	0,97287 —,51513 ,47338 0,93112	0,030278 —,125039 ,154681 0,059920	$r = 0,99711$ $r'' = 0,96494$ $r + r'' = 1,96205$ $c = 0,24479$	9,760 5 195 9,960
(224)	Hypothesis IV. Add $\frac{1}{400}$ or 0,00088 $\rho = 0,35438$	0,97663 —,10763 ,12559 0,99459	0,97287 —,51642 ,47575 0,93220	0,030278 —,125352 ,155455 0.060381	$r = 0,99730$ $r'' = 0,96550$ $r + r'' = 1,96280$ $c = 0,24573$	9,760 — 233 10,000

Col. 7.	Col. 8.	Col. 9.	Col. 10.
Coefficients of ρ.			
	0,303724	1,457243	0,353719
$\frac{1}{3}$ $\frac{1}{20}$	0,101241 5062	0.485748 24287	0,117906 5895
$\frac{1}{100}$	0,106303 1063	0,510035 5100	0,123801 1238
$\frac{1}{400}$	0,107366 268	0,51513 1288	0,125039 313
	0,107634	0,51642	0,125352
Coefficients of ρ^2.			
	1,000000	3,788166	1,237818
$\frac{1}{9}$ $\frac{1}{10}$ $\frac{1}{40}$	0.111111 11111 278	0,420907 42091 1052	0,137535 13754 344
$\frac{1}{50}$ $\frac{1}{200}$	0,122500 2450 12	0,464050 9281 46	0,151633 3033 15
$\frac{1}{200}$ $\frac{1}{800}$	0,124962 625 1	0,473377 2367 3	0,154681 773 1
	0,125588	0.475747	0.155455

(225) Hence $\rho = 0,35438$, $r = 0,99730$, $r'' = 0,96550$; which agree with Mr Ivory's calculation, excepting a unit in the last decimal place.

EXAMPLE IV.

These equations are equivalent to those given by Mr Ivory, in the Transactions of the Royal Society for 1814, page 165; and refer to the comet of 1769.

(226) $$r^2 = 1,012347 - 0,778609.\rho + 1,000000.\rho^2; \qquad (A)$$

(227) $$r''^2 = 1,010107 - 1,297813.\rho + 1,033677.\rho^2; \qquad (B)$$

(228) $$c^2 = 0,004678 - 0,027518.\rho + 0,139619.\rho^2; \qquad (C)$$

(229) $$r^2 + r''^2 = 2,022454 - 2,076422.\rho + 2,033677.\rho^2. \qquad (D)$$

(230) Interval between the extreme observations $T = 4^{\text{days}}$.

	ρ	r^2	r''^2	c^2	$r,\ r'',\ c$	T
(231)	Hypothesis I. $\rho = \frac{1}{4} = 0,25$ as in (148).	1,012 —,194 62 0,880	1,010 —,324 64 0,750	0,0046 —,0068 87 0,0065	$r = 0,94$ $r'' = 0,86$ $r + r'' = 1,80$ $c = 0,080$	3,119
(232)	Hypothesis II. $\rho = \frac{1}{3} = 0,33333$	1,01235 —,25954 ,11111 0,86392	1,01011 —,43260 ,11485 0,69236	0,004678 — 9173 15513 0,011018	$r = 0,92947$ $r'' = 0,83209$ $r + r'' = 1,76156$ $c = 0,10497$	3,856 2 ,191 4,049
(233)	Hypothesis III. Sub. $\frac{1}{80}$ or 0,00417 $\rho = 0,32916$	1,01235 —,25629 ,10835 0,86441	1,01011 —,42720 ,11200 0,69491	0,004678 — 9058 15127 0,010747	$r = 0,92974$ $r'' = 0,83362$ $r + r'' = 1,76336$ $c = 0,10367$	3,856 3 ,141 4,000

Coefficients of ρ.			
	0,778609	1,297813	0,027518
$\frac{1}{4}$	0,194	0,324	0,0068
$\frac{1}{3}$ $\frac{1}{80}$	0,259536 — 3244	0,432604 — 5408	0,009173 — 115
	0,256292	0,427196	0,009058
Coefficients of ρ^2.			
	1,000000	1,033677	0,139619
$\frac{1}{16}$	0,062	0,064	0,0087
$\frac{1}{9}$ $\frac{1}{40}$ $\frac{1}{165}$	0,111111 — 2778 17	0,114853 — 2871 18	0,015513 — 388 2
	0,108350	0,112000	0,015127

(234) Hence $\rho = 0,32916$, $r = 0,92974$, $r'' = 0,83362$.

The true values being $\rho = 0,32911$, $r = 0,92974$, $r'' = 0,83361$.

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EXAMPLE V.

The following equations were computed by Dr Olbers in his *Abhandlung*, &c., page 59; they correspond to the comet of 1681, and were computed from Halley's elements, and not deduced from actual observations.

$$r^2 = 0{,}96754 - 0{,}59292\,\rho + 1{,}24328.\rho^2 \qquad (A)$$

$$r''^2 = 0{,}96941 - 0{,}40185.\rho + 2{,}20087\,\rho^2 \qquad (B)$$

$$c^2 = 0{,}019726 - 0{,}122756.\rho + 0{,}265982.\rho^2 \qquad (C) \qquad (235)$$

$$r^2 + r''^2 = 1{,}93695 - 0{,}99477.\rho + 3{,}44415.\rho^2 \qquad (D) \qquad (236)$$

Interval between the extreme observations $T = 8^{\text{days}}{,}047$.

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6
ρ	r^2	r''^2	c^2	$r,\ r'',\ c$	T
Hypothesis I.	0,96754	0,96941	,019726	r = 0,99091	6,362
$\rho = 0{,}5$	−,29646	−,20092	−,061378	r'' = 1,14835	1,4
as in (149).	,31082	,55022	,066495	$r + r''$ = 2,13926	,323
	0,98190	1,31871	,024843	c = 0,15762	6,690
Hypothesis II.	0,96754	0,96941	,019726	r = 1,00872	7,740
Add $\frac{1}{10}$ or ,05	−,32611	−,22102	−,067516	r'' = 1,18918	1,
$\rho = 0{,}55$	,37609	,66576	,080459	$r + r''$ = 2,19790	32
	1,01752	1,41415	,032669	c = 0,18074	7,786
Hypothesis III.	0,96754	0,96941	,019726	r = 1,01302	7,776
Add $\frac{1}{50}$ or 0,011	−,33263	−,22544	−,068866	r'' = 1,19859	2
$\rho = 0{,}561$	,39129	,69266	,083709	$r + r''$ = 2,21161	,256
	1,02620	1,43663	,034569	c = 0,18593	8,035
Hypothesis IV.	0,96754	0,96941	,019726	r = 1,01324	7,776
Add $\frac{1}{1000}$=0,000561	−,33296	−,22566	−,068935	r'' = 1,19908	4
$\rho = 0{,}561561$	,39207	,69404	,083876	$r + r''$ = 2,21232	,267
	1,02665	1,43779	,034667	c = 0,18619	8,047

Col. 7.	Col. 8.	Col. 9.	Col. 10.
Coefficients of ρ.			
	0,59292	0,40185	0,122756
$\frac{1}{2}$	0,296460	0,200925	0,061378
$\frac{1}{10}$	29646	20092	6138
	0,326106	0,221017	0,067516
$\frac{1}{50}$	6522	4420	1350
	0,332628	0,225437	0,068866
$\frac{1}{1000}$	332	225	69
	0,332960	0,225662	0,068935	(237)

Coefficients of ρ^2.			
	1,24328	2,20087	0,265982
$\frac{1}{4}$	0,310820	0,550217	0,066495
$\frac{2}{10}$	62164	,110043	13299
$\frac{1}{20}$	3108	5502	665
	0,376092	0,665762	0,080459
$\frac{2}{50}$	15044	26631	3218
$\frac{1}{100}$	150	266	32
	0,391286	0,692659	0,083709
$\frac{2}{1000}$	782	1385	167
	0,392068	0,694044	0,083876

Hence $\rho = 0{,}561561$, $r = 1{,}01324$, $r'' = 1{,}19908$. The actual valnes, according to Halley's theory, upon which the proposed equations are founded, are $r = 1{,}0144$, $r'' = 1{,}2000$; which agree, very nearly, with the preceding result. (238)

EXAMPLE VI.

These equations correspond to the comet of 1805, in the calculation of Mr. Ivory in the Transactions of the Royal Society for 1814, page 175.

$$r^2 = 0{,}988192 - 1{,}271721.\rho + 1{,}000000.\rho^2\,;$$

$$r''^2 = 0{,}981987 - 2{,}311644.\rho + 1{,}881447\,\rho^2\,;$$

$$c^2 = 0{,}043371 - 0{,}074489.\rho + 0{,}485838\,\rho^2\,; \qquad (239)$$

$$r^2 + r''^2 = 1{,}970179 - 3{,}583365.\rho + 2{,}881447.\rho^2. \qquad (240)$$

Interval between the extreme observations $T = 12^{\text{days}}{,}036$.

[5994]

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
ρ	r^2	r''^2	c^2	$r,\ r'',\ c$	T
Hypothesis I. $\rho = 1$	0,98 −1,27 +1,00	0,98 −2,31 +1,88	0,043 −,074 +,485	$r = 0,84$ $r'' = 0,74$ $r + r'' = 1,58$	24days
as in (241).	0,71	0,55	0,454	$c = 0,67$	
(241) Hypothesis II.	0,98819 −,63580 ,25000	0,98199 −1,15582 ,47030	0,043371 − 37244 ,121459	$r = 0,77610$ $r'' = 0,54455$ $r + r'' = 1,32065$	11,65 3 ,238
$\rho = \frac{1}{2}$	0,60233	0,29653	0,127586	$c = 0,35719$	11,891
Hypothesis III. Add $\frac{1}{100}$ or 0,05	0,98819 −,64222 ,25502	0,98199 −1,10738 ,47982	0,043371 − 37616 ,123900	$r = 0,77523$ $r'' = 0,54262$ $r + r'' = 1,31785$	11,938 36 3
$\rho = 0,505$	0,60099	0,29443	0,129655	$c = 0,36008$	11,977
Hypothesis IV. Add $\frac{1}{150}$ or, 0 033	0,98819 −,64656 ,25844	0,98199 −1,17516 ,48623	0,043371 − 37867 ,125557	$r = 0,77468$ $r'' = 0,54135$ $r + r'' = 1,31603$	11,938 28 07
$\rho = 0,5083$	0,60013	0,29306	0,131061	$c = 0,36202$	12,03
Hypothesis V. Add $\frac{1}{3000} = 0,0017$	0,98819 −,64672 ,25861	0,98199 −1,17555 ,48656	0,043371 − 37880 ,125641	$r = 0,77465$ $r'' = 0,54129$ $r + r'' = 1,31594$	11,938 28 70
$\rho = 0,50847$	0,60008	0,29300	0,131132	$c = 0,36212$	12,036

Col. 7	Col. 8.	Col. 9.	Col. 10.
	Coefficients of ρ.		
	1,271721	2,311644	0,074489
$\frac{1}{2}$ $\frac{1}{100}$	0,635860 6359	1,155822 11558	0,037244 372
$\frac{1}{150}$	0,642219 4281	1,167380 7783	0,037616 251
$\frac{1}{7000}$	0,646500 215	1,175163 392	0,037867 13
	0,646715	1,175555	0,037880

	Coefficients of ρ^2		
	1,000000	1,881447	0,485838
$\frac{1}{4}$ $\frac{1}{50}$ $\frac{1}{200}$	0,250000 5000 25	0,470362 9407 47	0,121459 2429 12
$\frac{2}{150}$ $\frac{1}{300}$	0,255025 3400 11	0,479816 6398 21	0,123900 1652 5
$\frac{1}{1500}$	0,258436 172	0,486235 324	0,125557 84
	0,258608	0,486559	0,125641

(242) Hence $\rho = 0,50847$, $r = 0,77465$, $r'' = 0,54129$.
Mr Ivory makes $\rho = 0,5081$, $r = 0,77472$, $r'' = 0,54144$.

From these examples we see that the interval of time T, between the extreme observations, is found in Table II, with a sufficient degree of accuracy, and that the results agree with the calculations by logarithms of other astronomers, although the table is only carried to the nearest unit in the third decimal place. While treating upon this subject, it may not be amiss to recall to mind the remarks of La Lande, in the third volume, page 259, of the third edition of his astronomy, relative to the degree of accuracy in the cometary calculations. He has there given a table of the elements of the orbits of those comets which had been previously computed, giving the longitudes and angles to seconds, and the logarithms of the perihelion distances to five or six decimals; but at the same time observing, that though he has inserted the seconds, no confidence could be placed in them; neither could we depend on the correctness of the logarithms of the

(242)

Col. 1.	Col. 2.	Col. 3.	Col. 4.
Examples	Time T by Observation.	Time T by Table II.	Errors.
I.	8days,000	8days,020	+ 0days,020
II.	11days,973	11days,981	+ 0days,008
III.	10days,000	9days,998	− 0days,002
IV.	4days,000	3days,999	− 0days,001
V.	8days,047	8days,060	+ 0days,013
VI.	12days,036	12days,129	+ 0days,093

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perihelion distances in the fourth decimal place, as is abundantly manifest, by comparing the results of the calculations of different astronomers. To estimate the degree of accuracy with which the time T can be ascertained, by entering Table II, with the values of $r^2+r''^2$ at the bottom of the table and c^2 at the right hand side; we have computed the value of T, for the six preceding examples; as in the third column of the annexed table; the times by observation being given in the second column, and their differences or errors respectively, in the fourth column. These errors being very small, it is evident, that the method of combining the equations (C), (D), or the values of $r^2+r''^2$, c^2; by means of Table II, must generally give a very close approximation to the value of ρ. (242)

Gauss varied the forms of the equations (31, 32, 33), by the introduction of several auxiliary numbers $A, B, B'\ b, b'', c, c''$, &c. which are deduced from the co-efficients of the terms in the original equations; changing also the unknown quantity ρ into u; so as to reduce the expression of c^2 (33), to the form in (244). The object of these transformations is to render the calculations more convenient for computation by logarithms, by putting them under the following forms; (243)

(244)

$$r^2=\left(\frac{u+c}{b}\right)^2+B^2;\qquad r''^2=\left(\frac{u+c''}{b''}\right)^2+B''^2;\qquad c^2=u^2+A^2.$$

When the equations are given in this form, we may determine u, by means of Tables I, II, or by successive approximations, in the same manner as we have found ρ in the preceding examples; using in Table II the arguments, $r+r''$ at the top, with c at the side; and it is evident, on account of the decrease of the number of terms in the expression of c^2 (244), that the calculation of u is more simple than that of finding ρ in the former examples; but the saving of labor is nowise sufficient for the trouble of reducing the equations to the forms (244), when the time is deduced from Table II, in the manner we have here pointed out. We may also use the equations (C), (D), or the values of $r^2+r''^2$ and c^2, in finding the first rough estimate of u; in like manner as we have proceeded with the similar expressions in terms of ρ in (136—150). This process may be illustrated, by the two following examples. Thus if we put $u=0$, in (248, 247), we shall have $r^2+r''^2=2{,}49$, $c^2=0{,}028$, whence we obtain, by inspection in Table II, $T=7^{\text{days}},3$ nearly; which is less than the time by observation $14^{\text{days}},0493$. We also observe by inspecting the same vertical column, corresponding to $r^2+r''^2=2{,}49$; that this last mentioned time corresponds very nearly in the margin to $c^2=0{,}11$; substituting this in (247) we get $0{,}11=0{,}028+u^2$, whence we obtain $u=0.28$, or nearly $u=\frac{1}{4}$, which is assumed in (249). In like manner, in Example VIII, we have, by putting $u=0$, in (254, 253) $r^2+r''^2=12{,}53$, $c^2=0{,}051$; which correspond in Table II, to $14^{\text{days}},8$. If we suppose $u=\frac{1}{10}$, we get $r^2+r''^2=23{,}2$, $c^2=0{,}062$; corresponding in Table II to $18^{\text{days}},9$. As the actual time by observation falls nearly midway between these two times, we may assume, for an approximate value, $u=\frac{1}{20}$, as in (255). (244) (245) (245) (246)

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EXAMPLE VII.

The following equations correspond to the second comet of 1813. They are equivalent to those given by Gauss in vol. 28, page 509, of the Monatliche Correspondenz; or by Encke, in the Jahrbuch, for 1833, page 284.

$$r^2 = 1{,}24415 + 1{,}92565.u + 3{,}06973.u^2; \qquad (A)$$

$$r''^2 = 1{,}24837 + 1{,}51429.u + 0{,}79331.u^2; \qquad (B)$$

(247) $$c^2 = 0{,}028219 + u^2; \qquad (C)$$

(248) $$r^2 + r''^2 = 2{,}49252 + 3{,}43994.u + 3{,}86304.u^2; \qquad (D)$$

Interval between the extreme observations $T = 14^{\text{days}},0493$.

(249)

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
u	r^2	r''^2	c^2	$r,\ r'',\ c$	T
Hypothesis I. $u = \frac{1}{4} = 0{,}25$ as in (245).	1,24415 ,48141 ,19186	1,24837 ,37857 ,04958	0,02822 ,06250	$r = 1{,}38471$ $r'' = 1{,}29480$ $r + r'' = 2{,}67951$	14,241 26 57
	1,91742	1,67652	0,09072	$c = 0{,}30120$	14,324
Hypothesis II. Sub. $\frac{1}{50}$ makes $u = 0{,}245$	1,24415 ,47178 ,18426	1,24837 ,37100 ,04762	0,02822 ,06002	$r = 1{,}37847$ $r'' = 1{,}29112$ $r + r'' = 2{,}66959$	13,741 25 ,333
	1,90019	1,66699	0,08824	$c = 0{,}29705$	14,099
Hypothesis III. Sub. $\frac{1}{200}$ makes $u = 0{,}24375$	1,24415 ,46943 ,18242	1,24837 ,36915 ,04714	0,02822 0,05943	$r = 1{,}37695$ $r'' = 1{,}29022$ $r + r'' = 2{,}66717$	13,741 19 ,287
	1,89600	1,66466	0,08765	$c = 0{,}29606$	14,047
Hypothesis IV. Add $\frac{1}{4000}$ makes $u = 0{,}243836$	1,24415 ,46954 ,18251	1,24837 ,36924 ,04717	0,028219 0,059456	$r = 1{,}37702$ $r'' = 1{,}29026$ $r + r'' = 2{,}66728$	13,741 19 ,289
	1,89620	1,66478	0,087675	$c = 0{,}29610$	14,049

Col. 7.	Col. 8.	Col. 9.	Col. 10.
Coefficients of u.			
		1,92565	1,51429
$\frac{1}{4}$		0,481412	0,378572
$\frac{1}{50}$		−9628	−7571
		0,471784	0,371001
$\frac{1}{200}$		−2358	−1855
		0,469426	0,369146
$\frac{1}{4000}$		117	92
		0,469543	0,369238

Coefficients of u^2.			
	3,06973	0,79331	1,00000
$\frac{1}{4}$	,767432	0,198328	,250000
$\frac{1}{16}$	,191858	0,049582	0,062500
$\frac{2}{50}$	−7674	−1983	−2500
$\frac{1}{100}$	77	20	25
	0,184261	0,047619	0,060025
$\frac{2}{200}$	−1843	−476	−600
$\frac{1}{400}$	5	1	1
	0,182423	0,047144	0,059426
$\frac{2}{4000}$	91	24	30
	0,182514	0,047168	0,059456

Hence we have	$u = 0{,}243836$,	$r = 1{,}37702$,	$r'' = 1{,}29026$;
According to Gauss,	$u = 0{,}24388$,	$r = 1{,}37708$,	$r'' = 1{,}29027$;
(250) According to Encke,		$r = 1{,}37705$,	$r'' = 1{,}29027$.

We may observe, that the last, or fourth hypothesis, may be dispensed with, by interpolating between the values of ρ, r, r'', given in the second and third hypothesis, so as to make T correspond to the proposed interval $14^{\text{days}},0493$.

EXAMPLE VIII.

The following equations correspond to the comet of 1825, calculated by Nicolai in the tenth volume of the *Astronomische Nachrichten*, page 238.

(251) $$r^2 = 6{,}20536 + 43{,}23445\,u + 80{,}07556.u^2; \qquad (A)$$

(252) $$r''^2 = 6{,}33213 + 46{,}41411.u + 93{,}50610.u^2; \qquad (B)$$

(253) $$c^2 = 0{,}05158 + u^2; \qquad (C)$$

(254) $$r^2 + r''^2 = 12{,}53749 + 89{,}64856.u + 173{,}58166.u^2 \qquad (D)$$

Interval between the extreme observations $T = 16^{\text{days}},7821$.

[5994]

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
u	r^2	r''^2	c^2	$r,\ r'',\ c$	T
Hypothesis I.	6,20536	6,33213	0,05158	$r = 2{,}927$	16,237
	2,16172	2,32071		$r'' = 2{,}981$	11
$u = \frac{1}{20} = 0{,}05$	,20019	,23376	,00250	$r + r'' = 5{,}908$	,180
as in (246).	8,56727	8,88660	0,05408	$c = 0{,}23255$	16,428
Hypothesis II.	6,20536	6,33213	0,05158	$r = 3{,}0000$	16,374
	2,52201	2,70749		$r'' = 3{,}0590$	82
Add $\frac{1}{6}$ makes	,27248	,31817	,00340	$r + r'' = 6{,}0590$	,319
$u = 0{,}058333$	8,99985	9,35779	0,05498	$c = 0{,}23448$	16,775
Hypothesis III.	6,20536	6,33213	0,05158	$r = 3{,}00253$	16,374
Add $\frac{1}{200}$ or	2,53462	2,72103		$r'' = 3{,}06179$	89
0,0002916 makes	,27520	,32135	343	$r + r'' = 6{,}06432$	,323
$u = 0{,}058625$	9,01518	9,37451	0,05501	$c = 0{,}23454$	16,786

Col. 7.	Col. 8.	Col. 9.	Col. 10.
	Coefficients of u.		
	43,23445	46,41411	
$\frac{1}{20}$	2,16172	2,32071	
$\frac{1}{6}$	,36029	38678	
	2,52201	2,70749	
$\frac{1}{200}$	1261	1354	
	2,53462	2,72103	

(255)

	Coefficients of u^2.		
	80,07556	93,50010	1,00000
$\frac{1}{400}$	0,20019	0,23376	0,0025[illegible]
$\frac{1}{3}$	6673	7792	8[illegible]
$\frac{1}{12}$	556	649	[illegible]
	0,27248	0,31817	0,0034[illegible]
$\frac{2}{200}$	272	318	[illegible]
	0,27520	0,32135	0,003[illegible]

The value of T by observation, falls between the results of these two last hypotheses, and by taking parts of the corresponding variations of the values of ρ, r, r'', we get the final values corresponding to the actual value of T;

$$\rho = 0{,}05852\,; \qquad r = 3{,}00163\,; \qquad r'' = 3{,}06080.$$ (256)

Analytical investigation of Olbers's method.

This manner of finding the orbit of a comet has an imperfection, which obtains in several other methods; namely, that it fails in accuracy in the particular case where the value of M (30, or 92) appears under the form $M = \frac{0}{0}$; which happens when the *apparent path of the comet is in the ecliptic, or in any other great circle passing through the sun.* For in this (257) case, as the points A, B, C, figure 85, page 795, are situated in the same great circle, passing (258) through S, we shall have all three of the angles b, b', b'' (64), equal to each other, and then the expression (92) becomes $M = \frac{0}{0}$. Hence it is evident that this method can be (259) most successfully applied, in cases where the arc $B\,H$, is considerable, in comparison with the arc $A\,C$. When the ratio of these arcs, $B\,H$, $A\,C$, is small, there may be instances (260) in which the method, without actually failing, becomes somewhat uncertain, on account of the inaccuracy in the estimated value of M, in consequence of the neglected terms (93′), which have a more important influence than usual, and it is an object of interest, to obtain (261) a more correct estimate of the value of M. We shall therefore proceed to investigate the complete value, by the analytical methods, used by Gauss, Ivory, Encke, &c., without (262) neglecting any terms and we shall obtain in (306, &c.), the correction to be made to the approximate value, which is given in (30). Finally we shall give, in (355, &c.), the process to be used in the excepted case mentioned in (257).

[5994] Using the same notation as in (100—101), we have, identically,

(263) $$0 = (x'y'' - x''y').x + (x''y - xy'').x' + (xy' - x'y).x''.$$

(264) For the *first* term is balanced by the *fourth*, the *second*, by the *fifth*, and the *third* by the *sixth*; so that the second member is identically equal to nothing. We shall now represent the double of the area of any one of the plane triangles sab, sbc, sac, figure 84, page 792,
(265) by including the corresponding radii in brackets; so that we shall have,

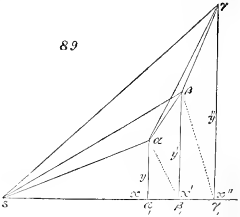
(266) $$[rr'] = 2.\text{ area of the triangle } sab\,; \qquad [r'r''] = 2.\text{ area of the triangle } sbc\,;$$
$$[rr''] = 2.\text{ area of the triangle } sac.$$

(266′) The plane of the comet's orbit being inclined to the ecliptic by the angle φ (21); it is evident, by the principles of the orthographic projection, that the double of the projections
(267) of the areas of these triangles, upon the plane of the ecliptic, will be obtained by multiplying the expressions (266) by $\cos.\varphi$, so that we shall have,

(268) $$[rr'].\cos.\varphi = 2.\text{projection of } sab\,; \qquad [r'r''].\cos.\varphi = 2.\text{projection of } sbc\,;$$
$$[rr''].\cos.\varphi = 2.\text{projection of } sac.$$

We shall represent the co-ordinates, of the projection of the point a, by x, y; those of the
(269) point b, by x', y'; and those of the point c, by x'' y'' (100, &c.); as in figure 89, where α, β, γ, represent respectively the projection of the points a, b, c, of figure 84, upon the plane of the ecliptic. Now we evidently have,

(270) $$\text{area } s\alpha\beta_1 = \tfrac{1}{2}s\beta_1 \times \alpha\alpha_1 = \tfrac{1}{2}x'y\,; \qquad \text{area } \beta\alpha\beta_1 = \tfrac{1}{2}\beta\beta_1 \times \alpha_1\beta_1 = \tfrac{1}{2}y'.(x'-x)\,;$$
$$\text{area } s\beta\beta_1 = \tfrac{1}{2}s\beta_1 \times \beta\beta_1 = \tfrac{1}{2}x'y'\,;$$



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subtracting the sum of the two first expressions from the third, we evidently get the value of the triangle,

(271) $$s\alpha\beta = \tfrac{1}{2}x'y' - \tfrac{1}{2}x'y - \tfrac{1}{2}y'.(x'-x)\,;$$

and if we neglect the terms $\frac{1}{2}x'y' - \frac{1}{2}x'y'$, which mutually destroy each other, it becomes as in the first of the expressions (273). If we change the accents
(272) on $x\,y$, so as to correspond to the other triangles $s\beta\gamma$, $s\alpha\gamma$, we shall obtain their values, as in (273).

(273) $$\text{Triangle} \quad s\alpha\beta = \tfrac{1}{2}.(xy' - x'y)\,; \qquad \text{triangle} \quad s\beta\gamma = \tfrac{1}{2}.(x'y'' - x''y')\,;$$
$$\text{triangle} \quad s\alpha\gamma = \tfrac{1}{2}.(xy'' - x''y).$$

Substituting these in (268), we get the following system of equations depending on the

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principle that the three observed places of the comet a, b, c, figure 84, are in the same (273')
plane passing through the sun; this plane being inclined to the ecliptic by the angle φ;

$$xy'-x'y=[rr'].\cos.\varphi\,; \qquad (x'y''-x''y')=[r'r''].\cos.\varphi\,; \qquad (xy''-x''y)=[rr''].\cos.\varphi. \qquad (274)$$

Introducing these values into the equation (263), and then dividing by $\cos.\varphi$, we get
the equation (277). This equation must be satisfied, whatever be the position of the axis of
x; and if we change this axis into that of y, we shall find that the values x, x', x'', will (275)
become y, y', y'', respectively, without altering $[r'r'']$, $[rr'']$, $[rr']$; hence we get (278).
In like manner, by changing the axis of x into that of z, we get (279). (276)

$$0=[r'r''].x-[rr''].x'+[rr'].x''\,; \qquad (277)$$

$$0=[r'r''].y-[rr''].y'+[rr'].y''\,; \qquad (278)$$

$$0=[r'r''].z-[rr''].z'+[rr'].z''\,; \qquad (279)$$

We may remark, that *the whole number of accents on each of the terms of these equations, is* (279')
three; and this symmetry obtains in many other of the equations of this article. The
recollection of this circumstance will sometimes assist in distinguishing the symbols from each
other. If we substitute $A=180^d+\odot$, (10) in (108), we shall obtain, for the co-ordinates (280)
x, y, z, at the first observation, the expressions (281), and, by accenting the letters, we get
the values corresponding to the other observations as in (282, 283);

$$x=\rho.\cos.\alpha-R.\cos.\odot\,; \qquad y=\rho.\sin.\alpha-R.\sin.\odot\,; \qquad z=\rho.\mathrm{tang}.\theta\,; \qquad (281)$$

$$x'=\rho'.\cos.\alpha'-R'.\cos.\odot'\,; \qquad y'=\rho'.\sin.\alpha'-R'.\sin.\odot'\,: \qquad z'=\rho'.\mathrm{tang}.\theta'\,; \qquad (282)$$

$$x''=\rho''.\cos.\alpha''-R''.\cos.\odot''\,; \qquad y''=\rho''.\sin.\alpha''-R''.\sin.\odot''\,; \qquad z''=\rho''.\mathrm{tang}.\theta''. \qquad (283)$$

Substituting these in (277—279), we obtain,

$$\begin{aligned}0=[r'r''].\{\rho.\cos.\alpha-R.\cos.\odot\}-[rr''].\{\rho'.\cos.\alpha'-R'.\cos.\odot'\}&\\ +[rr'].\{\rho''.\cos.\alpha''-R''.\cos.\odot''\}\,;&\end{aligned} \qquad (284)$$

$$\begin{aligned}0=[r'r''].\{\rho.\sin.\alpha-R.\sin.\odot\}-[rr''].\{\rho'.\sin.\alpha'-R'.\sin.\odot'\}&\\ +[rr'].\{\rho''.\sin.\alpha''-R''.\sin.\odot''\}\,;&\end{aligned} \qquad (285)$$

$$0=[r'r''].\rho.\mathrm{tang}.\theta-[rr''].\rho'.\mathrm{tang}.\theta'+[rr'].\rho''.\mathrm{tang}.\theta''. \qquad (286)$$

If we divide (284, 285, 286), by any one of the areas $[r'r'']$, $[rr'']$, $[rr']$, we shall find,
that these three equations contain *five* unknown quantities; namely, the *two* ratios of the
areas, and the *three* radii ρ, ρ', ρ''; any two of which, may be eliminated. In doing this,
we may observe, that the equations (284, 285), are wholly independent of each other; and
we may, in either of them, change at pleasure the direction of the axis of x. If we decrease (287)
the angles in (284), by the quantity $\odot'$, we shall get (292); if we decrease the angles in
(285) by α', and then change the signs of all the terms, we shall get (293); lastly, if we (288)
decrease the angles in (285), by $\odot'$, we shall get (294). The same results may also be
obtained by combining the equations (284, 285) by the usual methods; thus, if we multiply (289)

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(290) (284) by $\cos.\odot'$, and (285) by $\sin.\odot'$, then take the sum of the products, reducing them by [24], Int. we shall get (292). Again, multiplying (284), by $\sin.\alpha'$, also (285) by $-\cos.\alpha'$, then adding the products, we get (293) by reduction, and using [22], Int. Lastly,
(291) multiplying (285), by $\cos.\odot'$, and (284), by $-\sin.\odot'$, then adding the products, we get (294). The equation (295) is the same as (286).

(292)
$$0=[r'r''].\{\rho.\cos.(\alpha-\odot')-R.\cos.(\odot-\odot')\}-[rr''].\{\rho'.\cos.(\alpha'-\odot')-R'\}$$
$$+[rr'].\{\rho''.\cos.(\alpha''-\odot')-R''.\cos.(\odot''-\odot')\};$$

(293)
$$0=[r'r''].\{\rho.\sin.(\alpha'-\alpha)+R.\sin.(\odot-\alpha')\}-[rr''].R'.\sin.(\odot'-\alpha')$$
$$-[rr'].\{\rho''.\sin.(\alpha''-\alpha')-R''.\sin.(\odot''-\alpha')\};$$

(294)
$$0=[r'r''].\{\rho.\sin.(\alpha-\odot')+R.\sin.(\odot'-\odot)\}-[rr''].\rho'.\sin.(\alpha'-\odot')$$
$$+[rr'].\{\rho''.\sin.(\alpha''-\odot')-R''.\sin.(\odot''-\odot')\};$$

(295)
$$0=[r'r''].\rho.\text{tang}.\theta-[rr''].\rho'.\text{tang}.\theta'+[rr'].\rho''.\text{tang}.\theta''.$$

Multiplying (294) by $\text{tang}.\theta'$, and (295) by $-\sin.(\alpha'-\odot')$; then taking the sum of the two products, we find that the terms multiplied by ρ' vanish, and we get,

(296)
$$0=[r'r''].\rho.\{\text{tang}.\theta'.\sin.(\alpha-\odot')-\text{tang}.\theta.\sin.(\alpha'-\odot')\}+[r'r''].R.\text{tang}.\theta'.\sin.(\odot'-\odot)$$
$$+[rr'].\rho''.\{\text{tang}.\theta'.\sin.(\alpha''-\odot')-\text{tang}.\theta''.\sin.(\alpha'-\odot')\}-[rr'].R''.\text{tang}.\theta'.\sin.(\odot''-\odot').$$

Dividing by the coefficient of ρ'', we finally obtain,

(297)
$$\rho''=\frac{[r'r'']}{[rr']}\cdot\frac{\{\text{tang}.\theta'.\sin.(\alpha-\odot')-\text{tang}.\theta.\sin.(\alpha'-\odot')\}}{\{\text{tang}.\theta''.\sin.(\alpha'-\odot')-\text{tang}.\theta'.\sin.(\alpha''-\odot')\}}\cdot\rho$$
$$+\frac{\text{tang}.\theta'}{[rr']}\cdot\frac{\{[r'r''].R.\sin.(\odot'-\odot)-[rr'].R''.\sin.(\odot''-\odot')\}}{\text{tang}.\theta''.\sin.(\alpha'-\odot')-\text{tang}.\theta'.\sin.(\alpha''-\odot')}.$$

In like manner, the plane triangles $sa'b'$, $sb'c'$, $sa'c'$, figure 84, page 792, corresponding to the earth's orbit, give by using a notation like that in (266),

(298) $[RR']=2.$area of the triangle $sa'b'$; $[R'R'']=2.$area of the triangle $sb'c'$;
$[RR'']=2.$area of the triangle $sa'c'$.

The area of any one of these triangles, as $sa'b'$, is found by multiplying its base $sa'=R$, by half the perpendicular let fall upon it from its vertex b', or by $\frac{1}{2}R'.\sin.a'sb'$; therefore, this area is represented by $\frac{1}{2}RR'.\sin.a'sb'$; and as the angle $a'sb'=A'-A=\odot'-\odot$,
(299) the area becomes $\frac{1}{2}RR'.\sin.(\odot'-\odot)$. Substituting this in the first expression (298), we get the first of the equations (300); in like manner, the second and third of the formulas (298), become like those in (300). In exactly the same way, we get the expression [300′];
(299′) observing, that the angle $asb=v'-v$; the angle $csb=v''-v'$; the angle $asc=v''-v$;

(300)
$$[RR']=RR'.\sin.(\odot'-\odot);\quad [R'R'']=R'R''.\sin.(\odot''-\odot');$$
$$[RR'']=RR''.\sin.(\odot''-\odot);$$

(300′)
$$[rr']=rr'.\sin.(v'-v);\qquad [r'r'']=r'r''.\sin.(v''-v');\qquad [rr'']=rr''.\sin.(v''-v).$$

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The second of the equations (300), gives the first expression (301); multiplying its numerator and denominator by $R.\sin.(☉'-☉)$, we get its second expression; substituting in its denominator the value, $[RR']$ (300), we get the last of the formulas (301);

$$R''.\sin.(☉''-☉')=\frac{[R'R'']}{R'}=\frac{[R'R''].R\sin.(☉'-☉)}{RR'.\sin.(☉'-☉)}=\frac{[R'R''].R.\sin.(☉'-☉)}{[RR']};$$ (301)

substituting this last expression, in the numerator of the second line of the second member of (297), we get,

$$\begin{aligned}\rho''=&\frac{[r'r'']}{[rr']}\cdot\frac{\text{tang.}\theta'.\sin.(\alpha-☉')-\text{tang.}\theta.\sin.(\alpha'-☉')}{\text{tang.}\theta''.\sin.(\alpha'-☉')-\text{tang.}\theta'.\sin.(\alpha''-☉')}\cdot\rho\\&+\left\{\frac{[r'r'']}{[rr']}-\frac{[R'R'']}{[RR']}\right\}\cdot\frac{R.\text{tang.}\theta'.\sin.(☉'-☉)}{\text{tang.}\theta''.\sin.(\alpha'-☉')-\text{tang.}\theta'.\text{tang.}(\alpha''-☉')}.\end{aligned}$$ (302)

Now putting for brevity,

$$M_1=\frac{\text{tang.}\theta'.\sin.(\alpha-☉')-\text{tang.}\theta.\sin(\alpha'-☉')}{\text{tang.}\theta''.\sin.(\alpha'-☉')-\text{tang.}\theta'.\sin.(\alpha''-☉')};$$ (303)

$$M_2=\frac{\text{tang.}\theta'.\sin.(☉'-☉)}{\text{tang.}\theta''.\sin.(\alpha'-☉')-\text{tang.}\theta'.\sin.(\alpha''-☉')};$$ (304)

the preceding expression of ρ'' (302), or $M.\rho$ (29), becomes of the following form; *in which nothing is neglected;* (305)

$$\rho''=M.\rho=\frac{[r'r'']}{[rr']}\cdot M_1.\rho+\left\{\frac{[r'r'']}{[rr']}-\frac{[R'R'']}{[RR']}\right\}\cdot M_2.R.$$ (306)

Dividing this last expression by ρ, we get the correct value of M. If we suppose, as in Olbers's hypothesis (53), that, (306)

$$\frac{[r'r'']}{[rr']}=\frac{[R'R'']}{[RR']}=\frac{t''-t'}{t'-t};$$ (307)

the term depending on M_2 will vanish from (306), and we shall have, very nearly,

$$\rho''=M.\rho=\frac{t''-t'}{t'-t}\cdot M_1.\rho;$$ (308)

hence,

$$M=\frac{t''-t'}{t'-t}\cdot M_1=\frac{t''-t'}{t'-t}\cdot\frac{\text{tang.}\theta'.\sin.(\alpha-☉')-\text{tang.}\theta.\sin.(\alpha-☉')}{\text{tang.}\theta''.\sin.(\alpha'-☉')-\text{tang.}\theta'.\sin.(\alpha''-☉')}.$$ (309)

This expression of M is the same as the approximate value, assumed by Dr. Olbers (30); as is evident, by substituting in it the value of m (28), and making a slight reduction. To estimate the value of the neglected terms in the value of M, we may proceed in the

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(310) following manner. Taking the rectangular co-ordinates of the comet, *in the plane of its orbit*, and representing them in the three observations, by x, y, x′, y′, x″, y″; putting
(311) $\mu=1$, or neglecting the mass of the comet, in comparison with that of the sun, as in [760 xiii], we obtain from [761], by accenting the symbols, the following equations,

(312)
$$\frac{d^2x'}{dt^2}+\frac{x'}{r'^3}=0; \qquad \frac{d^2y'}{dt^2}+\frac{y'}{r'^3}=0.$$

(313) Now if we take, for the origin of the time t, the moment of the second observation, when the co-ordinates are x′, y′; and suppose that at the end of the time t, these co-ordinates become x″, y″, respectively; we shall have by Taylor's or Maclaurin's theorem [607*a*] the expression (315). Substituting in this the value of d^2x', and of its differentials, deduced
(314) from the first of the equations (312), we shall get (316); which is easily reduced to the form (317);

(315)
$$x''=x'+\frac{dx'}{dt}\cdot t+\tfrac{1}{2}\cdot\frac{d^2x'}{dt^2}\cdot t^2+\tfrac{1}{6}\cdot\frac{d^3x'}{dt^3}\cdot t^3+\&c.$$

(316)
$$=x'+\frac{dx'}{dt}\cdot t-\tfrac{1}{2}\cdot\frac{x'}{r'^3}\cdot t^2-\tfrac{1}{6}\cdot t^3.\left\{\frac{dx'}{dt}\cdot\frac{1}{r'^3}-\frac{dr'}{dt}\cdot\frac{3x'}{r'^4}\right\}+\&c.$$

(317)
$$=x'.\left\{1-\tfrac{1}{2}\cdot\frac{t^2}{r'^3}+\tfrac{1}{2}\cdot\frac{t^3}{r'^4}\cdot\frac{dr'}{dt}+\&c.\right\}+\frac{dx'}{dt}\cdot\left\{t-\tfrac{1}{6}\frac{t^3}{r'^3}+\&c.\right\}$$

In like manner, we can obtain the similar expression of y″. The intervals of the times
(318) between the observations, namely, $t'-t, t''-t', t''-t$, are to be reduced to parts of the radius, by multiplying them by k [5987(8)]; and we shall, for brevity, express these products by τ, τ', τ''; as in (319); observing that these symbols have the same symmetry as in
(318′) (279′,) namely, that the number of accents in each of the equations (319) is *three*. We shall also use the abridged expressions (320—323).

(319)
$$\tau''=k.(t'-t); \qquad \tau=k.(t''-t'); \qquad \tau'=k.(t''-t); \qquad \tau'=\tau+\tau'';$$

(320)
$$w_{,}=1-\tfrac{1}{2}\cdot\frac{\tau''^2}{r'^3}-\tfrac{1}{2}\cdot\frac{\tau''^3}{r'^4}\cdot\frac{dr'}{dt}+\&c.;$$

(321)
$$w_{,,}=\tau''-\tfrac{1}{6}\cdot\frac{\tau''^3}{r'^3}-\&c.;$$

(322)
$$w'=1-\tfrac{1}{2}\cdot\frac{\tau^2}{r'^3}+\tfrac{1}{2}\cdot\frac{\tau^3}{r'^4}\cdot\frac{dr'}{dt}+\&c.;$$

(323)
$$w''=\tau-\tfrac{1}{6}\cdot\frac{\tau^3}{r'^3}+\&c.$$

While the body moves from the second point b, to the third point c, figure 84, the time increases from t' to t'', the increment being $t''-t'$, or τ (319), expressed in parts of the
(324) radius. Substituting this for t in (317), we get the expression of x″, (328), using the symbols (322, 323); in like manner we get the similar expression of y″ (329). If we

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change, in this calculation, t'' into t, the quantity τ will change into $-\tau''$ (319); by (325) which means w' (322), changes into $w_{,}$ (320), and w'' (323) into $-w_{,,}$ (321); making these changes in x'', y'' (328, 329), we get x, y (326, 327). Finally as the plane of the orbit is taken for the plane of projection (310), we shall have $z=0$, $z''=0$, as in (327′, 329′).

$$x = w_{,}.x' - w_{,,}.\frac{dx'}{dt}; \qquad (326)$$

$$y = w_{,}.y' - w_{,,}.\frac{dy'}{dt}; \qquad (327)$$

$$z = 0; \qquad (327')$$

$$x'' = w'.x' + w''.\frac{dx'}{dt}; \qquad (328)$$

$$y'' = w'.y' + w''.\frac{dy'}{dt}. \qquad (329)$$

$$z'' = 0 \qquad (329')$$

Multiplying (326) by y', and (327), by $-x'$, then taking the sum of the products, we get the first expression (331). Again, multiplying (329) by x', and (328) by $-y'$; then taking the sum of the products, we get the first expression (332). Lastly, multiplying (330) (326) by (329), also, (327) by (328), and subtracting the last product from the preceding, we get the first expression (333). The second form of either of these expressions, is easily deduced from the first, by the substitution of

$$\frac{x'dy' - y'dx'}{dt} = \sqrt{a.(1-e^2)} = \sqrt{p}, \qquad (330')$$

which is easily deduced from [366, 596c], using (311), and [5985(5)].

$$xy' - x'y = w_{,,}.\frac{(x'dy' - y'dx')}{dt} = w_{,,}.\sqrt{p}; \qquad (331)$$

$$x'y'' - x''y' = w''.\frac{(x'dy' - y'dx')}{dt} = w''.\sqrt{p}; \qquad (332)$$

$$xy'' - x''y = w_{,}.w''.\frac{(x'dy' - y'dx')}{dt} + w'.w_{,,}.\frac{(x'dy' - y'dx')}{dt} = (w_{,}.w'' + w'.w_{,,}).\sqrt{p}. \qquad (333)$$

Now the expressions (320—323), give successively, by using $\tau' = \tau + \tau''$, (319),

$$w_{,}.w'' = \tau - \tfrac{1}{6}.\frac{\tau^3}{r'^3} - \tfrac{1}{2}.\tau.\frac{\tau''^2}{r'^3} + \&c.; \qquad w'.w_{,,} = \tau'' - \tfrac{1}{6}\frac{\tau''^3}{r'^3} - \tfrac{1}{2}.\tau''.\frac{\tau^2}{r'^3} + \&c.; \qquad (334)$$

$$w_{,}.w'' + w'.w_{,,} = \tau + \tau'' - \frac{1}{6r'^3}.\{\tau^3 + 3\tau^2.\tau'' + 3\tau.\tau''^2 + \tau''^3\} + \&c. \qquad (335)$$

$$= \tau' - \tfrac{1}{6}.\frac{\tau'^3}{r'^3} + \&c.; \qquad (336)$$

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Substituting in the first members of (331—333), the following expressions, which are deduced from (274), by putting $\varphi = 0$, as in (310).

(337) $$xy' - x'y = [rr'] \; ; \qquad x'y'' - x''y' = [r'r''] \; ; \qquad xy'' - x''y = [rr''] \; ;$$

and in their last members, the values (321, 323, 336), we get,

(338) $$[rr'] = \left\{ \tau'' - \frac{1}{6r'^3} \cdot \tau''^3 - \&c. \right\} . \sqrt{p} = \tau'' . \left\{ 1 - \frac{1}{6r'^3} \cdot \tau''^2 - \&c. \right\} . \sqrt{p} \; ;$$

(339) $$[r'r''] = \left\{ \tau - \frac{1}{6r'^3} \cdot \tau^3 + \&c. \right\} . \sqrt{p} = \tau . \left\{ 1 - \frac{1}{6r'^3} \cdot \tau^2 + \&c. \right\} . \sqrt{p} \; ;$$

(340) $$[rr''] = \left\{ \tau' - \frac{1}{6r'^3} \cdot \tau'^3 + \&c. \right\} . \sqrt{p} = \tau' . \left\{ 1 - \frac{1}{6r'^3} \cdot \tau'^2 + \&c. \right\} . \sqrt{p}.$$

Dividing these expressions, the one by the other, we obtain,

(341) $$\frac{[r'r'']}{[rr']} = \frac{\tau}{\tau''} . \left\{ 1 - \frac{1}{6r'^3} \cdot (\tau^2 - \tau''^2) + \&c. \right\} ;$$

(342) $$\frac{[rr'']}{[rr']} = \frac{\tau'}{\tau''} . \left\{ 1 - \frac{1}{6r'^3} \cdot (\tau'^2 - \tau''^2) + \&c. \right\} ;$$

(343) $$\frac{[rr'']}{[r'r'']} = \frac{\tau'}{\tau} . \left\{ 1 - \frac{1}{6r'^3} \cdot (\tau'^2 - \tau^2) + \&c. \right\} .$$

As these formulas may be used for any of the heavenly bodies, we shall obtain the expressions (344—346), corresponding to the earth's orbit, by merely changing r, r', r'', into R, R', R''; respectively,

(344) $$\frac{[R'R'']}{[RR']} = \frac{\tau}{\tau''} . \left\{ 1 - \frac{1}{6R'^3} \cdot (\tau^2 - \tau''^2) + \&c. \right\} ;$$

(345) $$\frac{[RR'']}{[RR']} = \frac{\tau'}{\tau''} . \left\{ 1 - \frac{1}{6R'^3} \cdot (\tau'^2 - \tau''^2) + \&c. \right\} ;$$

(346) $$\frac{[RR'']}{[R'R'']} = \frac{\tau'}{\tau} . \left\{ 1 - \frac{1}{6R'^3} \cdot (\tau'^2 - \tau^2) + \&c. \right\} .$$

(347) If the intervals between the three observations be equal, or $\tau'' = \tau$, we shall have $\tau^2 - \tau''^2 = 0$, and then the expressions (341, 344), will give, *by neglecting terms of the fourth order in* τ, τ'', (338—340), or of the *third* order in the factors of $\frac{\tau}{\tau'}$, (341, 344);

(348) $$\frac{[r'r'']}{[rr']} = \frac{[R'R'']}{[RR']} = \frac{\tau}{\tau''} = \frac{t''-t'}{t'-t} \quad (319),$$

which agrees with the supposition of Dr. Olbers (307). *Hence we see the great advantage*
(348') *of having the intervals of time between the observations equal to each other, in computing the*

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orbit of a comet, by this method; because it makes the factor of $M_2\, R$, (306), *nearly insensible; and gives a more accurate value of the expression* $M.\rho$, *than it would if the intervals were unequal.* If observations cannot be obtained, in which the intervals τ, τ'', are equal to each other, we must select those which are nearly equal; in order to diminish as much as possible the effect of the factor $\tau^2 - \tau''^2$. If $R' = r'$, the expressions (341, 344), become equal; hence it is evident, that if r' be nearly equal to R, and the intervals τ, τ'' differ considerably; it will be rather more accurate to compute $\frac{[RR'']}{[RR']}$, from the solar tables, and put $\frac{[r'r'']}{[rr']}$, equal to it, than to put each of these quantities equal to $\frac{\tau}{\tau''}$ (348). Finally, we may observe, that after we have computed, by a first approximation, the values of ρ, r, r'', we may, by interpolation, find an approximate value of r', by supposing the values to increase uniformly; by which means we shall have,

Great advantage of having the (349) intervals of time equal between the observations. (350)

(351)

(352)

$$r' = r + \frac{t'-t}{t''-t} \cdot (r'' - r). \qquad (353)$$

With these we may obtain the corrected value of the function (341), to be substituted in (306), to get a more accurate value of M; with which the calculation can be repeated, in any extreme case, where it shall be found necessary. (354)

In the case where the value of M (309), appears under the form of $M = \frac{0}{0}$, we may deduce the value of $\rho'' = M.\rho$ from the equation (293), instead of (294, 295), which are used in finding (297). Then as radius ρ' does not occur in (293), we shall have, (355)

$$\rho'' = \frac{[r'r'']}{[rr']} \cdot \frac{\sin.(\alpha' - \alpha)}{\sin.(\alpha'' - \alpha')} \cdot \rho + \frac{[r'r''].R.\sin.(\odot - \alpha') - [rr''].R'.\sin.(\odot' - \alpha') + [rr'].\sin.(\odot'' - \alpha').R''}{[rr'].\sin.(\alpha'' - \alpha')}. \qquad (356)$$

If we divide the expression (341) by (344), we get, by a slight reduction, the expression (357); in like manner, from (342, 345), we get (358), lastly, from (343, 346), we obtain (359). The equation (360), is evidently identical;

$$\frac{[r'r'']}{[rr']} = \frac{[R'R'']}{[RR']} \cdot \left\{ 1 + \tfrac{1}{6} \cdot (\tau^2 - \tau''^2) \cdot \left(\frac{1}{R'^3} - \frac{1}{r'^3} \right) + \&c. \right\} : \qquad (357)$$

$$\frac{[rr'']}{[rr']} = \frac{[RR'']}{[RR']} \cdot \left\{ 1 + \tfrac{1}{6} \cdot (\tau'^2 - \tau''^2) \cdot \left(\frac{1}{R'^3} - \frac{1}{r'^3} \right) + \&c. \right\} : \qquad (358)$$

$$\frac{[rr'']}{[r'r'']} = \frac{[RR'']}{[R'R'']} \left\{ 1 + \tfrac{1}{6} \cdot (\tau'^2 - \tau^2) \cdot \left(\frac{1}{R'^3} - \frac{1}{r'^3} \right) + \&c. \right\} : \qquad (359)$$

$$\frac{[rr']}{[rr']} = \frac{[RR']}{[RR']}. \qquad (360)$$

[5994] Taking, as in (269), the ecliptic for the plane of projection; we shall represent the
(361) rectangular co-ordinates of the earth, by X, Y, at the first observation; X', Y', at the second observation; X'', Y'' at the third observation; hence the identical equations in the earth's orbit, corresponding to (277, 278), in the comet's orbit; becomes,

(362) $$0=[R'R''].X-[RR''].X'+[RR'].X'';$$

(363) $$0=[R'R''].Y-[RR''].Y'+[RR'].Y''.$$

If we take for the axis of X, the line whose longitude is $180^d+\alpha'$, we shall evidently have,

(364) $$Y=R.\sin.(\odot-\alpha'); \qquad Y'=R'.\sin.(\odot'-\alpha'); \qquad Y''=R''.\sin.(\odot''-\alpha').$$

Substituting these in the numerator of the second line of (356), it becomes,

(365) $$[r'r''].Y-[rr''].Y'+[rr'].Y''.$$

If we substitute, in this expression, the values of $[r'r'']$, $[rr'']$, $[rr']$, (357, 358, 360) and neglect, for a moment, the terms depending on the factor $\frac{1}{R'^3}-\frac{1}{r'^3}$, it will become,

(366) $$\frac{[rr']}{[RR']}.\{[R'R''].Y-[RR''].Y'+[RR'].Y''\};$$

and as this vanishes, by means of the equation (363), it will be only necessary to retain the
(367) terms of (357, 358), which are multiplied by that factor $\frac{1}{R'^3}-\frac{1}{r'^3}$. In the case now under consideration, this factor is very small, because when the apparent motion of the
(368) comet is in a great circle, we shall have $r'=R'$ [780v]; and if the intervals $t'-t$,
(369) $t''-t'$, or τ'', τ, be nearly equal, we shall have $\tau^2-\tau''^2=0$; and we may therefore neglect the product of this quantity, by the preceding factor in (357); putting also $\tau=2\tau'$ in the factor $\tau'^2-\tau''^2$ (358), by which means we get $\frac{1}{6}.(\tau'^2-\tau''^2)=\frac{1}{2}\tau'^2$; hence the term of (358), depending on this factor, becomes,

$$\frac{[RR']}{[RR']}\cdot\tfrac{1}{2}.\tau'^2.\left(\frac{1}{R^3}-\frac{1}{r'^3}\right)=\tfrac{1}{2}\tau'\tau''.\left(\frac{1}{R^3}-\frac{1}{r'^3}\right)=\tfrac{1}{2}\tau\tau'.\left(\frac{1}{R'^3}-\frac{1}{r'^3}\right),$$

nearly; as is evident by using only the first term of the second member of (345). Substituting this in the numerator of the second line of (356, or 365, &c.), and putting in its first line,

(371) $$\frac{[r'r'']}{[rr']}=\frac{\tau}{\tau''}=\frac{t''-t'}{t'-t} \qquad (341, 369, 319),$$

we finally obtain the following value of ρ'', *which can be used in the case now under consideration, when the geocentric longitudes* α, α', α'', *vary from each other much more than the geocentric latitudes* θ, θ', θ'';

(372) $$\rho''=\frac{t''-t'}{t'-t}\cdot\frac{\sin.(\alpha'-\alpha)}{\sin.(\alpha''-\alpha')}.\rho+\tfrac{1}{2}.\tau\tau'.\frac{R'.\sin.(\alpha'-\odot')}{\sin.(\alpha''-\alpha')}.\left(\frac{1}{R'^3}-\frac{1}{r'^3}\right).$$

We may obtain another form of the expression of ρ'' by eliminating ρ' from (292,295); this is done by multiplying (292) by $-\text{tang.}\theta'$, and (295) by $\cos.(\alpha'-\odot')$, and taking the sum of the products, by which means we get, [5994]

$$0=[r'r''].\left\{\begin{array}{r}-\rho.\text{tang.}\theta'.\cos.(\alpha-\odot')+R.\text{tang.}\theta'.\cos.(\odot-\odot')\\ +\rho.\text{tang.}\theta.\cos.(\alpha'-\odot')\end{array}\right\}-[rr''].R'.\text{tang.}\theta'$$

$$+[rr'].\left\{\begin{array}{r}-\rho''.\text{tang.}\theta'.\cos.(\alpha''-\odot')+R''.\text{tang.}\theta'.\cos.(\odot''-\odot')\\ +\rho''.\text{tang.}\theta''.\cos.(\alpha'-\odot')\end{array}\right\}.$$

Dividing this by the coefficient of ρ'', we obtain,

$$\rho''=\frac{[r'r'']}{[rr']}.\left\{\frac{\text{tang.}\theta'.\cos.(\alpha-\odot')-\text{tang.}\theta.\cos.(\alpha'-\odot')}{\text{tang.}\theta''.\cos.(\alpha'-\odot')-\text{tang.}\theta'.\cos.(\alpha''-\odot')}\right\}.\rho$$

$$-\left\{\frac{[r'r''].R.\text{tang.}\theta'.\cos.(\odot-\odot')-[rr''].R'.\text{tang.}\theta'+[rr'].R''.\text{tang.}\theta'.\cos.(\odot''-\odot')}{[rr'].\{\text{tang.}\theta''.\cos.(\alpha'-\odot')-\text{tang.}\theta'.\cos.(\alpha''-\odot')\}}\right\}$$

The second line of this expression may be reduced, by a process similar to that in (364 &c.). Taking for the axis of X the line whose longitude is $180^d+\odot'$, we shall have, in like manner as in (364),

$$X=R.\cos.(\odot-\odot'),\qquad X'=R'.\cos.(\odot'-\odot')=R';\qquad X''=R''.\cos.(\odot''-\odot'):$$

and then the numerator of the expression in the second line of (374), becomes.

$$\{[r'r''].X-[rr''].X'+[rr'].X''\}.\text{tang.}\theta'.$$

If we substitute in this, the parts of $[r'r'']$, $[rr'']$, $[rr']$, (357, 358, 360), which depend on the first term of the second members, it becomes,

$$\frac{[rr']}{[RR']}.\{[R'R''].X-[RR''].X'+[RR'].X''\};$$

which vanishes, by means of (362). Hence we obtain the same result as in (367); namely, that it is only necessary to notice the terms depending on the factor $\frac{1}{R'^3}-\frac{1}{r'^3}$; and by supposing the intervals $t'-t$, $t''-t'$ to be nearly equal, we shall find as in (370), that the only part of this numerator, which it is necessary to notice, arises from that part of $\frac{[rr'']}{[rr']}$, which is denoted by $+\frac{1}{2}.\tau\tau'.\left(\frac{1}{R'^3}-\frac{1}{r'^3}\right)$ (370). Substituting this in the second line of (374), and putting, in the first line, the value (371), we finally obtain the following expression of ρ'', *which can be used, in this excepted case, when the geocentric latitudes* θ, θ', θ'', *vary from each other more than the geocentric longitudes* α, α', α'';

[5994

$$\rho'' = \frac{t''-t'}{t'-t} \cdot \left\{ \frac{\text{tang.}\theta'.\cos.(\alpha - \odot') - \text{tang.}\theta.\cos.(\alpha' - \odot')}{\text{tang.}\theta\ .\cos.(\alpha''- \odot') - \text{tang.}\theta''.\cos.(\alpha'' - \odot')} \right\} \cdot \rho$$

$$+ \tfrac{1}{2}\,\tau\tau' \cdot \left\{ \frac{R'.\,\text{tang.}\theta'}{\text{tang.}\theta''.\,\cos.(\alpha' - \odot') - \text{tang.}\theta'.\cos.(\alpha'' - \odot')} \right\} \cdot \left(\frac{1}{R'^3} - \frac{1}{r'^3} \right).$$

For convenience in the calculations we have arranged the formulas (372, 380), as in the table (387—392). If we neglect the term of ρ'' (372), depending on $\tau\tau'$, and use the symbol M' (387), it becomes $\rho'' = M'.\rho$, so that M' represents an approximate value of M, (29). With this we may compute the equations (31—33), and from thence deduce, as in (192), the approximate values r', r'', ρ. This value of ρ we shall represent by (ρ); and from r, r'', we may find the approximate value of r' (353), to be used in computing the term of the order $\tau\tau'.\left\{ \frac{1}{R'^3} - \frac{1}{r'^3} \right\}$, which occurs in (372). Substituting $\tau = \tau''.\left(\frac{t''-t'}{t'-t}\right)$ (319), in the second term of (372), and then dividing the whole of the second member, by the expression of M' (387); we find that the quotient becomes equal to F' (388); consequently, this expression of ρ'', will become as in (389). In like manner, by using the abridged values of M'', F'' (390, 391), we find that the expression of ρ'' (380), becomes as in (392); (ρ) being as before, the value of ρ, deduced from the first approximation, in which F'' is supposed to be equal to unity.

(387) $$M' = \frac{t''-t'}{t''-t} \cdot \frac{\sin.(\alpha'' - \alpha')}{\sin.(\alpha'' - \alpha\ )};$$

(388) $$F' = 1 + \tfrac{1}{2}\tau''\,\tau \cdot \frac{\sin.(\alpha - \odot')}{\sin.(\alpha' - \alpha)} \cdot \frac{R'}{(\rho)} \cdot \left(\frac{1}{R'^3} - \frac{1}{r'^3} \right);$$

(389) $$\rho'' = M.\rho = M'.F'.\rho; \qquad \text{or} \qquad M = M'.F'.$$

To be used when the longitudes $\alpha, \alpha', \alpha''$, vary faster than the latitudes $\theta, \theta', \theta''$.

(390) $$M'' = \frac{t''-t'}{t'-t} \cdot \frac{\text{tang.}\theta'.\cos.(\alpha - \odot') - \text{tang.}\theta.\cos.(\alpha' - \odot')}{\text{tang.}\theta''.\cos.(\alpha' - \odot') - \text{tang.}\theta'.\cos.(\alpha'' - \odot')};$$

(391) $$F'' = 1 + \tfrac{1}{2}\tau''\tau' \cdot \frac{\text{tang.}\theta'}{\text{tang.}\theta'.\cos.(\alpha - \odot') - \text{tang.}\theta.\cos.(\alpha' - \odot')} \cdot \frac{R'}{(\rho)} \cdot \left\{ \frac{1}{R'^3} - \frac{1}{r'^3} \right\};$$

(392) $$\rho'' = M.\rho = M''.F''.\rho; \qquad \text{or} \qquad M = M''.F''.$$

To be used when the longitudes $\alpha, \alpha', \alpha''$, vary slower than the latitudes $\theta, \theta', \theta''$.

(393) If we compare the correct value of $M = \frac{\rho''}{\rho}$ (306), with its approximate values (309, 387, 390), we shall find, that the first, or general form, is by far the most accurate; especially when the intervals of the observations are nearly equal, or $\tau^2 - \tau''^2 = 0$; since in this case, the value of M (309), is correct in terms of the second order, in

(394) τ, τ'', inclusively (306, 347, &c.) On the contrary, the values of M (389, 392), are found by multiplying the assumed values M', M'' (387, 390), by the factors F', F''

(395) (388, 391), which contains terms of the second order in τ, τ''; so that these expressions

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of M may be considered as less accurate than that in (30) or (309), by at least, terms of one order, in τ, τ'. Now from the mere inspection of the approximate values of M, (395)
given in (309, 387, 390), it is evident, that when the apparent path of the comet is near the ecliptic, and the latitudes θ, θ', θ'' differ but little from each other, the expressions (309, 390), will have very small numerators and denominators; therefore the resulting value of M or M may be considerably affected by the imperfections of the observations; but this would not be the case with the expression (387), supposing the longitudes of the comet to vary rapidly. On the other hand, when these longitudes vary slowly, the expression $\sin.(\alpha'-\alpha)$, $\sin.(\alpha''-\alpha')$, are small; consequently, the numerator and denominator of (387), may be so small that the errors of the observations can have an important influence on the resulting value of M'. Hence it follows, that when the expression (309) becomes uncertain, on account of the smallness of its numerator and denominator, we can use the expressions (387—389), if the longitudes of the comet vary more rapidly than the latitudes; or the expressions (390—392), if these longitudes vary slowly in comparison with the latitudes. The method of using the formulas (387—392), is so similar to that in the preceding examples (173, &c.), that it is unnecessary to give any examples for illustration. We shall, therefore, close our remarks on this method, by observing, that after the approximate values of the elements have been obtained, we may correct them by taking more distant observations, as we have already observed in [820''', &c., 849a, &c.].

Lubbock's method of computing the orbit of a comet.

Since the preceding article was prepared for this appendix, a new method of computing the orbit of a comet has been proposed by Mr Lubbock, and published in the fourth volume of the Memoirs of the Astronomical Society of London, and in a separate pamphlet "On the determination of the distance of a comet, &c.;" in which he has reduced the question to the solution of a quadratic equation. As we have not made any numerical computations by this process; we shall restrict ourselves to the explanation of the principles of the method, with such illustrations as may be necessary.

If we suppose the intervals of time $t'-t$, $t''-t'$, between the observations to be equal, we shall have $\tau''=\tau$ (319), and by neglecting terms of the order τ^3, we (405)
shall have, as in (320—323),

$$w_{,}=w'=1-\tfrac{1}{2}\cdot\frac{\tau^2}{r'^3}; \qquad w_{,,}=w''=\tau. \qquad (406)$$

Substituting these in (326—329'), we get, by taking the differences of the resulting expressions,

$$x''-x=2\tau\cdot\frac{dx'}{dt}; \qquad y''-y=2\tau\cdot\frac{dy'}{dt}; \qquad z''-z=2\tau\cdot\frac{dz'}{dt}=0. \qquad (407)$$

The sum of the squares of these three equations, produces the *first* and *second* of the

[5994] following expressions of c^2; from the *second* we easily deduce the *third* by means of the
(407') formula [572, line 5], putting $\mu=1$, as in (311);

$$c^2=(\mathrm{x}''-\mathrm{x})^2+(\mathrm{y}''-\mathrm{y})^2+(\mathrm{z}''-\mathrm{z})^2=4\tau^2.\left\{\frac{d\mathrm{x}'^2+d\mathrm{y}'^2+d\mathrm{z}'^2}{dt^2}\right\}$$

(408)

$$=4\tau^2.\left\{\frac{2}{r'}-\frac{1}{a}\right\}.$$

The values of r^2, r''^2, may be deduced from r'^2 and its differentials, by Maclaurin's theorem [607a], in the same manner as we have obtained x'' from x' in (315, &c.); and we shall have,

(409)
$$r^2=r'^2-\tau.\frac{d.(r'^2)}{dt}+\tfrac{1}{2}\tau^2.\frac{d^2.(r'^2)}{dt^2}-\&c.$$

(410)
$$r''^2=r'^2+\tau.\frac{d.(r'^2)}{dt}+\tfrac{1}{2}\tau^2.\frac{d^2.(r'^2)}{dt^2}+\&c.$$

Subtracting $2r'^2$ from the sum of these values of r^2, r''^2; neglecting the terms depending on τ^3, and the higher powers of τ, we get,

(411)
$$r^2-2r'^2+r''^2=\tau^2.\frac{d^2.(r'^2)}{dt^2}.$$

The second member of this equation may be reduced, by means of [595]. For if we put
(411') for a moment $\mathrm{r}=r^2$, the expression [595], becomes, by supposing as in (407') $\mu=1$,

(412)
$$2\,\mathrm{r}^{\frac{1}{2}}-\frac{1}{a}.\mathrm{r}-\frac{d\,\mathrm{r}^2}{4\,dt^2}=h^2.$$

Taking its differential, and dividing by $d\mathrm{r}$, we get,

(412')
$$\mathrm{r}^{-\frac{1}{2}}-\frac{1}{a}-\frac{dd\,\mathrm{r}}{2\,dt^2}=0.$$

Re-substituting the value of r, and making a slight transposition in the order of the terms, we get,

(413)
$$\frac{d^2.(r^2)}{2\,dt^2}=\frac{1}{r}-\frac{1}{a};$$

hence, the equation (411), becomes,

(414)
$$r^2-2r'^2+r''^2=2\tau^2.\left(\frac{1}{r'}-\frac{1}{a}\right).$$

Mr Lubbock's method is grounded on the two equations (408, 414); by substituting the values of c, r, r', r'', in terms of ρ', and assuming the following expressions of ρ, ρ'',

[5994]

$$\rho = \lambda_1 \cdot \rho' ; \qquad (415)$$

$$\rho'' = \lambda_2 \cdot \rho' . \qquad (416)$$

The values of λ_1, λ_2, may be deduced from the equations (294,295), by the elimination of ρ''. For if we multiply (294) by $\mathrm{tang.}\theta''$, also, (295) by $-\mathrm{sin.}(\alpha''-\odot')$, and take the sum of the products, we shall find that the terms depending on ρ'' will vanish, and we shall have, (417)

$$\begin{aligned}0 = [r'r''].\rho.\{\mathrm{tang.}\theta''.\,\mathrm{sin.}(\alpha-\odot') - \mathrm{tang.}\theta.\mathrm{sin.}(\alpha''-\odot')\} + [r'r''].R.\mathrm{tang.}\theta''.\,\mathrm{sin.}(\odot'-\odot)\\ +[rr''].\rho'.\{-\mathrm{tang.}\theta''.\mathrm{sin.}(\alpha'-\odot')+\mathrm{tang.}\theta'.\mathrm{sin.}(\alpha''-\odot')\}-[rr'].R''.\mathrm{tang.}\theta''.\mathrm{sin.}(\odot''-\odot').\end{aligned} \qquad (418)$$

Dividing by the co-efficient of ρ, we obtain (419). In like manner, if we multiply (294) by $\mathrm{tang.}\theta$, also (295) by $-\mathrm{sin.}(\alpha-\odot')$, then take the sum of the products, and divide by the co-efficient of ρ'', we shall get (420);

$$\begin{aligned}\rho = \frac{[rr'']}{[r'r'']} \cdot \left\{ \frac{\mathrm{tang.}\theta'.\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\mathrm{sin.}(\alpha-\odot')}{\mathrm{tang.}\theta.\,\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\mathrm{sin.}(\alpha-\odot')} \right\} \cdot \rho' \\ + \frac{\mathrm{tang.}\theta''}{[r'r'']} \cdot \frac{\{[r'r''].R.\mathrm{sin.}(\odot'-\odot) - [rr'].R''.\,\mathrm{sin.}(\odot''-\odot')\}}{\mathrm{tang.}\theta.\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\,\mathrm{sin.}(\alpha-\odot')} ;\end{aligned} \qquad (419)$$

$$\begin{aligned}\rho'' = \frac{[rr'']}{[rr']} \cdot \left\{ \frac{\mathrm{tang.}\theta.\,\mathrm{sin.}(\alpha'-\odot') - \mathrm{tang.}\theta'.\,\mathrm{sin.}(\alpha-\odot')}{\mathrm{tang.}\theta.\,\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\,\mathrm{sin.}(\alpha-\odot')} \right\} \cdot \rho' \\ + \frac{\mathrm{tang.}\theta}{[rr']} \cdot \frac{\{[rr'].R''.\mathrm{sin.}(\odot''-\odot') - [r'r''].R.\mathrm{sin.}(\odot'-\odot)\}}{\mathrm{tang.}\theta.\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\,\mathrm{sin.}(\alpha-\odot')} .\end{aligned} \qquad (420)$$

Substituting in the last term of each of these expressions, the value of $R''.\,\mathrm{sin.}(\odot''-\odot')$ (301), we get,

$$\begin{aligned}\rho = \frac{[rr'']}{[r'r'']} \cdot \left\{ \frac{\mathrm{tang.}\theta'.\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\,\mathrm{sin.}(\alpha'-\odot')}{\mathrm{tang.}\theta.\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\mathrm{sin.}(\alpha-\odot')} \right\} \cdot \rho' \\ + \left\{ \frac{[r'r'']}{[rr']} - \frac{[R'R'']}{[RR']} \right\} \frac{[rr']}{[r'r'']} \cdot \frac{R.\mathrm{tang.}\theta''.\mathrm{sin.}(\odot'-\odot)}{\mathrm{tang.}\theta.\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\,\mathrm{sin.}(\alpha-\odot')} ;\end{aligned} \qquad (421)$$

$$\begin{aligned}\rho'' = \frac{[rr'']}{[rr']} \cdot \left\{ \frac{\mathrm{tang.}\theta.\mathrm{sin.}(\alpha'-\odot') - \mathrm{tang.}\theta'.\mathrm{sin.}(\alpha-\odot')}{\mathrm{tang.}\theta.\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\,\mathrm{sin.}(\alpha-\odot')} \right. \cdot \rho' \\ - \left\{ \frac{[r'r'']}{[rr']} - \frac{[R'R'']}{[RR']} \right\} \cdot \frac{R.\mathrm{tang.}\theta.\mathrm{sin.}(\odot'-\odot)}{\mathrm{tang.}\theta.\mathrm{sin.}(\alpha''-\odot') - \mathrm{tang.}\theta''.\,\mathrm{sin.}(\alpha-\odot')} .\end{aligned} \qquad (422)$$

If we neglect those terms of the second members of these equations, which are multiplied by the extremely small quantity $\frac{[r'r'']}{[rr']} - \frac{[R'R'']}{[RR']}$ (307), we shall have,

[5994]

42. $$\rho = \frac{[rr'']}{[r'r'']} \cdot \left\{ \frac{\text{tang.}\delta'.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha'-\odot')}{\text{tang.}\delta.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha-\odot')} \right\} . \rho' ;$$

$$\rho'' = \frac{[rr'']}{[rr']} \cdot \left\{ \frac{\text{tang.}\delta.\sin.(\alpha'-\odot') - \text{tang.}\delta'.\sin.(\alpha-\odot')}{\text{tang.}\delta.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha-\odot')} \right\} . \rho' .$$

Comparing these with (415, 416), we get,

$$\lambda_1 = \frac{[rr'']}{[r'r'']} \cdot \left\{ \frac{\text{tang.}\delta'.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha'-\odot')}{\text{tang.}\delta.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha-\odot')} \right\} ;$$

$$\lambda_2 = \frac{[rr'']}{[rr']} \cdot \left\{ \frac{\text{tang.}\delta.\sin.(\alpha'-\odot') - \text{tang.}\delta'.\sin.(\alpha-\odot')}{\text{tang.}\delta.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha-\odot')} \right\} :$$

in which we must substitute the value of the factors $\frac{[rr'']}{[r'r'']}$ and $\frac{[rr'']}{[rr']}$. Now if we use the abridged symbols A, γ_1, γ_2 (428, 429, 430), and suppose the intervals $t'-t$, $t-t''$ to be equal, or, $\tau' = 2\tau = 2\tau''$ (319); we shall find from (342, 343), that both these factors become equal to $2A$ (431), and the values of λ_1, λ_2, ρ, ρ'' (425, 426, 415, 416) become as in (432, 433);

$$A = 1 - \frac{\tau^2}{2r'^3} ;$$

$$\gamma_1 = 2 . \left\{ \frac{\text{tang.}\delta'.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha'-\odot')}{\text{tang.}\delta.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha-\odot')} \right\} :$$

(430) $$\gamma_2 = 2 . \left\{ \frac{\text{tang.}\delta.\sin.(\alpha'-\odot') - \text{tang.}\delta'.\sin.(\alpha-\odot')}{\text{tang.}\delta.\sin.(\alpha''-\odot') - \text{tang.}\delta''.\sin.(\alpha-\odot')} \right\} :$$

(431) $$\frac{[rr'']}{[r'r'']} = \frac{[rr'']}{[rr']} = 2 . \left(1 - \frac{\tau^2}{2r'^3}\right) = 2A ;$$

(432) $$\lambda_1 = A.\gamma_1 ; \qquad \text{whence,} \qquad \rho = A.\gamma_1 . \rho' ;$$

(433) $$\lambda_2 = A.\gamma_2 ; \qquad \text{whence,} \qquad \rho'' = A.\gamma_2 . \rho' .$$

Hence it appears that each of the values of λ_1, λ_2 (432, 433) contains the *unknown factor* $A = 1 - \frac{\tau^2}{2r'^3}$; which is an inconvenience that Olbers's method does not suffer; since his

(434) value of M, deduced from $\rho'' = M.\rho$ (29), by the substitution of ρ, ρ'' (432, 433), does not contain this factor; for by using the value of ρ, ρ'' (432, 433), we have $M = \frac{\rho''}{\rho} = \frac{\gamma_2}{\gamma_1}$. Substituting this last value of M, also, $\rho = A.\gamma_1.\rho'$ (432), in (31, 32) we get (436, 438). The expression of r'^2 (437), is similar to (31). The same values

(435) of M, ρ, being substituted in (33), give the first expression of c^2 (439), and the second expression is the same as in (408). Lastly, substituting the values of r^2, r'^2, r''^2 (436—438) in (414), we get (440); observing that terms of the order τ^4 are neglected

in the second member of (439, 440); but may be introduced, by noticing the terms of a higher order, which are neglected in (406 &c.); [5994]

$$r^2 = R^2 - 2.\gamma_1.R.A.\rho'.\cos.(\odot - \alpha) + \gamma_1^2.A^2.\rho'^2.\sec^2.\theta; \tag{436}$$

$$r'^2 = R'^2 - 2.R'.\rho'.\cos.(\odot' - \alpha') + \rho'^2.\sec^2.\theta'; \tag{437}$$

$$r''^2 = R''^2 - 2.\gamma_2.R''.A.\rho'.\cos.(\odot'' - \alpha'') + \gamma_2^2.A^2.\rho'^2.\sec^2.\theta''; \tag{438}$$

$$\left.\begin{array}{l} r^2 + r''^2 - 2.RR''.\cos.(\odot'' - \odot) \\ + \{2.\gamma_1.R''.\cos.(\odot'' - \alpha) + 2.\gamma_2.R.\cos.(\odot - \alpha'')\}.A.\rho' \\ + \{-2.\gamma_1.\gamma_2.\cos.(\alpha'' - \alpha) - 2.\gamma_1.\gamma_2.\mathrm{tang}.\theta.\mathrm{tang}.\theta''\}.A^2.\rho'^2 \end{array}\right\} = 4.\tau^2.\left\{\frac{2}{r'} - \frac{1}{a}\right\}; \quad \left[\begin{array}{c}\text{Expression of} \\ c^2\end{array}\right] \tag{439}$$

$$\left.\begin{array}{l} R^2 - 2.R'^2 + R''^2 \\ + \left\{\begin{array}{r} -2.\gamma_1.R.\cos.(\odot - \alpha) + \frac{4}{A}.R'.\cos.(\odot' - \alpha') \\ -2.\gamma_2.R''.\cos.(\odot'' - \alpha'') \end{array}\right\}.A.\rho' \\ + \{\gamma_1^2.\sec^2.\theta - \frac{2}{A^2}.\sec^2.\theta' + \gamma_2^2.\sec^2.\theta''\}.A^2.\rho'^2 \end{array}\right\} = 2.\tau^2.\left\{\frac{1}{r'} - \frac{1}{a}\right\}. \left[\begin{array}{c}\text{Expression of} \\ r^2 - 2r'^2 + r''^2\end{array}\right] \tag{440}$$

Multiplying the equation (440) by -4, and adding the product to (439); after substituting the values of r^2, r''^2 (436, 438), we get the fundamental equation of Mr. Lubbock's method,

$$A' + B'.(A.\rho') + C'.(A.\rho')^2 = \frac{4.\tau^2}{a}. \tag{441}$$

In this equation, A', B', C' are functions of the given quantities R, R', R'', $\odot$, $\odot'$, $\odot''$, α, α', α'', θ, θ', θ''; and of the unknown quantity A (428). If we put $A = 1$, in the first operation, we shall obtain the approximate values of A', B', C'; and then putting $\frac{1}{a} = 0$, to correspond to a parabolic orbit, we shall finally obtain the *quadratic* equation, (442)

$$A' + B'.\rho' + C'.\rho'^2 = 0; \tag{443}$$

for the determination of an approximate value of ρ', or $A.\rho'$. With this value of ρ', we may find an approximate value of r' by means of (437), and this is to be used in finding A (428). This last value of A must be substituted in (439, 440), in order to get a more accurate expression of the equation (441, or 443); and thence a corrected value of $A.\rho'$. (444)
The same process is to be repeated till the true value of $A.\rho'$ is found; and then from (436 &c.) we get r, r', r'', &c. What we have said, will serve to explain the principle of this method, which is illustrated by examples, in the works of Mr. Lubbock, mentioned (445)
at the commencement of this article.

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If we compare these two methods together, we shall see that the peculiar advantage of
(446) Mr. Lubbock's method is, that the determination of ρ' is reduced to the solution of a quadratic equation (443); but the accuracy of this equation, is considerably impaired, in
(447) the first operation, by putting $A=1$ (442); and this defect can be remedied only by successive operations, with repeated solutions of the quadratic equations after correcting the
(448) coefficients, which increases the labor considerably, and sometimes alters very essentially the coefficients of the equations, so that it changes materially the successively approximating
(449) values of ρ'. This is evident by the inspection of the coefficient of $A.\rho'$, in the second and third lines of the first member of (440); where we see that when the interval of time is small, the term which is to be divided by A *is nearly equal to the sum of the other two*
(450) *terms of this coefficient, and has a different sign*; so that the resulting coefficient, arising from the difference of these expressions, is frequently so small as to be materially affected by the divisor A, which affects the largest term of this coefficient. Similar rermaks may be made
(451) relative to the three terms of the coefficient of $A^2.\rho'^2$, in the fourth line of the first member
(452) of the equation (440). Moreover *the intervals between the observations are required to be equal* in the equation (444); and the peculiar form of the second member of this equation is founded upon this circumstance; so that this method could not be applied, without some
(453) modification, when the intervals are unequal. Neither of these objections apply to the method of Dr. Olbers, because the fundamental equations (31,32,33), contain only the *known* coefficients of ρ, ρ^2, and the equations may be used whether the intervals be equal or
(454) unequal; the equal intervals being however the best. Finally, in consequence of introducing
(455) the three radii r, r', r'', into the equation (444), we are under the necessity of computing the coefficient of the equation (437), in Mr. Lubbock's method, as well as the value of A,
(456) neither of which are wanted in Dr. Olbers's method, or in the similar method of Mr. Ivory. Thus, we see, that these methods, which are the best now known by astronomers, have each
(457) their peculiar advantages and disadvantages. They are short and simple in their application; taking into view the difficulties of the problem; and, by either of them, an astronomer can obtain the elements of the orbit, in a few hours, instead of being employed several days, or
(458) weeks, as in the early calculations of the orbits of comets.

[5995] METHOD OF COMPUTING THE ELEMENTS OF THE ORBIT OF ANY HEAVENLY BODY; THERE BEING GIVEN THE TWO RADII r, r', THE INCLUDED ANGLE $v'-v=2f$, AND THE TIME $t'-t$ OF DESCRIBING THE ANGLE $2f$.

This is a very important problem, in the computation of the elements of the orbits of the planetary bodies; and the method of Gauss, which we shall give in [5999] depends
(1) essentially upon it. He has given two different solutions; the one by the process of quadratures; the other, by developing the quantities in series, and reducing them to tables,
(2) as in Tables VIII, IX, X. We shall restrict ourselves to this last method; which has different forms in the ellipsis, parabola, and hyperbola; and it is therefore necessary to
(3) consider each of them separately.

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TO FIND THE ELEMENTS OF AN ELLIPTICAL ORBIT.

In the first place we shall suppose the orbit to be *elliptical* and shall use the following symbols (6—16) which are similar to those in [5985]. For convenience of reference we (4) shall also insert in the table (17—67), most of the formulas which are used in this method; and shall afterwards give the demonstrations in (68&c.); (5)

r, r' the radii vectores; (6)

v, v' the mean anomalies; (7)

u, u' the excentric anomalies; (8)

the semi-parameter $p = a.(1-e^2) = a.\cos^2.\varphi = b.\cos.\varphi$; (9)

$a =$ the mean distance; *that of the sun from the earth being unity*; (10)

$b =$ the semi-conjugate axis $= a.\sqrt{1-e^2} = a.\cos.\varphi = \frac{p}{\cos.\varphi} = \sqrt{ap}$ [5985 (5), 378m]; (11)

$e =$ the excentricity $= \sin.\varphi$; $\sqrt{1-e^2} = \cos.\varphi$; (12)

$2f = v'-v$; $v = F - f$; (13)

$2F = v'+v$; $v' = F+f$; (14)

$2g = u'-u$; $u = G - g$; (15)

$2G = u'+u$; $u' = G+g$. (16)

$b.\sin.g = \sin.f.\sqrt{rr'}$; (17)

$b.\sin.G = \sin.F.\sqrt{rr'}$; (18)

$\sin.f.\sin.G = \sin.g.\sin.F$ (19)

$p.\cos.g = (\cos.f + e.\cos.F).\sqrt{rr'}$; (20)

$p.\cos.G = \{\cos.F + e.\cos.f\}.\sqrt{rr'}$; (21)

$\cos.f.\sqrt{rr'} = \{\cos.g - e.\cos.G\}.a$; (22)

$\cos.F.\sqrt{rr'} = \{\cos.G - e.\cos.g\}.a$; (23)

$\sqrt[4]{\frac{r'}{r}} = \text{tang}.(45^d + w)$; $\sqrt[4]{\frac{r}{r'}} = \text{tang}.(45^d - w)$; [Assumed value of w.] (24)

$r'-r = 2ae.\sin.g.\sin.G = \frac{4.\text{tang}.2w}{\cos.2w}.\sqrt{rr'}$; (25)

$r'+r = 2a - 2ae.\cos.g.\cos.G = 2a.\sin^2.g + 2.\cos.f.\cos.g.\sqrt{rr'}$. (26)

$= (2+4.\text{tang}^2.2w).\sqrt{rr'} = 2.\cos.f.(1+2l).\sqrt{rr'}$; (27)

$$\frac{\sqrt{\frac{r'}{r}} + \sqrt{\frac{r}{r'}}}{2.\cos.f} = 1 + 2l;$$ [Assumed value of l.] (28)

$\sqrt{\frac{r'}{r}} + \sqrt{\frac{r}{r'}} = (2+4.\text{tang}^2.2w)$; (29) When $\cos.f$ is positive.

$\sqrt{\frac{r'}{r}} - \sqrt{\frac{r}{r'}} = \frac{4.\text{tang}.2w}{\cos.2w}$; (30)

$l = \frac{\sin^2.\frac{1}{2}f}{\cos.f} + \frac{\text{tang}^2.2w}{\cos.f}$; (31)

$a = \frac{2.(l+\sin^2.\frac{1}{2}g).\cos.f.\sqrt{rr'}}{\sin^2.g}$; (32)

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(33) $\sqrt{a} = \pm \frac{\sqrt{\{2.(l+\sin.^2\frac{1}{2}g).\cos.f.\sqrt{rr'}\}}}{\sin.g}$; [Upper sign, if sin.$g$ be positive. Lower sign, if sin.g be negative.]

(34) $\frac{kt}{a^{\frac{3}{2}}} = u' - e.\sin.u' - u + e.\sin.u$;

(35) $= 2g - 2e.\sin.g.\cos.G$;

(36) $= 2g - \sin.2g + 2.\cos.f.\sin.g.\frac{\sqrt{rr'}}{a}$;

(37) $m = \frac{kt}{2^{\frac{3}{2}}.\cos.^{\frac{3}{2}}f.(rr')^{\frac{3}{4}}}$; [Assumed value of m.]

(38) $\log.m^2 = 5{,}5680729 + 2.\log.t - 3.\log.\cos.f - \frac{3}{2}.\log.(rr')$;

(39) $\pm m = (l+\sin.^2\frac{1}{2}g)^{\frac{1}{2}} + (l+\sin.^2\frac{1}{2}g)^{\frac{3}{2}}.\left(\frac{2g-\sin.2g}{\sin.^3g}\right)$; [Upper sign, if sin.$g$ be positive. Lower sign, if sin.g be negative.]

(40) $m = (l+x)^{\frac{1}{2}} + \frac{(l+x)^{\frac{3}{2}}}{\frac{3}{4}-\frac{9}{10}.(x-\xi)} = y.\sqrt{l+x}$; [Used when sine g and cosine f are positive.]

(41) $x = \sin.^2\frac{1}{2}g = \frac{1}{2}.(1-\cos.g) = \frac{1}{2}$ versed $\sin.g$; [Assumed value of x.]

(42) $X = \frac{2g-\sin.2g}{\sin.^3g} = \frac{1}{\frac{3}{4}-\frac{9}{10}.(x-\xi)}$; [Assumed value of X.]

(43) $\xi = x - \frac{5}{6} + \frac{10}{9X} = \frac{\sin.^3g - \frac{3}{4}.(2g-\sin 2g).(1-\frac{6}{5}.\sin.^2\frac{1}{2}g)}{\frac{9}{10}.(2g-\sin.2g)}$; [Assumed value of ξ.]

(44) $y = 1 + \frac{l+x}{\frac{3}{4}-\frac{9}{10}.(x-\xi)} = \frac{m}{\sqrt{l+x}}$; [Assumed value of y.]

(45) $h = \frac{m^2}{\frac{5}{6}+l+\xi}$; [Assumed value of h.]

(46) $h = \frac{(y-1).y^2}{y+\frac{1}{9}}$;

(47) $x = \frac{m^2}{y^2} - l$.

(48) $\frac{\sqrt{\frac{r'}{r}}+\sqrt{\frac{r}{r'}}}{2.\cos.f} = 1 - 2L$; [Assumed value of L.]

When cos.f is negative.

(49) $L = -\frac{\sin.^2\frac{1}{2}f}{\cos.f} - \frac{\tang.^2 2w}{\cos.f}$;

(50) $M = \frac{kt}{2^{\frac{3}{2}}.(-\cos.f)^{\frac{3}{2}}.(rr')^{\frac{3}{4}}}$; [Assumed value of M.]

(51) $a = \frac{-2.(L-\sin.^2\frac{1}{2}g).\cos.f.\sqrt{rr'}}{\sin.^2g}$;

(52) $\pm M = -(L-\sin.^2\frac{1}{2}g)^{\frac{1}{2}} + (L-\sin.^2\frac{1}{2}g)^{\frac{3}{2}}.\left(\frac{2g-\sin.2g}{\sin.^3g}\right)$; [Upper sign if sin.$g$ be positive. Lower sign if sin.g be negative.]

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$$M=-(L-x)^{\frac{1}{2}}+\frac{(L-x)^{\frac{3}{2}}}{\frac{3}{4}-\frac{9}{10}.(x-\xi)}=Y.\sqrt{L-x}\,; \qquad (53)$$

$$Y=-1+\frac{L-x}{\frac{3}{4}-\frac{9}{10}.(x-\xi)}=\frac{M}{\sqrt{L-x}}\,; \qquad \left[\begin{smallmatrix}\text{Assumed}\\ \text{value of } Y.\end{smallmatrix}\right] \qquad (54)$$

$$H=\frac{M^2}{L-\frac{5}{6}-\xi}\,; \qquad \left[\begin{smallmatrix}\text{Assumed}\\ \text{value of } H.\end{smallmatrix}\right] \qquad (55)$$

$$H=\frac{(Y+1).Y^2}{Y-\frac{1}{9}}\,; \qquad (56)$$

$$x=L-\frac{M^2}{Y^2}\,; \qquad (57)$$

$$a=2.\frac{m^2}{y^2}.\frac{\cos.f.\sqrt{rr'}}{\sin^2.g}\,; \qquad (58)$$

$$a=-2.\frac{M^2}{Y^2}.\frac{\cos.f.\sqrt{rr'}}{\sin^2.g}\,; \qquad (59)$$

$$p=\left(\frac{y.rr'.\sin.2f}{kt}\right)^2; \qquad (60)$$

$$p=\left(\frac{Y.rr'.\sin.2f}{kt}\right)^2; \qquad (61)$$

$\log.k=8{,}2355814\ldots.$ [5987 (8)]; (62)

$\log.k$ in seconds $=3{,}55000657\ldots.$ [5987 (14)]; (63)

with $a,p,$ we get $\cos.\varphi=\sqrt{1-e^2}=\sqrt{\frac{p}{a}}$; (11); (64)

$$\cos.G=\frac{\cos.g}{e}-\frac{\sqrt{rr'}.\cos.f}{ae}=\cos.g.\mathrm{cosec}.\varphi-\frac{\sqrt{rr'}}{a}\cos.f.\mathrm{cosec}.\varphi\,; \qquad (65)$$

$$\sin.F=\frac{\sin.f.\sin.G}{\sin.g}=\sin.f.\sin.G.\mathrm{cosec}.g\,; \qquad (66)$$

mean daily motion $=ka^{-\frac{3}{2}}$; or, (66)

log. mean daily motion in seconds $=3{,}55000657-\frac{3}{2}\log.a.$ (67)

Other formulas of a similar nature may be deduced from these, particularly the expressions of

$$\sin.(\tfrac{1}{2}f\mp\tfrac{1}{2}g)\,; \qquad \cos.(\tfrac{1}{2}f\mp\tfrac{1}{2}g)\,; \qquad \sin.(\tfrac{1}{2}F\mp\tfrac{1}{2}G)\,; \qquad \cos.(\tfrac{1}{2}F\mp\tfrac{1}{2}G)\,;$$

which may be conveniently used in logarithmic computations. In general, however, the use (67) of these auxiliary angles requires more labor than the common processes of spherical trigonometry; and the formulas we have given are all that are necessary. We shall now proceed to the demonstration of these formulas (17—67).

If we select the last values of $\sin.\frac{1}{2}u$, $\cos.\frac{1}{2}u$ [5985(12,13)], and then accent the symbols $r,\ v,\ u,$ we shall get the corresponding values of $\sin.\frac{1}{2}u'$, $\cos.\frac{1}{2}u'$; substituting these in (68) the first member of (69), it becomes as in its second member;

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(69) $\sin.\frac{1}{2}u'.\cos.\frac{1}{2}u \mp \cos.\frac{1}{2}u'.\sin.\frac{1}{2}u = \left\{\frac{rr'}{a^2.(1-e^2)}\right\}^{\frac{1}{2}}. \ \{\sin.\frac{1}{2}v'.\cos.\frac{1}{2}v \mp \cos.\frac{1}{2}v'.\sin.\frac{1}{2}v\}.$

Multiplying this by $b = a.(1-e^2)^{\frac{1}{2}}$ (11), and reducing, by means of [21, 22] Int. we get,

(70) $$b.\sin.\tfrac{1}{2}.(u' \mp u) = (rr')^{\frac{1}{2}}.\sin.\tfrac{1}{2}.(v' \mp v);$$

substituting the values (13—16) we get (17, 18); the upper sign giving (17), the lower, (18). Multiplying crosswise the two equations (17, 18), and dividing by $b.\sqrt{rr'}$, we get (19). In like manner, if we substitute the third values of [5985(12,13)], in the first member of (71), we obtain its second form, and by connecting together the terms depending on e, and reducing, by means of [23, 24] Int. we get (72),

(71) $$p.\{\cos.\tfrac{1}{2}u'.\cos.\tfrac{1}{2}u \pm \sin.\tfrac{1}{2}u'.\sin.\tfrac{1}{2}u\} = \{(1+e).\cos.\tfrac{1}{2}v'.\cos.\tfrac{1}{2}v \pm (1-e).\sin.\tfrac{1}{2}v'.\sin.\tfrac{1}{2}v\}.\sqrt{rr'}$$
$$= \left\{\begin{matrix}(\cos.\tfrac{1}{2}v'.\cos.\tfrac{1}{2}v \pm \sin.\tfrac{1}{2}v'.\sin.\tfrac{1}{2}v) \\ +e.(\cos.\tfrac{1}{2}v'.\cos.\tfrac{1}{2}v \mp \sin.\tfrac{1}{2}v'.\sin.\tfrac{1}{2}v)\end{matrix}\right\}.\sqrt{rr'}$$

(72) $$p.\cos.(\tfrac{1}{2}u' \mp \tfrac{1}{2}u) = \{\cos.(\tfrac{1}{2}v' \mp \tfrac{1}{2}v) + e.\cos.(\tfrac{1}{2}v' \pm \tfrac{1}{2}v)\}.\sqrt{rr'}.$$

Substituting (13—16), we find that the *upper* sign of this last expression gives (20), the *lower* (21). Multiplying (21) by $-e$, and adding the product to (20), we get,

(73) $$p.\{\cos.g - e.\cos.G\} = \sqrt{rr'}.(1-e^2).\cos.f;$$

substituting $p = a.(1-e^2)$ (9), and dividing by $1-e^2$, we get (22). In like manner, if we multiply (20) by $-e$, and add the product to (21), we get,

(74) $$p.\{\cos.G - e.\cos.g\} = \sqrt{rr'}.(1-e^2).\cos.F;$$

substituting the same value of p, and dividing by $1-e^2$ we obtain (23).

(75) We have, in [5985(9)], $r = a.(1-e.\cos.u)$, $r' = a.(1-e.\cos.u')$; taking the sum, and the difference of these quantities, we get, by means of [27, 28] Int.,

(76) $$r'-r = ae.\{\cos.u - \cos.u'\} = 2ae.\sin.\tfrac{1}{2}.(u'+u).\sin.\tfrac{1}{2}.(u'-u);$$

(77) $$r'+r = 2a - ae.\{\cos.u' + \cos.u\} = 2a - 2ae.\cos.\tfrac{1}{2}.(u'+u).\cos.\tfrac{1}{2}.(u'-u);$$

substituting the values (15, 16), we obtain the first forms of the values of $r'-r$, $r'+r$ (25, 26). The second expression (26), is deduced from the first, by changing the term $2a$ into $2a.(\sin^2.g + \cos^2.g)$, by which means we obtain,

(78) $$r'+r = 2a.\sin^2.g + \{\cos.g - e.\cos.G\}.a.2.\cos.g = 2a.\sin^2.g + \{\cos.f.\sqrt{rr'}\}.2.\cos.g \quad (22).$$

These admit of further reductions, by the introduction of the symbol w (24); and

(79) if we put for a moment $45^d + w = \mathrm{w}$, we shall have $\sqrt{\frac{r'}{r}} = \tan g^2.\mathrm{w}$; substituting this in the first member of (80), and successively reducing, by means of [34′, 32, 31] Int., we finally get the expression (81), which is the same as (29),

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$$\sqrt{\frac{r'}{r}}+\sqrt{\frac{r}{r'}}=\text{tang}^2.\text{w}+\text{cotan}^2.\text{w}=2+\{\text{tang.w}-\text{cotan.w}\}^2 \qquad (80)$$

$$=2+\left\{\frac{\text{sin.w}}{\text{cos.w}}-\frac{\text{cos.w}}{\text{sin.w}}\right\}^2=2+\left\{\frac{\sin^2.\text{w}-\cos^2.\text{w}}{\text{sin.w.cos.w}}\right\}=2+\left\{\frac{-\text{cos.}2\text{w}}{\frac{1}{2}.\text{sin.}2\text{w}}\right\}^2$$

$$=2+\{-2.\text{cotan.}2\text{w}\}^2=2+4.\text{tang}^2.2w. \qquad (81)$$

Multiplying this last expression by $\sqrt{rr'}$, we obtain the first value in (27); finally, if we multiply the assumed value of $1+2l$ (28), by $2.\sqrt{rr'}.\cos.f$, we shall get the second expression in (27); and, we may incidentally observe, that the comparison of (28) with (81) evidently shows that *l is positive.* The same expression (79), gives, (81)

$$\sqrt{\frac{r'}{r}}-\sqrt{\frac{r}{r'}}=\text{tang}^2.\text{w}-\cot^2.\text{w}=\frac{\sin^2.\text{w}}{\cos^2.\text{w}}-\frac{\cos^2.\text{w}}{\sin^2.\text{w}}=\frac{\sin^4.\text{w}-\cos^4.\text{w}}{\sin^2.\text{w}.\cos^2.\text{w}}; \qquad (82)$$

the numerator of this expression is easily reduced to the form,

$$(\sin^2.\text{w}+\cos^2.\text{w}).(\sin^2.\text{w}-\cos^2.\text{w})=\sin^2.\text{w}-\cos^2.\text{w}=-\text{cos.}2\text{w}=\text{sin.}2w; \qquad (82)$$

and the denominator is,

$$(\text{sin.w.cos.w})^2=(\tfrac{1}{2}.\text{sin.}2\text{w})^2=(\tfrac{1}{2}.\text{cos.}2w)^2=\tfrac{1}{4}.\cos^2.2w; \qquad (82')$$

hence we easily deduce the expression (30). Multiplying this by $\sqrt{rr'}$, we obtain the second form of (25). From the assumed value of $1+2l$ (28), we get, by substituting (81), the first expression of l (83); reducing by means of [1] Int., we get the last form in (83), which is the same as (31), and is composed of the given quantities $f,\ w$,

$$l=\frac{2+4.\text{tang}^2.2w}{4.\cos.f}-\tfrac{1}{2}=\frac{1-\cos.f}{2.\cos.f}+\frac{\text{tang}^2.2w}{\cos.f}=\frac{\sin^2.\frac{1}{2}f}{\cos.f}+\frac{\text{tang}^2.2w}{\cos.f}. \qquad (83)$$

Transposing the last term of the second expression (26), and dividing by $2.\sin^2.g$, we get successively, by using the last of the formulas (27);

$$a=\frac{r+r'-2.\cos.f.\cos.g.\sqrt{rr'}}{2.\sin^2.g}=\frac{2.\cos.f.(1+2l).\sqrt{rr'}-2.\cos.f.\cos.g.\sqrt{rr'}}{2.\sin^2.g} \qquad (84)$$

$$=\frac{\{2l+1-\cos.g\}.2.\cos.f.\sqrt{rr'}}{2.\sin^2.g}=\frac{\{2l+2.\sin^2.\frac{1}{2}g\}.2.\cos.f.\sqrt{rr'}}{2.\sin^2.g}. \qquad (85)$$

This last expression is easily reduced to the form (32); and its square root is as in (33); to which the double sign $\pm$ is prefixed, so that $\frac{\sqrt{l+\sin^2.\frac{1}{2}g}}{\sin.g}$, or, $\frac{\sqrt{l+x}}{\sin.g}$ (41), *may be considered as a positive quantity.* (85')

Substituting n [5987(12)] in [5985(7)], and neglecting the mass m, on account of its smallness, we get the first formula (87); the second is deduced from the first, by accenting $t',\ u'$. (86)

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(87) $$\frac{kt}{a^{\frac{3}{2}}} = u - e.\sin.u\,; \qquad \frac{kt'}{a^{\frac{3}{2}}} = u' - e.\sin.u'.$$

Subtracting the first of these expressions from the second, and for $t'-t$, which represents the interval of time between the observations, putting simply t, we get the expression (34).

(88) This is easily reduced to the form (35) by substituting $u'-u=2g$ (15), and,

(89) $$\sin.u' - \sin.u = 2.\sin.(\tfrac{1}{2}u' - \tfrac{1}{2}u).\cos.(\tfrac{1}{2}u' + \tfrac{1}{2}u) = 2.\sin.g.\cos.G \quad (15, 16)\,;$$

but from (22), we have,

(90) $$e.\cos.G = \cos.g - \cos.f.\,\frac{\sqrt{rr'}}{a}\,;$$

substituting this in (35), and putting $2.\sin.g.\cos.g = \sin.2g$, it becomes as in (36). The symbol m (37), is used for brevity, and when $\cos.f$ is positive, the expression of m,

(90') will be a *real and positive quantity*; being a function of the given quantities $r, r' f, t, k$; and its equivalent logarithmic expression is given in (38); using the value of $\log.k$ [5987(8)]. Multiplying (37) by the denominator of its second member, we get,

(91) $$kt = m.2^{\frac{3}{2}}.\cos.^{\frac{3}{2}}f.(rr')^{\frac{3}{4}}\,;$$

substituting this in (36), and then multiplying by $a^{\frac{3}{2}}$, we obtain,

(92) $$m.2^{\frac{3}{2}}.\cos.^{\frac{3}{2}}f.(rr')^{\frac{3}{4}} = (2g - \sin.2g).a^{\frac{3}{2}} + 2.\cos.f.\sin.g.(rr')^{\frac{1}{2}}.a^{\frac{1}{2}}.$$

Using the value of $\sqrt{a}$ (33), we find that each term of the expression contains the factor $\cos.^{\frac{3}{2}}f.(rr')^{\frac{3}{4}}$, and by rejecting it, we get,

(93) $$m.2^{\frac{3}{2}} = \pm\,(2g - \sin.2g).\frac{\{2.(l+\sin.^2\tfrac{1}{2}g)\}^{\frac{3}{2}}}{\sin.^3 g} \pm 2.\{2.(l+\sin.^2\tfrac{1}{2}g)\}^{\frac{1}{2}}\,;$$

dividing this by $\pm 2^{\frac{3}{2}}$, we get the expression of $\pm m$ (39); the order of the terms being changed. This equation contains the *known quantities* l, m; and from it we may determine the *unknown quantity* g. In the case which most frequently occurs, g is so

(94) small that the common tables of logarithms do not give the factor $\frac{2g-\sin.2g}{\sin.^3 g} = X$ (42), with a sufficient degree of accuracy. In this case, we must develop it, in a series, ascending according to the powers of $\sin.\frac{1}{2}g$; and then the value of the factor, which is represented

(95) by the assumed symbol X, can be obtained with accuracy, in the following manner.

Changing y into $\sin.\frac{1}{2}g$ in [46] Int. we get the value of the arc $\frac{1}{2}g$, in terms of

(96) $\sin.\frac{1}{2}g$; multiplying this by 4, we get the expression of $2g$ (98). Moreover,

(97) $\sin.2g = 2.\sin.g.\cos.g$; $\sin.g = 2.\sin.\frac{1}{2}g.\cos.\frac{1}{2}g$; $\cos.g = 1 - 2.\sin.^2\frac{1}{2}g$; hence,

$$\sin.2g = 4.\sin.\tfrac{1}{2}g.(1-2.\sin.^2\tfrac{1}{2}g).\cos.\tfrac{1}{2}g\,; \qquad [5995]\ (97)$$

and since,

$$\cos.\tfrac{1}{2}g = (1-\sin.^2\tfrac{1}{2}g)^{\frac{1}{2}} = 1-\tfrac{1}{2}.\sin.^2\tfrac{1}{2}g-\tfrac{1}{8}.\sin.^4\tfrac{1}{2}g-\&c.,$$

we find, that $\sin.2g$ becomes as in (99); subtracting this from (98), we get $2g-\sin.2g$ (100), being the numerator of the value of X (94),

$$2g = 4.\sin.\tfrac{1}{2}g+\tfrac{2}{3}.\sin.^3\tfrac{1}{2}g+\tfrac{3}{10}.\sin.^5\tfrac{1}{2}g+\&c.\,;$$

$$\sin.2g = 4.\sin.\tfrac{1}{2}g-10.\sin.^3\tfrac{1}{2}g+\tfrac{7}{2}.\sin.^5\tfrac{1}{2}g-\&c.\,;$$

$$2g-\sin.2g = \tfrac{32}{3}.\sin.^3\tfrac{1}{2}g-\tfrac{32}{10}.\sin.^5\tfrac{1}{2}g-\&c. = \tfrac{32}{3}.\sin.^3\tfrac{1}{2}g.\{1-\tfrac{3}{10}.\sin.^2\tfrac{1}{2}g-\&c.\}.$$

The denominator of X (94) is,

$$\sin.^3g = (2.\sin.\tfrac{1}{2}g.\cos.\tfrac{1}{2}g)^3 = 8.\sin.^3\tfrac{1}{2}g.\{1-\tfrac{3}{2}.\sin.^2\tfrac{1}{2}g-\&c\}\,; \qquad (101)$$

dividing the expression of the numerator (100), by that of the denominator (101), we get,

$$X = \tfrac{4}{3}.\{1+\tfrac{6}{5}.\sin.^2\tfrac{1}{2}g+\&c.\}\,;$$

expressed in a series ascending according to the powers of $\sin.^2\tfrac{1}{2}g = x$ (41). To obtain the law of this series, we shall resume the expression of X (94), which gives,

$$X.\sin.^3g = 2g-\sin.2g.$$

Taking its differential, and dividing by dg, we obtain,

$$\frac{dX}{dg}.\sin.^3g+3X.\sin.^2g.\cos.g = 2-2.\cos.2g = 4.\sin.^2g.$$

The differential of $x=\sin.^2\tfrac{1}{2}g$ (41), gives,

$$dx = dg.\sin.\tfrac{1}{2}g\ \cos.\tfrac{1}{2}g = \tfrac{1}{2}dg.\sin.g, \quad \text{or} \quad dg = \frac{2dx}{\sin.g}\,;$$

substituting this in (104), and dividing by $\tfrac{1}{2}.\sin.^4g$, we obtain,

$$\frac{dX}{dx} = \frac{8-6X.\cos.g}{\sin.^2g}\,;$$

but, from (41), we get,

$$\cos.g = 1-2x; \quad \sin.^2g = 1-\cos.^2g = 1-(1-2x)^2 = 4x-4x^2;$$

substituting these in (105′), and multiplying by $2x-2xx$, we finally obtain,

$$(1-x).2x.\frac{dX}{dx} = 4-3.(1-2x).X. \qquad (107)$$

Now if we assume for X, an expression of the form (108), c_1, c_2, &c., being constant; we shall find, that its differential, divided by dx, will become as in (109). Substituting these in (107), we get (110);

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(108) $$X = \tfrac{4}{3}.\{1 + c_1.x + c_2.x^2 + c_3.x^3 + c_4.x^4 + \&c.\};$$

(109) $$\frac{dX}{dx} = \tfrac{4}{3}.\{c_1 + 2c_2.x + 3c_3.x^2 + 4c_4.x^3 + \&c.\};$$

(110) $$\tfrac{8}{3}.\{c_1.x + (2c_2 - c_1).x^2 + (3c_3 - 2c_2).x^3 + (4c_4 - 3c_3).x^4 + \&c.\} = (8 - 4c_1).x + (8c_1 - 4c_2).x^2 + (8c_2 - 4c_3).x^3 + \&c.$$

Putting the coefficients of the different powers of x equal to nothing, we get, successively,

(111) $$c_1 = \tfrac{6}{5}; \qquad c_2 = \tfrac{8}{7}.c_1; \qquad c_3 = \tfrac{10}{9}.c_2; \qquad c_4 = \tfrac{12}{11}.c_3 \ \&c.;$$

the law of continuation being manifest; substituting these in (108), we finally obtain,

(112) $$X = \tfrac{4}{3} + \frac{4.6}{3.5}.x + \frac{4.6.8}{3.5.7}.x^2 + \frac{4.6.8.10}{3.5.7.9}.x^3 + \frac{4.6.8.10.12}{3.5.7.9.11}.x^4 + \&c.$$

This value of X may be computed by means of a table, with the argument x; but it is much more convenient to find and use the small quantity ξ (43), *of the order* x^2 (115), *or of the fourth order in* g, instead of X (112), which contains terms of the order
(113) x. If we divide the fraction $\frac{10}{9}$, by the expression of X (112), we shall get,

(114) $$\frac{10}{9X} = \tfrac{5}{6} - x + \tfrac{2}{35}.x^2 + \tfrac{52}{1575}.x^3 + \&c.;$$

substituting this in the assumed form of ξ (43), namely, $\xi = x - \frac{5}{6} + \frac{10}{9X}$; we get,

(115) $$\xi = \tfrac{2}{35}.x^2 + \tfrac{52}{1575}.x^3 + \&c.$$

With this formula we may compute the values of ξ, as in table IX, for the small values of x, when the usual tables would not be sufficiently accurate. The numbers in this
(116) table are given for the values of x, from $x = 0{,}001$, to $x = 0{,}300$. This last
(117) value corresponds to $g = 66^d\,25^m$; and for greater values, if any should occur in practice, we may use the indirect method of solving the equation (39), in its present form without making any reduction; assuming a value of g, and repeating the process,
(118) till we obtain an expression which will satisfy that equation. From the first expression of ξ (43), we easily deduce the second value of X (42). Finally, if we substitute the
(119) assumed value of X (94), in the first value of ξ (43), it becomes successively, by using x (41),

(120) $$\xi = x - \tfrac{5}{6} + \tfrac{10}{9}.\frac{\sin.^3 g}{2g - \sin.2g} = \sin.^2\tfrac{1}{2}g - \tfrac{5}{6} + \frac{\sin.^3 g}{\frac{9}{10}.(2g - \sin.2g)};$$

and this last expression is easily reduced to the second form in (43).

(121) In the case now under consideration, $\sin.g$ is positive; so that we must use the upper sign of the value of m (39); and by substituting $\sin.^2\frac{1}{2}g = x$ (41); also the second

value of X (42), it becomes as in the first expression of m (40); the second form [5995]
is deduced from the first, by the substitution of the first assumed value of y (44). The (122)
second form y (44), is easily deduced from the second expression of m (40). Squaring this, we get,

$$l+x=\frac{m^2}{y^2}\,; \qquad \text{whence,} \qquad x=\frac{m^2}{y^2}-l \quad \text{as in (47)}\,; \tag{123}$$

and if we use the assumed value of h (45), which gives,

$$\tfrac{5}{6}+l+\xi=\frac{m^2}{h}\,; \tag{123}$$

we shall get successively,

$$\begin{aligned}\tfrac{3}{4}-\tfrac{9}{10}\cdot(x-\xi)=\tfrac{9}{10}\cdot(\tfrac{5}{6}-x+\xi)=\tfrac{9}{10}\cdot\left(\tfrac{5}{6}+l+\xi-\frac{m^2}{y^2}\right)&=\tfrac{9}{10}\cdot\left\{\frac{m^2}{h}-\frac{m^2}{y^2}\right\}\\ &=\frac{9m^2}{10y^2}\cdot\left\{\frac{y^2}{h}-1\right\}.\end{aligned} \tag{124}$$

Substituting this, and $l+x$ (123), in the first expression of y (44), we obtain,

$$y=1+\frac{\frac{10}{9}}{\frac{y^2}{h}-1}\,; \qquad \text{or,} \qquad (y-1)\cdot\frac{y^2}{h}-(y-1)=\tfrac{10}{9}\,; \tag{125}$$

whence we easily deduce the expression of h (46).

When the heliocentric motion is between 180^d and 360^d; or generally when $\cos.f$ is
negative, the value of m deduced from (37) becomes imaginary, and l (31) is negative. (126)
To avoid this we must change,

l into $-L$; m into $-M.\sqrt{-1}$ or $M.(-1)^{\frac{3}{2}}$; y into $-Y$, and h into H; (127)

by this means, we find that (28) changes into (48); (31) into (49); (37) into (50), after
dividing by $(-1)^{\frac{3}{2}}$; (32) into (51); (39) into (52), after dividing by $(-1)^{\frac{3}{2}}$; (40) (128)
into (53), divided in the same manner; (44) into (54), after dividing by -1; (45) into (55), changing the signs of the numerator and denominator; (46) into (56), with the same changes of the signs; lastly, (47) into (57).

To determine the value of y, or rather of $\log.yy$, from the cubic equation (46), a
table was computed by Gauss, being the same as Table VIII, of the present collection. (129)
This table answers also for computing $\log.YY$ from H, as is evident from the consideration, that if we change y into $-Y$, yy changes into YY, the equation (46) for finding
y, changes into that in (56) for finding Y, and $\log.yy$ changes into $\log.YY$. This table (130)
is calculated from $h=0$, to $h=0{,}6$. From 0 to 0,04 the intervals in the values of h are (131)
taken equal to 0,0001, which do not require the use of second differences; and this is by

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far the most important part of the table; from 0,04 to 0,60 the intervals are 0,001, and then
(132) it is necessary to notice the second differences, if we wish to have the logarithms correct in the last figure of the decimals. If h exceed the limit of the table, we may obtain the
(133) solution of the cubic equation (46 or 56), by any indirect process, or by some one of the well known methods of solution.

The values of l, m, h (31, 37, 45) are positive; and as it is supposed in the equations (49, 50) that $\cos. f$ is negative, (126), we shall also have L and M positive. We have,
(134) by [32] Int. $-\sin^2.\frac{1}{2}f = \cos. f - \cos^2.\frac{1}{2}f$; substituting this in the value of L (49) it
(135) becomes $L = 1 + \frac{\cos^2.\frac{1}{2}f}{(-\cos. f)} + \frac{\text{tang}^2.2w}{(-\cos. f)}$; and as each term is positive, we shall have $L > 1$; therefore H (55) is also positive, ξ being small (115 &c.); moreover as
(136) m, $\sqrt{l+x}$ (90′, 85′) are positive, we shall have $y = \frac{m}{\sqrt{l+x}}$ (44) positive; and for similar
(137) reasons $Y = \frac{M}{\sqrt{L-x}}$ (54) is positive. If we now trace the successive values of h, while y decreases from ∞ positive, to 0, we shall see, by the mere inspection of the
(138) second member of the formula (46), that h decreases with y; becomes 0, when $y = 1$; and is negative, when y falls between 1 and 0; so that there is always one
(139) positive value of y, which exceeds 1, and will satisfy the equation (46), for any positive
(140) value of h, from $h = 0$, to $h = \infty$. In like manner, by the inspection of the equation (56), we find that while Y decreases from ∞ positive to $Y = \frac{1}{9}$, H will remain
(141) positive; and that it will become negative when Y falls between 0 and $\frac{1}{9}$; so that we
(142) have always one positive value of Y, which exceeds $\frac{1}{9}$, and satisfies the equation (56),
(143) for all positive values of H, from $H = \infty$, to its least limit. After this digression on the nature of the roots of the equations (46, 56), we shall now proceed to the explanation of the
(144) manner in which these roots are obtained by approximation.

If ξ be known we shall have the value of h (45) or H (55); and then from the cubic
(145) equation (46 or 56) we can obtain y, or Y; and finally, from (47 or 57), the value of
(146) x. Now as ξ is a very small quantity of the fourth order in g (113), *we may at first*
(147) *neglect it in the values of* h or H (45 or 55), putting $h = \frac{mm}{\frac{5}{6}+l}$, or $H = \frac{MM}{L-\frac{5}{6}}$. With this value of h or H, we find, from Table VIII, the corresponding value of $\log.\, yy$,
(148) or $\log.\, YY$; whence we obtain, from (47 or 57) the value of x, and with this we get, in Table IX, the corresponding value of ξ. Having obtained ξ, we may repeat the
(149) calculation, using (45 or 55), to obtain a corrected value of x; and generally, one operation will be sufficient to get the true result. *Having found* x, *we get* g *from the equation*
(150) (41), $x = \sin^2.\frac{1}{2}g$, or versed sine $g = 2x$. We may here remark, that both of the angles $u' - u = 2g$, and $v' - v = 2f$, (13, 15) fall between 0^d and 360^d; or between the
(151) same multiples of 360^d; consequently the angles g, f, fall between the same multiples of 180^d.

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Now considering g *as a known quantity, we shall proceed in the investigation of the formulas* (58—67), *for the determination of the elements of the orbit.* We have, from the (152)
equations (40, 41) $l + x = l + \sin^2.\tfrac{1}{2}g = \frac{m^2}{y^2}$; substituting this expression of $l+\sin^2.\tfrac{1}{2}g$, (153)
in (32), we get the value of a (58). In like manner, from (53, 41), we have,

$$L - x = L - \sin^2.\tfrac{1}{2}g = \frac{M^2}{Y^2}; \qquad (154)$$

substituting this in (51), we get the value of a (59). Dividing the square of the equation (17) by the expression of a (58), and rejecting the factor $\sin^2.g$, which occurs in both members of the equation, we get the first expression (155). Substituting the value of m^2 (37), we get its second form; and the third form is easily deduced from this, by using $2.\sin.f.\cos.f = \sin.2f$;

$$\frac{b^2}{a} = \frac{y^2.\sin^2.f.(rr')^{\frac{1}{2}}}{2m^2.\cos.f} = \frac{y^2.(rr')^2.(2.\sin.f.\cos.f)^2}{k^2t^2} = \left\{\frac{y.rr'.\sin.2f}{kt}\right\}^2; \qquad (155)$$

now we have $\frac{b^2}{a} = p$ (11); hence we get the expression of p (60). In like manner, (156)
by squaring the equation (17), then dividing by the expression of a (59), and substituting M^2 (50), we get (61). Now if a planet revolve about the sun, in a circular orbit, at the distance a; the angular motion in the time t will be represented by $nt = \frac{tk}{a^{\frac{3}{2}}}$ [5987(12)], (157)
neglecting the mass of the planet, on account of its smallness. Multiplying this by $\tfrac{1}{2}a^2$, we get the area of the circular sector $\tfrac{1}{2}.\sqrt{a}.kt$, described by the radius vector, in the time (158)
t, in this circular orbit, whose mean distance, or semi-parameter is a. If we retain the same mean distance, and suppose the orbit to be an ellipsis, whose semi-parameter is p (9), (159)
the area described by the radius vector, will be decreased in the ratio of the square roots of the parameters of $\sqrt{p}$ to $\sqrt{a}$ [383''], and it will therefore become $\tfrac{1}{2}.\sqrt{p}.kt$ (158); (160)
which may represent in figure 84, page 792, the area of the sector sab; included between the radii $Sa = r$, $Sb = r'$, and the elliptic arc ab. On the other hand, the area of the (161)
triangle Sab, included between the radii $Sa = r$, $Sb = r'$, and the chord ab, is (162)
represented, in [5994(300')], by

$$\tfrac{1}{2}.[rr'] = \tfrac{1}{2}rr'.\sin.(v'-v) = \tfrac{1}{2}rr'.\sin.2f \ (13). \qquad (163)$$

Dividing the area of the sector (160), by that of the triangle (163), we obtain the ratio of these two areas as in the first of the following expressions; and by comparing it with the value of y, deduced from (60), or that of Y from (61); we find that they are equal (163)
to each other, as in the third and fourth expressions (164);

$$\frac{\text{area of the sector } sab}{\text{area of the triangle } sab} = \frac{\tfrac{1}{2}.\sqrt{p}.kt}{\tfrac{1}{2}rr'.\sin.2f} = y = Y. \qquad (164)$$

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(165) *Hence it appears that* y *or* Y *represents the ratio of the area of the elliptical sector* sab, *to that of the triangle* sab. If we substitute,

(165) $$\sqrt{l+\sin^2\tfrac{1}{2}g}=\sqrt{l+x}=\frac{m}{y}\quad(41,40),$$

(166) and X (42) in (39), we get the expression of m (168), corresponding to figure 84, page 792; $\sin.g$ being supposed positive. In like manner, if we substitute,

(167) $$\sqrt{L-\sin^2\tfrac{1}{2}g}=\sqrt{L-x}=\frac{M}{Y}\quad(53),$$

in (52), we get the value of M (169), corresponding to $\sin.g$ positive,

(168) $$m=\frac{m}{y}+\frac{m^3}{y^3}.X;$$

(169) $$M=-\frac{M}{Y}+\frac{M^3}{Y^3}.X.$$

Now if we suppose the quantity m, which is proportional to the time t (37), to represent the area of the sector sab; the quantity $\frac{m}{y}$ (164), will represent the area of the triangle
(170) sab (164); and their difference, which is $\frac{m^3}{y^3}.X$ (168), will therefore represent the area of the segment, included between the chord ab, and the elliptic arc ab. Similar remarks
(171) may be made relative to M (169), observing that when the angle bsa exceeds 180^d, we have the sector equal to the difference between the segments and the triangle. Hence it is manifest that the quantities m, $(l+x)^{\frac{1}{2}}$, $(l+x)^{\frac{3}{2}}.\frac{X}{y^3}$, in the equation (39 or 40);
(172) and the quantities M, $(L-x)^{\frac{1}{2}}$, $(L-x)^{\frac{3}{2}}.\frac{X}{Y^3}$ in (52 or 53), are respectively proportional
(173) to the sector, the triangle, and the segment; and these geometrical considerations serve very
(174) much to illustrate this part of the calculation. We shall now show the use of these formulas, by the following examples, given by Gauss.

EXAMPLE I.

Given, $\log.r=0{,}1394892$, $\log.r'=0{,}3978794$, $v'-v=2f=224^d$, $t=206^{\text{days}}{,}80919$; to find the
(175) elements of the orbit a, p, e; the true anomalies v, v'; and the excentric anomalies u, u'. In this example, the value of Y exceeds the limits of Table VIII; we must, therefore, in this case, deduce Y from the original cubic equation (56), instead of using that table. We have computed G, in (181), by the formula (65); we may also determine $\sin.G$ by (25); and we find, from these formulas, that $\sin.G$ and $\cos.G$, are positive, therefore G (182), falls in the first quadrant of the circle, [5990, (23, 24)]. In like manner, we have computed $\sin.F$ (183) from (66); we may also compute $\cos.F$ from (23), and as both expressions are positive, F must also fall in the first quadrant.

To find x.

r'	log.	0,3978794		0,3978794
r	log.	0,1394892		0,1394892
$\frac{r'}{r} = \text{tang}^4.(45^d+w)$	log.	0,2583902	sum	0,5373686
$45^d+w=49^d 14^m 43^s,78$	tang.	0,0645975	half	0,2686843
$w=4^d\ 14^m\ 43^s,78$			$(rr')^{\frac{3}{2}}$log	0,8060529
			ar.co.	9,1939471
$2w=8^d\ 29^m\ 27^s,56$			tang.	9,1740314
			same	9,1740314
$-\frac{\text{tang}^2.2w}{\cos f}$	=	0,0594959	ar.co.cos.	$0,4264246_n$
			log.	$8,7744874_n$
$f=112^d$			ar.co.cos.	$0,4264246_n$
$\frac{1}{2}f=56^d$			sine	9,9185742
			same	9,9185742
$-\frac{\sin^2.\frac{1}{2}f}{\cos f}$	=	1,8347335	log.	$0,2635730_n$
sum is L	=	1,8942294		
$\frac{5}{6}$	=	0,8333333		
$L-\frac{5}{6}$	=	1,0608961	log.	0,0256728
MM (176)			log.	0,6724334
Approx. H			log.	0,6467606

Hence from the cubic equation (56), we get, Approximate Y = 1,591432

			YY log.	0,4035702
			MM log.	0,6724334
$\frac{MM}{YY}$	=	1,8571935	log.	0,2688572
L	=	1,8942294		
Approximate x	=	0,0370359		
Corresponding ξ	=	0,0000801	in Table IX.	
$L-\frac{5}{6}$	=	1,0608961		
$L-\frac{5}{6}-\xi$	=	1,0608160	log.	0,0256401
			MM log.	0,6724334
Corrected H			log.	0,6467933

Hence we get from (56), corrected Y = 1,591511

			YY log.	0,4036192
			MM log.	0,6724334
$\frac{MM}{YY}$	=	1,8570098	log.	0,2688142
L	=	1,8942294		
Corrected x	=	0,0372196		
Corresponding ξ	=	0,0000809	in Table IX.	
$L-\frac{5}{6}$	=	1,0608961		
$L-\frac{5}{6}-\xi$	=	1,0608152	log.	0,0256397
			MM log.	0,6724334
Corrected H			log.	0,6467937

Hence we get from (56), corrected Y = 1,5915124

			YY log.	0,4036200
			MM log.	0,6724334
$\frac{MM}{YY}$	=	1,8570064	log.	0,2688134
L	=	1,8942294		
$x=\sin^2.\frac{1}{2}g$	=	0,0372230		

To find M.M. (50) [5995]

	constant log.	5,5680729	
$t=206^{\text{days}},80919$	log.	2,3155698	
	same	2,3155698	
arith. co. log. $(-\cos.f)\times 3$		1,2792738	
$\frac{3}{2}$ log. $r\ r'$	arith. comp.	9,1939471	
MM	log.	0,6724334	(176)

To find a. (59)

$x=\sin^2.\frac{1}{2}g$		log.	8,5708114	
$\frac{1}{2}g$	$11^d\ 07^m\ 26^s,3$	sin.	9,2854057	
g	$22^d\ 14^m\ 52^s,6$	cosec.	0,4218017	(177)
		same	0,4218017	
$\frac{MM}{YY}$		log.	0,2688134	
2		log.	0,3010300	
$-\cos.f$		log.	9,5735754	
$\sqrt{rr'}$		log.	0,2686843	
a		log	1,2557065	178

To find p, and $e=\sin.\varphi$. (61, 64)

k	ar. co.	log.	1,7644186	
t	ar. co.	log.	7,6844302	
rr'		log.	0,5373686	
$2f$		sin.	$9,8417713_n$	
$-Y$		log.	$0,2018100_n$	
$\sqrt{p}$		log.	0,0097987	
$\sqrt{a}$		log.	0,6278532	
$\varphi=75^d\ 23^m\ 07^s,3$		cos.	9,4016355	(179)

To find F, G, v, v', u, u'. (65, 66)

φ	cosec.	0,0142840	(180)
g	cos.	9,9669018	
$\cos.\ g.\ \text{cosec}.\ \varphi = 0,9565018$	log.	9,9811858	
φ	cosec.	0,0142840	
a	ar. co. log.	8,7442935	
f	cos.	$9,5735754_n$	
$-\sqrt{rr'}$	log.	$0,2686843_n$	
$-\frac{\sqrt{rr'}}{a}\cos.f.\ \text{cosec}.\ \varphi = 0,0398875$	log.	8,6008372	
$\cos.\ G = 0,9963893$	log.	9,9984291	(181)
G $4^d\ 52^m\ 13^s$	sin.	8,9289080	(182)
f	sin.	9,9671659	
g	cosec.	0,4218017	
$F=\ 12^d$	sin.	9,3178756	(183)

$f=\ 112^d$

$v=F-f=-100^d$

$v'=F+f=\ 124^d$

$G\ =\ 4^d\ 52^m\ 13^s$

$g\ =\ 22^d\ 14^m\ 53^s$

$u=G-g=-17^d\ 22^m\ 40^s$

$u'=G+g=\ 27^d\ 07^m\ 06^s$ (184)

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EXAMPLE II.

(185) Given $\log. r = 0{,}3307640$, $\log. r' = 0{,}3222239$, $v' - v = 2f = 7^d\, 34^m\, 53^s{,}73$, $t = 21^{\text{days}}{,}93391$; to find the elements of the orbit a, p, $e = \sin.\phi$; the true anomalies v, v'; and the excentric anomalies u, u'.

(186) A considerable part of the calculation of this example, is given in the introduction to tables VIII, IX; and it is unneccessary to repeat it here; we shall merely give some of the results of this part of the process; namely,

$$m = -\,8^m\, 27^s;\qquad l = 0{,}0011205685;\qquad \log.\frac{m^2}{y^2} = 7{,}2715133;\qquad \log.yy = 0{,}0021633;$$

(187)

$$\log.m^2 = 7{,}2736766;\qquad \log.\sqrt{rr'} = 0{,}3264940;\qquad x = \sin^2.\tfrac{1}{2}g = 0{,}0007480186.$$

With these we shall compute a by the formula (58); p from (60); ϕ or e from (64); G from (65); F from (66); then v, v', u, u', from (13—16).

To find a

$x = \sin^2.\frac{1}{2}g$		log.	6,8739124
(188) $\frac{1}{2}g$	$1^d\, 34^m\, 02^s{,}03$	sin.	8,4369562
g	$3^d\, 08^m\, 04^s{,}06$	cosec.	1,2621764
		same	1,2621764
$\frac{m^2}{y^2}$		log.	7,2715133
2		log.	0,3010300
f		cos.	9,9990488
(189) $\sqrt{rr'}$		log.	0,3264940
a		log.	0,4224389

(190) *To find* p, *and* $e = \sin.\phi$.

k	ar.co.log.	1,7644186
(191) t	ar.co.log.	8,6588840
rr'	log.	0,6529879
$2f$	sin.	9,1203696
η	log.	0,0010816
(192) $\sqrt{p}$	log.	0,1977417
$\sqrt{a}$	log.	0,2112194
(193) $\phi = 14^d\, 12^m\, 02^s{,}0$	cos.	9,9865223
$\log.e = \log.\sin.\phi$		9,3897273

To find v, v', u, u'.

g		cos.	9,9993498
ϕ		cosec.	0,6102727
$\cos.g.\text{cosec}.\phi =$	4,0700635	log.	0,6096225
ϕ		cosec.	0,6102727
a		ar.co.log.	9,5775611
f		cos.	9,9990488
$-\sqrt{rr'}$		log.	$0{,}3264940_n$
$-\frac{\sqrt{rr'}}{a}.\cos.f.\text{cosec}.\phi =$	$-3{,}2611940$	log.	$0{,}5133766_n$
$\cos.G =$	0,8088695	log.	9,9079858
$G =$	$324^d\, 00^m\, 18^s{,}4$	sin.	$9{,}7691653_n$
$f =$	$3^d\, 47^m\, 26^s{,}865$	sin.	8,8202909
g		cosec.	1,2621764
$F =$	$314^d\, 42^m\, 54^s{,}95$	sin.	$9{,}8516326_n$
$f =$	$3^d\, 47^m\, 26^s{,}86$		
$v = F - f =$	$310^d\, 55^m\, 28^s$		
$v' = F + f =$	$318^d\, 30^m\, 22^s$		
$G =$	$324^d\, 00^m\, 18^s{,}4$		
$g =$	$3^d\, 08^m\, 04^s{,}1$		
$u = G - g =$	$320^d\, 52^m\, 14^s$		
$u' = G + g =$	$327^d\, 08^m\, 23^s$.		

In this example, $\cos.\ G$ is positive (189); but $\sin.\ G$ (25) is negative, because $r' - r$ is negative; therefore G must fall in the fourth quadrant [5990, (23, 24)]. Again, $\sin.\ F$ (190) is negative, and $\cos.\ F$, deduced from (22), is positive; therefore F falls in the fourth quadrant.

These examples will suffice for illustrating the calculations in an elliptic orbit; we shall now proceed to explain the similar calculations in a parabolic orbit.

TO FIND THE ELEMENTS OF A PARABOLIC ORBIT, THERE BEING GIVEN $r, r', v'-v=2f$. [5996]

In a parabolic orbit, we shall use the symbols (2—10), most of them being similar to those in an ellipsis [5995(6, &c.)]. We shall also insert in the same table (11—25), several formulas which are useful in these calculations; and shall afterwards give the demonstration in (26—60). (1)

r, r', the radii vectores; (2)

Symbols.

v, v', the mean anomalies; (3)

$p=2D$, the semi-parameter; [5986(2)]. (4)

$D=\frac{1}{2}p$, the perihelion distance; (5)

$2f=v'-v$; $\qquad v=F-f$; (6)

$2F=v'+v$; $\qquad v'=F+f$; (7)

$r'=r.\text{tang}^2.z$; (8)

$\cos.y=\cos.f.\sin.2z$; (9)

$Ck=1-\frac{2}{3}.\sin^2.\frac{1}{2}y$; $\qquad \log.k=8,2355814\ldots$ [5987(8)]; (10)

$$\sqrt{\frac{p}{2r}}=\cos.(\tfrac{1}{2}F-\tfrac{1}{2}f)=\cos.\tfrac{1}{2}v;$$ (11)

Formulas in a parabolic orbit.

$$\sqrt{\frac{p}{2r'}}=\cos.(\tfrac{1}{2}F+\tfrac{1}{2}f)=\cos.\tfrac{1}{2}v';$$ (12)

$$\frac{p}{\sqrt{rr'}}=\cos.F+\cos.f;$$ (13)

$$\frac{p.(r+r')}{2rr'}=1+\cos.F.\cos.f;$$ (14)

$$p=\frac{2rr'.\sin^2.f}{r+r'-2.\cos.f.\sqrt{rr'}}=r.\left(\frac{\sin.z.\sin.f}{\sin.\frac{1}{2}y}\right)^2;$$ (15)

$$kt=\frac{2.\sin.f.\cos.f.rr'}{\sqrt{p}}+\frac{4.\sin^3.f.(rr')^{\frac{3}{2}}}{3p^{\frac{3}{2}}}=\frac{\sqrt{2}}{3}.\left\{r+r'+\cos.f.\sqrt{rr'}\right\}.\left\{r+r'-2.\cos.f.\sqrt{rr'}\right\}^{\frac{1}{2}}$$ (16)

$$=Ck.\left\{\frac{\sqrt{r}}{\cos.z}\right\}^3.\sin.\tfrac{1}{2}y.$$ (16)

$$\frac{\sqrt{\frac{r'}{r}}+\sqrt{\frac{r}{r'}}}{2.\cos.f}=1+2l;$$ [Assumed value of l.] (17)

$$m=\frac{kt}{2^{\frac{3}{2}}.(\cos.f)^{\frac{3}{2}}.(rr')^{\frac{3}{4}}};$$ [Assumed value of m.] (18)

$\log.m^2=5,5680729+2.\log.t-3.\log.\cos.f-\frac{3}{2}.\log.(rr')$; (19)

[5996]

(20) $p=\frac{\sin^2.f.\sqrt{rr'}}{2l.\cos.f}$;

(21) $m=l^{\frac{1}{2}}+\frac{4}{3}l^{\frac{3}{2}}$;

(22) $\frac{\sqrt{\frac{r'}{r}}+\sqrt{\frac{r}{r'}}}{2.\cos.f}=1-2L$; [Assumed value of L.]

(23) $M=\frac{kt}{2^{\frac{3}{2}}.(-\cos.f)^{\frac{3}{2}}.(rr')^{\frac{3}{4}}}$; [Assumed value of M.]

(24) $p=\frac{\sin^2.f.\sqrt{rr'}}{-2L.\cos.f}$;

(25) $M=-L^{\frac{1}{2}}+\frac{4}{3}L^{\frac{3}{2}}$.

The formulas in the preceding table are easily demonstrated in the following manner.

(26) Substituting $D=\frac{1}{2}p$ (5), in the first expression of r [5986(4)], we get $r=\frac{p}{2.\cos^2.\frac{1}{2}v}$; whence,

(26) $$\sqrt{\frac{p}{2r}}=\cos.\tfrac{1}{2}v=\cos.(\tfrac{1}{2}F-\tfrac{1}{2}f) \quad (6);$$

and in like manner,

(27) $$\sqrt{\frac{p}{2r'}}=\cos.\tfrac{1}{2}v'=\cos.(\tfrac{1}{2}F+\tfrac{1}{2}f) \quad (7);$$

these agree with (11, 12). Multiplying the product of the two formulas (11, 12), by 2, and then reducing the second member, by means of [20] Int., we get (13). Taking the sum of the squares of the two expressions (11, 12), and reducing, by means of [6, 27] Int., we get, as in (14);

(28) $$\frac{p.(r+r')}{2rr'}=\cos^2.(\tfrac{1}{2}F-\tfrac{1}{2}f)+\cos^2.(\tfrac{1}{2}F+\tfrac{1}{2}f)=1+\tfrac{1}{2}.\cos.(F-f)+\tfrac{1}{2}.\cos.(F+f)=1+\cos.F.\cos.f.$$

Multiplying (13) by $-\cos.f$, and adding the product to (14), we eliminate $\cos.F$, and obtain,

(29) $$\frac{p.(r+r')-2p.\cos.f.\sqrt{rr'}}{2rr'}=1-\cos^2.f=\sin^2.f;$$

which is easily reduced to the first form (15). If we substitute, in this, the value of r' (8), we get the first of the following formulas, and by successive reductions, using y (9), we finally reduce it to the second of the forms (15);

[5996]

$$p=r\,.\,\frac{2.\text{tang}^2.z.\sin^2.f}{1+\text{tang}^2.z-2.\cos.f.\text{tang}.z}=r.\frac{2.\sin^2.z.\sin^2.f}{1-2.\cos.f.\sin.z.\cos.z}=r.\frac{2.\sin^2.z.\sin^2.f}{1-\cos.f.\sin.2z}$$

$$=r\,.\,\frac{2.\sin^2.z.\sin^2.f}{1-\cos.y}=r.\frac{2.\sin^2.z.\sin^2.f}{2.\sin^2.\frac{1}{2}y}=r\,.\left(\frac{\sin.z.\sin.f}{\sin.\frac{1}{2}y}\right)^2. \qquad (29)$$

Substituting $D=\frac{1}{2}p$ (5) in [5986(6)], we get, (30)

$$t=\frac{p^{\frac{3}{2}}}{2k}\,.\left\{\text{tang}.\tfrac{1}{2}v+\tfrac{1}{3}.\text{tang}^3.\tfrac{1}{2}v\right\}: \qquad (31)$$

and by accenting the letters,

$$t'=\frac{p^{\frac{3}{2}}}{2k}\,.\left\{\text{tang}.\tfrac{1}{2}v'+\tfrac{1}{3}.\text{tang}^3.\tfrac{1}{2}v'\right\}. \qquad (31)$$

Subtracting the first of these expressions from the second, and changing $t'-t$ into t, in conformity with the notation of this article, we shall get, by multiplying by k, the expression (32). The second member is easily reduced into two factors, as in (33 or 34).

$$kt=\tfrac{1}{2}p^{\frac{3}{2}}.\left\{(\text{tang}.\tfrac{1}{2}v'-\text{tang}.\tfrac{1}{2}v)+\tfrac{1}{3}.(\text{tang}^3.\tfrac{1}{2}v'-\text{tang}^3.\tfrac{1}{2}v)\right\} \qquad (32)$$

$$=\tfrac{1}{2}p^{\frac{3}{2}}.\left\{\text{tang}.\tfrac{1}{2}v'-\text{tang}.\tfrac{1}{2}v\right\}.\left\{1+\tfrac{1}{3}.\text{tang}^2.\tfrac{1}{2}v'+\tfrac{1}{3}.\text{tang}.\tfrac{1}{2}v'.\,\text{tang}.\tfrac{1}{2}v+\tfrac{1}{3}.\text{tang}^2.\tfrac{1}{2}v\right\} \qquad (33)$$

$$=\tfrac{1}{2}p^{\frac{3}{2}}.\left\{\text{tang}.\tfrac{1}{2}v'-\text{tang}.\tfrac{1}{2}v\right\}.\left\{1+\text{tang}.\tfrac{1}{2}v'.\,\text{tang}.\tfrac{1}{2}v+\tfrac{1}{3}.(\text{tang}.\tfrac{1}{2}v'-\text{tang}.\tfrac{1}{2}v)^2\right\}. \qquad (34)$$

Now we have,

$$\text{tang}.\tfrac{1}{2}v'-\text{tang}.\tfrac{1}{2}v=\frac{\sin.\frac{1}{2}v'}{\cos.\frac{1}{2}v'}-\frac{\sin\frac{1}{2}v}{\cos.\frac{1}{2}v}=\frac{\sin.\frac{1}{2}v'.\cos.\frac{1}{2}v-\cos.\frac{1}{2}v'.\sin.\frac{1}{2}v}{\cos.\frac{1}{2}v'.\cos.\frac{1}{2}v}=\frac{\sin.(\frac{1}{2}v'-\frac{1}{2}v)}{\cos.\frac{1}{2}v'.\cos.\frac{1}{2}v}$$

$$=\frac{\sin.f}{\cos.\frac{1}{2}v'.\cos.\frac{1}{2}v};$$

and the product of the expressions (11, 12), gives $\frac{p}{2.\sqrt{rr'}}=\cos.\frac{1}{2}v'.\cos.\frac{1}{2}v$: hence the (35) preceding expression becomes,

$$\text{tang}.\tfrac{1}{2}v'-\text{tang}.\tfrac{1}{2}v=\frac{2.\sin.f.\sqrt{rr'}}{p}. \qquad (36)$$

By similar substitutions, we obtain,

$$1+\text{tang}.\tfrac{1}{2}v'.\,\text{tang}.\tfrac{1}{2}v=\frac{\cos.\frac{1}{2}v'.\,\cos.\frac{1}{2}v+\sin.\frac{1}{2}v'.\,\sin.\frac{1}{2}v}{\cos.\frac{1}{2}v'.\,\cos.\frac{1}{2}v}=\frac{\cos.(\frac{1}{2}v'-\frac{1}{2}v)}{\cos.\frac{1}{2}v'.\cos.\frac{1}{2}v}=\frac{\cos.f}{\cos.\frac{1}{2}v'.\cos.\frac{1}{2}v}$$

$$=\frac{2.\cos.f.\sqrt{rr'}}{p}. \qquad (37)$$

Substituting (36, 37) in (34), we get

[5996]

(38) $$kt = p^{\frac{1}{2}}.\sin.f.\sqrt{rr'}\,.\left\{ \frac{2.\cos.f.\sqrt{rr'}}{p} + \tfrac{1}{3}.\left(\frac{2.\sin.f.\sqrt{rr'}}{p}\right)^2 \right\}$$

(39) $$= \frac{2.\sin.f.\cos.f.rr'}{\sqrt{p}} + \frac{4.\sin^3.f.(rr')^{\frac{3}{2}}}{3p^{\frac{3}{2}}}.$$

This last expression is the same as the first of the formulas (16). If we multiply the last term of the second member of (39), by p, and divide it by the first value of p (15), we get,

(40) $$kt = \frac{2.\sin.f.\cos.f.rr'}{\sqrt{p}} + \frac{2.\sin.f\sqrt{rr'}.\{r+r'-2.\cos.f\sqrt{rr'}\}}{3\sqrt{p}} = \frac{2.\sin f\sqrt{rr'}.\{r+r'+\cos.f\sqrt{rr'}\}}{3\sqrt{p}}.$$

Substituting in this last expression, the first value of $\sqrt{p}$ (15), we get the second expression (16). These two forms of Gauss, are reduced to the form (16′), by Burckhardt, in the
(41) following manner. Substituting the assumed value $r' = r.\text{tang}^2.z$ (8), in the second expression (16), we get (42); and by successive reductions, using the symbols z, y, C (8,9,10), we finally obtain the expression (43), which is the same as (16′),

(42) $$kt = \frac{\sqrt{2}}{3}.r^{\frac{3}{2}}.\{1+\text{tang}^2.z+\cos.f.\text{tang}.z\}.\{1+\text{tang}^2.z-2.\cos.f.\text{tang}.z\}^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{3}.r^{\frac{3}{2}}.\sec.^3z.\{1+\cos.f.\sin.z.\cos.z\}.\{1-2.\cos.f.\sin.z.\cos.z\}^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{3}.r^{\frac{3}{2}}.\sec.^3z.\{1+\tfrac{1}{2}.\cos.f.\sin.2z\}.\{1-\cos.f.\sin.2z\}^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{3}.r^{\frac{3}{2}}.\sec.^3z.\{1+\tfrac{1}{2}.\cos.y\}.\{1-\cos.y\}^{\frac{1}{2}} = \frac{\sqrt{2}}{3}.r^{\frac{3}{2}}.\sec.^3z.\{1+\tfrac{1}{2}.(1-2.\sin^2.\tfrac{1}{2}y)\}.\{2.\sin^2.\tfrac{1}{2}y\}^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{3}.r^{\frac{3}{2}}.\sec.^3z.\{\tfrac{3}{2}-\sin^2.\tfrac{1}{2}y\}.\,2^{\frac{1}{2}}.\sin.\tfrac{1}{2}y = r^{\frac{3}{2}}.\sec.^3z.\{1-\tfrac{2}{3}.\sin^2.\tfrac{1}{2}y\}.\sin.\tfrac{1}{2}y$$

(43) $$= r^{\frac{3}{2}}.\sec.^3z.Ck.\sin.\tfrac{1}{2}y = Ck.\left(\frac{\sqrt{r}}{\cos.z}\right)^3.\sin.\tfrac{1}{2}y.$$

To facilitate the use of this last formula, Burckhardt computed Table VII of this collection, which contains the values of the logarithms of $C = \frac{1-\frac{2}{3}.\sin^2.\frac{1}{2}y}{k}$, for intervals of ten
(44) minutes in the value of y, from $y = 0^d$ to $y = 20^d$; and by means of it, we can very easily compute the time t, corresponding to the radii r, r', and the included arc $2f = v' - v$; as may be seen in (53), or in the example which is given on the same page with the table. The assumed values of l, m, L, M (17,18,22,23), are precisely the
(45) same as in the ellipsis [5995(28,37,48,50)]. Multiplying (17) by $2.\cos.f\sqrt{rr'}$, we get,

$$r' + r = 2.\cos.f\sqrt{rr'} + 4l.\cos.f\sqrt{rr'},$$

[5996]

hence the denominator of the first expression in (15), becomes $4l.\cos.f.\sqrt{rr'}$; and the value of p is reduced to the form (20). Again, since (17) is reduced to the form (22), by changing l into $-L$, we may, in the same way, get (21) from (20). Substituting the value of p (20), in the first expression of kt (16), we get, (46)

$$kt=l^{\frac{1}{2}}.(2.\cos.f)^{\frac{3}{2}}.(rr')^{\frac{3}{4}}+\tfrac{4}{3}.l^{\frac{3}{2}}.(2.\cos.f)^{\frac{3}{2}}.(rr')^{\frac{3}{4}}=2^{\frac{3}{2}}.\cos.^{\frac{3}{2}}f.(rr')^{\frac{3}{4}}.\{l^{\frac{1}{2}}+\tfrac{4}{3}.l^{\frac{3}{2}}\}. \qquad (47)$$

Substituting this in the value of m (18), it becomes of the very simple form (21). In a similar manner, the substitution of the value of p (21), in kt (16), and then in M (23), gives (25); and this may be derived from (21), by changing, as in [5995(127)], l into $-L$, and m into $M.(-1)^{\frac{3}{2}}$. If we compare the equations (21, 25) with the similar ones in an ellipsis, [5995(40, 53)], we shall find that they agree, if we suppose $x=0$, or $\sin^2.\frac{1}{2}g=0$; which makes $\xi=0$ [5995(115)]. Hence it is evident, that in calculating an orbit, upon the supposition that it is an ellipsis; if we obtain $x=0$, that is to say $\frac{m^2}{y^2}-l=0$, or $\frac{M^2}{Y^2}-L=0$, [5995(47, 57)], we may immediately conclude that the orbit is a parabola, and we can then calculate the elements of the orbit, by any of the formulas in the preceding table (11—25). Thus we may find p from (15 or 20), also, $D=\frac{1}{2}p$, and then we may obtain F from (13 or 14). We shall illustrate these formulas by the following example. (48) (49) (50)

EXAMPLE.

Given in a parabolic orbit $\log.r=0{,}2476368$, $\log.r'=0{,}2929648$, and $v'-v=2f=30^d\ 18^m\ 42^s$, to find the elements $D,\ p$; the anomalies $v,\ v'$; and the time of describing the arc t. (51)

To find t.

	$\frac{1}{2}\log.r'$	0,1464824
	$\frac{1}{2}\log.r$	0,1238184
$z=46^d\ 29^m\ 39^s{,}6$	tang.	0,0226640
$2z=92^d\ 59^m\ 19^s{,}2$	sine	9,999[illegible]89
$f=15^d\ 09^m\ 21^s$	cos.	9,9846256
$y=15^d\ 26^m\ 27^s{,}2$	cos.	9,9840345
	$\frac{1}{2}\log.r$	0,1238184
z	cos.	9,8378575
$\sqrt{r}.\sec.z$	log.	0,2859609
	Multiplied by 3	0,8578827
	Table VII. log. C	1,7591607
$\frac{1}{2}y=7^d\ 43^m\ 13^s{,}6$	sine	9,1282047
$t=55^{\text{days}},6222$	log.	1,7452481

To find p, D, v.

f	sine	9,4173807
z	sine	9,8605214
$\frac{1}{2}y$	ar. co. sin.	0,8717913
	sum	0,1496934
	doubled	0,299[illegible]8
r	log.	0,2476368
l	log.	0,5470316
2		0,3010300
$D=\frac{1}{2}p$	log.	0,2460016
r	log.	0,2476368
$\sqrt{\frac{p}{2r}}=\cos^2.\frac{1}{2}v$	log.	9,9983648
$\frac{1}{2}v=3^d\ 30^m\ [illegible]$	cos.	9,9991824
$v=7^d\ 01^m\ 46^s$	Table III.	0,7[illegible]928
$D^{\frac{3}{2}}$	log.	0,3690024
Time from the perihelion corresponding to $r,\ v$. $11^{\text{days}},8248$	log.	1,07[illegible]

(52) (53)

[5997] TO FIND THE ELEMENTS OF A HYPERBOLIC ORBIT; THERE BEING GIVEN THE RADII r, r', THE ANGLE $v'-v=2f$, AND THE TIME t OF DESCRIBING THE ANGLE $2f$.

We shall here use the same symbols as in the elliptical orbit [5995(6, &c.)], changing
(1) u into $\frac{C}{c}$, and u' into Cc; using also the auxiliary angle ψ [5988 (3)].
For convenience of reference, we shall insert these symbols in the following table (3—9, &c.), together with the formulas which are used in this method (9—59), and
(2) their demonstrations in (60—172).

Symbols.

(3) r, r' the radii vectores;

(4) v, v' the mean anomalies;

(5) $a =$ the semi-transverse axis $= b.\text{cot}.\psi$;

(6) $b =$ the semi-conjugate axis $= a.\sqrt{(e^2-1)} = \frac{\sin.f.\sqrt{rr'}}{\text{tang}.2n}$;

(7) $p = a\,(e^2-1) = b.\sqrt{e^2-1} = a.\text{tang}.^2\psi = b.\text{tang}.\psi =$ semi-parameter;

(8) $e = \frac{1}{\cos.\psi} =$ secant $\psi =$ excentricity;

Formulas for a hyperbolic orbit.

(9) $\sqrt{e^2-1} = \text{tang}.\psi = \frac{\text{tang}.f.\text{tang}.2n}{2.(l-z)} = -\frac{\text{tang}.f.\text{tang}.2n}{2.(L+z)}$;

(10) $u = \frac{C}{c}$; [Corresponding to r, v.]

(11) $u' = C\,c$; [Corresponding to r', v'.]

(12) $c = \text{tang}.(45^d + n)$;

(13) $z = \tfrac{1}{4}.\left\{\sqrt{c} - \sqrt{\frac{1}{c}}\right\}^2$;

(14) $C = \text{tang}.(45^d + N)$;

(15) $Z = \dfrac{c^2 - \frac{1}{c^2} - 4.\log.c}{\frac{1}{4}.(c-\frac{1}{c})^3}$;

(16) $\text{tang}.2n = 2.\sqrt{(z+z^2)}$;

(17) $\text{tang}.2N = \dfrac{2.\sin.L.\text{tang}.2w}{\sin.f.\cos.2w}$;

(18) $2f = v' - v$; $v = F - f$;

(19) $2F = v' + v$; $v' = F + f$;

(20) $\sin.\tfrac{1}{2}v = \tfrac{1}{2}.\left\{\sqrt{\frac{C}{c}} - \sqrt{\frac{c}{C}}\right\}.\sqrt{\left\{\frac{(e+1).a}{r}\right\}}$;

(21) $\cos.\tfrac{1}{2}v = \tfrac{1}{2}.\left\{\sqrt{\frac{C}{c}} + \sqrt{\frac{c}{C}}\right\}.\sqrt{\left\{\frac{(e-1).a}{r}\right\}}$;

(22) $\text{tang}.\tfrac{1}{2}v = \dfrac{C-c}{(C+c).\text{tang}.\frac{1}{2}\psi} = \dfrac{\sin.(N-n)}{\cos.(N+n).\text{tang}.\frac{1}{2}\psi}$;

[5997]

$$\sin.\tfrac{1}{2}v'=\tfrac{1}{2}.\left\{\sqrt{Cc}-\sqrt{\frac{1}{Cc}}\right\}.\sqrt{\left\{\frac{(e+1).a}{r'}\right\}}\,; \qquad (23)$$

$$\cos.\tfrac{1}{2}v'=\tfrac{1}{2}.\left\{\sqrt{Cc}+\sqrt{\frac{1}{Cc}}\right\}.\sqrt{\left\{\frac{(e-1).a}{r'}\right\}}\,; \qquad (24)$$

$$\text{tang}.\tfrac{1}{2}v'=\frac{Cc-1}{(Cc+1).\text{tang}.\frac{1}{2}\psi}=\frac{\sin.(N+n)}{\cos.(N-n).\text{tang}.\frac{1}{2}\psi}\,; \qquad (25)$$

$$\sin.f=\tfrac{1}{2}a.\left\{c-\frac{1}{c}\right\}.\left\{\frac{e^2-1}{rr'}\right\}^{\frac{1}{2}}; \qquad (26)$$

$$\cos.f=\tfrac{1}{2}a.\left\{e.\left(C+\frac{1}{C}\right)-\left(c+\frac{1}{c}\right)\right\}.\left(\frac{1}{rr'}\right)^{\frac{1}{2}}; \qquad (27)$$

$$\sin.F=\tfrac{1}{2}a.\left\{C-\frac{1}{C}\right\}.\left\{\frac{e^2-1}{rr'}\right\}^{\frac{1}{2}}; \qquad (28)$$

$$\cos.F=\tfrac{1}{2}a.\left\{e\left(c+\frac{1}{c}\right)-\left(C+\frac{1}{C}\right)\right\}.\left(\frac{1}{rr'}\right)^{\frac{1}{2}}; \qquad (29)$$

$$\frac{r}{a}=\tfrac{1}{2}e.\left\{\frac{C}{c}+\frac{c}{C}\right\}-1\,; \qquad (30)$$

$$\frac{r'}{a}=\tfrac{1}{2}e.\left\{Cc+\frac{1}{Cc}\right\}-1\,; \qquad (31)$$

$$\frac{r'-r}{a}=\tfrac{1}{2}e.\left\{C-\frac{1}{C}\right\}.\left\{c-\frac{1}{c}\right\}; \qquad (32)$$

$$\frac{r'+r}{a}=\tfrac{1}{2}e.\left\{C+\frac{1}{C}\right\}.\left\{c+\frac{1}{c}\right\}-2\,; \qquad (33)$$

$$Z=\frac{(1+2z).(z+z^2)^{\frac{1}{2}}-\log.\{\sqrt{1+z}+\sqrt{z}\}}{2.(z+z^2)^{\frac{3}{2}}}; \qquad (34)$$

$$Z=\frac{1}{\frac{3}{4}+\frac{9}{10}.(z+\zeta)}; \qquad (35)$$

$$\sqrt[4]{\frac{r'}{r}}=\text{tang}.(45^d+w)\,; \qquad \sqrt[4]{\frac{r}{r'}}=\text{tang}.(45^d-w)\,; \qquad (36)$$ [Assumed value of w.]

$$\frac{\sqrt{\frac{r'}{r}}+\sqrt{\frac{r}{r'}}}{2.\cos.f}=1+2l\,; \qquad (37)$$ [When $\cos.f$ is positive.] [Assumed value of l.]

$$l=\frac{\sin^2.\frac{1}{2}f}{\cos.f}+\frac{\text{tang}^2.2w}{\cos.f}; \qquad (38)$$

$$m=\frac{kt}{2^{\frac{3}{2}}.(\cos.f)^{\frac{3}{2}}.(rr')^{\frac{3}{4}}}\,; \qquad (39)$$ [Assumed value of m.]

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(40) $m=(l-z)^{\frac{1}{2}}+(l-z)^{\frac{3}{2}}.Z=y.(l-z)^{\frac{1}{2}};$

(41) $y=1+(l-z).Z=\frac{m}{(l-z)^{\frac{1}{2}}};$ [Assumed value of y.]

(42) $h=\frac{m^2}{\frac{5}{6}+l+\zeta};$ [Assumed value of h.]

(43) $h=\frac{(y-1).y^2}{y+\frac{1}{9}};$

(44) $z=l-\frac{m^2}{y^2};$

(45) $\frac{\sqrt{\frac{r'}{r}}+\sqrt{\frac{r}{r'}}}{2.\cos.f}=1-2L;$ [When $\cos. f$ is negative.] [Assumed value of L.]

(46) $L=-\frac{\sin.^2\frac{1}{2}f}{\cos.f}-\frac{\text{tang}^2.2w}{\cos.f};$

(47) $M=\frac{kt}{2^{\frac{3}{2}}(-\cos.f)^{\frac{3}{2}}.(rr')^{\frac{3}{4}}};$ [Assumed value of M.]

(48) $M=-(L+z)^{\frac{1}{2}}+(L+z)^{\frac{3}{2}}.Z=Y.(L+z)^{\frac{1}{2}};$

(49) $Y=-1+(L+z).Z=\frac{M}{(L+z)^{\frac{1}{2}}};$ [Assumed value of Y.]

(50) $H=\frac{M^2}{L-\frac{5}{6}-\zeta};$ [Assumed value of H.]

(51) $H=\frac{(Y+1).Y^2}{Y-\frac{1}{9}};$

(52) $z=\frac{M^2}{Y^2}-L;$

(53) $T=\frac{a^{\frac{3}{2}}}{k}.\left\{\frac{c.\text{tang}.2N}{\cos.2n}-\text{hyp.}\log.\text{tang}.(45^d+N)\right\}$

(53′) $=\frac{a^{\frac{3}{2}}}{\lambda k}.\left\{\frac{\lambda c.\text{tang}.2N}{\cos.2n}-\text{comm.}\log.\text{tang}.(45^d+N)\right\};$

(54) $\frac{1}{2}t=\frac{a^{\frac{3}{2}}}{k}.\left\{\frac{c.\text{tang}.2n}{\cos.2N}-\text{hyp.}\log.\text{tang}.(45^d+n)\right\}$

(54′) $=\frac{a^{\frac{3}{2}}}{\lambda k}.\left\{\frac{\lambda c.\text{tang}.2n}{\cos.2N}-\text{comm.}\log.\text{tang}.(45^d+n)\right\};$

(54″) $\log.k=8{,}2355814\ldots;$ $\log.\lambda=9{,}6377843\ldots;$ $\log.\frac{1}{\lambda k}=2{,}1266342\ldots;$

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$$a=\frac{r'+r-\left(c+\frac{1}{c}\right).\cos.f.\sqrt{rr'}}{\frac{1}{2}.\left(c-\frac{1}{c}\right)^2}=\frac{8.\left\{l-\frac{1}{4}.\left(\sqrt{c}-\frac{1}{\sqrt{c}}\right)^2\right\}.\cos.f.\sqrt{rr'}}{\left(c-\frac{1}{c}\right)^2} \qquad (55)$$

$$=\frac{-8.\left\{L+\frac{1}{4}.\left(\sqrt{c}-\frac{1}{\sqrt{c}}\right)^2\right\}.\cos.f.\sqrt{rr'}}{\left(c-\frac{1}{c}\right)^2} \qquad (56)$$

$$=\frac{2.(l-z).\cos.f.\sqrt{rr'}}{\tan g^2.2n}=\frac{2m^2.\cos.f.\sqrt{rr'}}{y^2.\tan g^2.2n}=\frac{k^2t^2}{4y^2.rr'.\cos.^2f.\tan g^2.2n} \qquad (57)$$

$$=\frac{-2.(L+z).\cos.f.\sqrt{rr'}}{\tan g^2.2n}=\frac{-2.M^2.\cos.f.\sqrt{rr'}}{Y^2.\tan g^2.2n}=\frac{k^2t^2}{4Y^2.rr'.\cos.^2f.\tan g^2.2n}; \qquad (58)$$

$$p=\frac{\sin.f.\tan g.f.\sqrt{rr'}}{2.(l-z)}=\frac{y^2.\sin.f.\tan g.f.\sqrt{rr'}}{2m^2}=\left(\frac{y.rr'.\sin.2f.}{kt}\right)^2 \qquad (59)$$

$$=\frac{-\sin.f.\tan g.f.\sqrt{rr'}}{2.(L+z)}=\frac{-Y^2.\sin.f.\tan g.f.\sqrt{rr'}}{2.M^2}=\left(\frac{Y.rr'.\sin.2f}{kt}\right)^2. \qquad (60)$$

We shall now give the explanations and demonstrations of the formulas in this table, taking them generally, in the order in which they occur. The symbols (3—9) are similar to those in the table, page 767, or like those for the ellipsis, [5995(6—11)], page 831, changing as usual $1-e^2$ into e^2-1, &c.; the formulas in (6, 9, 17), depending on f will be noticed in (149, 150). We have in [5988(13)],

$$u=\tan g.\ (45^d+\tfrac{1}{2}\varpi), \qquad (61)$$

and in like manner,

$$u'=\tan g.\ (45^d+\tfrac{1}{2}\varpi'). \qquad (62)$$

When the quantities $\varpi,\ \varpi'$ have been obtained, from the times $t,\ t'$, by means of [5988(6 or 7)], we can easily deduce $u,\ u'$. Instead of the symbols $\varpi,\ \varpi'$, Gauss uses the quantities $c,\ C$, putting,

$$c=\left\{\frac{\tan g.(45^d+\frac{1}{2}\varpi')}{\tan g.(45^d+\frac{1}{2}\varpi)}\right\}^{\frac{1}{2}}=\left(\frac{u'}{u}\right)^{\frac{1}{2}};\ C=\{\tan g.(45^d+\tfrac{1}{2}\varpi).\tan g.(45^d+\tfrac{1}{2}\varpi')\}^{\frac{1}{2}}=(uu')^{\frac{1}{2}}; \qquad (63)$$

these values give,

$$\frac{C}{c}=\tan g.(45^d+\tfrac{1}{2}\varpi)=u\quad(61'); \qquad Cc=\tan g.(45^d+\tfrac{1}{2}\varpi')=u'\quad(62); \qquad (64)$$

being the same as in (10, 11). In the course of the calculations, the new symbols

[5997]

$n, N, z, Z,$ are introduced, depending on c, C. These assumed values are given in (12—15), in terms of c, C. If we put, in [5989(12, 14)],

(65) $$u = c, \qquad \varpi = 2n;$$

the first of these expressions will become as in (12); and the last form of [5989(14)] will give,

(66) $$\text{tang.}2n = \frac{c^2-1}{2c} = \tfrac{1}{2}\left(c - \frac{1}{c}\right).$$

Now the assumed form of z (13) gives,

(67) $$\sqrt{z} = \tfrac{1}{2}.\{c^{\frac{1}{2}} - c^{-\frac{1}{2}}\}; \qquad \sqrt{1+z} = \tfrac{1}{2}.\{c^{\frac{1}{2}} + c^{-\frac{1}{2}}\}; \qquad \sqrt{1+z} + \sqrt{z} = c^{\frac{1}{2}};$$

(68) $$\sqrt{z}\,.\sqrt{1+z} = \sqrt{z+z^2} = \tfrac{1}{4}.(c - c^{-1}); \qquad z = \tfrac{1}{4}.(c - 2 + c^{-1}); \qquad 1 + 2z = \tfrac{1}{2}.(c + c^{-1}).$$

Substituting the first of the expressions (68) in tang.$2n$ (66), we get (16). Dividing the numerator and denominator of (15) by 8, it becomes,

(69) $$Z = \frac{\frac{1}{8}.(c^2 - c^{-2}) - \log.c^{\frac{1}{2}}}{\frac{1}{32}.(c - c^{-1})^3}.$$

Now the product of the first and third of the equations (68) gives,

(70) $$\tfrac{1}{8}.(c^2 - c^{-2}) = (1 + 2z)\,.\,(z + z^2)^{\frac{1}{2}};$$

moreover the third power of the first of the equations (68), being multiplied by 2, produces,

(71) $$\tfrac{1}{32}.(c - c^{-1})^3 = 2.(z + z^2)^{\frac{3}{2}};$$

substituting these and the value of $c^{\frac{1}{2}}$, given by the third of the equation (67), in (69), we get (34); which is reduced to the form (35) in (119 &c.) The assumed values of f, F (18, 19) are similar to those in the ellipsis [5995(13, 14)]. If we divide the
(72) last of the expressions of $\sin.\frac{1}{2}v$, $\cos.\frac{1}{2}v$ [5988(18, 20)], by $\sqrt{r}$, and substitute the corresponding values of $u = \frac{C}{c}$ (10), we shall get (20, 21). The similar values of $\sin.\frac{1}{2}v'$, $\cos.\frac{1}{2}v'$ (23, 24) are found in the same manner, by merely accenting the
(73) letters r', v', and using $u' = Cc$ (11), instead of the value of u (10). Dividing (20) by (21), we get, without any reduction,

(74) $$\text{tang.}\tfrac{1}{2}v = \frac{C^{\frac{1}{2}}c^{-\frac{1}{2}} - C^{-\frac{1}{2}}c^{\frac{1}{2}}}{C^{\frac{1}{2}}c^{-\frac{1}{2}} + C^{-\frac{1}{2}}c^{\frac{1}{2}}}\,.\,\left(\frac{c+1}{c-1}\right)^{\frac{1}{2}}.$$

Multiplying the numerator and denominator, of the first factor of the second member of this expression, by $C^{\frac{1}{2}}c^{\frac{1}{2}}$, it becomes $\frac{C-c}{C+c}$; and we have as in [5988(3)],

[5997]

$$\left(\frac{c+1}{c-1}\right)^{\frac{1}{2}}=\frac{1}{\text{tang.}\,\frac{1}{2}\,\psi}\,; \qquad (74')$$

hence we get the first of the expressions of $\text{tang.}\frac{1}{2}v$ (22). The second expression can be deduced from the first, by substituting the values of c, C (12, 14). For if we put, for a moment, $45^d+n=n'$, $45^d+N=N'$, the expressions (12, 14) become $c=\text{tang.}n'$; $C=\text{tang.}N'$; hence we get, (75)

$$C\mp c=\text{tang.}N'\mp\text{tang.}n'=\frac{\sin.N'}{\cos.N'}\mp\frac{\sin.n'}{\cos.n'} \qquad (76)$$

$$=\frac{\sin.N'.\cos.n'\mp\cos.N'.\sin.n'}{\cos.N'.\cos.n'}=\frac{\sin.(N'\mp n')}{\cos.N'.\cos.n'}\,; \qquad (76')$$

and if we divide this expression of $C-c$, by that of $C+c$, we obtain,

$$\frac{C-c}{C+c}=\frac{\sin.(N'-n')}{\sin.(N'+n')}=\frac{\sin.(N-n)}{\sin.(90^d+N+n)}=\frac{\sin.(N-n)}{\cos.(N+n)}\,; \qquad (77)$$

substituting this in the first expression (22), we get its second form. In like manner, by dividing (23) by (24), we get the first expression (25); hence we may obtain its second form, by substituting the values of c, C (12, 14). It is, however, easier to derive (25) from (22); observing that if we change c into c^{-1}, in (20, 21), we shall obtain the formulas (23, 24) respectively; moreover the change of c (12), into c^{-1}, requires that (78)

we should change $\text{tang.}(45^d+n)$ into $\frac{1}{\text{tang.}(45^d+n)}$, or $\text{tang.}(45^d-n)$; which is (79)

equivalent to a change in the sign of n; making these changes in (22), we obtain (25) by a slight reduction. Multiplying (21) by (23) we get (80); also (20) by (24) gives (81); (21) by (24), gives (82); and (20) by (23) gives (83),

$$\sin.\tfrac{1}{2}v'.\cos.\tfrac{1}{2}v=\tfrac{1}{4}a.\left\{C-\frac{1}{C}+c-\frac{1}{c}\right\}.\left\{\frac{c^2-1}{rr'}\right\}^{\frac{1}{2}}; \qquad (80)$$

$$\cos.\tfrac{1}{2}v'.\sin.\tfrac{1}{2}v=\tfrac{1}{4}a.\left\{C-\frac{1}{C}-c+\frac{1}{c}\right\}.\left\{\frac{c^2-1}{rr'}\right\}^{\frac{1}{2}}; \qquad (81)$$

$$\cos.\tfrac{1}{2}v'.\cos.\tfrac{1}{2}v=\tfrac{1}{4}a.\left\{C+\frac{1}{C}+c+\frac{1}{c}\right\}.\frac{c-1}{(rr')^{\frac{1}{2}}}\,; \qquad (82)$$

$$\sin.\tfrac{1}{2}v'.\sin.\tfrac{1}{2}v=\tfrac{1}{4}a.\left\{C+\frac{1}{C}-c-\frac{1}{c}\right\}.\frac{c+1}{(rr')^{\frac{1}{2}}}\,. \qquad (83)$$

Subtracting (81) from (80), and substituting in the first member for,

$$\sin.\tfrac{1}{2}v'.\cos.\tfrac{1}{2}v-\cos.\tfrac{1}{2}v'.\sin.\tfrac{1}{2}v, \qquad (84)$$

its value, $\sin.(\frac{1}{2}v'-\frac{1}{2}v)=\sin.f$ (18), we get (26). In like manner, the sum of

[5997] (82,83), gives by substituting for the first member, its value $\cos.(\frac{1}{2}v'-\frac{1}{2}v)=\cos.f$, the expression (27). The sum of (80,81), substituting $\sin.F=\sin.(\frac{1}{2}v'+\frac{1}{2}v)$ (19), gives (28); lastly, by subtracting (83) from (82), and substituting $\cos.F=\cos.(\frac{1}{2}v'+\frac{1}{2}v)$, we get (29). Dividing the last expression of r [5988(12)] by a, and substituting the value of u (10), we get (30); accenting r, u, and substituting u' (11), we get (31). Subtracting (30) from (31), we get (32); and the sum of (30,31) gives (33). The assumed values of w, l, m (36,37,39), corresponding to the case of $\cos.f$ positive, are the same as in the ellipsis [5995(24,28,37)] respectively; and the resulting value of l [5995(31)], is the same as in (38). The similar values of L, M, (45—47), corresponding to the case of $\cos.f$ negative, are also the same as in the ellipsis [5995(48—50)]. If we substitute the value of $\tan g.\varpi$ [5989(14)], and $\tan g.(45^{\circ}+\frac{1}{2}\varpi)$ [5989(12)] in [5988(6)], it becomes,

$$\frac{k}{a^{\frac{3}{2}}}.t=\tfrac{1}{2}c.\left\{u-\frac{1}{u}\right\}-\text{hyp.}\log.u,$$

and by accenting t, u, we get,

$$\frac{k}{a^{\frac{3}{2}}}.t'=\tfrac{1}{2}c.\left\{u'-\frac{1}{u'}\right\}-\text{hyp.}\log.u'.$$

Subtracting the first of these expressions from the second, then changing $t'-t$ into t, to conform to the notation in this article, we get (90); which is easily reduced to the form (91), by the substitution of the values of u, u' (10,11); eliminating c by means of (27), which gives,

(89) $$\tfrac{1}{2}c.\left\{C+\frac{1}{C}\right\}=\frac{\sqrt{rr'}}{a}.\cos.f+\tfrac{1}{2}.\left(c+\frac{1}{c}\right),$$

we get (92),

(90) $$\frac{k}{a^{\frac{3}{2}}}.t=\tfrac{1}{2}c.\left\{u'-\frac{1}{u'}-u+\frac{1}{u}\right\}-\log.\frac{u'}{u}$$

(91) $$=\tfrac{1}{2}c.\left\{C+\frac{1}{C}\right\}.\left\{c-\frac{1}{c}\right\}-2\log.c$$

(92) $$=\frac{\left\{c-\frac{1}{c}\right\}.\cos.f.\sqrt{rr'}}{a}+\tfrac{1}{2}.\left\{c^2-\frac{1}{c^2}\right\}-2.\log.c.$$

Eliminating C, from (33) by means of (89), we get, by making a slight reduction,

(93) $$\frac{r+r'}{a}=\left\{c+\frac{1}{c}\right\}.\left\{\frac{\sqrt{rr'}}{a}.\cos.f+\tfrac{1}{2}.\left(c+\frac{1}{c}\right)\right\}-2$$

(93) $$=\left(c+\frac{1}{c}\right).\frac{\sqrt{rr'}}{a}.\cos.f+\tfrac{1}{2}.\left(c-\frac{1}{c}\right)^2;$$

whence we easily deduce the first value of a (55). Multiplying (37) by $2.\cos.f.\sqrt{rr'}$, [5997]
we get, $r'+r=(2+4l).\cos.f.\sqrt{rr'}$, substituting this in the preceding value of a
(55), we obtain,

$$a=\frac{(2+4l).\cos.f.\sqrt{rr'}-\left(c+\frac{1}{c}\right).\cos.f.\sqrt{rr'}}{\frac{1}{2}.\left(c-\frac{1}{c}\right)^2}=\frac{8.\left\{l-\frac{1}{4}.\left(c-2+\frac{1}{c}\right)\right\}\cos.f.\sqrt{rr'}}{\left(c-\frac{1}{c}\right)^2}; \qquad 95$$

which is easily reduced to the second form (55). The third form is easily found, from (45) by a similar process; or it may be easily derived from the second form, by changing l into $-L$, as in (37, 45). If we substitute the value of z (13), in the second and third forms of (55), we get,

$$a=\frac{8.(l-z).\cos.f.(rr')^{\frac{1}{2}}}{\left(c-\frac{1}{c}\right)^2}=\frac{-8.(L+z).\cos.f.(rr')^{\frac{1}{2}}}{\left(c-\frac{1}{c}\right)^2}.$$

Multiplying (45) by $\frac{1}{8}.\left(c-\frac{1}{c}\right)^3$ we get,

$$\tfrac{1}{2}.\left(c^2-\frac{1}{c^2}\right)-2.\log.c=\tfrac{1}{8}.\left(c-\frac{1}{c}\right)^3.\,Z;$$

substituting this in the second member of (92), and then multiplying by $a^{\frac{3}{2}}$, we get (100). Now the square root of the first expression of a (97), being multiplied by $c-\frac{1}{c}$, gives,

$$\left(c-\frac{1}{c}\right).a^{\frac{1}{2}}=2^{\frac{3}{2}}.(l-z)^{\frac{1}{2}}.\,(\cos.f)^{\frac{1}{2}}.(rr')^{\frac{1}{4}};$$

substituting this and its cube in (100), it becomes as in (101).

$$kt=\left(c-\frac{1}{c}\right).a^{\frac{1}{2}}.\ \cos.f.(rr')^{\frac{1}{2}}+\tfrac{1}{8}.\left(c-\frac{1}{c}\right)^3.a^{\frac{3}{2}}.\,Z$$

$$=2^{\frac{3}{2}}.(\cos.f)^{\frac{3}{2}}.(rr')^{\frac{3}{4}}.\left\{(l-z)^{\frac{1}{2}}+(l-z)^{\frac{3}{2}}.\,Z\right\};$$

hence the value of m (39) becomes as in the first form of (40); and by substituting in it, the assumed value of $y=1+(l-z).Z$ (41), it becomes $m=y.(l-z)^{\frac{1}{2}}$, as in the second expressions (40, 41). Squaring this value of m, and dividing by y^2, we obtain z (44). By a similar process, using the second value of a (97), we may reduce the value of M (47) to the first form in (48); and by substituting the assumed value of $Y=-1+(L+z).Z$ (49), we get the second forms of $M,\ Y$ (48, 49); finally, from (103)

[5997] these we easily deduce z (52). We may also obtain (48) from (40), by the same process of derivation which is used in [5995(127)], namely, by changing,

(104) l into $-L$; m into $M.(-1)^{\frac{3}{2}}$; y into $-Y$; and h into H.

By developing in series, we obtain,

(104′) $$\sqrt{(z+z^2)} = z^{\frac{1}{2}} + \tfrac{1}{2}z^{\frac{3}{2}} - \tfrac{1}{8}z^{\frac{5}{2}} + \&c.;$$

multiplying this by $1+2z$, we get,

(105) $$(1+2z).\sqrt{(z+z^2)} = z^{\frac{1}{2}} + \tfrac{5}{2}z^{\frac{3}{2}} + \tfrac{7}{8}z^{\frac{5}{2}} + \&c.$$

Moreover,

(106) $$\sqrt{1+z} + \sqrt{z} = 1 + z^{\frac{1}{2}} + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \&c.;$$

whose hyp. log., by (58) Int. is,

(107) $$\text{hyp. log.}\{\sqrt{1+z}+\sqrt{z}\} = (z^{\frac{1}{2}} + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \&c.) - \tfrac{1}{2}.(z^{\frac{1}{2}} + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \&c.)^2$$
$$+ \tfrac{1}{3}.(z^{\frac{1}{2}} + \tfrac{1}{2}z - \&c.)^3 - \tfrac{1}{4}.(z^{\frac{1}{2}} + \tfrac{1}{2}z - \&c.)^4 + \&c.$$
$$= (z^{\frac{1}{2}} + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \&c.) - \tfrac{1}{2}.(z + z^{\frac{3}{2}} + \tfrac{1}{4}z^2 - \tfrac{1}{4}z^{\frac{5}{2}} + \&c.)$$
$$+ \tfrac{1}{3}.(z^{\frac{3}{2}} + \tfrac{3}{2}z^2 + \tfrac{3}{4}z^{\frac{5}{2}} + \&c.) - \tfrac{1}{4}.(z^2 + \tfrac{4}{2}z^{\frac{5}{2}} + \&c.) + \tfrac{1}{5}z^{\frac{5}{2}} + \&c.$$

(108) $$= z^{\frac{1}{2}} - \tfrac{1}{6}z^{\frac{3}{2}} + \tfrac{3}{40}z^{\frac{5}{2}} - \&c.$$

Subtracting (108) from (105), we get,

(109) $$(1+2z).\sqrt{(z+z^2)} - \text{hyp. log.}\{\sqrt{1+z}+\sqrt{z}\} = \tfrac{8}{3}z^{\frac{3}{2}} + \tfrac{4}{5}z^{\frac{5}{2}} + \&c.,$$

moreover, the cube of (104′) is,

$$(z+z^2)^{\frac{3}{2}} = z^{\frac{3}{2}} + \tfrac{3}{2}z^{\frac{5}{2}} + \&c.;$$

substituting these expressions in (34), we get,

(110) $$Z = \frac{\tfrac{8}{3}z^{\frac{3}{2}} + \tfrac{4}{5}z^{\frac{5}{2}} + \&c.}{2.(z^{\frac{3}{2}} + \tfrac{3}{2}z^{\frac{5}{2}} + \&c.)} = \frac{\tfrac{4}{3} + \tfrac{2}{5}z + \&c.}{1 + \tfrac{3}{2}z + \&c.} = \tfrac{4}{3} - \tfrac{8}{5}z + \&c.$$

To obtain the law of this progression, we shall multiply the value of Z (34), by $2.(z+z^2)^{\frac{3}{2}}$, which gives,

(111) $$2.(z+z^2)^{\frac{3}{2}}.Z = (1+2z).(z+z^2)^{\frac{1}{2}} - \log.\{\sqrt{1+z}+\sqrt{z}\}.$$

The differential of this expression, being divided by dz, gives, without any reduction.

[5997]

$$3.(z+z^2)^{\frac{1}{2}}.(1+2z).Z+2.(z+z^2)^{\frac{3}{2}}.\frac{dZ}{dz}$$ (112)

$$=2.(z+z^2)^{\frac{1}{2}}+\tfrac{1}{2}.(1+2z)^2.(z+z^2)^{-\frac{1}{2}}-\frac{\tfrac{1}{2}.\left\{\frac{1}{\sqrt{1+z}}+\frac{1}{\sqrt{z}}\right\}}{\sqrt{1+z}+\sqrt{z}}.$$ (112')

The last term of the second member being reduced, by rejecting the factor $\sqrt{1+z}+\sqrt{z}$ which occurs in its numerator and denominator, becomes,

$$-\frac{\tfrac{1}{2}}{\sqrt{z}\cdot\sqrt{1+z}} \qquad \text{or} \qquad -\frac{1}{2.\sqrt{(z+z^2)}};$$ (113)

hence that second member may be put under the following form, by taking the terms in the same order as in (112′), and bringing the factor $\tfrac{1}{2}.(z+z^2)^{-\frac{1}{2}}$, without the braces:

$$\tfrac{1}{2}.(z+z^2)^{-\frac{1}{2}}.\{4.(z+z^2)+(1+2z)^2-1\}=\tfrac{1}{2}.(z+z^2)^{-\frac{1}{2}}\{8.(z+z^2)\}=4.(z+z^2)^{\frac{1}{2}}.$$ (114)

Substituting this in (112′), and then dividing the whole equation by $(z+z^2)^{\frac{1}{2}}$, we get, by transposing the term depending on Z,

$$(2z+2z^2).\frac{dZ}{dz}=4-(3+6z).Z.$$ (115)

If we compare this equation with that in [5995(107)], we find that the former may be derived from the latter, by changing X into Z, and x into $-z$; making the same changes in [5995(112)] which is deduced from [5995(107)], we get,

$$Z=\tfrac{4}{3}-\frac{4\,.\,6}{3\,.\,5}z+\frac{4\,.\,6\,.\,8}{3\,.\,5\,.\,7}z^2-\frac{4\,.\,6\,.\,8\,.\,10}{3\,.\,5\,.\,7\,.\,9}z^3+\frac{4\,.\,6\,.\,8\,.\,10\,.\,12}{3\,.\,5\,.\,7\,.\,9\,.\,11}.z^4-\&c.$$ (116)

Making the same changes in [5995(114,115)], and writing ζ for ξ, we obtain (117.118); substituting the second of these expressions, in the first, we get (119), [illegible]

$$\frac{10}{9Z}=\frac{5}{6}+z+\frac{2}{35}.z^2-\frac{52}{1575}.z^3+\&c.;$$ (117)

$$\zeta=\frac{2}{35}.z^2-\frac{52}{1575}.z^3+\&c.;$$ (118)

$$\frac{10}{9Z}=\frac{5}{6}+z+\zeta.$$ (119)

From the last equation we obtain the value of Z (35). In Table X, are given the values of ζ (118), corresponding to z; from $z=0{,}001$ to $z=0{,}300$; which are to be used in solving the equation (40 or 48), as we shall see hereafter, (130—134). The comparative magnitudes of z, ζ, in Table X, have a striking analogy with those of (12[illegible]

[5997] r, ξ, in Table IX; as is easily seen by the inspection of the tables; moreover, in
(122) consequence of the smallness of ζ, in comparison with z, *we may in the first approximation towards the values of* z *neglect* ζ, *as we have neglected* ξ *in*
(123) [5995(146, &c.)]. If we now assume for h the value (42), we shall get,

(124)
$$\tfrac{5}{6}+l+\zeta=\frac{m^2}{h}.$$

Substituting this, and the value of z (44), in the expression of Z^{-1} (35), we get by successive reductions,

$$Z^{-1}=\tfrac{1}{4}+\tfrac{9}{10}\cdot(z+\zeta)=\tfrac{9}{10}\cdot(\tfrac{5}{6}+z+\zeta)=\tfrac{9}{10}\cdot\left(\tfrac{5}{6}+l+\zeta-\frac{m^2}{y^2}\right)=\tfrac{9}{10}\cdot\left(\frac{m^2}{h}-\frac{m^2}{y^2}\right)$$

$$=\tfrac{9}{10}\cdot\left(\frac{y^2-h}{h}\right)\cdot\frac{m^2}{y^2}=\tfrac{9}{10}\cdot\left(\frac{y^2-h}{h}\right)\cdot(l-z):$$

whence we obtain,

$$(l-z)\,.\,Z=\tfrac{10}{9}\cdot\left(\frac{h}{y^2-h}\right):$$

and by substitution in the assumed value of y (102 or 41), we get,

$$y-1=\tfrac{10}{9}\cdot\left(\frac{h}{y^2-h}\right)\qquad\text{or}\qquad(y-1)\,.\,(y^2-h)=\tfrac{10}{9}\,.\,h;$$

whence we easily deduce the value of h (43). In like manner we may obtain, from the assumed values of Y, H, (49, 50), the expression (51). This may also be very easily deduced from (43), by the principle of derivation (101); observing that if we change the signs of the numerator and denominator of (42), and then make the changes, which are indicated in (104), it becomes as in (50).

We may deduce the value of z, from the cubic equation (43 or 51), in the same manner as r is obtained from [5995(46 or 56)], in [5995(145, &c.)]; by first neglecting ζ, on account of its smallness, and putting, $h=\dfrac{m^2}{\frac{5}{6}+l}$ (42), or $H=\dfrac{M^2}{L-\frac{5}{6}}$ (50). With this value of h or H, we find in Table VIII, the corresponding value of log. yy or log. YY; and then from (44 or 52), the approximate value of z; also from Table X, the corresponding value of ζ. *This operation is to be repeated till the assumed and computed values of* ζ *agree, and in general, it will be found that one single operation is sufficient to give a very close approximation to the true value.* Hence we see that the calculation for finding z, in a hyperbolic orbit, is nearly the same as that for finding x
(134) in the ellipsis; and we may observe that the *quantities* $\dfrac{m^2}{y^2}-l$, $L-\dfrac{M^2}{Y^2}$ [5995(47,57)],
(135) *which are positive in the ellipsis* [5995(47, 57, 41)], *become negative in the hyperbola,* (44, 52, 13), *and vanish in the parabola* [5996(49)]; *so that the sign of these functions,*
(135′) *determines the nature of the conic section.*

Having thus computed the value of z, *we may now consider it as one of the data of the problem, to be used in finding the elements of the orbit.* The value of e may be found from the formula, [5997]

$$e=1+2z+2\sqrt{(z+z^2)};$$ (136)

which is easily deduced from the first and third of the equations (68); by multiplying the first of these equations by 2, and adding the product to the third equation. We may also obtain e, from the formulas (16,12), namely,

$$\text{tang.}2n=2\sqrt{(z+z^2)}\,; \qquad e=\text{tang.}(45^d+n).$$ (137)

The remarks in [5995(131—141)], relative to the roots of the cubic equation in y or Y, corresponding to the ellipsis, may be applied also, with proper modifications, to the hyperbola, as is evident by considering that the formulas, [5995(46,56)], in the ellipsis, are of the same (138) forms as those in the hyperbola (43,51). Finally, we may observe, that if z exceed the limits of Table X, we may use the indirect methods of solution, without changing the (139) form of the equation (40 or 48). In this last case, if we suppose the elements of the orbit to be known approximatively, we may determine very nearly, the value of n, by means of the formula,

$$\text{tang.}2n=\frac{\sin.f.\sqrt{rr'}}{a\sqrt{e^2-1}}\,;$$ (140)

which is easily deduced from (26), by the substitution of,

$$\tfrac{1}{2}.\left(e-\frac{1}{e}\right)=\text{tang.}2n \quad (66).$$ (141)

Then z may be deduced from n, by the following expression of its value,

$$z=\frac{\sin^2.n}{\cos.2n}\,;$$ (142)

which is easily deduced from (16); for if we square (16), and add 1 to both members of the resulting equation, we get,

$$1+\text{tang}^2.2n=1+4z+4z^2 \quad \text{or} \quad \sec^2.2n=(1+2z)^2:$$

whence $\sec.2n=1+2z$, and,

$$z=\frac{\sec.2n-1}{2}=\frac{1-\cos.2n}{2.\cos.2n}=\frac{2.\sin^2.n}{2.\cos.2n}=\frac{\sin^2.n}{\cos.2n}.$$ (143)

This value of z, is to be used in finding ζ in Table X; and then a corrected value of h or H (42,50), may be obtained, which must be substituted in (43 or 51), to obtain

[5997] a more accurate value of y or Y. *These operations are to be repeated till we obtain a*
(144) *value of* y *or* Y, *which will satisfy the equation* (43 or 51); *and then from* (44 or 52),
(145) *we get the true value of* z, *to be used in computing the elements of the orbit.* We shall now give the demonstrations of the remaining formulas in the preceding table, which are used in this part of the computation.

Comparing the first of the equations (68) with (16), we get

$$c - \frac{1}{c} = 1.\sqrt{(z+z^2)} = 2.\text{tang}.2n;$$

and we have, in (13),

$$\tfrac{1}{4}.\left(\sqrt{c} - \frac{1}{\sqrt{c}}\right)^2 = z;$$

substituting these in the second and third of the formulas (55, 55′), we get the first of the formulas (56, 57) respectively. Substituting in these, the value $l - z = \frac{m^2}{y^2}$ (11), and $L + z = \frac{M^2}{Y^2}$ (52), we get the second expressions in (56, 57). Substituting the value of m^2, (39), in the second form of (56), we get its third form; and in like manner, by using M^2 (17), we may reduce the second form of (57) to its third form. The value of
(149) $e\sqrt{e^2-1}$, deduced from (140), is the same as the last of the formulas (6). Dividing this
(150) by the first of the expressions of a (56), we get the second form of $\sqrt{e^2-1}$ (9); and in like manner, by using the first value of a (57), we get the third, or last of the formulas (9). Multiplying the equations (26, 32) together crosswise, and dividing the product by $\frac{1}{2}.\left(c - \frac{1}{c}\right)$, which occurs in both members, we get,

(151)
$$e.\left(C - \frac{1}{C}\right).\sin.f = (r' - r).\left(\frac{e^2-1}{rr'}\right)^{\frac{1}{2}}.$$

(152) If we change, in (12), c into C, and n into N, it becomes as in (14), and by making the same changes in (66), which is derived from (12), we get,

(153)
$$\text{tang}.2N = \tfrac{1}{2}.\left(C - \frac{1}{C}\right).$$

We have also, as in [5995(30)],

(154)
$$\frac{r'-r}{(rr')^{\frac{1}{2}}} = \sqrt{\frac{r'}{r}} - \sqrt{\frac{r}{r'}} = \frac{4.\text{tang}.2w}{\cos.2w}.$$

Substituting these and

(155)
$$e = \sec.\psi; \qquad (e^2-1)^{\frac{1}{2}} = \text{tang}.\psi \quad (8.9),$$

in (151), it becomes,

(155′)
$$2.\sec.\psi.\text{tang}.2N.\sin.f = \frac{4.\text{tang}.2w}{\cos.2w}.\text{tang}.\psi;$$

[5997]

dividing this by $2.\sec.\psi.\sin.f$, we obtain the expression of $\tan g.2N$ (17). The third expressions of p (58,59), are the same as those in the ellipsis [5995(60,61)]; they can be easily deduced from (6), by squaring it, and then dividing by a, by which means we get, for $a.(e^2-1)$, or p (7), the following expression;

$$p=\frac{\sin.^2 f.rr'}{a.\tan g^2.2n};$$

substituting in this, the last of the values of a (56,57), and using $2.\sin.f.\cos.f=\sin.2f$, we get the last of the values of p (58,59). From these we easily obtain the second forms (58,59), by putting $\sin.2f=2.\sin.f.\cos.f$, and using, in (58),

$$(kt)^2=8.(\cos.f)^3.(rr')^{\frac{3}{2}}.m^2 \quad (39);$$

and in (59),

$$(kt)^2=8.(-\cos.f)^3.(rr')^{\frac{3}{2}}.M^2 \quad (47).$$

Lastly, substituting in the second form (58), the value $\frac{y^2}{m^2}=\frac{1}{l-z}$ (41), we get its first form; in like manner, by using $\frac{Y^2}{M^2}=\frac{1}{L+z}$ (52), we reduce the second form of (59) to its first form. Instead of representing the times from the perihelion of the first and second observations by t, t', as in (87,88), we shall now represent them by $T-\frac{1}{2}t$, and $T+\frac{1}{2}t$, and then the two expressions (87,88) will become,

$$\frac{k}{a^{\frac{3}{2}}}.(T-\tfrac{1}{2}t)=\tfrac{1}{2}c.\left(u-\frac{1}{u}\right)-\text{hyp. log.}\,u; \qquad \frac{k}{a^{\frac{3}{2}}}.(T+\tfrac{1}{2}t)=\tfrac{1}{2}c.\left(u'-\frac{1}{u'}\right)-\text{hyp. log.}\,u'.$$

The half sum and the half difference of these two expressions, being multiplied by $\frac{a^{\frac{3}{2}}}{k}$ give (163, 165), and by the substitution of the values of u, u' (10, 11), we get their second forms (164, 166);

$$T=\frac{a^{\frac{3}{2}}}{k}.\left\{\tfrac{1}{4}c.\left(u'+u-\frac{1}{u'}-\frac{1}{u}\right)\right\}-\tfrac{1}{2}\,\text{hyp. log.}\,u'u$$

$$=\frac{a^{\frac{3}{2}}}{k}.\left\{\tfrac{1}{4}c.\left(Cc+\frac{C}{c}-\frac{1}{Cc}-\frac{c}{C}\right)-\text{hyp. log.}\,C\right\};$$

$$\tfrac{1}{2}t=\frac{a^{\frac{3}{2}}}{k}.\left\{\tfrac{1}{4}c.\left(u'-u-\frac{1}{u'}+\frac{1}{u}\right)\right\}-\tfrac{1}{2}\,\text{hyp. log.}\,\frac{u'}{u}$$

$$=\frac{a^{\frac{3}{2}}}{k}.\left\{\tfrac{1}{4}c.\left(Cc-\frac{C}{c}-\frac{1}{Cc}+\frac{c}{C}\right)-\text{hyp. log.}\,c\right\}$$

Now if we use the values,

[5997] (167) $$c = \text{tang.}\,(45^d + n) = \text{tang.}\,n'\,; \qquad C = \text{tang.}\,(45^d + N) = \text{tang.}\,N'.$$

(12, 14, 75), we shall have as in (66, 153),

(168) $$c - \frac{1}{c} = 2.\text{tang.}\,2n\,; \qquad C - \frac{1}{C} = 2.\text{tang.}\,2N\,;$$

and by using [30″] Int. we get,

(169) $$\frac{2c}{1+c^2} = \frac{2.\text{tang.}\,n'}{1+\text{tang.}^2 n'} = \sin.2n' = \cos.\,2n\,;$$

(169′) $$\frac{2C}{1+C^2} = \frac{2.\text{tang.}\,N'}{1+\text{tang.}^2 N'} = \sin.2N' = \cos.\,2N\,;$$

substituting these in the first members of (170, 171), and making successive reductions, we finally obtain,

(170) $$Cc + \frac{C}{c} - \frac{1}{Cc} - \frac{c}{C} = \frac{1+c^2}{c}\cdot\left(C - \frac{1}{C}\right) = \frac{2}{\cos.\,2n}\,.\,2.\text{tang.}\,2N = \frac{4.\text{tang.}\,2N}{\cos.\,2n}\,;$$

(171) $$Cc - \frac{C}{c} - \frac{1}{Cc} + \frac{c}{C} = \frac{1+C^2}{C}\cdot\left(c - \frac{1}{c}\right) = \frac{2}{\cos.\,2N}\,.\,2.\text{tang.}\,2n = \frac{4.\text{tang.}\,2n}{\cos.\,2N}\,.$$

Substituting the last expressions (170, 171), and also c, C (167) in (164, 166), we get the
(172) first values of $T, \frac{1}{2}t$ (53, 54), adapted to the use of hyperbolic logarithms; the second forms (53′, 54′), are adapted to common logarithms, by using the factors λ, λk, (54″), which are the same as in [5988(8, 9)].

To illustrate the preceding formulas, we shall give the following example, from Gauss.

EXAMPLE

(173) Given $\log.\, r = 0{,}0333585$, $\log.\, r' = 0{,}2008541$, $v' - v = 2f = 48^d\, 12^m$, $t = 51^{\text{days}}{,}49788$. To find the elements of the orbit a, p, e, and the true anomalies v, v'.

We have given the calculation of z, in the introduction to Table X, and it is not necessary to repeat it
(174) here. The results of this calculation are $w = 2^d\, 45^m\, 28^s{,}47$ $l = 0{,}0579660388$; $\log.\,\frac{m^2}{y^2} = 8{,}7030725$,
(175) $\log.\, yy = 0{,}0560846$, $\log.\, m^2 = 8{,}7591571$ $\log.\sqrt{rr'} = 0{,}1171063$, $z = 0{,}00748583$. With these we shall compute n from (16), $\frac{1}{4}$ from (9), b from the last of the formulas (6). From this we shall deduce
(176) a and p, by means of (5, 7); N is deduced from (17), v, v' from the last of the formulas (22, 25): lastly, T, t from the formulas (53′, 54′). The computation of t, is made merely for the purpose of verification,
(177) as it is one of the data of the problem.

[5997]

To find n. (16).

$z=$	0,00748583	log.	7,8742399
$1+z=$	1,00748583	log.	0,0032389
$z+z^2$		log.	7,8774768
$\sqrt{(z+z^2)}=\frac{1}{2}.\text{tang.}\,2n$		log.	8,9387394
2		log.	0,3010300
$2n=$	$9^d\,51^m\,11^s,816$	tang.	9,2397694
$n=$	$4^d\,55^m\,35^s,908$		

To find ψ. (9).

$l=$	0,057960388		
$z=$	0,00748583		
$l-z=$	0,050474558	log. co.	1,2969275
$f=$	$24^d\,6^m$	tang.	9,6506199
$\frac{1}{2}$		log.	9,6989700
tang. $2n$ (as above)			9,2397694
$\psi=$	$37^d\,34^m\,59^s,77$	tang.	9,8862868

To find v. (22).

$n=$	$4^d\,55^m\,35^s,908$		
$\mathcal{N}=$	$8^d\,04^m\,53^s,127$		
$\mathcal{N}-n=$	$3^d\,09^m\,17^s,219$	sin.	8,7406274
$\mathcal{N}+n=$	$13^d\,00^m\,29^s,035$	ar.co.cos.	0,0112902
$\frac{1}{2}\psi=$	$18^d\,47^m\,29^s,885$	cot.	0,4681829
$\frac{1}{2}v=$	$9^d\,25^m\,29^s,97$	tang.	9,2201005
$v=$	$18^d\,50^m\,59^s,94$		

To find T. (53′).

$(\lambda k)^{-1}$		constant log.	2,1266342
a (in column 2)		log.	0,6020619
$a^{\frac{1}{2}}$		log.	0,3010309
Factor		log.	3,0297270
λe (in column 2)		log.	9,7388027
$2n$		sec.	0,0064539
$2\mathcal{N}$ (in column 2)		tang.	9,4621341
First term of $T=$	$172^{\text{days}},63056$	log.	2,2371177
Factor (above)		log.	3,0297270
$45^d+\mathcal{N}$ log.tang.	0,1241703	log.	9,0940177
Second term of $T=$	$-132^{\text{days}},96725$	log.	2,1237447
$T=$	$39^{\text{days}},66331$		
$\frac{1}{2}t=$	$25^{\text{days}},74891$		

To find b, a, p. (6, 5, 7).

f		sin.	9,6110118
$\sqrt{rr'}$		log.	0,1171063
. . . .		arith. co.	0,7602306 (178)
b		log.	0,4883487
ψ	(in the first column)	tang.	9,8862868
$a=b.\text{cot.}\,\psi$	(5)	log.	0,6020619
$p=b.\text{tang.}\,\psi$		log.	0,3746355

To find $\mathcal{N}$. (17).

. . . f		arith. co. sin.	0,3889882
. . . 2		log.	0,3010300
. . . ψ		sin.	9,7652685 (179)
$2w$	$5^d\,30^m\,56^s,94$	sec.	20156
$2w$		tang.	8,9848318
$2\mathcal{N}$	$16^d\,09^m\,46^s,253$	tang.	9,4621341

To find v'. (25).

. . . .		ar. co. cos.	0,0006567
		sin.	9,3523527
. . . .		same	0,4681829
$\frac{1}{2}v'=$	$33^d\,31^m\,29^s,93$	tang.	9,8211943
$v'=$	$67^d\,02^m\,59^s,86$		(180)

To find $\frac{1}{2}t$, (54′); *for verification.*

λ		constant log.	9,6377843
$e=\text{sec.}\,\psi$		log.	0,1010184
λe		log.	9,7388027
. . . .			3,0297270
. . . . . $2n$ (as in column 1)		tang.	9,2397694
$2\mathcal{N}$		sec.	17514?
First term of $\frac{1}{2}t=$	$106^{\text{days}},12393$	log.	2,0258133 (181)
. . . .		log.	3,0297270
45^d+n log. tang.	0,0750575	log.	8,8753941
Second term of $\frac{1}{2}t=$	$80^{\text{days}},37502$	log.	1,9051211
$\frac{1}{2}t=$	$25^{\text{days}},74891$		
$\frac{1}{2}t=$	$25^{\text{days}},74894$ by observation.		(182)
	$0^{\text{days}},00003$ difference.		

$T-\frac{1}{2}t=\quad 13^{\text{days}},91440=$ time from the perihelion of the first observation.

$T+\frac{1}{2}t=\quad 65^{\text{days}},41222=$ time from the perihelion of the second observation. (183)

[5998]

GAUSS'S METHOD OF CORRECTING FOR THE EFFECT OF THE PARALLAX AND ABERRATION OF ANY NEWLY DISCOVERED PLANET OR COMET, IN COMPUTING ITS ORBIT, BY MEANS OF THREE GEOCENTRIC OBSERVATIONS, WITH THE INTERVALS OF TIME BETWEEN THEM.

In the computation of the orbit of a newly discovered planet, by the method in [5999], it
(1) becomes important to avoid the trouble of repeating, with much labor, the preliminary
(2) calculations, similar to those in [5999(300—379)], to correct for the effect of the planet's parallax, which at the commencement of the calculation is wholly unknown. This is
(3) effected in a very elegant manner by Gauss, by applying an equivalent correction to the
(4) place of the earth in the ecliptic; supposing at each observation, a fictious or *second observer*
(5) to make the observation of the planet. *The place of this second observer being in the plane of the ecliptic, at the point where the line drawn from the planet, through the actual place of*
(6) *observation on the surface of the earth, and continued beyond, intersects the plane of the*
(7) *ecliptic.* It being evident that the geocentric latitude and longitude of the planet is the same in both places of observation; but the distances of the planet from the two observers will be varied, by the distance of the two places of observation. In consequence of this change of place, we must apply a small correction to the distance of the earth's centre from
(8) the sun; and also to the longitude and latitude of the earth, so as to reduce them to the assumed situation of the second observer. After these reductions have been made, the rest
(9) of the calculation must be continued; supposing that the second observer is situated at the times of the three observations, in the three points of the ecliptic, deduced in the abovementioned manner, from the actual places of observation; since it is a matter of
(10) indifference, from what places the planet is observed, provided we carefully ascertain the
(11) assumed positions of the places of observation, which are used in the calculations.

We shall put, at the time of any observation,

(12) $A = 180^{\circ} + ☉ =$ the heliocentric longitude of the earth's centre:

(13) $L =$ the heliocentric latitude of the earth's centre;

(14) $R =$ the distance of the centres of the earth and sun.

(15) In like manner A_1, L_1, R_1, represent the heliocentric longitude, latitude, and distance from the sun's centre, of the place of the *first*, or actual observer, upon the surface of the earth.
(16) Also, A_2, L_2, R_2, the corresponding heliocentric longitude, latitude and distance of the second or fictious observer.

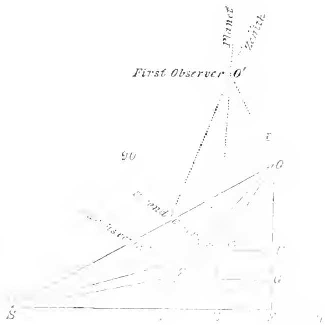
(17) α, θ, the geocentric longitude and latitude respectively of the planet; being the same for both observers;

(18) ρ_1 the distance of the planet from the first observer; $\rho_1 + \rho_2$ its distance from the second observer; ρ_2 the distance of the first and second observers from each other.

(19) Z the longitude, and z the latitude referred to the ecliptic, of the first, or actual observer, as seen from the centre of the earth; r, the distance of the first observer from the centre of the earth.

[5998]

We shall suppose that the plane of the annexed figure 90, is the plane of the ecliptic; S, the place of the sun; $S\Upsilon$, the line drawn from the sun towards the first point of aries; C', the centre of the earth; O', the actual place of the first observer; F, the corresponding place of the fictious or second observer; $C\,C'$, $O\,I\,O'$, perpendiculars let fall, upon the ecliptic, from the points C', O', respectively; $C\,A$, $F\,B$, $O\,E$, perpendiculars let fall upon $S\Upsilon$; also, $F\,H$, $C\,G$, perpendiculars let fall upon $O\,E$; lastly, $C'\,I$ is drawn parallel to $C\,O$. Then by the preceding notation we have, (20) (21) (22) (23) (24)



$$S\,C' = R; \qquad S\,O' = R_1; \qquad S\,F = R_2; \qquad C'\,O' = \mathrm{r}; \qquad (24)$$

$$\Upsilon S\,C = A; \qquad \Upsilon S\,O = A_1; \qquad \Upsilon S\,F = A_2; \qquad (25)$$

$$O\,C\,G = Z; \qquad O'\,C'\,I = z; \qquad O\,F\,H = \alpha; \qquad O'\,F\,O = \delta; \qquad (26)$$

and by the usual rules of plane trigonometry we have,

$$SC = SC'.\cos.L = R.\cos.L; \qquad CC' = IO = R.\sin.L; \qquad (27)$$

$$SA = SC.\cos.CSA = SC.\cos.A = R.\cos.L.\cos.A; \qquad (27')$$

$$CA = GE = R.\cos.L.\sin.A; \qquad C'I = CO = C'O'.\sin.O'C'I = \mathrm{r}.\cos.z; \qquad (27'')$$

$$IO' = C'O'.\sin.O'CI = \mathrm{r}.\sin.z; \qquad CG = AE = CO.\cos.OCG = CO.\cos.Z = \mathrm{r}.\cos.z.\cos.Z. \qquad (28)$$

$$OG = \mathrm{r}.\cos.z.\sin.Z; \qquad SB = SF.\cos.FSB = R_2.\cos.A_2; \qquad FB = EH = R_2.\sin.A_2; \qquad (29)$$

$$FO = FO'.\cos.O'FO = FO'.\cos.\delta = \rho_2.\cos.\delta; \qquad OO' = FO'.\sin.O'FO = \rho_2.\sin.\delta. \qquad (30)$$

$$FH = BE = FO.\cos.OFH = FO.\cos.\alpha = \rho_2.\cos.\delta.\cos.\alpha; \qquad (31)$$

$$OH = FO.\sin.OFH = \rho_2.\cos.\delta.\sin.\alpha. \qquad (31')$$

Now, by referring to the figure, we evidently have,

$$SB + BE = SA + AE; \qquad EH + OH = GE + OG; \qquad OO' = IO + IO'. \qquad (32)$$

Substituting the values (27—31′) in (32), we obtain the three following equations,

$$R_2.\cos.A_2 + \rho_2.\cos.\delta.\cos.\alpha = R.\cos.L.\cos.A + \mathrm{r}.\cos.z.\cos.Z; \qquad (33)$$

$$R_2.\sin.A_2 + \rho_2.\cos.\delta.\sin.\alpha = R.\cos.L.\sin.A + \mathrm{r}.\cos.z.\sin.Z; \qquad (34)$$

$$\rho_2.\sin.\delta = R.\sin.L + \mathrm{r}.\sin.z. \qquad (35)$$

[5998] If we assume the value of m (37), we shall get from (35), $\rho_2.\sin.\theta = \mathrm{m}.\mathrm{tang}.\theta$;
(36) whence we easily deduce ρ_2 (40); substituting this in the second terms of the equations (33,34), we obtain (38,39);

(37) $$\mathrm{m} = (R.\sin.L + \mathrm{r}.\sin.z).\mathrm{cotang}.\theta;$$

(38) $$R_2.\cos.A_2 = R.\cos.L.\cos.A + \mathrm{r}.\cos.z.\cos.Z - \mathrm{m}.\cos.\alpha;$$

(39) $$R_2.\sin.A_2 = R.\cos.L.\sin.A + \mathrm{r}.\cos.z.\sin.Z - \mathrm{m}.\sin.\alpha;$$

(40) $$\rho_2 = \mathrm{m}.\sec.\theta.$$

The equations (37—40) are perfectly accurate, and they give the values of R_2, A_2, ρ_2.
(41) This value of ρ_2, is used in (116, 117), in finding a corresponding correction of the time t,
(42) depending on the aberration. Multiplying the equation (38) by $\cos.A_2$, and (39) by
(43) $\sin.A_2$; then taking the sums of the products, and reducing by means of [24] Int., we get (44). In like manner, if we multiply (38) by $-\sin.A$, and (39) by $\cos.A$, we find that the sum of the products, reduced by [22] Int., becomes as in (45).

(44) $$R_2 = R.\cos L.\cos.(A_2 - A) + \mathrm{r}.\cos.z.\cos.(Z - A_2) - \mathrm{m}.\cos.(\alpha - A_2);$$

(45) $$R_2.\sin.(A_2 - A) = \mathrm{r}.\cos.z.\sin.(Z - A) - \mathrm{m}.\sin.(\alpha - A).$$

On account of the smallness of L and $A_2 - A$, we may put,

(46) $$\cos.L = 1, \quad \cos.(A_2 - A) = 1, \quad \sin.(A_2 - A) = A_2 - A;$$

also in (45), we may change R_2 into R; hence we finally obtain from (37, 44, 45, 40), the expressions (47—50);

(47) $$m = (RL + \mathrm{r}.\sin.z).\mathrm{cotang}.\theta;$$

(48) $$R_2 = R + \mathrm{r}.\cos.z.\cos.(Z - A) - \mathrm{m}.\cos.(\alpha - A);$$

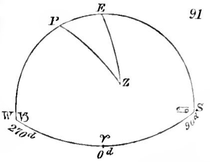
(49) $$A_2 - A = \frac{\mathrm{r}.\cos.z.\sin.(Z - A) - \mathrm{m}.\sin.(\alpha - A)}{R_2};$$

(50) $$\rho_2 = \mathrm{m}.\sec.\theta.$$

If r, m, are given in seconds, we must divide them by 206265^s, or multiply them by
(51) $\sin.1^s$, nearly. With these formulas, (47—50), we may compute the corrections of the place of the earth, for each of the three observations.

In making these calculations we must compute the longitude and latitude of the zenith;
(52) or as it is very commonly called, the longitude and the complement of the altitude of the
(53) nonagesimal degree of the ecliptic, for each of the three observations. The data in each of the observations being the obliquity of the ecliptic, the latitude of the place of observation
(54) reduced to the centre of the earth, on account of its elliptical figure, and the right ascension
(55) of the meridian. Various methods have been given for this purpose in books of astronomy
(56) and navigation; but that which is derived from Napier's formulas [1345 [48,49]], is as simple and short as any; it was published by me several years since, in a work on navigation, in

[5998]



nearly the following form. In the annexed figure, $W\,♈\,S$ is the equator, E its pole, P the pole of the ecliptic, Z (57)
the zenith of the observer. Then we have given, the side PE equal to the obliquity of the ecliptic, the side (58)
EZ equal to the complement of the reduced latitude of the place of the observer, and the angle PEZ equal to the difference between the right ascension of the meridian and 270^d, or the right ascension of the arch EPW; so that we have the sides PE, EZ, (59)
and the included angle PEZ, to find the angle EPZ, and the side PZ. Having computed this angle and side, we shall then have, by noticing the signs,

$$\text{longitude of the zenith} = 90^d - EPZ; \qquad \text{latitude of the zenith} = 90^d - PZ. \qquad (60)$$

We shall now put, for brevity,

$$2S = EZ + PE; \qquad 2D = EZ - PE; \qquad (61)$$

$$\text{angle } PZE = Z; \qquad \text{angle } EPZ = P; \qquad \text{angle } PEZ = E; \qquad (61)$$

$$A = \frac{\cos.D}{\cos.S}; \qquad B = \frac{\text{tang}.D}{\text{tang}.S}; \qquad C = \text{tang}.S. \qquad (62)$$

Then from Napier's formulas [1345 48, 49, 50], we have, by changing the letters A, B, C, into P, Z, E; and the arcs a, b, c, into EZ, PE, PZ, respectively;

$$\text{tang}.\tfrac{1}{2}(P+Z) = \frac{\cos.D}{\cos.S}\,.\,\text{cotang}.\tfrac{1}{2}E = A.\text{cotang}.\tfrac{1}{2}E; \qquad (63)$$

$$\text{tang}.\tfrac{1}{2}(P-Z) = \frac{\sin.D}{\sin.S}\,.\,\text{cotang}.\tfrac{1}{2}E = B.A.\text{cotang}.\tfrac{1}{2}E = B.\text{tang}.\tfrac{1}{2}(P+Z); \qquad (64)$$

$$\text{tang}.\tfrac{1}{2}PZ = \frac{\cos.\frac{1}{2}(P+Z)}{\cos.\frac{1}{2}(P-Z)}\,.\,\text{tang}.S = C.\,\frac{\cos.\frac{1}{2}(P+Z)}{\cos.\frac{1}{2}(P-Z)}. \qquad (65)$$

The values of D, S, do not vary sensibly, during the interval between the extreme observations, and we may put the preceding expressions under the following logarithmic forms; (66)

$2S$ = Polar Distance of the observer + Obliquity of the Ecliptic; (67)
$2D$ = Polar Distance of the observer — Obliquity of the Ecliptic; (67)
$\log.A = \log.\cos.D - \log.\cos.S$; $\log.B = \log.\text{tang}.D - \log.\text{tang}.S$; $\log.C = \log\,\text{tang}.S$. (68)
$\log.\text{tang}.\frac{1}{2}(P+Z) = \log.A + \log.\cot.\frac{1}{2}E$; (69)
$\log.\text{tang}.\frac{1}{2}(P-Z) = \log.B + \log.\text{tang}.\frac{1}{2}(P+Z)$; (70)
$\log.\text{tang}.\frac{1}{2}PZ = \log.C + \log.\cos.\frac{1}{2}(P+Z) - \log.\cos.\frac{1}{2}(P-Z)$. (71)

This method is peculiarly well adapted to this calculation, because it is short, simple, and

[5998] *requires only four openings of the table of logarithms for each observation; moreover the*
(72) *numbers A, B, C, do not sensibly vary in the time included between the extreme observations,*
so that the same numbers are used in all three of the observations. Thus in the example
(73) [5999(277—279)], the obliquity of the ecliptic varies only $0^s,42$, in the interval between
the extreme observations. To illustrate these formulas, we shall apply them to the three
(74) observations in the example [5999(277—285)]; and as the altitudes and longitudes of the
zenith are not required to any great degree of accuracy, we shall only use five places of
decimals in the logarithms. Then the co-latitude of the place of observation, [5999(281)],
(75) gives, $EZ = 38^d\,31^m\,21^s$; the obliquity of the ecliptic, $PE = 23^d\,27^m\,59^s$, [5999(277)].
(76) Their half sum, and half difference gives $S = 30^d\,59^m\,40^s$; $D = 7^d\,31^m\,41^s$. Then we
(77) have, from (68),

(78)	$D = 7^d\,31^m\,41^s$	cos. 9,99624	. . . . .	D	tang. 9,12107	
(79)	$S = 30^d\,59^m\,40^s$	cos. 9,93309		S	tang. 9,77868	= log. C
(80)	A	log. 0,06315		B	log 9,34239	

Subtracting 270^d from the observed right ascensions of Juno [5999(274—276)], which was
(81) observed on the meridian [5999(282)], we get the resulting values of E (84), corresponding
(82) to the three observations of the following table. Then by means of the formulas (69—71),
we obtain the values of the angle P, and the side PZ. Subtracting the angle P, from
90^d, we get the longitude of the zenith (88); and subtracting the side PZ from 90^d,
(83) we get the latitude of the zenith (89). The calculations for all three of these observations
are as in the following table.

First observation.

A. Merid $-270^d = E = 87^d\,10^m\,22^s$. (274).

A.	log. 0,06315		
$\frac{1}{2}E = 43^d\,35^m\,11^s$	cot. 0,02144		
[$\frac{1}{2}(P$]$+Z) = 50^d\,32^m\,42^s$	tan. 0,08459		cos. 9,80310
B.	log. 9,34239	C	log. 9,77868
[$\frac{1}{2}(P$]$-Z) = 14^d\,57^m\,53^s$	tan. 9,42698		sec. 0,01498
[S]um $= 65^d\,30^m\,35^s = EPZ$		$\frac{1}{2}PZ$	tan. 9,59676
[Co]mp. $= 24^d\,29^m\,25^s =$ long. zen.		$\frac{1}{2}PZ =$	$21^d\,33^m\,41^s$
		$PZ =$	$43^d\,07^m\,22^s$
		Latitude $=$	$46^d\,52^m\,38^s$

Second observation.

$E = 85^d\,43^m\,46^s$. (275).

A.	log. 0,06315		
$\frac{1}{2}E = 42^d\,51^m\,53^s$	cot. 0,03240		
$\frac{1}{2}(P+Z) = 51^d\,15^m\,09^s$	tan. 0,09555		cos. 9,79650
B.	log. 9,34239	C	log. 9,77868
$\frac{1}{2}(P-Z) = 15^d\,19^m\,46^s$	tan. 9,43794		sec. 0,01573
Sum $= 66^d\,34^m\,55^s = EPZ$		$\frac{1}{2}PZ$	tan. 9,59091
Comp. $= 23^d\,25^m\,05^s =$ long. zen.		$\frac{1}{2}PZ =$	$21^d\,17^m\,56^s$
		$PZ =$	$42^d\,35^m\,52^s$
		Latitude $=$	$47^d\,24^m\,08^s$

Third observation.

$E = 85^d\,11^m\,10^s$. (276).

A.	log. 0,06315		
$\frac{1}{2}E = 42^d\,35^m\,35^s$	cot. 0,03653		
$\frac{1}{2}(P+Z) = 51^d\,31^m\,06^s$	tan. 0,09968		cos. 9,79397
B.	log. 9,34239	C.	log. 9,77868
$\frac{1}{2}(P-Z) = 15^d\,28^m\,08^s$	tan. 9,44207		sec. 0,01602
Sum $= 66^d\,59^m\,14^s = EPZ$		$\frac{1}{2}PZ$	tan. 9,58867
Comp. $= 23^d\,00^m\,46^s =$ long. zen.		$\frac{1}{2}PZ =$	$21^d\,11^m\,57^s$
		$PZ =$	$42^d\,23^m\,54^s$
		Latitude $=$	$47^d\,36^m\,06^s$

[T]hese results are the same as in [5999(283—285)].

(90) We have used in (75), the same latitude of Greenwich as that given by Gauss, $51^d\,28^m\,39^s$;
but it would be rather more accurate to reduce it, on account of the oblateness of the earth;
(91) the difference is, however, of no importance, in the present example, on account of the
smallness of the parallax. In calculating the parallaxes in longitude and latitude, in a total
(92) or annular eclipse of the sun, the longitude and latitude of the zenith may be required at the
times of the *four* contacts of the limbs of the sun and moon; and during this interval the
value of A, B, C, remain unchanged. In fact, the numbers vary but very little in several
(93) years, so that we may compute a table for the obliquity $23^d\,27^m\,40^s$, like that in (96), with

[5998]

the variations corresponding to a change of 100^s in the obliquity, or in the latitude, and by (94) this means we can obtain, by inspection, for any places inserted in the table, the values of log. A, B, C; and can make any allowance for a small variation in the latitude of the (95) place of observation, arising from any correction in the observations, or in the reduction for the ellipticity.

Table computed for the obliquity $23^d\,27^m\,40^s$. (96)

PLACES.	Reduced latitudes. North.	log. A	Var. log. A + 100^s Lat.	Var. log. A + 100^s Obl.	log. B	Var. log. B + 100^s Lat.	Var. log. B + 100^s Obl.	log. C	Var. log. C + 100^s Lat.	Var. log. C + 100^s Obl.
	d m s		−	+		−	+		−	+
Albany,	42,27,13	0,079670	53	97	9,475733	293	739	9,853328	223	223
Berlin,	52,20,24	0,061608	49	75	9,324135	618	1099	9,771197	240	240
Cambridge, (E.)	52,01,28	0,062166	49	76	9,331054	600	1086	9,773925	240	240
Cambridge, (A.)	42,12,02	0,080150	52	97	9,478383	288	733	9,855355	222	222
Dublin, (Obs.)	53,12,07	0,060090	48	73	9,304166	670	1155	9,763705	242	242
Edinburgh,	55,46,02	0,055618	47	67	9,233401	878	1376	9,741011	249	249
Greenwich, (Obs.)	51,17,28	0,063466	49	77	9,346396	562	1038	9,780232	238	238
Havanna,	23,03,34	0,120000	64	148	9,597658	95	516	10,003045	210	210
Leon, I. (Obs.)	36,16,52	0,091680	55	112	9,520940	202	634	9,902005	216	216
London,	51,19,29	0,063406	49	77	9,345714	564	1040	9,779944	238	238
Oxford, (Obs.)	51,34,28	0,062963	50	77	9,340586	576	1054	9,777800	239	239
Paris,	48,38,51	0,068207	50	83	9,394413	452	918	9,802627	233	233
Philadelphia,	30,45,44	0,084828	53	104	9,501872	248	687	9,874738	219	219

We may observe that the same rules of Napier (63—65) may be used in finding the apparent longitude and latitude of a planet from its right ascension and declination, as in the (97) observations which are computed in [5999(277—285)]; supposing in the preceding figure 91, page 869, that the point Z represents the place of the planet; and using its right ascension, instead of the right ascension of the meridian; and its distance PZ from the north (98) pole of the equator, instead of the co-latitude of the place of observation. To illustrate this by an example, we shall take the first observation of Juno [5999(274, 277)], namely, right (99) ascension $357^d\,10^m\,22^s,35$; declination $6^d\,40^m\,08^s$ south; obliquity of the ecliptic (100) $23^d\,27^m\,59^s,48$. Hence we have,

$$\text{angle } PEZ = 357^d\,10^m\,22^s,35 - 270^d = 87^d\,10^m\,22^s,35 = E. \qquad (101)$$

$$PE = 23^d\,27^m\,59^s,48. \qquad EZ = 96^d\,40^m\,08^s. \qquad (102)$$

$$S = \tfrac{1}{2}(EZ + PE) = 60^d\,01^m\,03^s,74; \qquad D = \tfrac{1}{2}(EZ - PE) = 36^d\,36^m\,04^s,26. \qquad (102')$$

D sin. 9,7754222
S arith. co. sin. 0,0621735
$\tfrac{1}{2}E = 43^d\,35^m\,11^s,18$ cotan. 0,0214379
$\tfrac{1}{2}(P-Z) = 35^d\,51^m\,37^s,34$ tan. 9,8590336
$\tfrac{1}{2}(P+Z) = 59^d\,23^m\,28^s,44$
Sum $= 95^d\,15^m\,05^s,78 = EPZ$
90
$354^d\,44^m\,54^s,22$ = longitude of Juno.

cos. 9,9046102
arith. co. cos. 0,3019201
cotang. 0,0214379
$\tfrac{1}{2}(P+Z) = 59^d\,23^m\,28^s,44$ tan. 0,2279682

$\tfrac{1}{2}(P-Z)$ (103) ar.co.cos. 0,0912754
$\tfrac{1}{2}(P+Z)$ (103) cos. 9,7068655
S (102′) tan. 0,2397466
$\tfrac{1}{2}PZ = 47^d\,29^m\,45^s,79$ tan. 0,0378875 (103)
$PZ = 94^d\,59^m\,31^s,58$ (104)
90 (105)
Latitude $4^d\,59^m\,31^s,58$ south (106)

[5998] These results agree with those in [5999(283)]. After we have found the angle EPZ, we may compute PZ, by means of the formula [1345 [15]], which gives,

(107) $$\sin. PZ = \frac{\sin. PEZ . \sin. EZ}{\sin. EPZ};$$

but it is rather more accurate to determine PZ by means of the tangents, as in the formula (65).

The effect of the aberration of the planet cannot be so completely determined as that of
(108) the parallax in the preliminary part of the calculation of the orbit. Gauss adopts the usual
(109) method of correcting the observed places for the effect of that part of the aberration which is common to the fixed stars; namely, by adding $20^s,25$ *to the longitude of the sun, which is given by the solar tables, neglecting the small correction from the inequality of the motion of*
(110) *the earth, and applying to the observed places of the planet, the same corrections for the aberration in longitude and latitude, as if it were a fixed star. These corrected values are to*
(111) *be used throughout the whole calculation of the orbit.* Moreover, when the distance of the
(112) comet from the earth has been nearly determined, by the first approximation, as in the example [5999(426)], we must apply a correction for the remaining part of the aberration
(113) of the planet; by *decreasing* the time of observation, by the time t_1, which is required by the light, in passing from the planet to the earth, supposing it to take 493 seconds, or
(114) $0^{days},005706$, in passing from the sun to the earth, when at its mean distance. It being evident that this corrected time corresponds to the actual place of the planet, in its orbit, at
(115) the time that the particle of light quits the planet, which after the interval of time t_1, strikes the eye of the observer. Moreover, we may remark, that these reduced times corresponding to the orbit of the planet, are those which enter into the calculation of the orbit in [5999], and not the actual times at the place of the observer. Finally, the
(116) correction of the distance $\rho_2 = m . \sec. \theta$ (40), requires a corresponding correction in the aberration, which upon the same principles is represented by,

(117) $$493^s . \rho_2 = 493^s . m . \sec. \theta = 0^{days},005706 . m . \sec. \theta; \qquad \log. 0,005706 = 7,75633;$$

but this correction is generally insensible, as in (121), and may be neglected.

EXAMPLE.

(118) Given the geocentric longitude of the planet $\alpha = 354^d\, 44^m\, 54^s$; its geocentric latitude $\theta = -4^d\, 59^m\, 32^s$ (106); longitude of the zenith $Z = 24^d\, 29^m$ (88); latitude of the zenith $z = 46^d\, 53^m$ (89); heliocentric
(119) longitude of the earth $A = 12^d\, 28^m\, 54^s$ [5999(277)]; heliocentric latitude of the earth $L = +0^s,49$ [5999(277)]; distance of the earth from the sun $R = 0.9988839$ [5999(277)]; distance of the observer from the centre of the earth $r = 8^s,60$, being put equal to the sun's mean horizontal parallax, the mean distance of the earth from the sun being supposed 206265^s.

(120) From the above data we get $Z - A = 12^d\, 00^m$; $\alpha - A = 342^d\, 16^m$;

[5998]

To find m.

R	log.	9,99951
L	log.	9,69020
$R\,L = 0,48945$		9,68971
r	log.	0,93450
z	sin.	9,86330
r sin. $z = 6,27769$	log.	0,79780
Sum $= 6,76714$	log.	0,83040
θ	cotang.	$1,05873_n$
m	log.	$1,88913_n$

To find the correction of the time for the aberration.

m (col. 1),	log.	$1,88913_n$
1^s	sin.	4,68557
493^s	log.	2,69285
θ	sec.	0,00165 (121)
Correction of time $= -0^s,186$	log.	$9,26920_n$

As this correction of the time $-0^s,186$ is so very small it may be neglected. (122)

To find R_2.

.	log.	0,93450
z	cos.	9,83473
$Z - A$	cos.	9,99040
1^s	sin.	4,68557
$+0,0000279$	log	5,44520
$-$ m	log.	1,88913
$a - A$	cos.	9,97886
1^s	sin.	4,68557
$+0,0003577$	log.	6,55356

Sum $= 0,0003856 =$ correction,
Add $R = 0,9988839$ gives,
$R_2 = 0,9992695$ (126)

To find A_2.

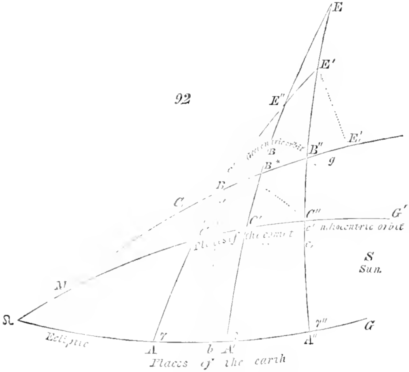
. . . .	same	0,93450
. . . .	same	9,83473
$Z - A$	sin.	9,31788 (123)
R_2	ar. co. log.	0,00032
$+1^s,22$	log.	0,08743
. . . .	log.	1,88913
$a - A$	sin.	$9,48371_n$ (124)
R_2	ar. co. log.	0,00032
$-23^s,61$	log.	$1,37316_n$

Sum $= -22^s,39 = A_2 - A$; hence,
$A_2 = A - 22^s,39$ (125)

GAUSS'S METHOD OF DETERMINING THE ORBIT OF A PLANET OR COMET, MOVING IN ANY CONIC SECTION, BY MEANS OF THREE OBSERVED GEOCENTRIC LONGITUDES AND LATITUDES, TOGETHER WITH THE TIMES OF OBSERVATION.

We shall here give the excellent method, published by Gauss, in his *Theoria Motus Corporum Cœlestium*, by which he determined the orbits of the newly discovered planets [5999] Ceres, Juno, Pallas, and Vesta; by means of three geocentric observations, with the times (1) of observations; the intervals between the observations being small, corresponding to an arc of a few degrees in the motion of the body. The importance of this method was exemplified several times in the computations of the orbits of these four planets, particularly Ceres, which was discovered by Piazzi, a few days before its conjunction with the sun. It remained obscured in the sun's rays above ten months; and after the conjunction, was sought for, in vain, during several weeks, by many European astronomers. It was feared by some that they would be unable to find it again, and that it might be considered as wholly lost. But when Gauss furnished the elements of its motion, they were able easily to distinguish this very small planet from the numerous little stars which appear so much like it; and on this account, he may be

[5999] considered as its second discoverer. The great simplicity of this method, as well as the rapidity with which Gauss performs such laborious calculations, was shown in the very remarkable instance, of his computing to a considerable degree of accuracy, in the period of about *ten* hours, the orbit of the planet Vesta, by observations embracing a period of nineteen days, with a geocentric motion of the planet of only *four* degrees.



(1′) The annexed figure 92, represents a portion of the concave surface of the starry heavens, the sun S, being the centre of this surface; ☊ $AA'A''G$ the ecliptic; ☊ $CC'C''G'$ the heliocentric orbit of the planet or comet, whose elements are to be computed.
(1′) A, A', A'', the heliocentric places of the earth, at the times of the three observations; C, C', C'', the corresponding
(2) heliocentric places of the planet; B, B', B'', the geocentric places; the arcs AB, $A'B'$, $A''B''$, being always less than 180^d. Then as the sun, earth, and planet are situated in a plane, which is projected in the heavens, in a great circle, it is evident that the arcs ACB, $A'C'B'$, $A''C''B''$, are portions of great circles, and we shall suppose them to be continued, till they intersect each other, in the points E, E', E''. Lastly, we shall suppose the points B'', B, to be connected, by a great circle, which intersects $A'B'$ in the point B^*, and the orbit
(2′) ☊ G', in the point M. From this construction, it is manifest, that the situation of the point B^*, will be *indeterminate*, if the arcs BB'', $A'B'$ coincide; or, in other words, if the points
(3) A', B, B', B'', fall in the same great circle. *This case we shall exclude from our calculations,*
First excepted case. with the remark, that we must select such observations as vary considerably from this situation; so that the slight errors of the observations may not materially effect the position of the point B^*, which is an object of importance in these calculations. Moreover, the situation
(3) of the point B^*, or of the arc BB'' is *indeterminate*, when the points B, B'', coincide; or
(4) are in opposite parts of the spherical surface; we must therefore, for the same reason, avoid the use of observations, where the geocentric positions, in the first and last observations, are
Second excepted case. very near to each other, or are very nearly in opposite parts of the heavens. *We shall also*
(5) *exclude this case from our calculations.* It is important to observe that the *geocentric and heliocentric places of the comet, in any particular observation, fall on the same side of the*

[5999]

ecliptic; the latitudes being either both north, or both south; moreover, the heliocentric place of the planet, is always situated in a point of that part of the arc of the great circle, (6)
which is included between the geocentric place of the planet and the heliocentric place of the earth. Thus, in the first observation, the heliocentric place of the planet C, is situated between the heliocentric place of the earth A, and the geocentric place of the planet B (2). This will be evident from the following considerations. If the planet be at an infinite distance from the earth, the point C will evidently fall infinitely near to B; and if that (7)
distance be infinitely small, the point C will fall infinitely near to A. Moreover, it is plain, that if we suppose the situations of the sun and earth to remain unaltered, while the distance of the planet from the earth aa', figure 81, page 792, increases in the direction of the line aa', from nothing to infinite, without altering the geocentric position of the planet in the (7)
heavens, or the position of the line aa'; the heliocentric place C, figure 92, will gradually move from A towards B; which are the two extreme points or limits corresponding to an infinitely small, or an infinitely great distance of the planet from the earth; therefore, the point C will always fall between A and B. Hence we shall have,

Heliocentric place of the planet.

$$CB < AB < 180^d; \qquad C'B' < A'B' < 180^d, \text{ or } z < \delta', \ (30, 21); \qquad C''B'' < A''B'' < 180^d. \qquad (8)$$

In the calculations of this article, we shall use the following symbols, which are similar to those in [5995—5997].

Symbols.

t, t', t'',	Times of observation;	(9)
$☉, ☉', ☉''$,	Longitudes of the Sun;	(10)
A, A', A'',	Longitudes of the earth, differing 180^d from $☉, ☉', ☉''$, respectively;	(11)
$\alpha, \alpha', \alpha''$,	Geocentric longitudes of the planet;	(12)
$\theta, \theta', \theta''$,	Geocentric latitudes of the planet; *southern latitudes being considered as negative;*	(13)
α^*, θ^*,	Geocentric longitude and latitude of the point B^*; *southern values of the latitude θ^* being negative;*	(13′)
R, R', R'',	Distances of the earth from the sun;	(14)
$\rho_{\prime}, \rho_{\prime}', \rho_{\prime}''$,	Distances of the planet from the earth;	(15)
ρ, ρ', ρ'',	Curtate distances of the planet from the earth;	(16)
r, r', r'',	Radii vectores of the orbit of the planet;	(17)
β, β', β'',	Heliocentric longitudes of the planet;	(18)
$\varpi, \varpi', \varpi''$,	Heliocentric latitudes of the planet; *southern latitudes being considered as negative;*	(19)
v, v', v'',	True anomalies of the planet;	(20)
u, u', u'',	Arguments of latitude of the planet, or distances from the ascending node, counted on the orbit;	(21)
C, C', C'',	C = angle $G'CB$; C' = angle $G'C'B'$; C'' = angle $G'C''E''$;	(22)
w, w', w'',	Arguments of latitude of the planet, reduced to the ecliptic, and counted from the ascending node;	(23)
$\delta, \delta', \delta''$,	δ = arc AB; δ' = arc $A'B'$; δ'' = arc $A''B'$;	(24)
δ^*	δ^* = arc $E'B^*$;	(25)
Ω,	Longitude of the ascending node of the orbit of the planet; $\mho = 180^d + \Omega$;	(26)
φ,	Inclination of the orbit of the planet to the ecliptic;	(27)
E, E', E'',	E = angle $A'EA''$; E' = angle $AE'A''$; E'' = angle $AE''A'$;	(28)
$2f, 2f', 2f''$,	$2f$ = arc $C'C'' = v'' - v'$; $2f'$ = arc $CC'' = v'' - v$; $2f''$ = arc $CC' = v' - v$;	(29)
z, z'	z = arc $C'B'$; z' = arc $C'B^*$ = arc $C'B'$ − arc $B'B^* = z - \delta^*$, (25);	(30)
ζ, ζ''	ζ = arc CE'; ζ'' = arc $C''E'$;	(31)

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Formulas used in these calculations.

(32) $$\mathrm{a} = \frac{R.\sin.\delta \sin.(A''E'-\delta'')}{R''.\sin.\delta''.\sin.(AE'-\delta)};$$ [Assumed value of a.]

(33) $$b = \frac{R'.\sin.\delta'.\sin.(A''E-\delta'')}{R''.\sin\delta''.\sin.(A'E-\delta'+\delta'')};$$ [Assumed value of b.]

(34) $$c = \frac{1}{2R'^3.\sin^3.\delta'.\sin.\delta''},$$ [Assumed value of c.]

(35) $$d = \frac{b.\sec.\delta'' - \mathrm{a}}{b.\sec.\delta'' - 1};$$ [Assumed value of d.]

(36) $$e = \frac{\mathrm{tang}.\delta''}{b.\sec.\delta'' - 1},$$ [Assumed value of e.]

(37) $[r\,r']$, $[r'\,r'']$, $[r\,r'']$, represent as in [5994(266)], the double of the areas of the plane triangles *sab*, *sbe*, *sac*, in figure 84, page 792, respectively. The radii, corresponding to any particular triangle, being included between the brackets;

(38) $$P = \frac{[r\,r']}{[r'r'']};$$ [First unknown quantity P.]

(39) $$Q = 2.\left\{\frac{[r\,r'] + [r'\,r'']}{[r\,r'']} - 1\right\}.r'^3;$$ [Second unknown quantity Q.]

(40) $$\mathrm{tang.\,w} = \frac{\sin.\delta''}{b.\left(\frac{P+1}{P+\mathrm{a}}\right) - \cos.\delta''} = \frac{(P+\mathrm{a}).e}{P+d}; \quad (130')$$ [Assumed value of w.]

(40') $$Q' = c\,Q.\sin.\,\mathrm{w};$$ [Assumed value of Q'.]

(41) $$Q'.\sin^4.z = \sin.\,(z - \mathrm{w} - \delta''); \quad \text{or}, \quad (126)$$

(41') $$0 = \log.Q' + 4.\log.\sin z - \log\ \sin.\,(z - \mathrm{w} - \delta'');$$

(41'') $$\frac{[r\,r'']}{[r'r'']}.r' = \frac{(P+\mathrm{a}).R'.\sin.\delta'}{b.\sin.(z-\delta'')} \quad (168)$$

(41''') $$\frac{[r\,r'']}{[r\,r']}\,r = \left\{\frac{[r\,r'']}{[r'\,r'']}.r'\right\}.\frac{1}{P}. \quad (169)$$

(42) $$b_{\prime} = \frac{R.\sin.\delta.\,\sin.\,(A'E''-\delta'+\delta'')}{R'.\,\sin.\delta'.\,\sin.(AE''-\delta)} = \frac{\mathrm{a}}{b}; \quad (133')$$ [Assumed value of $b_{\prime}$.]

(43) $$\mathrm{tang\ w}_{\prime} = \frac{b_{\prime}.\,\sin.\delta''}{P + (1 - b_{\prime}.\cos.\delta'')} \quad (139');$$ [Assumed value of $\mathrm{w}_{\prime}$.]

(44) $$\varkappa = \frac{R.\sin.\delta}{\sin.(AE'-\delta)},$$ [Assumed value of $\varkappa$.]

(45) $$\varkappa'' = \frac{R'.\sin.\delta''}{\sin.(A''E'-\delta'')}; \quad \text{whence} \quad \mathrm{a} = \frac{\varkappa}{\varkappa''}; \quad (32,\ 44,\ 45)$$ [Assumed value of $\varkappa''$.]

(46) $$\lambda = \frac{\cos.(AE'-\delta)}{R.\sin.\delta};$$ [Assumed value of λ.]

(47) $$\lambda'' = \frac{\cos.(A''E'-\delta'')}{R''.\sin.\delta''},$$ [Assumed value of λ''.]

(48) $$\mathrm{p} = \left\{\frac{[r\,r'']}{[r'r'']}.r'\right\}\frac{\sin.E}{\sin.E'}.\sin.(z + A'E - \delta'), \quad (182)$$ [Assumed value of p.]

(49) $$\mathrm{p}'' = \left\{\frac{[r\,r'']}{[r\,r']}.r'\right\}.\frac{\sin.E''}{\sin.E'}.\sin.(z + A'E'' - \delta'); \quad (181)$$ [Assumed value of p''.]

(50) $$\mathrm{q} = \varkappa.(\lambda\mathrm{p} - 1); \quad (195)$$ [Assumed value of q.]

(51) $$\mathrm{q}'' = \varkappa''.(\lambda''\mathrm{p}'' - 1); \quad (198)$$ [Assumed value of q''.]

(52) $$\mathrm{p} = r.\sin.\zeta; \quad \mathrm{q} = r.\cos.\zeta; \quad (182,\ 195); \quad \mathrm{tang}.\zeta = \frac{\mathrm{p}}{\mathrm{q}}; \quad r = \mathrm{p.cosec}.\zeta = \mathrm{q.sec}.\zeta.$$

(53) $$\mathrm{p}'' = r''.\sin.\zeta''; \quad \mathrm{q}'' = r''.\cos.\zeta''. \quad (181,\ 198); \quad \mathrm{tang}.\zeta'' = \frac{\mathrm{p}''}{\mathrm{q}''}; \quad r'' = \mathrm{p}''.\mathrm{cosec}.\zeta'' = \mathrm{q}''.\mathrm{sec}.\zeta''$$

(54) $\log.\,k = 8{,}2355814$; [5987(8)]

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The points B, B', B'', are given by observation, also the points A, A', A'', by the solar tables; and when they are connected by great circles, as in figure 92, we shall have several (54)
spherical triangles, whose sides and angles can be computed, by the common processes of spherical trigonometry; frequently using, with much advantage, the formulas of Napier (55)
[1345^{48,49,50,51}]. Gauss has given many other similar formulas, but it is not necessary to repeat them here, because the computations, by the usual methods, are in general more (56)
simple, short, and accurate than those in which many auxiliary angles are introduced; since the small fractional parts which are neglected in these auxiliary angles, may have a tendency to produce small errors in the results. We shall now give the enumeration of the triangles which are to be computed, inserting some of the formulas, to which we may have occasion (57)
to refer.

First. From the point B, draw the arc Bb, perpendicular to the arc AG; then in the rectangular triangle AbB, we have the perpendicular $Bb = \theta =$ the geocentric latitude of (58)
the planet at the first observation; and the base $Ab = \alpha - A =$ the difference of longitudes of the points B, A; whence we find the angle $BAb = \gamma$, as in the first of the formulas (62), (59)
which is the same as [1345^{31}]; and the hypothenuse δ, as in the first of the formulas (63), which (60)
corresponds to the second of [1345^{29}]. In like manner, by letting fall perpendicular arcs, from the points B', B'', upon the arc AG; we may form similar triangles, corresponding to First process.
the second and third observations; from which we may deduce the values of $\gamma', \gamma'', \delta', \delta''$, (60')
(62,63); or they may be more simply derived from the expressions of γ, δ, by merely accenting the letters, to correspond to the particular observations. The values $\delta, \delta', \delta''$, (61)
are always considered as positive.

$$\text{tang.}\gamma = \frac{\text{tang.}\theta}{\sin.(\alpha - A)}; \qquad \text{tang.}\gamma' = \frac{\text{tang.}\theta'}{\sin.(\alpha' - A')}; \qquad \text{tang.}\gamma'' = \frac{\text{tang.}\theta''}{\sin.(\alpha'' - A'')}; \tag{62}$$

$$\text{tang.}\delta = \frac{\text{tang.}(\alpha - A)}{\cos.\gamma}; \qquad \text{tang.}\delta' = \frac{\text{tang.}(\alpha' - A')}{\cos.\gamma'}; \qquad \text{tang.}\delta'' = \frac{\text{tang.}(\alpha'' - A'')}{\cos.\gamma''}. \tag{63}$$

We shall suppose, that neither of the expressions of tang.γ, tang.γ', tang.γ'', *appear under the form* $\frac{0}{0}$, *in the observations which have been selected for computing the orbit.* (64)

Second. In the triangle $AA'E''$, we have the angles $A'AE'' = \gamma$, $AA'E'' = 180^d - \gamma'$; Second process.
and the side $AA' = A' - A$; to find by Napier's formulas [1345^{50,51}], the sides AE'', $A'E''$; (65)
and then the angle E'', by [1345^{45}], or $\frac{1}{2}E''$ by [1345^{48} or 1345^{49}]. In like manner, in the triangle $AA''E'$, we have the angles $A''AE' = \gamma$, $AA''E' = 180^d - \gamma''$, and the side (66)
$AA'' = A'' - A$; to find by the same formulas, the sides AE', $A''E'$, and the angle $AE'A'' = E'$. Lastly, in the triangle $A'A''E$, we have the angles $A''A'E = \gamma'$, (67)
$A'A''E = 180^d - \gamma''$, and the side $A'A'' = A'' - A'$; to find, by the same formulas, the sides $A'E$, $A''E$, and the angle E.

[5999] Third process. *Third.* To find the point B^*; we have given, in the triangle $BE'B''$, the side
(68) $BE'=AE'-AB=AE'-\delta$, the side $B''E'=A''E'-A''B''=A''E'-\delta''$, and the
(68') angle $BE'B''=E'$; to find the angles $E'BB''$, $E'B''B$, by Napier's formulas [1345[48,49]], and the side BB'', by [1345[15]]. Then in the triangle $BE''B^*$, we have given, the angle
(69) $BE''B^*=E''$, the angle $E''BB^*=E'BB''$, and the side $BE''=AE''-AB=AE''-\delta$; to find the sides BB^*, B^*E'', by Napier's formulas [1345[50,51]], and the angle BB^*E'', by
(70) formulas [1345[15]]. Finally, we have $B^*B''=BB''-BB^*$;

(71)
$$B^*B'=B^*E''-E''B'=B^*E''-A'E''+A'B'=B^*E''-A'E''+\delta'.$$

In the plane triangle $STC_{,}$, figure 87, page 798, the sides $TC_{,}$, ST, $SC_{,}$, or the corresponding symbols $\rho_{,}$, R, r, are respectively proportional to the sines of the opposite
(72) angles $TSC_{,}$, $SC_{,}T$, $STC_{,}$; and these angles are represented in figure 92, page 874, by the arcs AC, CB, $180^{d}-AB$; as will evidently appear, if we suppose in figure 87, page
(73) 798, a line SB to be drawn through S, parallel to $TC_{,}$, and continued infinitely, in the heavens, towards this point which is marked B, in figure 92; so that we shall have, in
(74) figure 87, the angle $BSC_{,}=$ angle $SC_{,}T$; and the lines $SC_{,}$, ST, being continued infinitely, fall in the points C, A, figure 92. Hence we have,

(75)
$$\frac{\sin.AC}{\rho_{,}}=\frac{\sin.CB}{R}=\frac{\sin.AB}{r}.$$

From these we obtain the expressions of r, $\rho_{,}$ (77, 78); and by accenting the letters we
(76) get the similar quantities corresponding to the second and third observations, using the symbols (24, 30);

(77)
$$r=R.\frac{\sin.AB}{\sin.CB}=\frac{R.\sin.\delta}{\sin.CB};\qquad r'=R'.\frac{\sin.A'B'}{\sin.C'B'}=\frac{R'.\sin.\delta'}{\sin.C'B'}=\frac{R'.\sin.\delta'}{\sin.z};$$
$$r''=\frac{R''.\sin.A''B''}{\sin.C''B''}=\frac{R''.\sin.\delta''}{\sin.C''B''};$$

(78)
$$\rho_{,}=R.\frac{\sin.AC}{\sin.CB};\qquad \rho_{,}'=R'.\frac{\sin.A'C'}{\sin.C'B'}=\frac{R'.\sin.(\delta'-z)}{\sin.z};\qquad \rho_{,}''=R''.\frac{\sin.A''C''}{\sin.C''B''}.$$

Hence it is manifest, that *when the situations of the points* C, C', C'' *are known, we can*
(79) *determine the values of* r, r', r''; $\rho_{,}, \rho_{,}', \rho_{,}''$.

Fourth process. *Fourth. We shall now show these points* C, C', C'', *can be determined by means of the*
(80) *quantities* P, Q (38, 39). We shall suppose M to be the point of intersection of the great
(81) circles $B''B^*B$, $C''C'C$, and for brevity we shall put,

(82)
$$2f=\text{arc } C'C''=MC''-MC'=v''-v';\qquad 2f'=\text{arc } CC''=MC''-MC=v''-v;$$
$$2f''=\text{arc } CC'=MC'-MC=v'-v;$$

(82') observing that *these symbols have the same symmetry relative to the number of accents, in*

[5999]

f, f', f''; $C'C''$, CC'', CC'; *as in the similar expressions* [5994(279')]; moreover, the values of f, f', f'', in terms of v, v', v'', are the same as in [5995 (13 &c.)]. (82')

The equation [5994(278)] is founded upon the supposition that the three places of the planet or comet a, b, c, figure 84, page 792, are situated in the same plane, passing through the sun, which is the origin of the rectangular co-ordinates x, y, z; but the plane of xy, and the direction of the axis x, are wholly arbitrary. Now if we take the plane of the orbit for that of xy, it will be represented in figure 92, page 874, by the plane drawn through the centre of the sphere S, and the great circle $MCC'C''$; and if we take the line SM for the axis of x, and the line perpendicular to it, in the same plane, for the axis of y, we shall find that the radii r, r', r'', form with the axis of x, three angles which are represented by the arcs MC, MC', MC'', respectively; therefore, by the usual rules of trigonometry, we shall have, (82'') (83) (84) (85) (86) (86')

$$x = r.\cos.MC; \qquad x' = r'.\cos.MC'; \qquad x'' = r''.\cos.MC''; \qquad (87)$$

$$y = r.\sin.MC; \qquad y' = r'.\sin.MC'; \qquad y'' = r''.\sin.MC''. \qquad (88)$$

Substituting these values of y, y', y'' in [5994(278)], we get,

$$0 = [r'r''].r.\sin.MC - [rr''].r'.\sin.MC' + [rr'].r''.\sin.MC''; \qquad (89)$$

and by comparing [5994(300')], with (82), we obtain the following expressions, which have the same symmetry, in the accents as in (82');

$$[rr'] = rr'.\sin.2f''; \qquad [r'r''] = r'r''.\sin.2f; \qquad [rr''] = rr''.\sin.2f'. \qquad (90)$$

Substituting these last expressions in (89), and dividing by $rr'r''$, we get (91), which is the same as (92), using the values of $2f, 2f', 2f''$ (82);

$$0 = \sin.2f.\sin.MC - \sin.2f'.\sin.MC' + \sin.2f''.\sin.MC''; \qquad (91)$$

$$0 = \sin.C'C''.\sin.MC - \sin.CC''.\sin.MC' + \sin.CC'.\sin.MC''. \qquad (92)$$

Theorem in spherics.

This may be considered as *a theorem in spherics, signifying that the points* C, C', C'', *are situated in the same great circle* $MCC'C''$; M *being any point whatever of the circumference of this great circle.* If we suppose the point M to be placed on the continuation of the arc CM, of the great circle, so as to increase the distance CM, by the quantity 90^d, the term $\sin.MC$, will change into $\sin.(MC+90^d)$, or $\cos.MC$, and the other terms of the equations (91, 92), being changed in the same manner, we get. (92') (92'')

$$0 = \sin.2f.\cos.MC - \sin.2f'.\cos.MC' + \sin.2f''.\cos.MC'';$$

$$0 = \sin.C'C''.\cos.MC - \sin.CC''.\cos.MC' + \sin.CC'.\cos.MC''. \qquad (93)$$

which is merely another form of the theorem in spherics (92). We shall now suppose that perpendicular arcs of great circles are let fall from the points C, C', C'', E, E', E'', upon the great circle $MBB'E_{\prime}$; the arcs $C_{\prime}C$, $E_{\prime}'E'$, are the only ones, which are actually drawn in the figure; the others being omitted, to avoid confusion. We shall represents these arcs. Theorem. (94) (95)

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(96) by the Roman capital letters C, C′, C″, E, E′, E″, respectively. Then in the rectangular spherical triangle $MC_{\prime}C$, we have, as in [1315[28]], the first of the following equations, or

(97) the value of $\sin.C_{\prime}C$ or sin.C; the second and third of these equations correspond to the points C', C'', and are easily derived from the first, by increasing the number of accents;

(98) $$\sin.\mathrm{C} = \sin.CMC_{\prime}.\sin.MC\,; \quad \sin.\mathrm{C}' = \sin.CMC_{\prime}.\sin.MC'\,; \quad \sin.\mathrm{C}'' = \sin.CMC_{\prime}.\sin.MC''.$$

Substituting these in (89), after multiplying it by $\sin.CMC_{\prime}$, we get,

(99) $$0 = [r'r''].r.\sin.\mathrm{C} - [rr''].r'.\sin.\mathrm{C}' + [rr'].r''.\sin.\mathrm{C}''.$$

In the right angled spherical triangles $E'E_{\prime}'B$, $CC_{\prime}B$, we have, by [1345[28]],

(100) $$\sin.E'E_{\prime}' = \sin.E'BE_{\prime}'.\sin.BE'\,; \qquad \sin.CC_{\prime} = \sin.CBC_{\prime}.\sin.CB.$$

Dividing the first of these expressions, by the second, and observing that,

$$\sin.E'BE_{\prime}' = \sin.CBC_{\prime},$$

we get,

(101) $$\frac{\sin.E'E_{\prime}'}{\sin.CC_{\prime}} = \frac{\sin.BE'}{\sin.CB}\,;$$

whence we obtain,

(101′) $$\sin.CC_{\prime} = \frac{\sin.E'E_{\prime}'.\sin.CB}{\sin.BE'}\,;$$

substituting,

(102) $$BE' = AE' - AB = AE' - \delta\,; \qquad E'E_{\prime}' = \mathrm{E}'\,; \qquad CC_{\prime} = \mathrm{C},$$

(103) we get the first of the equations (105); and by adding another accent to the letters E', E′, we get the second expression (105), corresponding to the point E''. In exactly the same

(104) way, we obtain the values of sin.C′ (106), and sin.C″ (107).

(105) $$\sin.\mathrm{C} = \frac{\sin.\mathrm{E}'\,\sin.CB}{\sin.(AE'-\delta)} = \frac{\sin.\mathrm{E}''.\sin.CB}{\sin.(AE''-\delta)}\,;$$

(106) $$\sin.\mathrm{C}' = \frac{\sin.\mathrm{E}.\sin.C'B^{*}}{\sin.(A'E-\delta'+\delta^{*})} = \frac{\sin.\mathrm{E}''.\sin.C'B^{*}}{\sin.(A'E''-\delta'+\delta^{*})}\,;$$

(107) $$\sin.\mathrm{C}'' = \frac{\sin.\mathrm{E}.\sin.C''B''}{\sin.(A''E-\delta'')} = \frac{\sin.\mathrm{E}'.\sin.C''B''}{\sin.(A''E'-\delta'')}.$$

Dividing the first of the equations (105), by the second of (107), we get the first equation (108); in like manner, by dividing the first of the equations (106), by the first of (107), we get the second equation (108);

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$$\frac{\sin.C}{\sin.C''}=\frac{\sin.C\,B}{\sin.C''B''}\cdot\frac{\sin.(A''E'-\delta'')}{\sin.(AE'-\delta)};\qquad \frac{\sin.C'}{\sin.C''}=\frac{\sin.C'B^*}{\sin.C''B''}\cdot\frac{\sin.(A''E-\delta'')}{\sin.(A'E-\delta'+\delta^*)}.$$ (108)

Dividing the equation (99) by $r''.\sin.C''$, and substituting (108), we obtain,

$$0=[r'r'']\cdot\frac{r.\sin.CB}{r''.\sin.C''B''}\cdot\frac{\sin.(A''E'-\delta'')}{\sin.(AE'-\delta)}-[rr'']\cdot\frac{r'.\sin.C'B^*}{r''.\sin.C''B''}\cdot\frac{\sin.(A''E-\delta'')}{\sin.(A'E-\delta'+\delta^*)}+[rr'].$$ (109)

Substituting the values,

$$r.\sin.CB=R.\sin.\delta;\qquad r'.\sin.C'B'=R'.\sin.\delta';\qquad r''.\sin.C''B''=R''.\sin.\delta''\quad(77);$$ (110)

observing also that $\sin.C'B^*$ may be put under the form,

$$\sin.C'B^*=\sin.C'B'.\frac{\sin.C'B^*}{\sin.C'B'}=\sin.C'B'.\frac{\sin.z'}{\sin.z};\quad(30);$$ (111)

we get,

$$0=[r'r'']\cdot\frac{R.\sin.\delta}{R''.\sin.\delta''}\cdot\frac{\sin.(A''E'-\delta'')}{\sin.(AE'-\delta)}-[rr'']\cdot\frac{R'.\sin.\delta'}{R''.\sin.\delta''}\cdot\frac{\sin.(A''E-\delta'')}{\sin.(A'E-\delta'+\delta^*)}\cdot\frac{\sin.z'}{\sin.z}+[rr'];$$ (112)

and if we use the assumed values of a, b, (32,33), it becomes,

$$0=\mathrm{a}.[r'r'']-[rr''].b.\frac{\sin.z'}{\sin.z}+[rr'].$$ (113)

From the assumed value of P (38), we easily deduce,

$$[r'r'']=\frac{[rr']+[r'r'']}{P+1};\qquad [rr']=P.\frac{[rr']+[r'r'']}{P+1};$$ (114)

substituting these in (113), we obtain,

$$0=\{[rr']+[r'r'']\}.\frac{P+\mathrm{a}}{P+1}-[rr''].b.\frac{\sin.z'}{\sin.z};$$ (115)

hence we get,

$$\frac{[rr']+[r'r'']}{[rr'']}=\frac{P+1}{P+\mathrm{a}}.b.\frac{\sin.z'}{\sin.z}.$$ (116)

Substituting this, and the value of $r'=\frac{R'.\sin.\delta'}{\sin.z}$ (77), in the assumed value of Q (39), (117)
we obtain,

$$Q=2.\left\{\frac{P+1}{P+\mathrm{a}}.b.\frac{\sin.z'}{\sin.z}-1\right\}.\frac{R'^3.\sin^3.\delta'}{\sin^3.z}.$$ (118)

Multiplying this by $\frac{\sin^4.z}{2R'^3.\sin^3.\delta'}$, we get,

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(119) $$\frac{Q.\sin^4.z}{2R'^3.\sin^3.\delta'} = b.\frac{P+1}{P+a}.\sin.z' - \sin.z.$$

Now, from [21] Int., we have, by using $z = z' + \delta^*$ (30),

(120) $$\sin.z = \sin.(z' + \delta^*) = \sin.z'.\cos.\delta^* + \cos.z'.\sin.\delta^*;$$

substituting this in the last term of (119), we obtain,

(121) $$\frac{Q.\sin^4.z}{2R'^3.\sin^3.\delta'} = \left\{ b.\frac{P+1}{P+a} - \cos.\delta^* \right\}.\sin.z' - \sin.\delta^*.\cos.z'.$$

The assumed value of the first expression of tang.w (40), gives,

(122) $$b.\left(\frac{P+1}{P+a}\right) - \cos.\delta^* = \frac{\sin.\delta^*}{\text{tang.w}} = \frac{\sin.\delta^*.\cos.\text{w}}{\sin.\text{w}}.$$

Substituting this in (121), we get (123); thence by successive reductions, and the
(122) re-substitution of $z' = z - \delta^*$ (30), we obtain (125);

(123) $$\frac{Q.\sin^4.z}{2R'^3.\sin^3.\delta'} = \sin.\delta^*.\left\{ \frac{\cos.\text{w}}{\sin.\text{w}}.\sin.z - \cos.z' \right\}$$

(124) $$= \frac{\sin.\delta^*}{\sin.\text{w}}.\{\cos.\text{w}.\sin.z' - \sin.\text{w}.\cos.z'\} = \frac{\sin.\delta^*}{\sin.\text{w}}.\sin.(z' - \text{w})$$

(125) $$= \frac{\sin.\delta^*}{\sin.\text{w}}.\sin.(z - \text{w} - \delta^*).$$

Multiplying this last expression by $\frac{\sin.\text{w}}{\sin.\delta^*}$, and substituting, in its first member, the assumed value of c (31), we get,

(125') $$c\,Q.\sin.\text{w}.\sin^4.z = \sin.(z - \text{w} - \delta^*):$$

Fundamental equation for z.

and by using Q' (40), it becomes,

(126) $$Q'.\sin^4.z = \sin.(z - \text{w} - \delta^*);$$

or by using logarithms,

(126') $$\log.Q' + 4.\log.\sin.z - \log.\sin.(z - \text{w} - \delta^*) = 0;$$

from which we must find the value of the unknown quantity z. We may observe that the assumed value of,

(127) $$\text{tang.w} = \frac{\sin.\delta^*}{b.\left(\frac{P+1}{P+a}\right) - \cos.\delta^*} \quad (40),$$

may be rendered more convenient for calculation, in the following manner. Multiplying the numerator and denominator by $\frac{P+\mathrm{a}}{\cos.\delta^*}$, it becomes, [5999]

$$\mathrm{tang.w}=\frac{(P+\mathrm{a}).\mathrm{tang}.\delta^*}{b.\sec.\delta^*.(P+1)-(P+\mathrm{a})}=\frac{(P+\mathrm{a}).\mathrm{tang}.\delta^*}{P.(b.\sec.\delta^*-1)+(b.\sec.\delta^*-\mathrm{a})}. \qquad (128)$$

Substituting in the numerator, the expression,

$$\mathrm{tang}.\delta^* = e.(b.\sec.\delta^*-1), \qquad (129)$$

depending on the assumed value of e (36); and in the denominator,

$$b.\sec.\delta^*-\mathrm{a}=(b.\sec\,\delta^*-1).d, \qquad (130)$$

depending on the assumed value of d (35), we find that the whole numerator and denominator becomes divisible by $b.\sec.\delta^*-1$, and we finally obtain the second expression of tang.w (40), namely,

$$\mathrm{tang.w}=\frac{(P+\mathrm{a}).e}{P+d}. \qquad (130')$$

The calculation of the quantities a, b, c, d, e (32—36), *which depends on known quantities, constitutes the fourth operation.* The actual values of b, c, e, are not required, but merely their logarithms. If we put $\frac{\mathrm{a}}{b}=b_{\prime}$, and substitute the values of a, b (32,33), we find that the factor $R''.\sin.\delta''$ occurs in the numerator and denominator, and by rejecting it, we get the following expression, (130'') (131)

$$b_{\prime}=\frac{R.\sin.\delta}{R''.\sin.\delta''}\cdot\frac{\sin.(A''E'-\delta'')}{\sin.(A''E-\delta'')}\cdot\frac{\sin.(A'E-\delta'+\delta^*)}{\sin.(AE'-\delta)}. \qquad (131')$$

Now from the second and third forms of the equation (107), we get (132); from the second and third forms of (106), we get (132'); and from the second and third forms of (105), we get (132''):

$$\frac{\sin.(A''E'-\delta'')}{\sin.(A''E-\delta'')}=\frac{\sin.\mathrm{E}'}{\sin.\mathrm{E}}; \qquad (132)$$

$$\sin.(A'E-\delta'+\delta^*)=\frac{\sin.\mathrm{E}}{\sin.\mathrm{E}''}.\sin.(A'E''-\delta'+\delta^*); \qquad (132')$$

$$\frac{1}{\sin.(AE'-\delta)}=\frac{\sin.\mathrm{E}''}{\sin.\mathrm{E}'}.\frac{1}{\sin.(AE''-\delta)}. \qquad (132'')$$

Multiplying these three expressions together, and rejecting the factor $\sin.\mathrm{E}.\sin.\mathrm{E}'.\sin.\mathrm{E}''$, which occurs in the numerator and denominator of the second member, we get,

$$\frac{\sin.(A''E'-\delta'')}{\sin.(A''E-\delta'')}\cdot\frac{\sin.(A'E-\delta'+\delta^*)}{\sin.(AE'-\delta)}=\frac{\sin.(A'E''-\delta'+\delta^*)}{\sin.(AE''-\delta)}. \qquad (133)$$

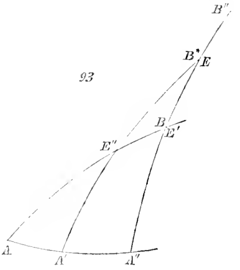
[5999] Substituting this in the second member of (131′), we obtain the value of $b_{,}$ (42), satisfying

(133′) the equation $b = \frac{\mathrm{a}}{b}$, or $\mathrm{a} = bb_{,}$ (131).

133) *There are two special cases, where some modification must be made in this calculation.*

Special cases. The *first* is when the great circles BB'', $A''B''$, coincide; as in the annexed figure 93; in which the point B coincides with E', and B^* with E. In this

(134) case, the quantities a, b (32,33), become infinite, because the factors,



(135) $$\sin.(A\,E' - \delta), \qquad \sin.(A'E - \delta' + \delta^*),$$

which occur in the denominators of these values of a, b, vanish. When this happens we must divide

(136) the equation (113), by b, and substitute the assumed value of $\frac{\mathrm{a}}{b} = b$ (133), and $z' = z - \delta^*$ (30),

(137) also $\frac{[rr']}{b} = 0$. Hence we get,

(138) $$0 = b_{,}.[r'r''] - \frac{\sin.(z - \delta^*)}{\sin.z}.[rr''].$$

(138′) Multiplying the numerator and denominator of $\tang.\mathrm{w}$ (40), by $b_{,}$, it becomes, by putting as in (133′) $bb_{,} = \mathrm{a}$,

(139) $$\tang.\mathrm{w} = \frac{b_{,}.\sin.\delta^*}{\frac{\mathrm{a}}{P + \mathrm{a}}.(P + 1) - b_{,}.\cos.\delta^*};$$

(139′) and as a is infinite, the denominator is equal to $P + 1 - b_{,}.\cos.\delta^*$; consequently this value of $\tang.\mathrm{w}$, becomes the same as the expression of $\tang.\mathrm{w}_{,}$ (43).

(140) The *second case* is where $\delta^* = 0$. Then the expression c (31), is infinite; and $\mathrm{w} = 0$ (40); hence it would seem that the factor $c.\sin.\mathrm{w}$ (125′), becomes *indeterminate*. But if we multiply together the expressions of c, $\tang.\mathrm{w}$ (31,40), and the product by

(141) $\cos.\mathrm{w} = 1$, we get, by rejecting the factor $\sin.\delta^*$, which occurs in the numerator and denominator,

(142) $$c.\sin.\mathrm{w} = \frac{1}{2R^3.\sin^3.\delta'.\left\{b.\left(\frac{P+1}{P+\mathrm{a}}\right) - \cos.\delta^*\right\}}.$$

Multiplying the numerator and denominator by $P + \mathrm{a}$, and substituting $\cos.\delta^* = 1$, we get,

[5999]

$$c.\sin.\mathrm{w}=\frac{P+\mathrm{a}}{2R^3.\sin^3.\delta'.\{b.(P+1)-P-\mathrm{a}\}}. \qquad (143)$$

Now when $\cos.\delta^*=1$, the expression of d (35) becomes $d=\frac{b-\mathrm{a}}{b-1}$, or $b-\mathrm{a}=d.(b-1)$; (144)
consequently,

$$b.(P+1)-P-\mathrm{a}=(b-1).P+(b-\mathrm{a})=(b-1).P+d.(b-1)=(b-1).(P+d). \qquad (145)$$

Substituting this in (143), and then multiplying by Q, we get for Q' (40′), the following definite expression;

$$Q'=cQ.\sin.\mathrm{w}=\frac{(P+\mathrm{a}).Q}{2R^3.\sin^3.\delta'.(b-1).(P+d)}. \qquad (146)$$

Lastly, substituting $\delta^*=0$, and $\mathrm{w}=0$ (140), in the second member of (126), we find that the whole equation becomes divisible by $\sin.z$; then substituting the expression (147)
of Q' (146), and extracting the cube root, we get, in this second case,

$$\sin.z=R'.\sin.\delta'.\sqrt[3]{\frac{2.(b-1).(P+d)}{(P+\mathrm{a}).Q}}. \qquad (148)$$

Fifth. When P, Q, are known, we can obtain w *from* (40), *and then* z *from the equation* (149)
(41 or 41′). In a first approximation we may assume for P, Q the values P', Q' (259); and by repeated processes, in the manner explained in (259—267) we can compute the true values of P, Q; from which we finally deduce the required value of (150)
z. If we develop the second member of (41), by [22] Int.; it may be put under the form,

Fifth process.

$$Q'.\sin^4.z-\cos.(\mathrm{w}+\delta^*).\sin.z=-\sin.(\mathrm{w}+\delta^*).\cos.z; \qquad (151)$$

squaring this equation and substituting $\cos^2.z=1-\sin^2.z$, it produces an equation of the *eighth* degree in $\sin.z$; which according to the general theory of equations may have *eight* (152)
roots, real or imaginary. Several of these roots must necessarily be real, and they may all be very quickly found, by supposing $\sin.z$ to increase gradually from 0 to 1, and (153)
selecting, by inspection, those values which nearly correspond to this equation; and then, by a few operations, correcting these first assumed quantities, so as to get the precise values of z which satisfy it. We may reject all the negative values of $\sin.z$, because they (154)
would make r' (77) negative, δ' being supposed positive (61); we must also reject those in which z exceeds δ' as is evident from (8); and also from the consideration (155)
that if $\sin.(\delta'-z)$ were negative, it would render $\rho'_{\prime}$ (78) negative. When the intervals of the times are moderate, it is generally found that there are four values of $\sin.z$, which satisfy the equation, of which one is most commonly negative, and can be rejected; (156)
sometimes there are three negative values, and only one positive value, consequently there is then no ambiguity as to that which is to be used. In case of having three positive values.

[5999] it commonly happens that one of them is very nearly equal to δ'. This value satisfies
(157) the *analytical conditions* of the problem, but not the *physical conditions*. The analytical conditions require that the planet should be situated, at the times of the three observations,
(158) somewhere on the lines $a'a$, $b'b$, $c'c$, fig. 84, page 792, respectively, and that the selected points a, b, c, should be situated in the same plane and at such distances as to make the
(159) areas of the sectors sab, sbc, proportional to the times. Now all these *analytical* conditions are completely satisfied by supposing the planet, at the times of the three observations, to
(160) be in the same places as the earth, so that the points C, C', C'' may coincide with A, A', A'',
(161) respectively, in fig. 92, page 874; in this case, we shall have $C'B' = A'B' = \delta'$. This result is evidently incompatible with the physical conditions of the problem, which require
(162) that the light in coming from the planet to the earth, should proceed from points a, b, c, fig. 84; which are at some distance from the eye of the observer at a', b', c', respectively.
(163) In most cases it will be found, that where there are three positive values of $\sin. z$, we can neglect one of them because it is nearly equal to δ' (161), another because z
(164) exceeds δ'; and then the remaining one can be used. If it should however happen that the equation admits of two solutions, which satisfy the proposed conditions of the problem,
(165) we shall thence obtain two different orbits. In this case the true orbit is to be determined, by comparing it with observations taken at greater intervals of time.

As soon as we have ascertained the value of z, we can find r', from the equation,
(166) $r' = \frac{R'.\sin.\delta'}{\sin.z}$ (77). Now we have, in (114), $[r\,r'] + [r'r''] = [r'\,r''].(P+1)$; substituting this in the first member of (116), and dividing by $P+1$, we get, by re-substituting $z' = z - \delta^*$ (30),

(167) $$\frac{[r'\,r'']}{[r\,r'']} = \frac{b}{P+a} \cdot \frac{\sin.(z-\delta^*)}{\sin.z}.$$

Dividing the value of r' (166), by the preceding expression, we get (168). The equation (169) is easily proved to be correct, by the substitution of the value of P (38); these expressions are the same as (41'' 41''');

(168) $$\frac{[r\,r'']}{[r'r'']} \cdot r' = \frac{(P+a).R'.\sin.\delta'}{b.\sin.(z-\delta^*)};$$

(169) $$\frac{[r\,r'']}{[r\,r']} \cdot r' = \frac{[r\,r'']}{[r'\,r'']} \cdot r' \cdot \frac{1}{P}.$$

(170) We shall suppose the arcs $C'c'$, $C''c''$, fig. 92, page 874, to be let fall from the points
(171) C', C'' respectively, upon the great circle ABE'; then in the right angled spherical triangle
(172) $C'c'E''$, we shall have, by [1345[23]] and (24, 30),

[5999]

$$\sin.C'c'=\sin.E''.\sin.C'E''=\sin.E''.\sin.(C'B'+B'E'')=\sin.E''.\sin.(C'B'+A'E''-A'B') \quad (173)$$
$$=\sin.E''.\sin.(z+A'E''-\delta') ; \quad (174)$$

in like manner, in the right angled spherical triangle $C''c''E'$, we have, by using (31);

$$\sin.C''c''=\sin.E'.\sin.C''E'=\sin.E'.\sin.\zeta''. \quad (175)$$

Now in the two right angled spherical triangles $C'c'C$, $C''c''C$, we have, by using C (22), the first of the four following expressions of $\sin.C'c'$, $\sin.C''c''$; from these we deduce the second and third forms, by using (29, 90); the last forms are the same as those in (174, 175); (176)

$$\sin.C'c'=\sin.C.\sin.CC'=\sin.C.\sin.2f''=\sin.C.\frac{[rr']}{rr'}=\sin.E''.\sin.(z+A'E''-\delta') ; \quad (177)$$

$$\sin.C''c''=\sin.C.\sin.CC''=\sin.C.\sin.2f'=\sin.C.\frac{[rr'']}{rr''}=\sin.E'.\sin.\zeta'' . \quad (178)$$

Dividing the two last of the expressions (178), by the corresponding ones in (177), we eliminate $\sin.C$, and, by a slight reduction, obtain the value of $r''.\sin.\zeta''$ (181); and this value is for brevity, put equal to p'', in (49, 53). In like manner, by supposing (179) perpendiculars $Cc_{,}$, $C''c'_{,}$ to be let fall from the points C, C'', upon the great circle $A'B'$, so as to form the right angled triangles $Cc_{,}C'$, $C''c'_{,}C''$, we get the expression of $r.\sin.\zeta$ (180) (182); which may also be derived from (181), by changing the quantities, relative to the point C, into those of the point C'', and the contrary. This value of $r.\sin.\zeta$, is, for abridgment, put equal to p in (48, 52), (180')

$$r''.\sin.\zeta''=\frac{[rr'']}{[rr']}.r'.\frac{\sin.E'}{\sin.E''}.\sin.(z+A'E''-\delta')=\mathrm{p}'' ; \quad (181)$$

$$r.\sin.\zeta=\frac{[r'r'']}{[rr'']}.r'.\frac{\sin.E}{\sin.E'}.\sin.(z+A'E-\delta')=\mathrm{p}. \quad (182)$$

In the last place, we shall suppose the arcs Cc_2, $C''c''_2$, to be let fall perpendicularly upon the great circle $A'B'$; though we have not actually marked these arcs, in the figure, to avoid confusion; then, from the right angled spherical triangle $CE''c_2$, we obtain (184): (183) and from the triangle $C''Ec''_2$, we obtain (185),

$$\sin.Cc_2=\sin.E''.\sin.CE''=\sin.E''.\sin.(CE'+AE''-AE')=\sin.E''.\sin.(\zeta+AE''-AE') ; \quad (184)$$
$$\sin.C''c''_2=\sin.E.\sin.C''E=\sin.E.\sin.(C''E'+A'E-A'E')=\sin.E.\sin.(\zeta''+A'E-A'E'). \quad (185)$$

Now by proceeding, as in (177, 178), we get, in the right angled spherical triangles $CC'c_2$, $C''C'c''_2$, the first of the expressions (187, 188); from these we deduce the second and (186) third forms, by using (29, 90); the last forms are the same as in (181, 182);

$$\sin.Cc_2=\sin.C'.\sin.CC'=\sin.C'.\sin.2f''=\sin.C'.\frac{[rr']}{rr'}=\sin.E''.\sin.(\zeta+AE''-AE') ; \quad (187)$$

[5999]

(188) $\sin.C''c''_2 = \sin.C'.\sin.C'C'' = \sin.C'.\sin.2f = \sin.C'.\frac{[r'r'']}{r'r''} = \sin.E.\sin.(\zeta''+A''E-A''E')$.

Dividing the two last expressions of (187), by those in (188), and substituting P (38), we get, by a slight reduction,

(189) $$r.\sin.(\zeta+AE''-AE') = r''.P.\frac{\sin.E}{\sin.E''}.\sin.(\zeta''+A''E-A''E');$$

(190) Substituting $CB=\zeta-AE'+\delta$, $C''B''=\zeta''-A''E'+\delta''$.(31, 24), in the values of r, r'' (77), we get,

(191) $$r.\sin.(\zeta-AE'+\delta) = R.\sin.\delta;$$

(192) $$r''.\sin.(\zeta''-A''E'+\delta'') = R''.\sin.\delta''.$$

Developing the first member of (191), by [22] Int.; and then dividing by $R.\sin.\delta$, we get (193); substituting, in this, the assumed values of λ, $\varkappa$, (46, 44,), we get (194),

(193) $$r.\sin.\zeta.\frac{\cos.(AE'-\delta)}{R.\sin.\delta} - r.\cos.\zeta.\frac{\sin.(AE'-\delta)}{R.\sin.\delta} = 1;$$

(194) $$r.\sin.\zeta.\lambda - r.\cos.\zeta.\frac{1}{\varkappa} = 1.$$

Substituting in (194), the expression $r.\sin.\zeta = \mathrm{p}$ (182), and then multiplying by $\varkappa$, we

(195) get, by using the symbol q (50); $r.\cos.\zeta = \varkappa.(\lambda\mathrm{p}-1) = \mathrm{q}$; as in (50, 52). Again if we develop, in the same manner, the expression (192), and divide by $R''.\sin.\delta''$, we shall obtain (196); and by substituting (47, 45), we get (197);

(196) $$r''.\sin.\zeta''.\frac{\cos.(A''E'-\delta'')}{R''.\sin.\delta''} - r''.\cos.\zeta''.\frac{\sin.(A''E'-\delta'')}{R''.\sin.\delta''} = 1;$$

(197) $$r''.\sin.\zeta''.\lambda'' - r''.\cos.\zeta''.\frac{1}{\varkappa''} = 1.$$

Substituting $r''.\sin.\zeta'' = \mathrm{p}''$ (181); then multiplying by $\varkappa''$, we get,

(198) $$r''.\cos.\zeta'' = \varkappa''.(\lambda''\mathrm{p}''-1) = \mathrm{q}'' \quad (51, 53).$$

Hence it appears, that we may deduce r, ζ, from the expressions of p, q, as in (52): and r'', ζ'', from p'', q'', as in (53). There can be no ambiguity in the values of

(199) ζ, ζ'', because r, r'', must necessarily be positive. The accuracy of the calculation can be verified by substituting these values in (189), to ascertain whether this equation is satisfied, by the results we have obtained. There are two cases in which other methods are to be followed. In the first place, when the point B coincides with E', or with its

(200) opposite point, in the spherical surface; or in other words, when $AE'-\delta$ is 0^d, or 180^d: because then the equations (182, 191) are identical; $\varkappa$ (44) becomes infinite, and,

[5999]

$$\lambda p - 1 = \frac{q}{\varkappa} = 0 \quad (50) ;$$ (201)

so that q (50) is indeterminate. In this case we must find r'', ζ'' from (53), as in the (202)
former method; then r, ζ, from the combination of (189), with (182 or 191); by
methods similar to the preceding, and which require no particular explanation. We may (203)
also observe that when $AE - \delta$ is very nearly equal to 0^d, or 180^d, the same method (204)
must be used, because the former is deficient in accuracy; adopting that combination of
(189) with (182), or with (191), which will give the best form, to the resulting equation. (205)
for the determination of r, ζ.

The second case which requires modification is where the point B'', very nearly coincides (206)
with E', or with its opposite point; in this case the determination of r'', ζ'', by the
preceding method would be impossible or inaccurate, on account of the smallness of
$\sin.(AE - \delta'')$, in the value of $\varkappa''$ (45). Then r, ζ, must be determined by the (207)
former method; but r'', ζ'', must be found by combining (189) with (181), or with (192), (208)
upon similar principles to those adopted in the preceding case, in (205). The case where (209)
the points B, B'', coincide with E', or with its opposite point, is excluded in (4).

Having found the arcs ζ, ζ''; the points C, C'', together with the point C', will be given (210)
in position; and the arc $CC'' = 2f'$, can be determined by means of the given arcs,

$$\zeta = CE ; \qquad \zeta'' = C''E'', \quad (31),$$ (211)

and the angle $CE'C'' = E'$ (28); using Napier's formulas [1345[48,49]], to find the angles (212)
$C''CE'$, $CC''E'$, and [1345[45]] to obtain the included side CC''. Moreover in the triangle
$C'EC''$, we have the angles $C''EC'$, $C'C''E$, and the side $C''E$, to find $C'C'' = 2f$:
also in the triangle $C'E''C$, we have the angles $C'E''C$, $C'CE''$, and the side CE'', to find
$CC' = 2f''$. These values of $2f$, $2f''$, are however much more easily obtained by the (213)
following formulas; observing that the logarithms of $\frac{[r'r'']}{[rr'']} \cdot \frac{1}{r'}$, $\frac{[rr']}{[rr'']} \cdot \frac{1}{r'}$, have been
obtained by a previous calculation in (168,169);

$$\sin.2f = \frac{[r'r'']}{[rr'']} \cdot \frac{r}{r'} \cdot \sin.2f' ;$$ (214)

$$\sin.2f'' = \frac{[rr']}{[rr'']} \cdot \frac{r''}{r'} \cdot \sin.2f'.$$ (215)

These formulas are easily proved to be correct, by the substitution of the values of, $[rr']$, (216)
$[r'r'']$, $[rr'']$, (90), in the second members, and making a slight reduction. Hence we have
a new confirmation of the previous calculations; because we ought to have $2f + 2f'' = 2f'$; (217)
and if any difference be found, we must re-examine the calculations. If the difference be
small, we may apportion it between $2f$, $2f''$, so that their log. sines may be equally (217')
increased or diminished, by which means the equation,

[5999]

(218) $$P=\frac{[rr']}{[r'r'']}=\frac{r.\sin.2f''}{r''.\sin.2f}\quad(38,90),$$

will be satisfied. If f, f'', differ but little, the error may be equally divided between $2f$ and $2f''$.

After we have obtained, in this manner, the position of the body in its orbit,
(219) we may compute the elements in two different ways; the one by combining the first observation with the second; the other by combining the second observation with the third;
(220) using the intervals corresponding to the times of observation; by the method given in [5995 &c.]. Before these operations are commenced, we must correct the observed times,
(221) for the effect of aberration, by *subtracting* from the times of observations, the number of seconds represented by, t_1, t_2, t_3, respectively, and computed by the following formulas,

(222) $$t_1=493^s.\rho_{\prime};\qquad t_2=493^s.\rho_{\prime}';\qquad t_3=493^s.\rho_{\prime}'';$$

(222') observing that 493 seconds is the time required for the light to pass from the sun to the earth, when at the mean distance, which is taken for unity. This, expressed in parts of a
(223) day, is $0^{day},005706$ [5998(114)], whose logarithm is 7,75633. The values of $\rho_{\prime}, \rho_{\prime}', \rho_{\prime}''$, are found, as in (78,77,190), to be,

(224) $$\rho_{\prime}=\frac{R.\sin.(AE'-\zeta)}{\sin.(\zeta-AE'+\delta)}=\frac{r.\sin.(AE'-\zeta)}{\sin.\delta};$$

(225) $$\rho_{\prime}'=\frac{R'.\sin.(\delta'-z)}{\sin.z}=\frac{r'.\sin.(\delta'-z)}{\sin.\delta'};$$

(226) $$\rho_{\prime}''=\frac{R''.\sin.(A''E'-\zeta'')}{\sin.(\zeta''-A''E'+\delta'')}=\frac{r''.\sin.(A''E'-\zeta'')}{\sin.\delta''}.$$

If the situations of the body, at the times of the three observations, be nearly known, by
(227) any previous calculations, we may immediately correct the observations for the effect of aberration, and suppress this part of the calculation. Using these corrected times of
(228) observation t, t', t'', and the value of k (51), we shall put, as in [5994(319)];

(229) $$\tau''=k.(t'-t);\qquad \tau=k.(t''-t');\qquad \tau'=k.(t''-t);\qquad \tau'=\tau+\tau''.$$

When we have gone through the calculation, as far as to find the value of y, or Y
(230) [5995(129 &c.)], which expresses the ratio of the area of the elliptical sector sab [5995(164)], to that of the corresponding triangle sab; we can use this value of y or Y, to compute more correct values of P, Q, by the formulas (235,256); and then a corrected
(231) value of z from (41,40',40). This part of the calculation is to be repeated till the assumed and computed values of P, Q, agree. As the values of y or Y, differ
(232) according as we use the different triangles or sectors, sbc, sac, sab, we shall denote them by $\mathrm{y}, \mathrm{y}', \mathrm{y}''$, respectively; so that we shall have by using the same notation as in (37 or 90);

(233) $$\text{sector } sbc=\tfrac{1}{2}\mathrm{y}.[r'r''];\qquad \text{sector } sac=\tfrac{1}{2}\mathrm{y}'.[rr''];\qquad \text{sector } sab=\tfrac{1}{2}\mathrm{y}''.[rr'];$$

[5999]

in which the accents have the same symmetry as in (82'). Now by Kepler's first law, the sectors sbc, sab [5994(47)], are proportional to the intervals of time $t''-t'$, $t'-t$ or τ, τ'' (229); hence we have,

$$\frac{\text{sector } sab}{\text{sector } sbc}=\frac{\tau''}{\tau}=\frac{\frac{1}{2}\mathrm{y}''.[rr']}{\frac{1}{2}\mathrm{y}.[r'r'']}=\frac{\mathrm{y}''}{\mathrm{y}}\,.\,P \quad (38);$$ (234)

consequently,

$$P=\frac{\mathrm{y}}{\mathrm{y}''}\cdot\frac{\tau''}{\tau};$$ (235) Correct value of P.

and as y, y'' are very nearly equal to unity [5995(44,31 &c.)], we shall have $P=\frac{\tau''}{\tau}$, (236)

for a very near approximation to the value of P; to be used in a first operation, as we shall see in (259 &c.). When the intervals τ'', τ, are nearly equal, the expressions y, y'', will commonly not differ much from each other, and then the assumed value of P (236), (236) is very near its true value. We shall now investigate the value of Q; putting it under such a form as will enable us to assume, at the commencement of the operation, a quantity, (237) which is very nearly equal to it. We have in [5985(10)], the following system of equations, in which the anomalies are counted from the perihelion, (238)

$$p=r.(1+e.\cos.v)\,; \qquad p=r'.(1+e.\cos.v')\,; \qquad p=r''.(1+e.\cos.v'').$$ (239)

Multiplying these three equations, by the values of $[r'r'']$, $-[rr'']$, $[rr']$ (90), respectively, and adding together the products, we get,

$$p.\{[r'r'']-[rr'']+[rr']\}=rr'r''.\{\sin.2f-\sin.2f'+\sin.2f''\}$$ (240)

$$+\,rr'r''.e.\{\sin.2f.\cos.v-\sin.2f'.\cos.v'+\sin.2f''.\cos.v''\}.$$ (241)

The coefficient of e (241), vanishes by means of the formula (93); the arbitrary position of the point M being taken so as to correspond to the position of the perihelion, from which (242) the angles v, v', v'', are counted (238); hence we have,

$$p.\{[r'r'']-[rr'']+[rr']\}=rr'r''.\{\sin.2f-\sin.2f'+\sin.2f''\}.$$ (243)

Now by [31, 26] Int., we have, by observing that $f'=f+f''$ (29), (244)

$$\sin.2f=2.\sin.f.\cos.f\,; \qquad \sin.2f''-\sin.2f'=2.\sin.(f''-f').\cos.(f''+f')$$ (245)

$$=-\,2.\sin.f.\cos.(f''+f').$$ (245')

Adding these two equations together, and reducing, by means of [28] Int. and (244), we get successively,

$$\sin.2f-\sin.2f'+\sin.2f''=2.\sin.f.\{\cos.f-\cos.(f''+f')\}$$

$$=2.\sin.f.\{2.\sin.\tfrac{1}{2}.(f+f'+f'').\sin.\tfrac{1}{2}.(f''+f'-f\}=4.\sin.f.\sin.f'.\sin.f''.$$ (246)

[5999] Substituting this in (243), and dividing by the coefficient of p, we get,

(247) $$p=\frac{4.rr'r''.\sin.f.\sin.f'.\sin.f''}{[r'r'']-[rr'']+[rr']}.$$

(248) If we substitute the value of $[rr']$ (90) in [5995(60)], we shall get $\surd p=\frac{\mathrm{y}''.[rr']}{\tau''}$, using

(249) $\mathrm{y}'', f'', t'-t$ &c., for y, f, t &c. as in (232 &c.), also τ'' for $k.(t'-t)$, as in (229). In like manner, in the triangle or sector corresponding to the radii r', r'', we have

(250) $\surd p=\frac{\mathrm{y}.[r'r'']}{\tau}$. The product of the two expressions of $\surd p$ (248, 250), gives,

(251) $$p=\frac{\mathrm{y}\,\mathrm{y}''.[rr'].[r'r'']}{\tau\tau''}.$$

Putting this expression of p equal to that in (247), we get,

(252) $$[r'r'']-[rr'']+[rr']=\frac{4\tau\tau''.rr'r''.\sin.f.\sin.f'.\sin.f''}{\mathrm{y}\mathrm{y}''.[rr'].[r'r'']}.$$

Multiplying the numerator and denominator of this expression by $2rr'r''.\cos.f.\cos.f'.\cos.f''$ we find that the numerator becomes,

(253) $$\tau\tau''.(2rr'.\sin.f''.\cos.f'').(2r'r''.\sin.f.\cos.f).(2rr''.\sin.f'.\cos.f')=\tau\tau''.[rr'].[r'r''].[rr''],$$

(254) as is evident from (90), observing that $2rr'.\sin.f''.\cos.f''=rr'.\sin.2f''=[rr']$, &c. Using this reduced value of the numerator, and rejecting the factor $[rr'], [r'r'']$, which is common to the numerator and denominator, we obtain the first of the following expressions; the second is derived from the assumed value of Q (39);

(255) $$[r'r'']-[rr'']+[rr']=\frac{\tau\tau''.[rr'']}{2\mathrm{y}\mathrm{y}''.rr'r''.\cos.f.\cos.f'.\cos.f''}=\frac{Q.[rr'']}{2r'^3}.$$

Dividing these two last expressions, by the coefficient of Q, we get,

(256) Correct value of Q. $$Q=\tau\tau''.\frac{r'^2}{rr''}.\frac{1}{\cos.f.\cos.f'.\cos.f''}.\frac{1}{\mathrm{y}\mathrm{y}''}.$$

Now the angles f, f', f'', being generally small; their cosines do not vary much from unity;

(257) moreover as the radius r' falls between r, r'', we shall have $\frac{r'^2}{rr''}$, nearly equal to unity, in most cases, in practice. Hence it is evident, that we may take, at the

(258) commencement of the operation, $Q=\tau\tau''$, for a very near approximation to the value of Q; it is not however so close an approximation as the assumed value of P (236), on account of the magnitude of the factor $\cos.f.\cos.f'.\cos.f''$. The success of Gauss's method essentially depends on this happy selection of the unknown quantities P, Q, whose

(258') values are so nearly known by means of the times t, t'; P being nearly proportional to the ratio of their times, and Q proportional to their products.

[5999]

We shall now show how, by means of the approximate values of P, Q (236, 258), namely,

Approximate values of P, Q.

$$P=\frac{\tau''}{\tau}\,; \qquad\qquad Q=\tau\tau''\,; \tag{259}$$

we may compute the elements of the orbit. The preliminary calculations for finding a, b, c, d, e, δ, δ', δ'', $\varkappa$, $\varkappa''$, λ, λ'' (32—36, 62, 63, &c., 44—47) being made; we may (260)
substitute in (40) the assumed value of P (259), and we shall get the value of w; then from (41′) we may obtain by a few trials the value of z; substituting this in (166) we get (261)
r'; also,

$$\frac{[rr'']}{[r'r'']}\cdot r'. \quad (168), \qquad\qquad \frac{[rr'']}{[rr']}\cdot r'. \quad (169)\,; \tag{262}$$

hence we deduce p, p″ (48, 49); q, q″ (50, 51); ζ, r (52); ζ'', r'' (53); then we obtain the arcs f, f', f'', as in (211—215). With these values of r, r', r'', f, f', f'', (263)
we may compute the corresponding values of $[rr']$, $[r'r'']$, $[rr'']$ (90); and with these we can obtain new values of P, Q (38, 39). If these last expressions are equal, respectively, (264)
to the assumed values (259), we may conclude that we have obtained the true expressions of r, r', r'', f, f', f'', &c. But if the assumed and computed values of P, Q, differ (265)
from each other, we must repeat the calculation, in the same manner as in (260—265); and the same process is to be continued, by assuming the last found values of P, Q, for (266)
a new operation; and when the assumed and computed values of P, Q agree, they must be taken for the correct expression of P, Q, to be used in the rest of the (267)
calculation, in finding the elements of the orbit.

Taking the extreme observations, for this purpose, we have, by the preceding calculations the values of r, r'', $2f'=v''-v$, and the corrected interval of time $t''-t$. With these (268)
we can find, by the precepts in [5995] for an elliptical orbit, the elements corresponding to the plane of the orbit; namely, the semi-major axis, and the excentricity e; also, the time and place of the perihelion in its orbit. If the orbit be a parabola we can use [5996], (269)
and if it be a hyperbola we must use [5997]. The place of the node and inclination of the orbit, to the ecliptic, may be obtained, by means of the triangle ΩAC, or $\Omega A''C''$, (270)
figure 92, page 874; and it may be useful, for the purpose of verification, to make the calculation in both triangles; and take the mean of the results, if there should be any slight difference. In the triangle ΩAC, we have given, the angle $\Omega CA=C$, the angle (271)
$\Omega AC=180^d-\gamma$, and the included side $AC=AE'-\zeta$, to find the sides ΩA, ΩC, by Napier's formulas $[1345^{50,51}]$, and the angle $A\Omega C=\varphi$, by $[1345^{15}]$. If we use the (272)
triangle $\Omega A''C''$, we have the angles $\Omega C''A''=C''$, $\Omega A''C''=180^d-\gamma''$, and the side $A''C''=A''E'-\zeta''$; to find, as above, the sides $\Omega A''$, $\Omega C''$, and the angle $A''\Omega C''=\varphi$. (273)

[5999]

EXAMPLE.

We shall take, for an example of this method of calculation, the following observations of the planet Juno, made by Dr. Maskelyne at Greenwich. The times of observation may be reduced to the meridian of Paris, by adding the difference of meridians, which Gauss puts equal to $9^m\,20^s,9 = 0^{day},006492$.

Data.	Observation.	Mean time at Greenwich.					App. Right Ascension.	App. Declination south.
(274)	I.	1804,	October	$5^d\,10^h\,51^m\,06^s$	or	$5^{days},452152$	$357^d\,10^m\,22^s,35$	$6^d\,40^m\,08^s$
(275)	II.	.	.	17 09 58 10	or 17	,415393	355 43 45 30	8 47 25
(276)	III.	.	.	27 09 16 41	or 27	,386585	355 11 10 95	10 02 28

At these times we have, from the solar tables, the following results,

	Observation.	☉'s longitude from app. Equinox.	Nutation of equin. point.	☉'s distance from Earth.	☉'s latitude.	App. Obliquity of the ecliptic.
(277)	I.	$194^d\,28^m\,53^s,71$	$+15^s,43$	0,9988839	$-0^s,49$	$23^d\,27^m\,59^s,48$
(278)	II.	204 20 21 54	+15 51	0,9953968	+0 79	23 27 59 26
(279)	III.	214 16 52 21	+15 60	0,9928340	−0 15	23 27 59 06

With these data we obtain the apparent longitudes and latitudes of Juno, at the times of observation, as in the following table; the latitudes being south are marked negative. Also the longitudes and latitudes of the zenith,
(280) which are equivalent to the longitudes and complements of the altitudes of the nonagesimal degree of the
(281) ecliptic; the latitude of the place of observation being $51^d\,28^m\,39^s$; and the right ascensions of the meridian being the same as the right ascensions of Juno, because the planet was observed in the meridian. This method
(282) of making these calculations is given in [5998(88, 89, 106)].

	Observation.	App. longitude of Juno.	App. latitude of Juno.	Longitude of the Zenith.	Latitude of the Zenith.
(283)	I.	$354^d\,44^m\,54^s,27$	$-4^d\,59^m\,31^s,59$	$24^d\,29^m$	$46^d\,53^m$
(284)	II.	352 34 44 51	−6 21 56 25	23 25	47 24
(285)	III.	351 34 51 57	−7 17 52 70	23 01	47 36

(286) The parallax of Juno being unknown, we must use the method explained in [5998]; by applying a correction to the sun's place, as in [5998(121—126)], where we have computed the corrections corresponding to the first
(287) observation, as in the first line of the following table; in which we have given the corrections for all three of
(288) the observations; the corrections of the time in the third column are so small that they may be neglected.

	Observation.	Reduction of ☉'s longitude.	Reduction of ☉'s distance.	Reduction of the time.
(289)	I.	$-22^s,39$	+0,0003856	$-0^s,19$
(290)	II.	−27 21	+0,0002329	−0 12
(291)	III.	−35 82	+0,0002085	−0 12

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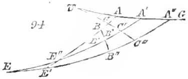


(292) These longitudes are reduced to the epoch of the mean vernal equinox, corresponding to the beginning of the
(293) year 1805, by adding the corrections for the precession as in the following table (301—312). We must also
(294) correct the longitudes and latitudes for the aberration, as in [5998(110, 111)]; by applying to the planet's
(295) longitudes and latitudes the same corrections as if it were a fixed star; these quantities being also contained in
(296) the same table. The correction for the aberration of the sun in longitude is made in (277—279), where the tabular numbers have been increased $20^s,25$.

	Observation.	Reduction of precession to January 1, 1805.	Juno's aberration in longitude.	Juno's aberration in latitude.
(297)	I.	$11^s,87$	$-19^s,11$	$+0^s,53$
(298)	II.	10 23	−17 11	+1 18
(299)	III.	8 86	−14 82	+1 75

[5999]

We shall now apply these corrections to the longitudes and latitudes, in order to obtain the values of A, A', A''; $\alpha, \alpha', \alpha''$; $\theta, \theta', \theta''$, R, R', R''; observing that the signs of the nutation of the equinoctial points (277—279), are such as are used in finding the apparent place from the mean; and these must be changed, in [301′, 303′] in finding the longitudes from the *mean* equinox. (300)

Data.

	Observation I.	Observation II.	Observation III.	
☉'s longitudes — 180^d,	$12^d28^m53^s,71$	$24^d20^m21^s,54$	$34^d16^m52^s,21$	(301)
Nutation of Equinoctial points,	$-15\ 43$	$-15\ 51$	$-15\ 60$	(301′)
Correction for parallax of Juno,	$-22\ 39$	$-27\ 21$	$-35\ 82$	
Precession to Jan. 1, 1805,	$+11\ 87$	$+10\ 23$	$+8\ 86$	
	$A=12^d28^m27^s,76$	$A'=24^d19^m49^s,05$	$A''=34^d16^m09^s,65$	(302)
Juno's longitude,	$354^d44^m54^s,27$	$352^d34^m44^s,51$	$351^d34^m51^s,57$	(303)
Nutation of the equinoctial points,	$-15\ 43$	$-15\ 51$	$-15\ 60$	(303′)
Precession to Jan. 1, 1805,	$+11\ 87$	$+10\ 23$	$+8\ 86$	
Aberration as a fixed star,	$-19\ 11$	$-17\ 11$	$-14\ 82$	
	$\alpha=354^d44^m31^s,60$	$\alpha'=352^d34^m22^s,12$	$\alpha''=351^d34^m30^s,01$	(304)
Juno's latitude,	$-4^d59^m31^s,59$	$-6^d21^m56^s,25$	$-7^d17^m52^s,70$	(305)
Aberration as a fixed star,	$+53$	$+1\ 18$	$+1\ 75$	
	$\theta=-4^d59^m31^s,06$	$\theta'=-6^d21^m55^s,07$	$\theta''=-7^d17^m50^s,95$	(306)
Sun's distance,	$0,9988839$	$0,9953968$	$0,9928340$	(307)
Correction for Juno's parallax,	$+0,0003856$	$+0,0002329$	$+0,0002085$	
Corrected distances R, R', R'',	$R=0,9992695$	$R'=0,9956297$	$R''=0,9930425$	(308)
Logarithms of these distances,	$\log. R=9,9996826$	$\log. R'=9,9980979$	$\log. R''=9,9969678$	(309)
Mean times of observation at Paris, found by adding $0^{days},006492$ to the times at Greenwich.	$t=$ Oct. $5^{days},458644$	$t'=$ Oct. $17^{days},421885$	$t''=$ Oct. $27^{days},393077$	(310)
From (302, 304) we get,	$A-\alpha=17^d43^m56^s,16$	$A'-\alpha'=31^d45^m26^s,93$	$A''-\alpha''=42^d41^m39^s,64$	(311)
	$A'-A=11\ 51\ 21\ 29$	$A''-A'=9\ 56\ 20\ 60$	$A''-A=21\ 47\ 41\ 89$	(312)

As all the latitudes have the *same* sign, we have considered them as *positive*, in the following calculations (312′—319, &c.), and have drawn the figure 94, page 894, to conform to this supposition, making the points B, B', B'', C, C', C'', &c., fall *below* AA'', instead of *above*, as in figure 92, page 874. The change of the directions in the lines AB, $A'B'$, $A''B''$, of the figure, are indicated by the signs. Thus if we had supposed θ to be negative, in finding γ (314), we should have tang.θ and tang.γ *negative*; but this negative (312′) value of γ merely indicates that the arc AB falls *below* AA'', as in figure 94, instead of *above*, as in figure 92, (312″) page 874. Hence we see that by a careful attention to the actual situations of the points of the figure, we may avoid, in a great degree, the trouble of noticing the signs in these preliminary calculations; and by referring to the figure, are less liable to mistakes, than we should be, if we restricted ourselves exclusively to the analytical method of computation.

Preliminary calculations.

To find $\gamma, \gamma', \gamma''$. (62).			*To find* $\delta, \delta', \delta''$. (63).			
θ (306)	tang.	8,9412495	$A-\alpha$ (311)	tang.	9,5048260	(312‴)
$A-\alpha$ (311)	subtract sin.	9,4836865	γ (314)	subtract cos.	9,9828366	(313)
$\gamma=16^d00^m08^s,38$	tang.	9,4575630	$\delta=AB=18^d23^m59^s,20$	tang.	9,5219894	(314)
θ' (306)	tang.	9,0474879	$A'-\alpha'$ (311)	tang.	9,7916902	(314′)
$A'-\alpha'$ (311)	subtract sin.	9,7212540	γ' (316)	subtract cos.	9,9904579	(315)
$\gamma'=11^d58^m00^s,33$	tang.	9,3262339	$\delta'=A'B'=32^d19^m24^s,93$	tang.	9,8012323	(316)
θ'' (306)	tang.	9,1074080	$A''-\alpha''$ (311)	tang.	9,9650091	(317)
$A''-\alpha''$ (311)	subtract sin.	9,8312855	γ'' (319)	subtract cos.	9,9923903	(318)
$\gamma''=10^d41^m40^s,27$	tang.	9,2761225	$\delta''=A''B''=43^d11^m42^s,05$	tang.	9,9726188	(319)

[5999]

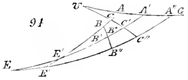
To find E, $A'E$, $A''E$, in the triangle $EA'A''$.

(320) $EA'A''=180^d-\gamma'=168^d01^m59^s,67$ (316). Using Napier's Rules [1345[50, 51]].

(321) $EA''A'=\gamma''=10\ 41\ 40\ 17$ (319)

(322) Sum $=2S_1=178\ 43\ 39\ 84$; $S_1=89^d21^m49^s,92$

(323) Difference $=2D_1=157\ 20\ 19\ 50$; $D_1=78\ 40\ 09\ 75$



Preliminary calculations.

D_1	sin.	9,9914519
S_1	arith. co. sin.	0,0000268
(324) $\frac{1}{2}(A''-A')=4^d58^m10^s,30$	tang.	8,9392834
$\frac{1}{2}(A''E-A'E)=4\ 52\ 24\ 38$	tang.	8,9307621
$\frac{1}{2}(A''E+A'E)=56\ 58\ 50\ 78$		
(325) Difference is $A'E=52\ 06\ 26\ 40$		
(326) Sum is $A''E=61\ 51\ 15\ 16$		

D_1	cos.	9,2932968
S_1	arith. co. cos.	1,9545834
$\frac{1}{2}(A''-A')$ (312)	tang.	8,9392834
$\frac{1}{2}(A''E+A'E)=56^d58^m50^s,78$	tang.	0,1871636
$A''-A'$ (312)	sin.	9,2370422
$A'E$ (325)	arith. co. sin.	0,1028335
γ'' (319)	sin.	9,2685128
$E=2^d19^m34^s,00$	sin.	8,6083885

To find E', AE', $A''E'$, in the triangle $E'AA''$.

(327) $E'AA''=180-\gamma=163^d59^m51^s,62$ (314)

(328) $E'A''A=\gamma''=10\ 41\ 40\ 17$ (319)

Sum $=2S_2=174\ 41\ 31\ 79$; $S_2=87^d20^m45^s,90$

Difference $=2D_2=153\ 18\ 11\ 45$; $D_2=76\ 39\ 05\ 73$

D_2	sin.	9,9881058
S_2	arith. co. sin.	0,0004660
(329) $\frac{1}{2}(A''-A)=10^d53^m50^s,95$	tang.	9,2844852
$\frac{1}{2}(A''E'-AE')=10\ 37\ 15\ 55$	tang.	9,2730570
$\frac{1}{2}(A''E'+AE')=43\ 49\ 45\ 33$		
(330) Difference is $AE'=33\ 12\ 29\ 78$		
(330) Sum is $A''E'=54\ 27\ 00\ 88$		
(331)		

D_2	cos.	9,3633710
S_2	arith. co. cos.	1,3343907
$\frac{1}{2}(A''-A)$ (312)	tang.	9,2844852
$\frac{1}{2}(A''E'+AE')=43^d49^m45^s,33$	tang.	9,9822469
$A''-A$ (312)	sin.	9,5697089
AE' (330)	arith. co. sin.	0,2614699
γ'' (326)	sin.	9,2685128
$E'=7^d13^m37^s,70$	sin.	9,0966916

To find E'', AE'', $A'E''$, in the triangle $E''AA'$.

(332) $E''AA'=180^d-\gamma=163^d59^m51^s,62$ (314)

(333) $E''A'A=\gamma'=11\ 58\ 00\ 33$ (316)

Sum $2S_3=175\ 57\ 51\ 95$; $S_3=87^d58^m55^s,97$

Difference $2D_3=152\ 01\ 51\ 29$; $D_3=76\ 00\ 55\ 64$

D_3	sin.	9,9869333
S_3	arith. co. sin.	0,0002694
(334) $\frac{1}{2}(A'-A)=5^d55^m40^s,64$	tang.	9,0163358
$\frac{1}{2}(A'E''-AE'')=5\ 45\ 25\ 19$	tang.	9,0035385
$\frac{1}{2}(A'E''+AE'')=35\ 28\ 32\ 49$		
(335) Difference is $AE''=29\ 43\ 07\ 30$		
(336) Sum is $A'E''=41\ 13\ 57\ 68$		

D_3	cos.	9,3832051
S_3	arith. co. cos.	1,4533373
(312)	tang.	9,0163358
$\frac{1}{2}(A'E''+AE'')=35^d28^m32^s,49$	tang.	9,8528782
$A'-A$ (312)	sin.	9,3127087
AE'' (335)	arith co. sin.	0,3047442
γ' (316)	sin.	9,3166918
$E''=4^d55^m46^s,22$	sin.	8,9341447

[5999]

To find the angles B, B'', in the triangle $E'BB''$, by [1345[48,49]].

$AE'=33^d12^m29^s,78$ (330)	$A''E'=54^d27^m00^s,88$ (330′)	$BE'B''=E'=7^d13^m37^s,70$ (331)	(337)
$AB=18\ 23\ 59\ 20$ (314)	$A''B''=43\ 11\ 42\ 05$ (319)		(338)
$E'B=14\ 48\ 30\ 58=AE'-\delta$;	$E'B''=11\ 15\ 18\ 83=A''E'-\delta''$;		(339)
$E'B''=11\ 15\ 18\ 83$ (339)			(340)

Sum $2S_4=26\ 03\ 49\ 41$; $S_4=13^d01^m54^s,71$ — Preliminary calculations.

Diff. $2D_4=3\ 33\ 11\ 75$; $D_4=1\ 46\ 35\ 88$

D_4	sin.	8,4914056	D_4	cos.	9,9997912
S_4	arith. co. sin.	0,6468671	S_4	arith. co. cos.	0,0113319
$\tfrac{1}{2}BE'B''=3^d36^m48^s,85$ (337)	cotan.	1,1996098	$\tfrac{1}{2}BE'B''$ (337)	cotang.	1,1996098 (341)
$\tfrac{1}{2}(B''-B)=65\ 19\ 46\ 66$	tang.	0,3378825	$\tfrac{1}{2}(B''+B)=86^d28^m39^s,26$	tang.	1,2107329

$\tfrac{1}{2}(B''+B)=86\ 28\ 39\ 26$

Sum is $B''=151\ 48\ 25\ 92$ (342)

Diff. is $B=21\ 08\ 52\ 60$ (343)

To find the side $E''B^*$ in the triangle $E''BB^*$, by [1345[50,51]].

$B=21^d08^m52^s,60$ (343)	$AE''=29^d43^m07^s,30$ (335)	(344)
$E''=4\ 55\ 46\ 22$ (336)	$AB=18\ 23\ 59\ 20$ (314)	(345)
Sum $2S_5=26\ 04\ 38\ 82$ — $S_5=13^d02^m19^s,41$	$BE''=11\ 19\ 08\ 10$	(345′)
Diff $2D_5=16\ 13\ 06\ 38$ — $D_5=8\ 06\ 33\ 19$		

D_5	sin.	9,1494055	D_5	cos.	9,9956356
S_5	arith. co. sin.	0,6466425	S_5	arith. co. cos.	0,0113439
$\tfrac{1}{2}BE''=5^d39^m34^s,05$ (345′)	tang.	8,9960679	$\tfrac{1}{2}BE''$ (345′)	tang.	8,9960679 (346)
$\tfrac{1}{2}(E''B^*-B'B^*)=3\ 32\ 43\ 98$	tang.	8,7921159	$\tfrac{1}{2}(E''B^*+B'B^*)=5^d45^m01^s,93$	tang.	9,0030474

$\tfrac{1}{2}(E''B^*+B'B^*)=5\ 45\ 01\ 93$

Sum $E''B^*=9\ 17\ 45\ 91$. The sum is taken because $E''B^*$ is opposite to the greatest of the two angles B, E''. (347)

To find δ^*, $AE'-\delta$, $AE''-\delta$, $A'E-\delta'+\delta^*$, $A'E''-\delta'+\delta^*$, &c.

$A'E''=(336)\ 41^d13^m57^s,68$	AE' (330)$=33^d12^m29^s,78$	AE''(335)$=29^d43^m07^s,30$	$A'E$(325)$=52^d06^m26^s,40$	(348)
$\delta'-\delta^*=A'B^*=31\ 56\ 11\ 77$	δ (314)$=18\ 23\ 59\ 20$	δ (314)$=18\ 23\ 59\ 20$	$\delta'-\delta^*$ (349)$=31\ 56\ 11\ 77$	(349)
$\delta'=A'B'=(316)\ 32\ 19\ 24\ 93$	$AE'-\delta=14\ 48\ 30\ 58$	$AE''-\delta=11\ 19\ 08\ 10$	$A'E-\delta'+\delta^*=20\ 10\ 14\ 63$	(350)
$\delta^*=B'B^*=0\ 23\ 13\ 16$	$AE'-\delta$ sin. 9,4075423	$AE''-\delta$ sin. 9,2928537	$A'E-\delta'+\delta^*$ sin. 9,5375909	351
	cos. 9,9853302			(351′)

$A'E$(325)$=52^d06^m26^s,40$	$A'E''$ (336)$=41^d13^m57^s,68$	$A'E''$(336)$=41^d13^m57^s,68$	$A''E$ (326)$=61^d51^m15^s,16$	$A''E'$ (330′)$=54^d27^m00^s,88$	(352)
δ' (316)$=32\ 19\ 24\ 93$	δ' (316)$=32\ 19\ 24\ 93$	$\delta'-\delta^*$(349)$=31\ 56\ 11\ 77$	δ'' (319)$=43\ 11\ 42\ 05$	δ'' (319)$=43\ 11\ 42\ 05$	(353)
$A'E-\delta'=19\ 47\ 01\ 47$	$A'E''-\delta'=8\ 54\ 32\ 75$	$A'E''-\delta'+\delta^*=9\ 17\ 45\ 91$	$A''E-\delta''=18\ 39\ 33\ 11$	$A''E'-\delta''=11\ 15\ 18\ 83$	(354)
		$A'E''-\delta+\delta^*$ sin. 9,2082704	$A''E-\delta''$ sin. 9,5050663	$A''E'-\delta''$ sin. 9,2904350	(355)
				cos. 9,9915661	(356)

To find $R.\sin.\delta$, $R'.\sin.\delta'$, $R''.\sin\delta''$

R	(309)	log.	9,9996826	R'	(309)	log.	9,9980979	R''	(309)	log.	9,9969678	(357)
δ	(314)	sin.	9,4991994	δ'	(316)	sin.	9,7281105	δ''	(319)	sin.	9,8353631	(358)
$R.\sin.\delta$		log.	9,4988820	$R'.\sin.\delta'$		log.	9,7262084	$R''.\sin.\delta''$		log.	9,8323309	(359)

[5999]

To find a, b, c, d, e. (32—36).

(360)	$A''E'-\delta''$ (355)	sin.	9,2904350	$A''E-\delta''$ (355)	sin.	9,5050663
(361)	$AE'-\delta$ (351)	arith. co. sin.	0,5924577	$A'E-\delta'+\delta^*$ (351)	arith. co. sin.	0,4624091
(362)	$R.\sin.\delta$ (359)	log.	9,4988820	$R'.\sin.\delta'$ (359)	log.	9,7262084
(363)	$R''\sin.\delta''$ (359)	arith. co. log.	0,1676691	$R''.\sin.\delta''$ (359)	arith. co. log.	0,1676691
(364)	a$=0,3543593$ (32)	log.	9,5494438	b (33)	log.	9,8613529
(365)	2	log.	0,3010300	δ^* (351)	secant	0,0000099
(366)	$3.\log.(R'.\sin.\delta')$ (359)		9,1786252	$b.\sec.\delta^*=0,7267128$	log.	9,8613628
(367)	δ^* (351)	sin.	7,8295726	a$=0,3543593$ (364)		
(368)	c^{-1} (34)	log.	7,3092278	$b.\sec.\delta^*-$a$=0,3723535$	log.	9,5709555
(369)	c	log.	2,6907722	$b.\sec.\delta^*-1=-0,2732872$	log. subtract	$9,4366192_n$
(370)				$d=-1,3624994$ (35)	log.	$0,1343363_n$
(371)				δ^* (351)	tang.	7,8295825
(372)				$b.\sec\,\delta^*-1$ (369)	log. subtract	$9,4366192_n$
(373)				e (36)	log.	$8,3929633_n$

To find $\varkappa$, $\varkappa''$, λ, λ''. (44—47).

(374)	$R.\sin.\delta$ (359)	log.	9,4988820	$R''.\sin.\delta''$ (359)	log.	9,8323309
(375)	$AE'-\delta$ (361)	arith. co. sin.	0,5924577	$A''E'-\delta''$ (355)	arith. co. sin.	0,7095650
(376)	$\varkappa=1,2340696$ (44)	log.	0,0913397	$\varkappa''=3,4825384$ (45)	log.	0,5418959
(377)	$AE'-\delta$ (351')	cos.	9,9853302	$A''E'-\delta''$ (356)	cos.	9,9915661
(378)	$R.\sin.\delta$ (359)	arith. co. log.	0,5011180	$R''.\sin.\delta''$ (359)	arith. co. log.	0,1676691
(379)	λ (46)	log.	0,4864482	λ'' (47)	log.	0,1592352

First approximation.

First Approximation to P, Q.

To find the first values of P, Q, w, *Q' and the equation in z.* (41').

(380)	$t'-t=11,963241$ (310)	log.	1,0778489	$t''-t'=9,971192$ (310)	log.	0,9987471
	k (54)	log.	8,2355814	k (54)	log.	8,2355814
(381)	τ'' (229)	log.	9,3134303	τ (229)	log.	9,2343285
	τ (381)	subtract log.	9,2343285	τ'' (381)	log.	9,3134303
(382)	$P=\dfrac{\tau''}{\tau}=$ 1,1997804 (259)	log.	0,0791018	$Q=\tau\tau''$ (259)	log.	8,5477588
	a$=$ 0,3543593 (364)			c (369)	log.	2,6907722
	$d=-1,3624994$ (370)			w (386)	sin.	9,3612404
				Q' (40')	log.	0,5997714
(383)	$P+$a$=$ 1,5541397	log.	0,1914901			
(384)	$P+d=-0,1627190$	log. co.	$0,7885618_n$			
(385)	e (373)	log	$8,3929633_n$			
(386)	w$=13^d16^m54^s,77$ (40)	tang.	9,3730152			
	$\delta^*=$ 23 13 16 (351)					
(387)	w$+\delta^*=13$ 40 07 93					

Hence the equation (41') becomes,

$$0,5997714+4.\log.\sin z-\log.\sin.(z-13^d40^m07^s,93)=0.$$

To find z *by approximation from the preceding equation,* (386.)

By a slight inspection of the table of log. sines, we find that $z=14^d$ may be assumed for a first process, in the following table; and $z=15^d$ for a second process. The errors of these assumed values leads to a third value 14^d45^m, and so on, by repeated operations as in the following table, till we get the correct value of z. In the same way we may find the other values of z, which satisfy this equation; as in the second example of the table.

[5999]

Assumed value of z,	14^d	15^d	14^d45^m	14^d30^m	14^d36^m	14^d35^m	$14^d35^m09^s$	32^d2^m	32^d3^m	$32^d2^m26^s$	(388)
Its log. sine,	9,384	9,413	9,406	9,3986	9,40152	9,40103	9,40111	9,72461	9,72481	9,72470	
Multiplied by 4,	7,536	7,652	7,624	7,5944	7,60608	7,60412	7,60444	8,89844	8,89924	8,89880	
Add log. Q',	0,600	0,600	0,600	0,5997	0,59977	0,59977	0,59977	0,59977	0,59977	0,59977	
Sum,	8,136	8,252	8,224	8,1941	8,20585	8,20389	8,20421	9,49821	9,49901	9,49857	
$(z-13^d40^m7^s,93)$ log sine.	7,762	8,366	8,276	8,1615	8,21086	8,20302	8,20421	9,49839	9,49877	9,49856	
Difference,	+0,374	−0,114	−0,052	+0,0326	−0,00501	+0,00087	0,00000	−0,00018	+0,00024	0,00001	(389)

Hence we find that the value of z, corresponding to this equation is $z=14^d35^m09^s$; the other value $z=32^d2^m26^s$ is nearly equal to $\delta'=32^d19^m24^s,93$ (316), and is to be neglected, as in (157, &c.) (390)

First Approximation.

To find r' (77), *and the factors* (41″, 41‴).

$R'.\sin.\delta'$ (359)	log.	9,7262084	
$z=14^d35^m09^s$ (390)	sub. sin.	9,4011076	
r' (77) . . .	log.	0,3251008	

δ^* (351) $23^m13^s,16$

$z-\delta^*$ $=14^d11^m55^s,84$

$A'E-\delta'$ (354) $=19^d47^m01^s,47$	$A'E''-\delta'$ (354) $=8^d54^m32^s,75$
$z=14\ 35\ 09\ 00$	(390) $z=14\ 35\ 09\ 00$
$z+A'E-\delta'=34\ 22\ 10\ 47$	$z+A'E''-\delta'=23\ 29\ 41\ 75$
Its log. sine $=9,7516861$	Its log. sin. $=9,6006113$

$R'.\sin\delta'$ (359)	log.	9,7262084	
$P+a$ (383)	log.	0,1914901	(391)
b (364)	log. co.	0,1386471	(392)
$z-\delta^*$ (394)	arith. co. sin.	0,6103240	(393)
$\left\{\frac{[r\,r'']}{[r'r'']}.r'\right\}$ (41″)	log.	0,6666696	(394)
P (382)	subtract log.	0,0791018	(395)
$\left\{\frac{[rr'']}{[r\,r']}.r'\right\}$ (41‴)	log.	0,5875678	(396)
			(397)
			(398)

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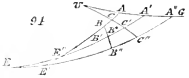

To find p, (p″). (48, 49); q, q″ (50, 51); ζ, ζ'', r, r'' (52, 53).

$\left\{\frac{[r\,r'']}{[r'r'']}.r'\right\}$ (394)	log.	0,6666696
$z+A'E-\delta'$ (398)	sin.	9,7516861
E (326)	sin.	8,6083885
E' (331)	arith. co. sin.	0,9003084
p (48)	log.	9,9270526
λ (379)	log.	0,4864482
$\varkappa=1,2340696$ (376)	log.	0,0913397
$\lambda\,p\,\varkappa=3,1977206$		0,5048405
$q=\lambda\,p\,\varkappa-\varkappa=1,9636510$ (50)	log.	0,2930643
p (403)	log.	9,9270526
$\frac{p}{q}=\text{tang}.\zeta$ (52); $\zeta=23^d17^m33^s,38$	tang.	9,6339883
$\zeta=CE'$ (409)	sec.	0,0369220
q (407)	log.	0,2930643
r (52)	log.	0,3299863

$\left\{\frac{[rr'']}{[r\,r']}.r'\right\}$ (396)	log.	0,5875678	(399)
$z+A'E''-\delta'$ (398)	sin.	9,6006113	(400)
E'' (336)	sin.	8,9341447	(401)
E' (331)	arith. co. sin.	0,9003084	(402)
p″ (49)	log.	0,0226322	(403)
λ'' (379)	log.	0,1592352	(404)
$\varkappa''=3,4825384$ (376)	log.	0,5418959	(405)
$\lambda''\,p''\,\varkappa''=5,2937488$	log.	0,7237633	(406)
$q''=\lambda''\,p''\,\varkappa''-\varkappa''=1,8112104$ (51)	log.	0,2579689	(407)
p″ (403)	log.	0,0226322	(408)
$\frac{p''}{q''}=\text{tang}.\zeta''$ (53); $\zeta''=30^d11^m04^s,25$	tang.	9,7646633	(409)
$\zeta''=C''E'$ (409)	sec.	0,0632798	(410)
q″ (407)	log.	0,2579689	(411)
r'' (53)	log.	0,3212487	(412)

[5999] First Approximation.

To find the arc $CC''=2f'$, *in the triangle* $CE'C''$, *by* [1345[48,49]].

(413)	$\zeta=CE'$	$=23^d17^m33^s,38$ (409)					
(413′)	$\zeta''=C''E'$	$=30\ 11\ 04\ 25$ (409)					
	Sum $2S_6$	$=53\ 28\ 37\ 63$;	$S_6=26^d44^m18^s,82$				
	Diff. $2D_6$	$=6\ 53\ 30\ 87$;	$D_6=3\ 26\ 45\ ,44$				
	D_6		sin.	8,7789252	D_6	cos.	9,9992141
	S_6		arith. co. sin.	0,3468646	S_6	arith. co. cos.	0,0491151
(414)	$\frac{1}{2}E'$	$=3^d36^m48^s,85$ (341)	cotan.	1,1996098	$\frac{1}{2}E'$ (341)	cotan.	1,1996098
	$\frac{1}{2}(C-C'')$	$=64\ 41\ 56\ 92$	tang.	0,3253996	$\frac{1}{2}(C+C'')=86^d45^m58^s,08$	tang.	1,2479390
	$\frac{1}{2}(C+C'')$	$=86\ 45\ 58\ 08$					
(415)	Sum is $E'CC''$	$=151\ 27\ 55\ 00$			$E'C''C$ (416)	arith. co. sin.	0,4251700
(416)	Diff. is $E'C''C$	$=22\ 04\ 01\ 16$			CE' (413)	sin.	9,5970663
					E' (331)	sin.	9,0996916
(417)					$2f'=CC''=7^d36^m32^s,42$	sin.	9,1219279

To find the arcs $CC'=2f''$, $C'C''=2f$, (214, 215).

	r (412)		log.	0,3299863	r'' (412)	log.	0,3212487
	$\left\{\frac{[rr'']}{[r'r'']}.r'\right\}$ (394)		arith. co. log.	9,3333304	$\left\{\frac{[rr'']}{[rr']}.r'\right\}$ (396)	arith. co. log.	9,4124322
(418)	$2f'=CC''$ (417)		sin.	9,1219279	$2f'$ (417)	sin.	9,1219279
(419)	$2f=C'C''$	$=3^d29^m47^s,50$	sin.	8,7852446	$2f''=CC'=4^d06^m44^s,95$	sin	8,8556088
(420)	$2f''=CC'$	$=4\ 06\ 44\ 95$ (419)					
(421)	Sum is $2f'=CC''$	$=7\ 36\ 32\ 45$					
	Computed CC''	$=7\ 36\ 32\ 42$ (417)					

To find $\rho_{\prime}$, $\rho_{\prime}'$, $\rho_{\prime}''$, *in order to correct* t, t', t'', τ, τ'', *for the aberration*, (222).

	To find $\rho_{\prime}$ *and* t_1 (224).		*To find* $\rho_{\prime}'$ *and* t_2 (225).		*To find* $\rho_{\prime}''$ *and* t_3 (226).	
(422)	$AE'=33^d12^m29^s,78$ (330)		$\delta'=32^d19^m24^s,93$ (316)		$A''E'=54^d27^m00^s,88$ (330′)	
(423)	$\zeta=23\ 17\ 33\ 38$ (409)		$z=14\ 35\ 09$ (390)		$\zeta''=30\ 11\ 04\ 25$ (409)	
(424)	$AE'-\zeta=9\ 54\ 56\ 40$	sin. 9,23603	$\delta'-z=17\ 44\ 15\ 93$	sin. 9,48382	$A''E'-\zeta''=24\ 15\ 56\ 63$	sin. 9,61381
	r (412)	log. 0,32999	r' (392)	log. 0,32510	r'' (412)	log. 0,32125
(425)	δ (358)	arith. co. sin. 0,50080	δ' (358)	arith. co. sin. 0,27189	δ'' (358)	arith. co. sin. 0,16464
(426)	$\rho_{\prime}$ (224)	log. 0,06682	$\rho_{\prime}'$ (225)	log. 0,08081	$\rho_{\prime}''$ (226)	log. 0,09970
(427)	Constant log. of aberration	7,75633	(223) Constant	log. 7,75633	Constant	log. 7,75633
(428)	Correction $t_1=0,006655$	log. 7,82315	(223) $t_2=0,006873$	log. 7,83714	$t_3=0,007178$	log. 7,85603
	Observ. Oct. 5,458644 (310)		Oct. 17,421885 (310)		Oct. 27,393077 (310)	
(429)	Corrected Oct. 5,451989 $=t$.		Oct. 17,415012 $=t'$, corrected.		Oct. 27,385899 $=t''$, corrected.	
			Oct. 5,451989 $=t$, corrected.		Oct. 17,415012 $=t'$, corrected.	
			Int. $t'-t=11,963023$	log. 1,0778409	Int. $t''-t'=9,970887$	log. 0,9987338
(430)			Constant k (54)	log. 8,2355814	Constant k (54)	log. 8,2355814
(431)			Corrected τ'' (229)	log. 9,3134223	Corrected τ (229)	log. 9,2343152

[5999]

To find y'' *from* $r, r', 2f'', t'-t$. *Like* [5995(187)].

r' (392)	log.	0,3251008 (392) . . .		0,3251008
r (412)	log.	0,3299863 (412) . . .		0,3299863
$\frac{r'}{r}=\text{tang}^4.(45^d+w)$	log.	9,9951145	sum	0,6550871
$45^d+w=44^d55^m09^s,957$	tang.	9,9987786	half	0,3275436
$w=-\;4^m50^s,043$			$(rr')^{\frac{3}{2}}$ log.	0,9826307
			arith. co.	9,0173693
$2w=-\;9^m40^s,086$			tang.	$7,4490\!7_n$
			same	$7,44907_n$
$f''=2^d03^m22^s,475$ (420)			sec.	0,00028
$\text{tang}^2.2w.\sec f''=0,00000791$			log.	4,89842
			constant log.	5,5680729
$t'-t$	11,963023 (430)		log.	1,0778409
			same	1,0778409
			$3\times\log.\sec.f''$ (439)	0,0008391
$\frac{3}{2}.\log.(rr')$	arith. co. (433)			9,0173693
			mm log.	6,7419631
$f''=2^d03^m22^s,475$ (435)			sec.	0,0002797
$\frac{1}{2}f''=1\;01\;41\;2375$			sine	8,2538985
			same	8,2538985
$\sin^2.\frac{1}{2}f''.\sec f''=0,00032216$			log.	6,5080767
$\text{tang}^2.2w.\sec f''=0,00000791$ (436)				
$l=0,00033007$				
$\frac{5}{6}=0,83333333$				
$l+\frac{5}{6}=0,83366340$			subtract log.	9,9209908
mm (438)			log.	6,7419631
$h=0,00066217$			log.	6,8209723

Corresponds in Table VIII, to app. log. $y''y''=0,0006383$

log. $y''=0,0003192$

To find P.

y''	(447)	arith co. log.	9,9996808
y	(447)	log.	0,0002285
τ''	(431)	log.	9,3134223
τ	(431)	arith. co. log.	0,7656848
Corrected $P=\frac{y\tau''}{y''\tau}$	(235)	log.	0,0790164
Assumed value of P	(382)	log.	0,0791018
		Difference	$-0,0000854$

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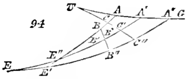

We may remark that the value of h (445) does not require, in this example, any correction for the quantity ξ [5995(147)], which is wholly insensible.

To find y *from* $r', r'', 2f, t''-t'$.

First Approximation.

r'' (412)	log.	0,3212487 (412) . . .		0,3212487	(432)
r' (392)	log.	0,3251008 (392) . . .		0,3251008	
$\frac{r''}{r'}=\text{tang}^4.(45^d+w)$	log.	9,9961479	sum	0,6463495	
$45^d+w=44^d56^m11^s,302$	tang.	9,9990369	half	0,3231748	
$w=-\;3^m48^s,698$			$(r'r'')^{\frac{3}{2}}$ log.	0,9695243	(433)
			arith.co.log.	9,0304757	(433')
$2w=-\;7^m37^s,396$			tang.	$7,34587_n$	(434)
			same	$7,34587_n$	
$f=1^d44^m53^s,75$ (419)			sec.	0,00020	(435)
$\text{tang}^2.2w.\sec.f=0,00000492$			log.	4,69194	(436)
[5995(38)]			constant log.	5,5680729	(436)
$t''-t'$	9,970887 (430)		log.	0,9987338	(437)
			same	0,9987338	
			$3\times\log.\sec.f$ (439)	0,0006066	
$\frac{3}{2}.\log.(r'r'')$	arith. co. log. (433')			9,0304757	
			mm log.	6,5966228	(438)
$f=1^d44^m53^s,75$ (435)			sec.	0,0002022	(439)
$\frac{1}{2}f=0\;52\;26\;875$			sine	8,1834375	(440)
			same	8,1834375	
$\sin^2.\frac{1}{2}f.\sec.f=0,00023285$			log.	6,3670772	(441)
$\text{tang}^2.2w,\sec.f=0,00000492$ (436)					(442)
$l=0,00023777$					(443)
$\frac{5}{6}=0,83333333$					
$l+\frac{5}{6}=0,83357110$			subtract log.	9,9209427	(444)
mm (438)			log.	6,5966228	
[5995(147)] $h=0,00047389$			log.	6,6756801	(445)

Corresponds in Table VIII, to app. log. $yy=0,0004570$ (446)

log. $y=0,0002285$ (447)

To find Q.

τ	(431)	log.	9,2343152	(447')
τ''	(431)	log.	9,3134223	(447'')
$2.\log.r'$	(392)		0,6502016	
r	(412)	arith. co. log.	9,6700137	(447''')
r''	(412)	arith. co. log.	9,6787513	
y	(447)	arith. co. log.	9,9997715	(448)
y''	(447')	arith. co. log.	9,9996808	(449)
f	(439)	secant	0,0002022	
f'	(417)	secant	0,0009581	(450)
f''	(439)	secant	0,0002797	
Corrected Q	(256)	log.	8,5475964	(451)
Assumed Q	(382)	log.	8,5477588	(452)
		Difference	$-0,0001624$	(453)

(453)

[5999] Second Approximation.

Second Approximation to P, Q.

(454) With the corrected values of τ, τ'', (431), and the computed values of P, Q (448, 451), we must repeat that part of the calculation, which is contained in (382—453), in order to obtain a nearer approximation to the
(455) values of P, Q. We shall give this calculation at full length, and in the same form as in the first process (382—453); but the part (422—431) relative to the aberration, is given with sufficient accuracy; and it is not necessary to make any correction in it. The labor of this re-computation is much decreased from the circumstance that the same form of calculation is retained, and the results are not much varied.

(456)	$P=\ \ 1{,}1995445$ (448)	log.	0,0790164	
	$a=\ \ 0{,}3543593$ (364)			
	$d=-1{,}3624994$ (370)			
(457)	$P+a=\ \ 1{,}5539038$	log.	0,1914242	
	$P+d=-0{,}1629549$	log. co.	$0{,}7879326_n$	
	e (373)	log.	$8{,}3929633_n$	
(458)	$w=13^d15^m41^s{,}00$	tang.	9,3723201	
(459)	$\delta^*=\ \ 23\ 13\ 16$ (351)			
(460)	$w+\delta^*=13\ 38\ 54\ 16$			

Q	(451)	log.	8,5475964
c	(369)	log.	2,6907722
w	(458)	sin.	9,3605818
Q'	(40')	log.	0,5989504

Hence the equation (41') becomes,

$0{,}5989504+4.\text{log.sin.}z-\text{log.sin.}(z-13^d38^m54^s{,}16)=0.$

To find z by approximation from the equation, (459).

(461)	Assumed value of z,	14^d35^m	14^d33^m	$14^d33^m23^s$
	Its log. sine,	9.40103	9,40006	9,40025
	Multiplied by 4,	7,60412	7,60024	7,60100
	Add log. Q',	0,59895	0,59895	0.59895
(462)	Sum,	8,20307	8,19919	8,19995
	$(z-13^d38^m54^s{,}16)$sine,	8,21265	8,19688	8,19995
(463)	Difference,	−0,00958	+0,00231	0,00000

This operation is much abridged, because we are able to assume, in the first operation, the value of z, computed in (390), which varies but very little from the result here found, namely $z=14^d\,33^m\,23^s$.

To find r' (77), and the factors $(41'', 41''')$.

(464)	$R'.\text{sin.}\delta'$ (359)		log.	9,7262084
(465)	z $14^d33^m23^s$	(462)	sub. sin.	9,4002490
(466)		r'	log.	0,3259594

(467)	δ^* (351) $\ 0^d23^m13^s{,}16$	
(468)	$z-\delta^*\ =14^d10^m09^s{,}84$	
(469)	$A'E-\delta'(395)=19^d47^m01^s{,}47$	$A'E''-\delta'(395)=8^d54^m32^s{,}75$
(470)	z (462)$=14\ 33\ 23\ 00$	$z=14\ 33\ 23\ 00$
(471)	$z+A'E-\delta'=34\ 20\ 24\ 47$	$z+A'E''-\delta'=23\ 27\ 55\ 75$
(472)	Its log. sine $=9{,}7513596$	Its log. sin. $=9{,}6000975$

$R'.\text{sin.}\delta'$	(359)	log.	9,7262084
$P+a$	(457)	log.	0,1914242
b	(392)	log. co.	0,1386471
$z-\delta^*$	(468)	arith.co.sin.	0,6112070
$\left\{\frac{[r\,r'']}{[r'r'']}.r'\right\}$ (41'')		log.	0,6674867
P	(456)	log. subtract	0,0790164
$\left\{\frac{[rr'']}{[r\,r']}.r'\right\}$ (41''')		log.	0,5884703

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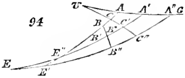


[5999] Second Approximation.

To find p, p'', (48, 49); q, q'' (50, 51); ζ, ζ''; r, r'' (52, 53).

$\left\{\frac{[rr'']}{[r'r'']}\,r'\right\}$ (468)	log.	0,6674867	$\left\{\frac{[rr'']}{[rr']}.r'\right\}$ (470)	log.	0,5884703	(473)
$z+A'E-\delta'$ (472)	sin.	9,7513596	$z+A'E''-\delta'$ (472)	sin.	9,6000975	(474)
E (401)	sin.	8,6083885	E'' (401)	sin.	8,9341447	(475)
E' (402)	arith. co. sin.	0,9003084	E' (402)	arith. co. sin.	0,9003084	(476)
p	log.	9,9275432	p''	log.	0,0230209	(477)
λ (404)	log.	0,4864482	λ'' (404)	log.	0,1592352	(478)
$\varkappa=1,2340696$ (405)	log.	0,0913397	$\varkappa''=3,4825384$ (405)	log.	0,5418959	(479)
$\lambda\,\mathrm{p}\,\varkappa=3,2013348$	log.	0,5053311	$\lambda''\,\mathrm{p}''\,\varkappa''=5,2984890$	log.	0,7241520	(480)
$\mathrm{q}=\lambda\,\mathrm{p}\,\varkappa-\varkappa=1,9672652$	log.	0,2938629	$\mathrm{q}''=\lambda''\,\mathrm{p}''\,\varkappa''-\varkappa''=1,8159506$	log.	0,2591040	(481)
p (477)	log.	9,9275432	p'' (477)	sub. log.	0,0230209	(482)
$\frac{\mathrm{p}}{\mathrm{q}}=\mathrm{tang.}\zeta$ (52); $\zeta=23^d16^m40^s,26$	tang.	9,6336803	$\frac{\mathrm{p}''}{\mathrm{q}''}=\mathrm{tang.}\zeta''$ (53); $\zeta''=30^d08^m30^s,24$	tang.	9,7639169	(483)
$\zeta=CE'$	sec.	0,0368739	$\zeta''=C''E'$	sec.	0,0630914	(484)
q (481)	tang.	0,2938629	q'' (481)	log.	0,2591040	(485)
r	log.	0,3307368	r''	log.	0,3221954	(486)

To find the arc $CC''=2f'$, *in the triangle* $CE'C''$.

$\zeta=CE'=23^d16^m40^s,26$ (483)						(487)
$\zeta''=C''E'=30\ 08\ 30\ 24$ (483)						
Sum $2S_7=53\ 25\ 10\ 50$; $S_7=26^d42^m35^s,25$						
Diff. $2D_7=6\ 51\ 49\ 98$; $D_7=3\ 25\ 54\ 99$						
D_7	sin.	8,7771576	D_7	cos.	9,9992204	
S_7	arith. co. sin.	0,3472977	S_7	arith. co. cos.	0,0490053	
$\frac{1}{2}E'$ (414)	cotan.	1,1996098	$\frac{1}{2}E'$ (414)	cotan.	1,1996098	(488)
$\frac{1}{2}(C-C'')$ $64^d37^m51^s,80$	tang.	0,3240651	$\frac{1}{2}(C+C'')=86^d45^m55^s,31$	tang.	1,2478355	
$\frac{1}{2}(C+C'')$ 86 45 55 31			$E'C''C$ (490)	arith. compl. sin.	0,4239133	
Sum is $E'CC''=151\ 23\ 47\ 11$			CE' (487)	sin.	9,5968064	(489)
Diff. is $E'C''C=22\ 08\ 03\ 51$			E' (416)	sin.	9,0996916	(490)
			$2f'=CC''=7^d34^m56^s,36$	sin.	9,1204113	(491)

To find the arcs $CC'=2f''$, $C'C''=2f$, (214, 215).

r (486)	log.	0,3307368	r'' (486)	log.	0,3221954	
$\left\{\frac{[rr'']}{[r'r'']}.r'\right\}$ (473)	arith. co. log.	9,3325133	$\left\{\frac{[rr'']}{[rr']}.r'\right\}$ (473)	arith. co. log.	9,4115297	
$2f'=CC''$ (491)	sin.	9,1204113	$2f'$ (491)	sin.	9,1204113	(492)
$2f=C'C''=3^d29^m01^s,64$	sin.	8,7836614	$2f''=CC'=4^d05^m54^s,75$	sin.	8,8541364	(493)
$2f''=CC'=4\ 05\ 54\ 75$ (493)						(494)
Sum is $2f'=CC''=7\ 34\ 56\ 39$						(495)
Computed $CC''=7\ 34\ 56\ 36$ (491)						

[5999]

To find y'' *from* $r, r', 2f'', t'-t$, (432—447).

(496) r'	log.=0,3259594 (466) . . . 0,3259594		
r	log.=0,3307368 (486) . . . 0,3307368		
Second Approximation. $\frac{r'}{r}=\text{tang}^4.(45^d+w)$	log. 9,9952226	sum	0,6566962
$45^d+w=44^d55^m16^s,37$	tang. 9,9988056	half	0,3283481
(497) $w=-4^m43^s,63$		$(rr')^{\frac{3}{2}}$ log.	0,9850443
(497')		arith. co.	9,0149557
(498) $2w=-9^m27^s,26$		tang.	$7,43936_n$
		same	$7,43936_n$
(499) $f''=2^d02^m57^s,375$ (493)		sec.	0,00028
(500) $\text{tang}^2.2w.\sec f''=0,00000757$		log.	4,87900
(436')		constant log.	5,5680729
(501) $t'-t$ (437)		log.	1,0778409
		same	1,0778409
		$3\times\log\sec.f''$ (503)	0,0008334
$\frac{3}{2}.\log.(rr')$		arith. co. (497')	9,0149557
(502)	$m\,m$	log.	6,7395438
(503) $f''=2^d02^m57^s,375$ (499)		sec.	0,0002778
(504) $\frac{1}{2}f''=1\ 01\ 28\ 688$		sine	8,2524236
		same	8,2524236
(505) $\sin^2.\frac{1}{2}f''.\sec.f''=0,00031998$		log.	6,5051250
(506) $\text{tang}^2.2w.\sec.f''=0,00000757$	(500)		
(507) $l=0,00032755$			
$\frac{5}{6}=0,83333333$			
(508) $l+\frac{5}{6}=0,83366088$		log. sub.	9,9209895
$m\,m$ (502)		log.	6,7395438
(509) $h=0,00065850$		log.	6,8185543
(510) Corresponds in Table VIII, to log. y''		y'' =	0,0006347
(511)		log. y'' =	0,0003173

To find P.

y'' (511)	arith. co. log.	9,9996827
y (511)	log.	0,0002271
τ'' (447'')	log.	9,3134223
τ (447''')	arith. co. log.	0,7656848
(512) Corrected $P=\frac{y\tau''}{y''\tau}$ (235)	log.	0,0790169
(513) Assumed value of P (456)	log.	0,0790164
(514)	Difference	+0,0000005

To find y *from* $r', r'', 2f, t''-t'$.

r''	log.=0,3221954 (486) . . . 0,3221954		
r'	log.=0,3259594 (466) . . . 0,3259594		
$\frac{r''}{r'}=\text{tang}^4.(45^d+w)$	log. 9,9962360	sum	0,6481548
$45^d+w=44^d56^m16^s,55$	tang. 9,9990590	half	0,3240774
$w=-3^m43^s,45$		$(r'r'')^{\frac{3}{2}}$ log.	0,9722322
		arith. co.	9,0277678
$2w=-7^m26^s,90$		tang.	$7,33578_n$
		same	$7,33578_n$
$f=1^d44^m30^s,82$ (493)		sec.	0,00020
$\text{tang}^2.2w.\sec.f=0,00000470$		log.	4,67176
(436')		constant log.	5,5680729
$t''-t'$ (437)		log.	0,9987338
		same	0,9987338
		$3\times\log.\sec.f$ (503)	0,0006021
$\frac{3}{2}.\log.(r'r'')$ arith. co.		(497')	9,0277678
	$m\,m$	log.	6,5939104
$f=1^d44^m30^s,82$ (499)		sec.	0,0002007
$\frac{1}{2}f=0^d52^m15^s,41$		sine	8,1818525
		same	8,1818525
$\sec^2.\frac{1}{2}f.\sec.f=0,00023116$		log.	6,3639057
$\text{tang}^2.2w.\sec.f=0,00000470$	(500)		
$l=0,00023586$			
$\frac{5}{6}=0,87333333$			
$l+\frac{5}{6}=0,83356919$		log. sub.	9,9209417
$m\,m$	(502)	log.	6,5939104
$h=0,00047094$		log.	6,6729687
Corresponds in Table VIII, to log.		yy	0,0004541
		log. y	0,0002271

To find Q.

τ (447')	log.	9,2343152
τ'' (447'')	log.	9,3134223
2.log. r' (466)		0,6519188
r (486)	arith. co. log.	9,6692632
r'' (486)	arith. co. log.	9,6778046
y (511)	arith. co. log.	9,9997729
y'' (511)	arith. co. log.	9,9996827
f (503)	sec.	0,0002007
f' (491)	sec.	0,0009514
f'' (503)	sec.	0,0002778
(515) Q	log.	8,5476096
(516) Assumed value of Q (456)	log.	8,5475964
(517) Difference	+	0,0000132

[5999]

Third Approximation.

Third Approximation to P, Q.

With the computed values of P, Q (512, 515), we must again repeat the operation, as in (456—517) to obtain the final values of P, Q. The form of calculation is the same as in the last process, and the numbers vary but very little, so that the calculation is repeated with great facility; and it serves as a verification of the process. (518)

$P=$ 1,1995459 (512)	log.	0,0790169	Q (515)	log.	8,5476096	(519)
a= 0,3543593 (364)			c (369)	log.	2,6907722	
$d=$—1,3624994 (370)			w (522)	sin.	9,3605857	
$P+$a= 1,5539052	log.	0,1914246	Q' (40′)	log.	0,5989675	(520)
$P+d=$—0,1629535	log. co.	$0,7879363_n$				(521)
e (373)	log.	$8,3929633_n$				
w=$13^d15^m41^s,44$	tang.	9,3723242	Hence the equation (41′) becomes,			(522)
$\delta^* =$ 23 13 16 (351)			$0,5989675+4.\log.\sin.z-\log.\sin.(z-13^d38^m54^s,60)=0.$			(523)
w$+\delta^*=$13 38 54 60						(524)

To find z by approximation from the equation, (523).

Assumed value of z,	$14^d33^m23^s$	$14^d33^m23^s,8$	$14^d33^m23^s,72$
Its log. sine,	9,40025	9,4002555	9,4002548
Multiplied by 4,	7,60100	7,6010220	7,6010192
Add log. Q',	0,59897	0,5989675	0,5989675
Sum,	8,19997	8,1999895	8,1999867
$(z-13^d38^m54^s,60)$sine,	8,19989	8,1999982	8,1999875
Difference,	+0,00008	—0,0000087	—0,0000008

(525)

The value of z, obtained in (462), is here assumed as the first operation, and by a very easy calculation we find $z=14^d33^m23^s,72$ nearly. (526)

(527)

To find r′ (77), and the factors (41″, 41‴).

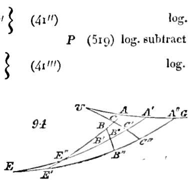
$R'.\sin.\delta'$ (464)		log.	9,7262084
z $14^d33^m23^s,72$ (525)		sub. sin.	9,4002548
	r'	log.	0,3259536

δ^* (351) $0^d23^m13^s,16$	
$z-\delta^*=14^d10^m10^s,56$	
$A'E-\delta'(469)=19^d47^m01^s,47$	$A'E''-\delta'(469)=8^d54^m32^s,75$
z (526)=14 33 23 72	$z=$14 33 23 72
$z+A'E-\delta'=$34 20 25 19	$z+A'E''-\delta'=$23 27 56 47
Its log. sine =9,7513618	Its log. sin. =9,6001010

$R'.\sin.\delta'$	(464)	log.	9,7262084	(528)
$P+$a	(520)	log.	0,1914246	(529)
b	(466)	arith.co.log.	0,1386471	(530)
$z-\delta^*$	(532)	arith.co.sin.	0,6112011	(531)
$\left\{\frac{[r\,r'']}{[r'r'']}.r'\right\}$ (41″)		log.	0,6674812	(532)
P	(519)	log. subtract	0,0790169	(533)
$\left\{\frac{[rr'']}{[r\,r']}.r'\right\}$ (41‴)		log.	0,5884643	(534)

(535)

536)



[5999] Third Approximation.

To find p, p″, (48, 49) ; q, q″ (50, 51) ; ζ, ζ'' ; r, r'' (52, 53).

(537)	$\left\{\frac{[r\,r'']}{[r'r'']}.r'\right\}$ (532)		log.	0,6674812	$\left\{\frac{[rr'']}{[r\,r']}.r'\right\}$ (534)		log.	0,5884643
(538)	$z+A'E-\delta'$	(536)	sin.	9,7513618	$z+A'E''-\delta'$	(536)	sin.	9,6001010
(539)	E	(475)	sin.	8,6083885	E''	(475)	sin.	8,9341447
(540)	E'	(476)	cosec.	0,9003084	E'	(476)	cosec.	0,9003084
(541)	p		log.	9,9275399	p″		log.	0,0230184
(542)	λ	(478)	log.	0,4864482	λ''	(478)	log.	0,1592352
(543)	$\varkappa=1,2340696$	(479)	log.	0,0913397	$\varkappa''=3,4825384$	(479)	log.	0,5418959
(544)	$\lambda\,\mathrm{p}\,\varkappa=3,2013104$		log.	0,5053278	$\lambda''\,\mathrm{p}''\,\varkappa''=5,2984585$		log.	0,7241495
(545)	$\mathrm{q}=\lambda\,\mathrm{p}\,\varkappa-\varkappa=1,9672408$		log. sub.	0,2938575	$\mathrm{q}''=\lambda''\,\mathrm{p}''\,\varkappa''-\varkappa''=1,8159201$		log. sub.	0,2590967
(546)	p	(541)	log.	9,9275399	p″	(541)	log.	0,0230184
(547)	$\frac{\mathrm{p}}{\mathrm{q}}=\text{tang.}\,\zeta=23^d16^m40^s,62$		tang.	9,6336824	$\frac{\mathrm{p}''}{\mathrm{q}''}=\text{tang.}\,\zeta''=30^d08^m31^s,23$		tang.	9,7639217
(548)	$\zeta=CE'$		sec.	0,0368743	$\zeta''=C''E'$		sec.	0,0630926
(549)	q	(545)	tang.	0,2938575	q″	(545)	log.	0,2590967
(550)	r		log.	0,3307318	r''		log.	0,3221893

To find the arc $CC''=2f'$, *in the triangle* $CE'C''$.

(551)	$\zeta=CE'=23^d16^m40^s,62$ (547)						
(552)	$\zeta''=C''E'=30\;08\;31\;23$ (547)						
(553)	Sum $2S_3=53\;25\;11\;85$;	$S_3=26^d42^m35^s,93$					
(554)	Diff. $2D_3=6\;51\;50\;61$;	$D_3=3\;25\;55\;31$					
(555)	D_3	sin.	8,7771688	D_3		cos.	9,9992204
(556)	S_3	arith. co. sin.	0,3472948	S_3		arith. co. cos.	0,0490060
(557)	$\frac{1}{2}E'$ (488)	cotan.	1,1996098	$\frac{1}{2}E'$	(488)	cotan.	1,1996098
	$\frac{1}{2}(C-C'')=64^d37^m53^s,31$	tang.	0,3240734	$\frac{1}{2}(C+C'')=86^d45^m55^s,33$		tang.	1,2478362
	$\frac{1}{2}(C+C'')=86\;45\;55\;33$			$E'C''C$	(559)	arith. compl. sin.	0,4239210
(558)	Sum is $E'CC''=151\;23\;48\;64$			CE'	(551)	sin.	9,5968081
(559)	Diff. is $E'C''C=22\;08\;02\;02$			E'	(490)	sin.	9,0996916
(560)				$2f'=CC''=7^d34^m56^s,96$		sin.	9,1204207

To find the arcs $CC'=2f''$, $C'C''=2f$, (214, 215).

	r	(550)	log.	0,3307318	r''	(550)	log.	0,3221893
	$\left\{\frac{[r\,r'']}{[r'r'']}.r'\right\}$	(537)	arith. co. log.	9,3325188	$\left\{\frac{[rr'']}{[rr']}.r'\right\}$	(537)	arith. co. log.	9,4115357
	$2f'=CC''$	(560)	sin.	9,1204207		(560)	sin.	9,1204207
(561)	$2f=3^d29^m01^s,92$		sin.	8,7836713	$2f''=4^d05^m55^s,07$		sin	8,8541457
(562)	$2f''=4\;05\;55\;07$	(561)						
(563)	Sum is $2f'=7\;34\;56\;99$							
(564)	Computed above $=7\;34\;56\;96$	(560)						

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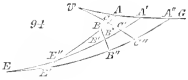

[5999]

To find y'' *from* $r, r', 2f'', t'-t$. (496—511).

r'	log. 0,3259536	(530) . . .	0,3259536
r	log. 0,3307318	(550) . . .	0,3307318
$\frac{r'}{r}=\text{tang}^4.(45^d+w)$	log. 9,9952218	sum	0,6566854
$45^d+w=44^d55^m16^s,32$	tang. 9,9988054 5	half	0,3283427
$w=-4^m43^s,68$		$(rr')^{\frac{3}{2}}$ log.	0,9850281
		arith. co.	9,0149719

$2w=-9^m27^s,36$	tang.	$7,43943_n$
	same	$7,43943_n$
$f''=2^d02^m57^s,535$ (562)	sec.	0,00028
$\text{tang}^2.2w.\sec f''=0,00000757$	log.	4,87914
(436')	constant log.	5,5680729
$t'-t$ (437)	log.	1,0778409
	same	1,0778409
	$3\times$log.sec. f'' (572)	0,0008337
$\frac{3}{2}$.log. (rr')	arith. co. (566')	9,0149719
mm	log.	6,7395603
$f''=2^d02^m57^s,535$ (568)	sec.	0,0002779
$\frac{1}{2}f''=1\ 01\ 28\ 768$	sine	8,2524331
	same	8,2524331
$\sin^2.\frac{1}{2}f''.\sec f''=0,00032000$	log.	6,5051441
$\text{tang}^2.2w.\sec f''=0,00000757$ (569)		
$l=0,00032757$		
$\frac{5}{6}=0,83333333$		
$l+\frac{5}{6}=0,83366090$	subtract log.	9,9209895
mm (571)	log.	6,7395603
$h=0,00065852$	log.	6,8185708

Corresponds in Table VIII, to app. log. $y''y''$=0,0006348
log. y''=0,0003174

To find P.

y''	(580)	arith co. log.	9,9996826
y	(580)	log.	0,0002271
τ''	(447'')	log.	9,3134223
τ	(447''')	log.	0,7656848
Corrected $P=\frac{y\tau''}{y''\tau}$	(235)	log.	0,0790168
Assumed value of P	(519)	log.	0,0790169
		Difference =	−0,0000001

To find y *from* $r', r'', 2f, t''-t'$.

Third Approximation.

r''	log. 0,3221893	(550) . . .	0,3221893	(565)
r'	log. 0,3259536	(530) . . .	0,3259536	
$\frac{r''}{r'}=\text{tang}^4.(45^d+w)$	log. 9,9962357	sum	0,6481429	
$45^d+w=44^d56^m16^s,53$	tang. 9,9990589 2	half	0,3240714	
$w=-3^m43^s,47$		$(r'r'')^{\frac{3}{2}}$ log.	0,9722143	(566)
		arith.co.log.	9,0277857	(566')

$2w=-7^m26^s,94$	tang.	$7,33582_n$	(567)
	same	$7,33582_n$	
$f=1^d44^m30^s,96$ (561)	sec.	0,00020	(568)
$\text{tang}^2.2w.\sec.f=0,00000470$	log.	4,67184	(569)
(436')	constant log.	5,5680729	
$t''-t'$ (437)	log.	0,9987338	(570)
	same	0,9987338	
	$3\times$log.sec. f (572)	0,0006021	
$\frac{3}{2}$.log. $(r'r'')$	arith. co. (566')	9,0277857	
mm	log.	6,5939283	(571)
$f=1^d44^m30^s,96$ (568)	sec.	0,0002007	(572)
$\frac{1}{2}f=0\ 52\ 15\ 48$	sine	8,1818622	(573)
	same	8,1818622	
$\sin^2.\frac{1}{2}f.\sec.f=0,00023117$	log.	6,3639251	(574)
$\text{tang}^2.2w.\sec.f=0,00000470$ (569)			(575)
$l=0,00023587$			(576)
$\frac{5}{6}=0,83333333$			
$l+\frac{5}{6}=0,83356920$	subtract log.	9,9209417	(577)
mm (571)	log.	6,5939283	
$h=0,00047096$	log.	6,6729866	(578)

Corresponds in Table VIII, to app. log. yy=0,0004541 (579)
log. y=0,0002271 (580)

To find Q.

τ	(447')	log.	9,2343152	
τ''	(447'')	log.	9,3134223	
2.log. r'	(530)		0,6519072	
r	(550)	arith. co. log.	9,6692682	
r''	(550)	arith. co. log.	9,6778107	(581)
y	(580)	arith. co. log.	9,9997729	(582)
y''	(580)	arith. co. log.	9,9996826	(583)
f	(572)	secant	0,0002007	
f'	(566)	secant	0,0009514	
f''	(572)	secant	0,0002779	
Corrected Q		log.	8,5476691	(584)
Assumed value of Q	(519)	log.	8,5476696	(585)
		Difference=	−0,0000005	(586)

The differences between the assumed and computed values of P, Q (583,586), are so very small that it will not be necessary to repeat the operation; and we may suppose the expressions of r, r'', f', deduced from this last calculation to be their true values; from which we may deduce the elements of the orbit, in the following manner. (587)

[5999]

To compute the elements of the orbit.

(588) We have for this purpose log. r=0,3307318 (550); log. r''=0,3221893 (550); $2f'=7^d34^m56^s,96$ (560);
(589) t=Oct. 5^{days},451989 (429); t''=Oct. 27^{days},385899 (429), or $t''-t=21^{days}$,933910. With these data we may determine the elements, by the method explained in [5995].

Computation of the elements.

To find $x=\sin^2.\frac{1}{2}g$. [5995(187)].

(590)	r'' log.=0,3221893		(588) . . .	0,3221893
(591)	r log.=0,3307318		(588) . . .	0,3307318
(591′)	$\frac{r''}{r}$=tang$^4.(45^d+w)$ log.	9.9914575	sum	0,6529211
(591″)	$45^d+w=44^d51^m32^s,85$ tang.	9,99786438	half	0,3264606
(592)	$w=-\;8^m27^s,15$		$(rr'')^{\frac{3}{2}}$ log.	0,9793817
(592′)			arith. co.	9,0206183
(593)	$2w=-16^m54^s,30$		tang.	7,6917447
(594)			same	7,6917447
(595)	$f'=3^d47^m28^s,48$ (588)		sec.	0,0009514
(595′)	tang$^2.2w$.secf'=0,00002423		log.	5,3844408
	constant log. (436′)			5,5680729
(596)	$t''-t$=21.933910 (589)		log.	1,3411160
			same	1,3411160
	3×log sec.f' (595)			0,0028542
	$\frac{3}{2}$.log. (rr'') (592′) arith.co.			9,0206183
(597)	$m\,m$		log.	7,2737774
(598)	f' (595)		sec.	0,0009514
(599)	$\frac{1}{2}f'=1^d53^m44^s,24$ (595)		sine	8,5195500
(600)			same	8,5195500
	sin$^2.\frac{1}{2}f'$.sec.f'=0,00109661		log.	7,0400514
(601)	tang$^2.2w$.sec.f'=0,00002423 (595′)			
(602)	l=0,00112084			
	$\frac{5}{6}$=0,83333333			
(603)	$l+\frac{5}{6}$=0,83445417		log. sub.	9,9214025
(604)	$m\,m$ (597)		log.	7,2737774
(605)	h = 0,0022510		log.	7,3523749
(606)	Corresponds in Table VIII, to $y'\,y'$ =			0,0021638
(607)	$m\,m$ (597)		log.	7,2737774
(608)	$\frac{m^2}{y'^2}$=0,00186902		log.	7,2716136
(609)	l=0,00112084 (602)			
(610)	$x=\frac{m^2}{y'^2}-l=\sin^2.\frac{1}{2}g$ =0,00074818		log.	6,8740061
(611)	$\frac{1}{2}g=1^d34^m02^s,64$		sin.	8,4370030
(612)	$g=3^d08^m05^s,28$			
(613)				
(614)	After finding a in the second column (594), we may find the mean daily motion in seconds from [5995(67)].			
(615)	a (594)		log. ar. co.	9,5775882
(616)			its half	9,7887941
(617)			constant log.	3,5500066
(618)	Daily motion $824^s,877$		log.	2,9163889

To find a, [5995(58)].

g	(612)	arith.compl.log.sin.	1,2621295
		same	1,2621295
$\frac{m^2}{y^2}$	(608)	log.	7,2716136
2		log.	0,3010300
f'	(595)	cos.	9,9990486
$\sqrt{rr'}$	(591″)	log.	0,3264606
a		log.	0,4224118

To find p *and* $e=\sin.\varphi$, [5995(60,12,9)].

k	(54)	arith. co. log.	1,7644186
$t''-t$	(596)	arith. co. log.	8,6588840
rr''	(591′)	log.	0,6529211
$2f'=7^d34^m56^s,96$	(588)	sin.	9,1204208
y'	(606)	log.	0,0010819
$\sqrt{p}$	[5995(60)]	log.	0,1977264
$\sqrt{a}$	(594)	log.	0,2112059
$\sqrt{\frac{p}{a}}=\sqrt{1-e^2}$=cos.$\varphi$; $\varphi=14^d12^m05^s,3$		cos.	9,9865205

To find G, F, v, v'', u, u'', [5995(65,66, &c.)].

φ	(600)	sub. sine	9,3897547
g	(612)	cos.	9,9993497
	cos.g.cosec.φ=4,0700056	log.	0,6095950
$-\sqrt{rr''}$	(591″)	log.	$0,3264606_n$
a	(594)	ar.co.log.	9,5775882
f'	(592′)	cos.	9,9990486
φ	(601)	cosec.	0,6102453
$-\sqrt{\frac{rr''}{a}}\cos f'$.cosec.$\varphi$=−3,2609391		log.	$0,5133427_n$
[5995(65)]	cos. G= 0,8090665	log.	9,9079842
[5995(41,47)]	$G=324^d00^m17^s,4$	sin.	$9,7691682_n$
f'	(595)	sin.	8,8203422
g	(612,591) $3^d08^m05^s,3$	cosec.	1,2621295
F=	$314^d42^m51^s,4$	sin.	$9,8516399_n$
f'=	3 47 28 5	(595)	
$v=F-f'$=	310 55 22 9	[5995(13)]	
$v''=F+f'$=	318 30 19 9	[5995(14)]	
$u=G-g$=	320 52 12 1	[5995(15)]	
$u''=G+g$=	327 08 22 7	[5995(16)]	

[5999]

We may remark that the expression of cos. $G=0{,}8090665$ (608), corresponds to $G=324^{d}00^{m}17^{s},4$ or to (619)
$G=35^{d}59^{m}42^{s},6$. The first of these expressions is to be used as in (610), because the corresponding values of (620)
v,v'' (615,616), are in the *fourth quadrant* of the true anomaly, where the radii r,r'' are *decreasing*, as in (588); (621)
but the other value of G, gives v,v'', in the *first quadrant* of the true anomaly, when the radii r,r'' are *increasing*. The mean anomalies $nt=u-e.\sin.u$, $nt''=u''-e.\sin.u''$ [5985(7)], corresponding to the first (622)
and third observations, may be found in the following manner.

To find nt, [5985(7)].

e (601)	log.	9,3897547
radius in seconds,	log.	5,3144251
$-\sin.u$ (617) $320^{d}52^{m}12^{s},1$	sin.	9,8000855
$-e.\sin.u=31934^{s},9=$ 8 52 14 9	log.	4,5042653
Mean anom. $nt=$329 44 27 0		

To find nt'', [5985(7)].

e (601)	log.	9,3897547 (623)
radius in seconds,	log.	5,3144251
$-\sin.u''$ (618) $327^{d}08^{m}22^{s},7$	sin.	9,7344746
$-e.\sin.u''=27457^{s},1=$ 7 37 37 1	log.	4,4386544 (624)
Mean anom. $nt''=$334 45 59 8		(625)

To find ☊, φ.

We may find the longitude of the node ☊, and the inclination φ, of the orbit to the ecliptic; by means of the (626)
triangle ☊AC, in which we have given,

the angle ☊$AC=\gamma=16^{d}00^{m}08^{s},38$ (314); the angle ☊$CA=E'CC''=151^{d}23^{m}48^{s},64$ (558); (627)

$AC=AE'-CE'=33^{d}12^{m}29^{s},78-23^{d}16^{m}40^{s},62=9^{d}55^{m}49^{s},16$ (330, 551); (628)

to find the angle ☊, and the sides ☊C, ☊A, by Napier's formulas [1345[50,51]].

☊$AC=$ $16^{d}00^{m}08^{s},38$ (629)
☊$CA=$151 23 48 64 (630)

Sum is $2S_9=$167 23 57 02		$S_9=$ $83^{d}41^{m}58^{s},51$	
Diff. is $2D_9=$135 23 40 26		$D_9=$ 67 41 50 13	
D_9		sin.	9,9662316
S_9		arith. co. sin.	0,0026310
$\frac{1}{2}AC$ = 4 57 54 58	(628)	tang.	8,9388998
$\frac{1}{2}$(☊$A-$☊C)= 4 37 23 46		tang.	8,9077624
$\frac{1}{2}$(☊$A+$☊C)= 16 43 18 36			
Diff. is ☊$C=$ 12 05 54 90		☊$C=$ $12^{d}05^{m}54^{s},9$	
Sum is ☊$A=$ 21 20 41 82		☊ (635) is =351 07 45 9	
Lon. A(302)= 12 28 27 76		Sum is long. of $C=$ 3 13 40 8	
Diff. is ☊=351 07 45 9		Sub. v (615) =310 55 22 9	
		Long. perihelion= 52 18 17 9	

D_9	cos.	9,5792122	
S_9	arith. co. cos.	0,9596291	
$\frac{1}{2}AC$ (628)	tang.	8,9388998	(631)
$\frac{1}{2}$(☊$A+$☊C)$=16^{d}43^{m}18^{s},36$	tang.	9,4777411	
☊C (632)	sin. arith. co.	0,6786204	
☊AC (629)	sin.	9,4403996	(632)
AC (628)	sin.	9,2366642	(633)
A☊$C=13^{d}06^{m}36^{s},0$	sin.	9,3556842	(634)

(635)
(636)

From the time of the last observation, corrected for aberration as in (429), October 27, 385899 to the epoch (637)
January 1, 1805, the interval is $64^{days},614101$. Multiplying this by the daily motion $824^{s},877$ (618), we get the (638)
mean motion in that interval, $14^{d}48^{m}18^{s},6$. Adding this to $nt''=334^{d}45^{m}59^{s},8$ (625); we get the mean (639)
anomaly at the epoch, equal to $349^{d}34^{m}18^{s},4$. This last expression being added to the longitude of the perihelion $52^{d}18^{m}17^{s},9$ (636), gives the mean longitude at the epoch $41^{d}52^{m}36^{s},3$. Hence we have the following (639')
elements of the orbit.

Elements of the orbit of Juno.

Log. of the mean distance	$a=$ 0,4224118	(594).	(640)
Log. of the semiparameter	$p=$ 0,3954528	(598).	(641)
Log. of the excentricity	$e=$ 9,3897547	(601).	(642)
Daily motion	$824^{s},877$	(618).	(643)
Inclination of the orbit to the ecliptic	$=13^{d}06^{m}36^{s},0$	(634).	(644)
Long. of the ascending node	$171^{d}07^{m}45^{s},9$	(635).	(645)
Long. of the perihelion in the orbit	$52^{d}18^{m}17^{s},9$	(636).	(646)
Mean longitude at the epoch	$41^{d}52^{m}36^{s},3$	(639').	(647)

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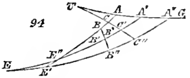

With the daily motion $824^{s},877$(618), the planet would describe the whole circumference 360^{d} in about 1571
days, which represents the time of revolution of the planet. If we compare these elements of the apparent orbit, (648)
corresponding to the epoch 1805, with those in [4079i], corresponding to the year 1831; we shall find that they

[5999]
(649) agree as well as could be expected, taking into consideration that all the calculations in this article are deduced
(650) from the motion of the planet in a geocentric arc of *less than four degrees*. These elements were sufficiently
accurate to trace the path of the planet for several days, until other more distant observations could be obtained,
for correcting them.

This method, like all others of a similar nature, requires some modification in particular cases. ***First.***
(651) When any one of the three geocentric places of the planet coincides with the heliocentric place of the
earth, or with its opposite point at that time; because then the arc, connecting this geocentric place of the
(652) planet, and the corresponding heliocentric place of the earth becomes indeterminate. *Second.* When the
(653) geocentric places of the planet in the first and third observations coincide. *Third.* When the three geocentric
(654) places of the planet are situated in a great circle, passing through the heliocentric place of the earth in the
second observation. In the first of these cases the situation of one of the great circles AB, $A'B'$, $A''B''$, remains
(655) indeterminate; in the second and third cases, the situation of the point B^* is indeterminate; and in these two
last cases, the defect is inherent in the problem itself, and cannot be rectified. We must, therefore, in selecting
(656) the observations, which are to be used, avoid those which are at the same time near the node, and near the
conjunction or opposition with the sun; we must also avoid those observations in which the geocentric place of
(657) the planet, in the third observation, is near to that in the first observation; finally, we must reject those in which
all three of the observed places of the planet lie nearly in a great circle passing through the heliocentric place
(658) of the earth, in the middle observation. We may easily rectify the rules in the first case (651), by supposing the
points E, E', E'', figure 92, page 874, to coincide, and then finding this point of coincidence by means of the ***two*** of
(659) the three arcs AB, $A'B'$ or $A''B''$, which are given in position and magnitude; supposing the other arc to be
(660) infinitely small, but taking it in the direction towards the common point E. For example, if the points A, B,
coincide, we may suppose the arc AB to be infinitely small, and that it is taken in the direction of the great circle
(661) ABE. It being evident that this small change in the place of the planet, at the time of the first observation,
(662) can produce no sensible effect in the result of the calculation. In this case the factor $\frac{\sin.\delta}{\sin.(AE'-\delta)}$, which
occurs in the expression of a (32) becomes, $\frac{\sin.\delta}{\sin.EB}$, which may be put equal to nothing, on account of
(663) the extreme smallness of $\sin.\delta$; hence we have $\mathrm{a}=0$ (32). This value of a is to be substituted in (35,40),
and we shall get the value of w, to be substituted in (41'); then the calculation is to be completed in the usual
manner. The method of proceeding is nearly the same, when the points A'', B'', coincide in the third observation;
(664) and as a, b, (32,33), become infinite, because $\sin.\delta''=0$, we must put as in (42) $\mathrm{a}=bb_{\prime}$; and tang.w
(40) changes into $\mathrm{tang.w}_{\prime}$ (43); also the factor, $\frac{P+\mathrm{a}}{b}$ (41''), changes into $\frac{\mathrm{a}}{b}=b_{\prime}$. When the points
(665) A', B', coincide, we have $b=0$ (33); hence (40) becomes, $\mathrm{tang.w}=-\frac{\sin.\delta^*}{\cos.\delta^*}=-\mathrm{tang.}\delta^*$, or $\mathrm{w}=-\delta^*$;
and so on for the other quantities. It is unnecessary to enter more minutely into the consideration of these
(666) uncommon cases, as the method of proceeding is sufficiently obvious.

In all the preceding calculations, we have supposed the orbit to be wholly unknown, at the commencement
(667) of the calculations; but it is evident that the same method can be observed for correcting the approximate
(668) elements, in a manner similar to that in [825—829]. Taking P and Q for the unknown quantities; and then
separately varying each of them, by a small quantity, in two successive operations, so as to obtain two equations,
similar to [829], for correcting the assumed values of P, Q. This method is so plain, that it requires no
particular illustration. We may however remark that when the arcs $2f$, $2f'$, $2f''$ are large, the assumed values
(669) of P, Q (259) may not be sufficiently accurate for the first operation, and then we may use the expressions
(670), computing the values roughly, by means of the approximate elements, which have been previously found.

(670) $$P=\frac{r\sin 2f''}{r''.\sin.f}; \qquad Q=\frac{4r'^4.\sin.f.\sin.f''}{p.\cos.f'}.$$

This value of P is easily deduced from (38, 90); and if we multiply the expression of Q (39), by that of p (247),
(671) and the product by $[r\,r'']$, we get $p.Q\,[r\,r'']=8\,r\,r'^4\,r''.\sin.f.\sin.f'.\sin.f''$. Substituting in the first member,
the value of $[r\,r'']$ (90), and then dividing by $2\,p.rr'.\sin.f'.\cos.f'$, we get Q (670).

TABLE I.—OF SQUARE ROOTS.

The proposed number is to be found, as far as the second decimal place, in the side column of the table, and the third decimal at the top of one of the vertical columns; the number corresponding is the required root.

	0	1	2	3	4	5	6	7	8	9
0.00	0.00000	03162	04472	05477	06325	07071	07746	08367	08944	09487
0.01	0.10000	10488	10954	11402	11832	12247	12649	13038	13416	13784
0.02	0.14142	14491	14832	15166	15492	15811	16125	16432	16733	17029
0.03	0.17321	17607	17889	18166	18439	18708	18974	19235	19494	19748
0.04	0.20000	20248	20494	20736	20976	21213	21448	21679	21909	22136
0.05	0.22361	22583	22804	23022	23238	23452	23664	23875	24083	24290
0.06	0.24495	24698	24900	25100	25298	25495	25690	25884	26077	26268
0.07	0.26458	26646	26833	27019	27203	27386	27568	27749	27928	28107
0.08	0.28284	28460	28636	28810	28983	29155	29326	29496	29665	29833
0.09	0.30000	30166	30332	30496	30659	30822	30984	31145	31305	31464
0.10	0.31623	31780	31937	32094	32249	32404	32558	32711	32863	33015
0.11	0.33166	33317	33466	33615	33764	33912	34059	34205	34351	34496
0.12	0.34641	34785	34928	35071	35214	35355	35496	35637	35777	35917
0.13	0.36056	36194	36332	36469	36606	36742	36878	37014	37148	37283
0.14	0.37417	37550	37683	37815	37947	38079	38210	38341	38471	38601
0.15	0.38730	38859	38987	39115	39243	39370	39497	39623	39749	39875
0.16	0.40000	40125	40249	40373	40497	40620	40743	40866	40988	41110
0.17	0.41231	41352	41473	41593	41713	41833	41952	42071	42190	42308
0.18	0.42426	42544	42661	42778	42895	43012	43128	43243	43359	43474
0.19	0.43589	43704	43818	43932	44045	44159	44272	44385	44497	44609
0.20	0.44721	44833	44944	45056	45166	45277	45387	45497	45607	45717
0.21	0.45826	45935	46043	46152	46260	46368	46476	46583	46690	46797
0.22	0.46904	47011	47117	47223	47329	47434	47539	47645	47749	47854
0.23	0.47958	48062	48166	48270	48374	48477	48580	48683	48785	48888
0.24	0.48990	49092	49193	49295	49396	49497	49598	49699	49800	49900
0.25	0.50000	50100	50200	50299	50398	50498	50596	50695	50794	50892
0.26	0.50990	51088	51186	51284	51381	51478	51575	51672	51769	51865
0.27	0.51962	52058	52154	52249	52345	52440	52536	52631	52726	52820
0.28	0.52915	53009	53104	53198	53292	53385	53479	53572	53666	53759
0.29	0.53852	53944	54037	54129	54222	54314	54406	54498	54589	54681
0.30	0.54772	54863	54955	55045	55136	55227	55317	55408	55498	55588
0.31	0.55678	55767	55857	55946	56036	56125	56214	56303	56391	56480
0.32	0.56569	56657	56745	56833	56921	57009	57096	57184	57271	57359
0.33	0.57446	57533	57619	57706	57793	57879	57966	58052	58138	58224
0.34	0.58310	58395	58481	58566	58652	58737	58822	58907	58992	59076
0.35	0.59161	59245	59330	59414	59498	59582	59666	59749	59833	59917
0.36	0.60000	60083	60166	60249	60332	60415	60498	60581	60663	60745
0.37	0.60828	60910	60992	61074	61156	61237	61319	61400	61482	61563
0.38	0.61644	61725	61806	61887	61968	62048	62129	62209	62290	62370
0.39	0.62450	62530	62610	62690	62769	62849	62929	63008	63087	63166
0.40	0.63246	63325	63403	63482	63561	63640	63718	63797	63875	63953
0.41	0.64031	64109	64187	64265	64343	64420	64498	64576	64653	64730
0.42	0.64807	64885	64962	65038	65115	65192	65269	65345	65422	65498
0.43	0.65574	65651	65727	65803	65879	65955	66030	66106	66182	66257
0.44	0.66332	66408	66483	66558	66633	66708	66783	66858	66933	67007
0.45	0.67082	67157	67231	67305	67380	67454	67528	67602	67676	67750
0.46	0.67823	67897	67971	68044	68118	68191	68264	68337	68411	68484
0.47	0.68557	68629	68702	68775	68848	68920	68993	69065	69138	69210
0.48	0.69282	69354	69426	69498	69570	69642	69714	69785	69857	69929
0.49	0.70000	70071	70143	70214	70285	70356	70427	70498	70569	70640
0.50	0.70711	70781	70852	70922	70993	71063	71133	71204	71274	71344
0.51	0.71414	71484	71554	71624	71694	71764	71833	71903	71972	72042
0.52	0.72111	72180	72250	72319	72388	72457	72526	72595	72664	72732
0.53	0.72801	72870	72938	73007	73075	73144	73212	73280	73348	73417
0.54	0.73485	73553	73621	73689	73756	73824	73892	73959	74027	74095
0.55	0.74162	74229	74297	74364	74431	74498	74565	74632	74699	74766
0.56	0.74833	74900	74967	75033	75100	75166	75233	75299	75366	75432
0.57	0.75498	75565	75631	75697	75763	75829	75895	75961	76026	76092
0.58	0.76158	76223	76289	76354	76420	76485	76551	76616	76681	76746
0.59	0.76811	76877	76942	77006	77071	77136	77201	77266	77330	77395
	0	1	2	3	4	5	6	7	8	9

When the quantity x whose root is to be found consists of several places of decimals and is less than 0, 1, it will be convenient to find the root of $100x$ and divide the result by 10, which is done by merely changing the decimal point two figures in finding $100x$, and one figure in dividing by 10.

	210	205	200	195	190	185	180	175	170	165	160	155	150
1	21	21	20	20	19	19	18	18	17	17	16	16	15
2	42	41	40	39	38	37	36	35	34	33	32	31	30
3	63	62	60	59	57	56	54	53	51	50	48	47	45
4	84	82	80	78	76	74	72	70	68	66	64	62	60
5	105	103	100	98	95	93	90	88	85	83	80	78	75
6	126	123	120	117	114	111	108	105	102	99	96	93	90
7	147	144	140	137	133	130	126	123	119	116	112	109	105
8	168	164	160	156	152	148	144	140	136	132	128	124	120
9	189	185	180	176	171	167	162	158	153	149	144	140	135

	148	146	144	142	140	138	136	134	132	130	128	126	124
1	15	15	14	14	14	14	14	13	13	13	13	13	12
2	30	29	29	28	28	28	27	27	26	26	26	25	25
3	44	44	43	43	42	41	41	40	40	39	38	38	37
4	59	58	58	57	56	55	54	54	53	52	51	50	50
5	74	73	72	71	70	69	68	67	66	65	64	63	62
6	89	88	86	85	84	83	82	80	79	78	77	76	74
7	104	102	101	99	98	97	95	94	92	91	90	88	87
8	118	117	115	114	112	110	109	107	106	104	102	101	99
9	133	131	130	128	126	124	122	121	119	117	115	113	112

	123	122	121	120	119	118	117	116	115	114	113	112	111
1	12	12	12	12	12	12	12	12	12	11	11	11	11
2	25	24	24	24	24	24	23	23	23	23	23	22	22
3	37	37	36	36	36	35	35	35	35	34	34	34	33
4	49	49	48	48	48	47	47	46	46	46	45	45	44
5	62	61	61	60	60	59	59	58	58	57	57	56	56
6	74	73	73	72	71	71	70	70	69	68	68	67	67
7	86	85	85	84	83	83	82	81	81	80	79	78	78
8	98	98	97	96	95	94	94	93	92	91	90	90	89
9	111	110	109	108	107	106	105	104	104	103	102	101	100

	110	109	108	107	106	105	104	103	102	101	100	99	98
1	11	11	11	11	11	11	10	10	10	10	10	10	10
2	22	22	22	21	21	21	21	21	20	20	20	20	20
3	33	33	32	32	32	32	31	31	31	30	30	30	29
4	44	44	43	43	42	42	42	41	41	40	40	40	39
5	55	55	54	54	53	53	52	52	51	51	50	50	49
6	66	65	65	64	64	63	62	62	61	61	60	59	59
7	77	76	76	75	74	74	73	72	71	71	70	69	69
8	88	87	86	86	85	84	83	82	82	81	80	79	78
9	99	98	97	96	95	95	94	93	92	91	90	89	88

	97	96	95	94	93	92	91	90	89	88	87	86	85	84	83	82	81
1	10	10	10	9	9	9	9	9	9	9	9	9	9	8	8	8	8
2	19	19	19	19	19	18	18	18	18	18	17	17	17	17	17	16	16
3	29	29	29	28	28	28	27	27	27	26	26	26	26	25	25	25	24
4	39	38	38	38	37	37	36	36	36	35	35	34	34	34	33	33	32
5	49	48	48	47	47	46	46	45	45	44	44	43	43	42	42	41	41
6	58	58	57	56	56	55	55	54	53	53	52	52	51	50	50	49	49
7	68	67	67	66	65	64	64	63	62	62	61	60	60	59	58	57	57
8	78	77	76	75	74	74	73	72	71	70	70	69	68	67	66	66	65
9	87	86	86	85	84	83	82	81	80	79	78	77	77	76	75	74	73

	80	79	78	77	76	75	74	73	72	71	70	69	68	67	66	65	64
1	8	8	8	8	8	8	7	7	7	7	7	7	7	7	7	7	6
2	16	16	16	15	15	15	15	15	14	14	14	14	14	13	13	13	13
3	24	24	23	23	23	23	22	22	22	21	21	21	20	20	20	20	19
4	32	32	31	31	30	30	30	29	29	28	28	28	27	27	26	26	26
5	40	40	39	39	38	38	37	37	36	36	35	35	34	34	33	33	32
6	48	47	47	46	46	45	44	44	43	43	42	41	41	40	40	39	38
7	56	55	55	54	53	53	52	51	50	50	49	48	48	47	46	46	45
8	64	63	62	62	61	60	59	58	58	57	56	55	54	54	53	52	51
9	72	71	70	69	68	68	67	66	65	64	63	62	61	60	59	59	58

TABLE I.—OF SQUARE ROOTS.

The proposed number is to be found, as far as the second decimal place, in the side column of the table, and the third decimal at the top of one of the vertical columns; the number corresponding is the required root.

	0	1	2	3	4	5	6	7	8	9
0.60	0.77460	77524	77589	77653	77717	77782	77846	77910	77974	78038
0.61	0.78102	78166	78230	78294	78358	78422	78486	78549	78613	78677
0.62	0.78740	78804	78867	78930	78994	79057	79120	79183	79246	79310
0.63	0.79373	79436	79498	79561	79624	79687	79750	79812	79875	79937
0.64	0.80000	80062	80125	80187	80250	80312	80374	80436	80498	80561
0.65	0.80623	80685	80747	80808	80870	80932	80994	81056	81117	81179
0.66	0.81240	81302	81363	81425	81486	81548	81609	81670	81731	81792
0.67	0.81854	81915	81976	82037	82098	82158	82219	82280	82341	82401
0.68	0.82462	82523	82583	82644	82704	82765	82825	82885	82946	83006
0.69	0.83066	83126	83187	83247	83307	83367	83427	83487	83546	83606
0.70	0.83666	83726	83785	83845	83905	83964	84024	84083	84143	84202
0.71	0.84261	84321	84380	84439	84499	84558	84617	84676	84735	84794
0.72	0.84853	84912	84971	85029	85088	85147	85206	85264	85323	85381
0.73	0.85440	85499	85557	85615	85674	85732	85790	85849	85907	85965
0.74	0.86023	86081	86139	86197	86255	86313	86371	86429	86487	86545
0.75	0.86603	86660	86718	86776	86833	86891	86948	87006	87063	87121
0.76	0.87178	87235	87293	87350	87407	87464	87521	87579	87636	87693
0.77	0.87750	87807	87864	87920	87977	88034	88091	88148	88204	88261
0.78	0.88318	88374	88431	88487	88544	88600	88657	88713	88769	88826
0.79	0.88882	88938	88994	89051	89107	89163	89219	89275	89331	89387
0.80	0.89443	89499	89554	89610	89666	89722	89778	89833	89889	89944
0.81	0.90000	90056	90111	90167	90222	90277	90333	90388	90443	90499
0.82	0.90554	90609	90664	90719	90774	90830	90885	90940	90995	91049
0.83	0.91104	91159	91214	91269	91324	91378	91433	91488	91542	91597
0.84	0.91652	91706	91761	91815	91869	91924	91978	92033	92087	92141
0.85	0.92195	92250	92304	92358	92412	92466	92520	92574	92628	92682
0.86	0.92736	92790	92844	92898	92952	93005	93059	93113	93167	93220
0.87	0.93274	93327	93381	93434	93488	93541	93595	93648	93702	93755
0.88	0.93808	93862	93915	93968	94021	94074	94128	94181	94234	94287
0.89	0.94340	94393	94446	94499	94552	94604	94657	94710	94763	94815
0.90	0.94868	94921	94974	95026	95079	95131	95184	95237	95289	95341
0.91	0.95394	95446	95499	95551	95603	95656	95708	95760	95812	95864
0.92	0.95917	95969	96021	96073	96125	96177	96229	96281	96333	96385
0.93	0.96437	96488	96540	96592	96644	96695	96747	96799	96850	96902
0.94	0.96954	97005	97057	97108	97160	97211	97263	97314	97365	97417
0.95	0.97468	97519	97570	97622	97673	97724	97775	97826	97877	97929
0.96	0.97980	98031	98082	98133	98184	98234	98285	98336	98387	98438
0.97	0.98489	98539	98590	98641	98691	98742	98793	98843	98894	98944
0.98	0.98995	99045	99096	99146	99197	99247	99298	99348	99398	99448
0.99	0.99499	99549	99599	99649	99700	99750	99800	99850	99900	99950
1.00	1.00000	00050	00100	00150	00200	00250	00300	00349	00399	00449
1.01	1.00499	00549	00598	00648	00698	00747	00797	00846	00896	00945
1.02	1.00995	01045	01094	01143	01193	01242	01292	01341	01390	01440
1.03	1.01489	01538	01587	01637	01686	01735	01784	01833	01882	01931
1.04	1.01980	02029	02078	02127	02176	02225	02274	02323	02372	02421
1.05	1.02470	02518	02567	02616	02665	02713	02762	02811	02859	02908
1.06	1.02956	03005	03053	03102	03150	03199	03247	03296	03344	03392
1.07	1.03441	03489	03537	03586	03634	03682	03730	03779	03827	03875
1.08	1.03923	03971	04019	04067	04115	04163	04211	04259	04307	04355
1.09	1.04403	04451	04499	04547	04594	04642	04690	04738	04785	04833
1.10	1.04881	04929	04976	05024	05071	05119	05167	05214	05262	05309
1.11	1.05357	05404	05451	05499	05546	05594	05641	05688	05736	05783
1.12	1.05830	05877	05925	05972	06019	06066	06113	06160	06207	06254
1.13	1.06301	06348	06395	06442	06489	06536	06583	06630	06677	06724
1.14	1.06771	06818	06864	06911	06958	07005	07051	07098	07145	07191
1.15	1.07238	07285	07331	07378	07424	07471	07517	07564	07610	07657
1.16	1.07703	07750	07796	07842	07889	07935	07981	08028	08074	08120
1.17	1.08167	08213	08259	08305	08351	08397	08444	08490	08536	08582
1.18	1.08628	08674	08720	08766	08812	08858	08904	08950	08995	09041
1.19	1.09087	09133	09179	09225	09270	09316	09362	09407	09453	09499
	0	1	2	3	4	5	6	7	8	9

	65	64	63
1	7	6	6
2	13	13	13
3	20	19	19
4	26	26	25
5	33	32	32
6	39	38	38
7	46	45	44
8	52	51	50
9	59	58	57

	62	61	60
1	6	6	6
2	12	12	12
3	19	18	18
4	25	24	24
5	31	31	30
6	37	37	36
7	43	43	42
8	50	49	48
9	56	55	54

	59	58	57
1	6	6	6
2	12	12	11
3	18	17	17
4	24	23	23
5	30	29	29
6	35	35	34
7	41	41	40
8	47	46	46
9	53	52	51

	56	55	54
1	6	6	5
2	11	11	11
3	17	17	16
4	22	22	22
5	28	28	27
6	34	33	32
7	39	39	38
8	45	44	43
9	50	49	49

	53	52	51
1	5	5	5
2	11	10	10
3	16	16	15
4	21	21	20
5	27	26	26
6	32	31	31
7	37	36	36
8	42	42	41
9	48	47	46

	50	49	48
1	5	5	5
2	10	10	10
3	15	15	14
4	20	20	19
5	25	25	24
6	30	29	29
7	35	34	34
8	40	39	38
9	45	44	43

	47	46	45
1	5	5	5
2	9	9	9
3	14	14	14
4	19	18	18
5	24	23	23
6	28	28	27
7	33	32	32
8	38	37	36
9	42	41	41

TABLE I.—OF SQUARE ROOTS.

The proposed number is to be found, as far as the second decimal place, in the side column of the table, and the third decimal at the top of one of the vertical columns; the number corresponding is the required root.

	0	1	2	3	4	5	6	7	8	9
1.20	1.09544	09590	09636	09681	09727	09772	09818	09864	09909	09955
1.21	1.10000	10045	10091	10136	10182	10227	10272	10318	10363	10408
1.22	1.10454	10499	10544	10589	10635	10680	10725	10770	10815	10860
1.23	1.10905	10950	10995	11041	11086	11131	11176	11221	11265	11310
1.24	1.11355	11400	11445	11490	11535	11580	11624	11669	11714	11759
1.25	1.11803	11848	11893	11937	11982	12027	12071	12116	12161	12205
1.26	1.12250	12294	12339	12383	12428	12472	12517	12561	12606	12650
1.27	1.12694	12739	12783	12827	12872	12916	12960	13004	13049	13093
1.28	1.13137	13181	13225	13270	13314	13358	13402	13446	13490	13534
1.29	1.13578	13622	13666	13710	13754	13798	13842	13886	13930	13974
1.30	1.14018	14061	14105	14149	14193	14237	14280	14324	14368	14412
1.31	1.14455	14499	14543	14586	14630	14673	14717	14761	14804	14848
1.32	1.14891	14935	14978	15022	15065	15109	15152	15195	15239	15282
1.33	1.15326	15369	15412	15456	15499	15542	15585	15629	15672	15715
1.34	1.15758	15802	15845	15888	15931	15974	16017	16060	16103	16146
1.35	1.16190	16233	16276	16319	16362	16404	16447	16490	16533	16576
1.36	1.16619	16662	16705	16748	16790	16833	16876	16919	16962	17004
1.37	1.17047	17090	17132	17175	17218	17260	17303	17346	17388	17431
1.38	1.17473	17516	17558	17601	17644	17686	17729	17771	17813	17856
1.39	1.17898	17941	17983	18025	18068	18110	18152	18195	18237	18279
1.40	1.18322	18364	18406	18448	18491	18533	18575	18617	18659	18701
1.41	1.18743	18786	18828	18870	18912	18954	18996	19038	19080	19122
1.42	1.19164	19206	19248	19290	19331	19373	19415	19457	19499	19541
1.43	1.19583	19624	19666	19708	19750	19791	19833	19875	19917	19958
1.44	1.20000	20042	20083	20125	20167	20208	20250	20291	20333	20374
1.45	1.20416	20457	20499	20540	20582	20623	20665	20706	20748	20789
1.46	1.20830	20872	20913	20955	20996	21037	21078	21120	21161	21202
1.47	1.21244	21285	21326	21367	21408	21450	21491	21532	21573	21614
1.48	1.21655	21696	21737	21778	21820	21861	21902	21943	21984	22025
1.49	1.22066	22107	22147	22188	22229	22270	22311	22352	22393	22434
1.50	1.22474	22515	22556	22597	22638	22678	22719	22760	22801	22841
1.51	1.22882	22923	22963	23004	23045	23085	23126	23167	23207	23248
1.52	1.23288	23329	23369	23410	23450	23491	23531	23572	23612	23653
1.53	1.23693	23734	23774	23814	23855	23895	23935	23976	24016	24056
1.54	1.24097	24137	24177	24218	24258	24298	24338	24378	24418	24459
1.55	1.24499	24539	24579	24619	24660	24700	24740	24780	24820	24860
1.56	1.24900	24940	24980	25020	25060	25100	25140	25180	25220	25260
1.57	1.25300	25340	25379	25419	25459	25499	25539	25579	25618	25658
1.58	1.25698	25738	25778	25817	25857	25897	25936	25976	26016	26056
1.59	1.26095	26135	26174	26214	26254	26293	26333	26372	26412	26452
1.60	1.26491	26531	26570	26610	26649	26689	26728	26768	26807	26846
1.61	1.26886	26925	26964	27004	27043	27083	27122	27161	27201	27240
1.62	1.27279	27318	27358	27397	27436	27475	27515	27554	27593	27632
1.63	1.27671	27711	27750	27789	27828	27867	27906	27945	27984	28023
1.64	1.28062	28102	28141	28180	28219	28258	28297	28335	28374	28413
1.65	1.28452	28491	28530	28569	28608	28647	28686	28725	28763	28802
1.66	1.28841	28880	28919	28957	28996	29035	29074	29112	29151	29190
1.67	1.29228	29267	29306	29345	29383	29422	29460	29499	29538	29576
1.68	1.29615	29653	29692	29730	29769	29808	29846	29885	29923	29962
1.69	1.30000	30038	30077	30115	30154	30192	30231	30269	30307	30346
1.70	1.30384	30422	30461	30499	30537	30576	30614	30652	30690	30729
1.71	1.30767	30805	30843	30882	30920	30958	30996	31034	31072	31111
1.72	1.31149	31187	31225	31263	31301	31339	31377	31415	31453	31491
1.73	1.31529	31567	31605	31643	31681	31719	31757	31795	31833	31871
1.74	1.31909	31947	31985	32023	32061	32098	32136	32174	32212	32250
1.75	1.32288	32325	32363	32401	32439	32476	32514	32552	32590	32627
1.76	1.32665	32703	32740	32778	32816	32853	32891	32929	32966	33004
1.77	1.33041	33079	33116	33154	33192	33229	33267	33304	33342	33379
1.78	1.33417	33454	33492	33529	33566	33604	33641	33679	33716	33754
1.79	1.33791	33828	33866	33903	33940	33978	34015	34052	34090	34127
	0	1	2	3	4	5	6	7	8	9

	46	45
1	5	5
2	9	9
3	14	14
4	18	18
5	23	23
6	28	27
7	32	32
8	37	36
9	41	41

	44	43
1	4	4
2	9	9
3	13	13
4	18	17
5	22	22
6	26	26
7	31	30
8	35	34
9	40	39

	42	41
1	4	4
2	8	8
3	13	12
4	17	16
5	21	21
6	25	25
7	29	29
8	34	33
9	38	37

	40	39
1	4	4
2	8	8
3	12	12
4	16	16
5	20	20
6	24	23
7	28	27
8	32	31
9	36	35

	38	37
1	4	4
2	8	7
3	11	11
4	15	15
5	19	19
6	23	22
7	27	26
8	30	30
9	34	33

TABLE I.—OF SQUARE ROOTS.

The proposed number is to be found, as far as the second decimal place, in the side column of the table, and the third decimal at the top of one of the vertical columns; the number corresponding is the required root.

	0	1	2	3	4	5	6	7	8	9
1.80	1.34164	34201	34239	34276	34313	34350	34387	34425	34462	34499
1.81	1.34536	34573	34611	34648	34685	34722	34759	34796	34833	34870
1.82	1.34907	34944	34981	35019	35056	35093	35130	35167	35204	35241
1.83	1.35277	35314	35351	35388	35425	35462	35499	35536	35573	35610
1.84	1.35647	35683	35720	35757	35794	35831	35868	35904	35941	35978
1.85	1.36015	36051	36088	36125	36162	36198	36235	36272	36308	36345
1.86	1.36382	36418	36455	36492	36528	36565	36602	36638	36675	36711
1.87	1.36748	36785	36821	36858	36894	36931	36967	37004	37040	37077
1.88	1.37113	37150	37186	37222	37259	37295	37332	37368	37405	37441
1.89	1.37477	37514	37550	37586	37623	37659	37695	37732	37768	37804
1.90	1.37840	37877	37913	37949	37985	38022	38058	38094	38130	38167
1.91	1.38203	38239	38275	38311	38347	38384	38420	38456	38492	38528
1.92	1.38564	38600	38636	38672	38708	38744	38780	38816	38852	38888
1.93	1.38924	38960	38996	39032	39068	39104	39140	39176	39212	39248
1.94	1.39284	39320	39356	39392	39427	39463	39499	39535	39571	39607
1.95	1.39642	39678	39714	39750	39786	39821	39857	39893	39929	39964
1.96	1.40000	40036	40071	40107	40143	40178	40214	40250	40285	40321
1.97	1.40357	40392	40428	40464	40499	40535	40570	40606	40641	40677
1.98	1.40712	40748	40784	40819	40855	40890	40926	40961	40996	41032
1.99	1.41067	41103	41138	41174	41209	41244	41280	41315	41351	41386
2.00	1.41421	41457	41492	41527	41563	41598	41633	41669	41704	41739
2.01	1.41774	41810	41845	41880	41915	41951	41986	42021	42056	42092
2.02	1.42127	42162	42197	42232	42267	42302	42338	42373	42408	42443
2.03	1.42478	42513	42548	42583	42618	42653	42688	42724	42759	42794
2.04	1.42829	42864	42899	42934	42969	43003	43038	43073	43108	43143
2.05	1.43178	43213	43248	43283	43318	43353	43388	43422	43457	43492
2.06	1.43527	43562	43597	43631	43666	43701	43736	43771	43805	43840
2.07	1.43875	43910	43944	43979	44014	44049	44083	44118	44153	44187
2.08	1.44222	44257	44291	44326	44361	44395	44430	44465	44499	44534
2.09	1.44568	44603	44637	44672	44707	44741	44776	44810	44845	44879
2.10	1.44914	44948	44983	45017	45052	45086	45121	45155	45190	45224
2.11	1.45258	45293	45327	45362	45396	45430	45465	45499	45534	45568
2.12	1.45602	45637	45671	45705	45739	45774	45808	45842	45877	45911
2.13	1.45945	45979	46014	46048	46082	46116	46151	46185	46219	46253
2.14	1.46287	46322	46356	46390	46424	46458	46492	46526	46561	46595
2.15	1.46629	46663	46697	46731	46765	46799	46833	46867	46901	46935
2.16	1.46969	47003	47037	47071	47105	47139	47173	47207	47241	47275
2.17	1.47309	47343	47377	47411	47445	47479	47513	47547	47580	47614
2.18	1.47648	47682	47716	47750	47784	47817	47851	47885	47919	47953
2.19	1.47986	48020	48054	48088	48122	48155	48189	48223	48257	48290
2.20	1.48324	48358	48391	48425	48459	48492	48526	48560	48593	48627
2.21	1.48661	48694	48728	48762	48795	48829	48862	48896	48930	48963
2.22	1.48997	49030	49064	49097	49131	49164	49198	49231	49265	49298
2.23	1.49332	49365	49399	49432	49466	49499	49533	49566	49599	49633
2.24	1.49666	49700	49733	49766	49800	49833	49867	49900	49933	49967
2.25	1.50000	50033	50067	50100	50133	50167	50200	50233	50266	50300
2.26	1.50333	50366	50399	50433	50466	50499	50532	50566	50599	50632
2.27	1.50665	50698	50732	50765	50798	50831	50864	50897	50930	50964
2.28	1.50997	51030	51063	51096	51129	51162	51195	51228	51261	51294
2.29	1.51327	51360	51394	51427	51460	51493	51526	51559	51592	51625
2.30	1.51658	51690	51723	51756	51789	51822	51855	51888	51921	51954
2.31	1.51987	52020	52053	52085	52118	52151	52184	52217	52250	52283
2.32	1.52315	52348	52381	52414	52447	52480	52512	52545	52578	52611
2.33	1.52643	52676	52709	52742	52774	52807	52840	52872	52905	52938
2.34	1.52971	53003	53036	53069	53101	53134	53167	53199	53232	53264
2.35	1.53297	53330	53362	53395	53428	53460	53493	53525	53558	53590
2.36	1.53623	53655	53688	53721	53753	53786	53818	53851	53883	53916
2.37	1.53948	53981	54013	54045	54078	54110	54143	54175	54208	54240
2.38	1.54272	54305	54337	54370	54402	54434	54467	54499	54532	54564
2.39	1.54596	54629	54661	54693	54726	54758	54790	54822	54855	54887
	0	1	2	3	4	5	6	7	8	9

	38
1	4
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3	11
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7	27
8	30
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7	25
8	29
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8	28
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5	17
6	20
7	24
8	27
9	31

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1	3
2	7
3	10
4	13
5	17
6	20
7	23
8	26
9	30

	32
1	3
2	6
3	10
4	13
5	16
6	19
7	22
8	26
9	29

TABLE I.—OF SQUARE ROOTS.

The proposed number is to be found, as far as the second decimal place, in the side column of the table, and the third decimal at the top of one of the vertical columns; the number corresponding is the required root.

	0	1	2	3	4	5	6	7	8	9
2.40	1.54919	54952	54984	55016	55048	55081	55113	55145	55177	55210
2.41	1.55242	55274	55306	55338	55371	55403	55435	55467	55499	55531
2.42	1.55563	55596	55628	55660	55692	55724	55756	55788	55820	55852
2.43	1.55885	55917	55949	55981	56013	56045	56077	56109	56141	56173
2.44	1.56205	56237	56269	56301	56333	56365	56397	56429	56461	56493
2.45	1.56525	56557	56589	56621	56652	56684	56716	56748	56780	56812
2.46	1.56844	56876	56908	56939	56971	57003	57035	57067	57099	57131
2.47	1.57162	57194	57226	57258	57290	57321	57353	57385	57417	57448
2.48	1.57480	57512	57544	57575	57607	57639	57671	57702	57734	57766
2.49	1.57797	57829	57861	57892	57924	57956	57987	58019	58051	58082
2.50	1.58114	58146	58177	58209	58240	58272	58304	58335	58367	58398
2.51	1.58430	58461	58493	58524	58556	58588	58619	58651	58682	58714
2.52	1.58745	58777	58808	58840	58871	58902	58934	58965	58997	59028
2.53	1.59060	59091	59123	59154	59185	59217	59248	59280	59311	59342
2.54	1.59374	59405	59437	59468	59499	59531	59562	59593	59625	59656
2.55	1.59687	59719	59750	59781	59812	59844	59875	59906	59937	59969
2.56	1.60000	60031	60062	60094	60125	60156	60187	60219	60250	60281
2.57	1.60312	60343	60375	60406	60437	60468	60499	60530	60562	60593
2.58	1.60624	60655	60686	60717	60748	60779	60810	60842	60873	60904
2.59	1.60935	60966	60997	61028	61059	61090	61121	61152	61183	61214
2.60	1.61245	61276	61307	61338	61369	61400	61431	61462	61493	61524
2.61	1.61555	61586	61617	61648	61679	61710	61741	61771	61802	61833
2.62	1.61864	61895	61926	61957	61988	62019	62049	62080	62111	62142
2.63	1.62173	62204	62234	62265	62296	62327	62358	62388	62419	62450
2.64	1.62481	62512	62542	62573	62604	62635	62665	62696	62727	62757
2.65	1.62788	62819	62850	62880	62911	62942	62972	63003	63034	63064
2.66	1.63095	63126	63156	63187	63218	63248	63279	63310	63340	63371
2.67	1.63401	63432	63463	63493	63524	63554	63585	63615	63646	63677
2.68	1.63707	63738	63768	63799	63829	63860	63890	63921	63951	63982
2.69	1.64012	64043	64073	64104	64134	64165	64195	64225	64256	64286
2.70	1.64317	64347	64378	64408	64438	64469	64499	64530	64560	64590
2.71	1.64621	64651	64682	64712	64742	64773	64803	64833	64864	64894
2.72	1.64924	64955	64985	65015	65045	65076	65106	65136	65167	65197
2.73	1.65227	65257	65288	65318	65348	65378	65409	65439	65469	65499
2.74	1.65529	65560	65590	65620	65650	65680	65711	65741	65771	65801
2.75	1.65831	65861	65892	65922	65952	65982	66012	66042	66072	66102
2.76	1.66132	66163	66193	66223	66253	66283	66313	66343	66373	66403
2.77	1.66433	66463	66493	66523	66553	66583	66613	66643	66673	66703
2.78	1.66733	66763	66793	66823	66853	66883	66913	66943	66973	67003
2.79	1.67033	67063	67093	67123	67153	67183	67212	67242	67272	67302
2.80	1.67332	67362	67392	67422	67451	67481	67511	67541	67571	67601
2.81	1.67631	67660	67690	67720	67750	67780	67809	67839	67869	67899
2.82	1.67929	67958	67988	68018	68048	68077	68107	68137	68167	68196
2.83	1.68226	68256	68285	68315	68345	68375	68404	68434	68464	68493
2.84	1.68523	68553	68582	68612	68642	68671	68701	68731	68760	68790
2.85	1.68819	68849	68879	68908	68938	68967	68997	69027	69056	69086
2.86	1.69115	69145	69174	69204	69234	69263	69293	69322	69352	69381
2.87	1.69411	69440	69470	69499	69529	69558	69588	69617	69647	69676
2.88	1.69706	69735	69765	69794	69823	69853	69882	69912	69941	69871
2.89	1.70000	70029	70059	70088	70118	70147	70176	70206	70235	70265
2.90	1.70294	70323	70353	70382	70411	70441	70470	70499	70529	70558
2.91	1.70587	70617	70646	70675	70704	70734	70763	70792	70822	70851
2.92	1.70880	70909	70939	70968	70997	71026	71056	71085	71114	71143
2.93	1.71172	71202	71231	71260	71289	71318	71348	71377	71406	71435
2.94	1.71464	71493	71523	71552	71581	71610	71639	71668	71697	71727
2.95	1.71756	71785	71814	71843	71872	71901	71930	71959	71988	72017
2.96	1.72047	72076	72105	72134	72163	72192	72221	72250	72279	72308
2.97	1.72337	72366	72395	72424	72453	72482	72511	72540	72569	72598
2.98	1.72627	72656	72685	72714	72743	72772	72800	72829	72858	72887
2.99	1.72916	72945	72974	73003	73032	73061	73090	73118	73147	73176
	0	1	2	3	4	5	6	7	8	9

	33	32	31	30	29	28
1	3	3	3	3	3	3
2	7	6	6	6	6	6
3	10	10	9	9	9	8
4	13	13	12	12	12	11
5	17	16	16	15	15	14
6	20	19	19	18	17	17
7	23	22	22	21	20	20
8	26	26	25	24	23	22
9	30	29	28	27	26	25

TABLE I.—OF SQUARE ROOTS.

The proposed number is to be found, as far as the second decimal place, in the side column of the table, and the third decimal at the top of one of the vertical columns; the number corresponding is the required root.

	0	1	2	3	4	5	6	7	8	9
3.00	1.73205	73234	73263	73292	73321	73349	73378	73407	73436	73465
3.01	1.73494	73522	73551	73580	73609	73638	73666	73695	73724	73753
3.02	1.73781	73810	73839	73868	73897	73925	73954	73983	74011	74040
3.03	1.74069	74098	74126	74155	74184	74213	74241	74270	74299	74327
3.04	1.74356	74385	74413	74442	74471	74499	74528	74557	74585	74614
3.05	1.74642	74671	74700	74728	74757	74786	74814	74843	74871	74900
3.06	1.74929	74957	74986	75014	75043	75071	75100	75129	75157	75186
3.07	1.75214	75243	75271	75300	75328	75357	75385	75414	75442	75471
3.08	1.75499	75528	75556	75585	75613	75642	75670	75699	75727	75756
3.09	1.75784	75812	75841	75869	75898	75926	75955	75983	76011	76040
3.10	1.76068	76097	76125	76153	76182	76210	76238	76267	76295	76324
3.11	1.76352	76380	76409	76437	76465	76494	76522	76550	76579	76607
3.12	1.76635	76664	76692	76720	76748	76777	76805	76833	76862	76890
3.13	1.76918	76946	76975	77003	77031	77059	77088	77116	77144	77172
3.14	1.77200	77229	77257	77285	77313	77341	77370	77398	77426	77454
3.15	1.77482	77511	77539	77567	77595	77623	77651	77679	77708	77736
3.16	1.77764	77792	77820	77848	77876	77904	77933	77961	77989	78017
3.17	1.78045	78073	78101	78129	78157	78185	78213	78241	78269	78298
3.18	1.78326	78354	78382	78410	78438	78466	78494	78522	78550	78578
3.19	1.78606	78634	78662	78690	78718	78746	78774	78802	78830	78857
3.20	1.78885	78913	78941	78969	78997	79025	79053	79081	79109	79137
3.21	1.79165	79193	79221	79248	79276	79304	79332	79360	79388	79416
3.22	1.79444	79471	79499	79527	79555	79583	79611	79639	79666	79694
3.23	1.79722	79750	79778	79805	79833	79861	79889	79917	79944	79972
3.24	1.80000	80028	80056	80083	80111	80139	80167	80194	80222	80250
3.25	1.80278	80305	80333	80361	80388	80416	80444	80472	80499	80527
3.26	1.80555	80582	80610	80638	80665	80693	80721	80748	80776	80804
3.27	1.80831	80859	80887	80914	80942	80970	80997	81025	81052	81080
3.28	1.81108	81135	81163	81191	81218	81246	81273	81301	81328	81356
3.29	1.81384	81411	81439	81466	81494	81521	81549	81576	81604	81631
3.30	1.81659	81687	81714	81742	81769	81797	81824	81852	81879	81907
3.31	1.81934	81962	81989	82016	82044	82071	82099	82126	82154	82181
3.32	1.82209	82236	82264	82291	82318	82346	82373	82401	82428	82455
3.33	1.82483	82510	82538	82565	82592	82620	82647	82675	82702	82729
3.34	1.82757	82784	82811	82839	82866	82893	82921	82948	82975	83003
3.35	1.83030	83057	83085	83112	83139	83167	83194	83221	83248	83276
3.36	1.83303	83330	83358	83385	83412	83439	83467	83494	83521	83548
3.37	1.83576	83603	83630	83657	83685	83712	83739	83766	83793	83821
3.38	1.83848	83875	83902	83929	83957	83984	84011	84038	84065	84092
3.39	1.84120	84147	84174	84201	84228	84255	84282	84310	84337	84364
3.40	1.84391	84418	84445	84472	84499	84526	84554	84581	84608	84635
3.41	1.84662	84689	84716	84743	84770	84797	84824	84851	84878	84905
3.42	1.84932	84959	84986	85014	85041	85068	85095	85122	85149	85176
3.43	1.85203	85230	85257	85284	85311	85338	85365	85391	85418	85445
3.44	1.85472	85499	85526	85553	85580	85607	85634	85661	85688	85715
3.45	1.85742	85769	85796	85822	85849	85876	85903	85930	85957	85984
3.46	1.86011	86038	86065	86091	86118	86145	86172	86199	86226	86253
3.47	1.86279	86306	86333	86360	86387	86414	86440	86467	86494	86521
3.48	1.86548	86574	86601	86628	86655	86682	86708	86735	86762	86789
3.49	1.86815	86842	86869	86896	86922	86949	86976	87003	87029	87056
3.50	1.87083	87110	87136	87163	87190	87216	87243	87270	87297	87323
3.51	1.87350	87377	87403	87430	87457	87483	87510	87537	87563	87590
3.52	1.87617	87643	87670	87697	87723	87750	87776	87803	87830	87856
3.53	1.87883	87910	87936	87963	87989	88016	88043	88069	88096	88122
3.54	1.88149	88175	88202	88229	88255	88282	88308	88335	88361	88388
3.55	1.88414	88441	88468	88494	88521	88547	88574	88600	88627	88653
3.56	1.88680	88706	88733	88759	88786	88812	88839	88865	88892	88918
3.57	1.88944	88971	88997	89024	89050	89077	89103	89130	89156	89182
3.58	1.89209	89235	89262	89288	89315	89341	89367	89394	89420	89447
3.59	1.89473	89499	89526	89552	89578	89605	89631	89658	89684	89710
	0	1	2	3	4	5	6	7	8	9

	29
1	3
2	6
3	9
4	12
5	15
6	17
7	20
8	23
9	26

	28
1	3
2	6
3	8
4	11
5	14
6	17
7	20
8	22
9	25

	27
1	3
2	5
3	8
4	11
5	14
6	16
7	19
8	22
9	24

	26
1	3
2	5
3	8
4	10
5	13
6	16
7	18
8	21
9	23

TABLE I.—OF SQUARE ROOTS.

The proposed number is to be found, as far as the second decimal place, in the side column of the table, and the third decimal at the top of one of the vertical columns; the number corresponding is the required root.

	0	1	2	3	4	5	6	7	8	9
3.60	1.89737	89763	89789	89816	89842	89868	89895	89921	89947	89974
3.61	1.90000	90026	90053	90079	90105	90132	90158	90184	90210	90237
3.62	1.90263	90289	90316	90342	90368	90394	90421	90447	90473	90499
3.63	1.90526	90552	90578	90604	90631	90657	90683	90709	90735	90762
3.64	1.90788	90814	90840	90866	90893	90919	90945	90971	90997	91024
3.65	1.91050	91076	91102	91128	91154	91181	91207	91233	91259	91285
3.66	1.91311	91337	91364	91390	91416	91442	91468	91494	91520	91546
3.67	1.91572	91599	91625	91651	91677	91703	91729	91755	91781	91807
3.68	1.91833	91859	91885	91911	91937	91964	91990	92016	92042	92068
3.69	1.92094	92120	92146	92172	92198	92224	92250	92276	92302	92328
3.70	1.92354	92380	92406	92432	92458	92484	92510	92536	92562	92588
3.71	1.92614	92640	92666	92691	92717	92743	92769	92795	92821	92847
3.72	1.92873	92899	92925	92951	92977	93003	93028	93054	93080	93106
3.73	1.93132	93158	93184	93210	93236	93261	93287	93313	93339	93365
3.74	1.93391	93417	93442	93468	93494	93520	93546	93572	93598	93623
3.75	1.93649	93675	93701	93727	93752	93778	93804	93830	93856	93881
3.76	1.93907	93933	93959	93985	94010	94036	94062	94088	94113	94139
3.77	1.94165	94191	94216	94242	94268	94294	94319	94345	94371	94397
3.78	1.94422	94448	94474	94499	94525	94551	94576	94602	94628	94654
3.79	1.94679	94705	94731	94756	94782	94808	94833	94859	94885	94910
3.80	1.94936	94962	94987	95013	95038	95064	95090	95115	95141	95167
3.81	1.95192	95218	95243	95269	95295	95320	95346	95371	95397	95423
3.82	1.95448	95474	95499	95525	95551	95576	95602	95627	95653	95678
3.83	1.95704	95729	95755	95780	95806	95832	95857	95883	95908	95934
3.84	1.95959	95985	96010	96036	96061	96087	96112	96138	96163	96189
3.85	1.96214	96240	96265	96291	96316	96342	96367	96392	96418	96443
3.86	1.96469	96494	96520	96545	96571	96596	96621	96647	96672	96698
3.87	1.96723	96749	96774	96799	96825	96850	96876	96901	96926	96952
3.88	1.96977	97003	97028	97053	97079	97104	97129	97155	97180	97205
3.89	1.97231	97256	97282	97307	97332	97358	97383	97408	97434	97459
3.90	1.97484	97509	97535	97560	97585	97611	97636	97661	97687	97712
3.91	1.97737	97762	97788	97813	97838	97864	97889	97914	97939	97965
3.92	1.97990	98015	98040	98066	98091	98116	98141	98167	98192	98217
3.93	1.98242	98267	98293	98318	98343	98368	98394	98419	98444	98469
3.94	1.98494	98520	98545	98570	98595	98620	98645	98671	98696	98721
3.95	1.98746	98771	98796	98822	98847	98872	98897	98922	98947	98972
3.96	1.98997	99023	99048	99073	99098	99123	99148	99173	99198	99223
3.97	1.99249	99274	99299	99324	99349	99374	99399	99424	99449	99474
3.98	1.99499	99524	99549	99575	99600	99625	99650	99675	99700	99725
3.99	1.99750	99775	99800	99825	99850	99875	99900	99925	99950	99975
4.00	2.00000	00025	00050	00075	00100	00125	00150	00175	00200	00225
4.01	2.00250	00275	00300	00325	00350	00375	00400	00425	00449	00474
4.02	2.00499	00524	00549	00574	00599	00624	00649	00674	00699	00724
4.03	2.00749	00774	00798	00823	00848	00873	00898	00923	00948	00973
4.04	2.00998	01022	01047	01072	01097	01122	01147	01172	01196	01221
4.05	2.01246	01271	01296	01321	01345	01370	01395	01420	01445	01470
4.06	2.01494	01519	01544	01569	01594	01618	01643	01668	01693	01718
4.07	2.01742	01767	01792	01817	01842	01866	01891	01916	01941	01965
4.08	2.01990	02015	02040	02064	02089	02114	02139	02163	02188	02213
4.09	2.02237	02262	02287	02312	02336	02361	02386	02410	02435	02460
4.10	2.02485	02509	02534	02559	02583	02608	02633	02657	02682	02707
4.11	2.02731	02756	02781	02805	02830	02855	02879	02904	02929	02953
4.12	2.02978	03002	03027	03052	03076	03101	03126	03150	03175	03199
4.13	2.03224	03249	03273	03298	03322	03347	03372	03396	03421	03445
4.14	2.03470	03494	03519	03544	03568	03593	c3617	03642	03666	03691
4.15	2.03715	03740	03765	03789	03814	03838	03863	03887	03912	03936
4.16	2.03961	03985	04010	04034	04059	04083	04108	04132	04157	04181
4.17	2.04206	04230	04255	04279	04304	04328	04353	04377	04402	04426
4.18	2.04450	04475	04499	04524	04548	04573	04597	04622	04646	04670
4.19	2.04695	04719	04744	04768	04793	04817	04841	04866	04890	04915
	0	1	2	3	4	5	6	7	8	9

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8	21
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2	5
3	8
4	10
5	13
6	15
7	18
8	20
9	23

	24
1	2
2	5
3	7
4	10
5	12
6	14
7	17
8	19
9	22

TABLE I.—OF SQUARE ROOTS.

The proposed number is to be found, as far as the first decimal place, in the side column of the table, and the second decimal at the top of one of the vertical columns; the number corresponding is the required root.

	0	1	2	3	4	5	6	7	8	9
4.2	2.04939	05183	05426	05670	05913	06155	06398	06640	06882	07123
4.3	2.07364	07605	07846	08087	08327	08567	08806	09045	09284	09523
4.4	2.09762	10000	10238	10476	10713	10950	11187	11424	11660	11896
4.5	2.12132	12368	12603	12838	13073	13307	13542	13776	14009	14243
4.6	2.14476	14709	14942	15174	15407	15639	15870	16102	16333	16564
4.7	2.16795	17025	17256	17486	17715	17945	18174	18403	18632	18861
4.8	2.19089	19317	19545	19773	20000	20227	20454	20681	20907	21133
4.9	2.21359	21585	21811	22036	22261	22486	22711	22935	23159	23383
5.0	2.23607	23830	24054	24277	24499	24722	24944	25167	25389	25610
5.1	2.25832	26053	26274	26495	26716	26936	27156	27376	27596	27816
5.2	2.28035	28254	28473	28692	28910	29129	29347	29565	29783	30000
5.3	2.30217	30434	30651	30868	31084	31301	31517	31733	31948	32164
5.4	2.32379	32594	32809	33024	33238	33452	33666	33880	34094	34307
5.5	2.34521	34734	34947	35160	35372	35584	35797	36008	36220	36432
5.6	2.36643	36854	37065	37276	37487	37697	37908	38118	38328	38537
5.7	2.38747	38956	39165	39374	39583	39792	40000	40208	40416	40624
5.8	2.40832	41039	41247	41454	41661	41868	42074	42281	42487	42693
5.9	2.42899	43105	43311	43516	43721	43926	44131	44336	44540	44745
6.0	2.44949	45153	45357	45561	45764	45967	46171	46374	46577	46779
6.1	2.46982	47184	47386	47588	47790	47992	48193	48395	48596	48797
6.2	2.48998	49199	49399	49600	49800	50000	50200	50400	50599	50799
6.3	2.50998	51197	51396	51595	51794	51992	52190	52389	52587	52784
6.4	2.52982	53180	53377	53574	53772	53969	54165	54362	54558	54755
6.5	2.54951	55147	55343	55539	55734	55930	56125	56320	56515	56710
6.6	2.56905	57099	57294	57488	57682	57876	58070	58263	58457	58650
6.7	2.58844	59037	59230	59422	59615	59808	60000	60192	60384	60576
6.8	2.60768	60960	61151	61343	61534	61725	61916	62107	62298	62488
6.9	2.62679	62869	63059	63249	63439	63629	63818	64008	64197	64386
7.0	2.64575	64764	64953	65141	65330	65518	65707	65895	66083	66271
7.1	2.66458	66646	66833	67021	67208	67395	67582	67769	67955	68142
7.2	2.68328	68514	68701	68887	69072	69258	69444	69629	69815	70000
7.3	2.70185	70370	70555	70740	70924	71109	71293	71477	71662	71846
7.4	2.72029	72213	72397	72580	72764	72947	73130	73313	73496	73679
7.5	2.73861	74044	74226	74408	74591	74773	74955	75136	75318	75500
7.6	2.75681	75862	76043	76225	76405	76586	76767	76948	77128	77308
7.7	2.77489	77669	77849	78029	78209	78388	78568	78747	78927	79106
7.8	2.79285	79464	79643	79821	80000	80179	80357	80535	80713	80891
7.9	2.81069	81247	81425	81603	81780	81957	82135	82312	82489	82666
8.0	2.82843	83019	83196	83373	83549	83725	83901	84077	84253	84429
8.1	2.84605	84781	84956	85132	85307	85482	85657	85832	86007	86182
8.2	2.86356	86531	86705	86880	87054	87228	87402	87576	87750	87924
8.3	2.88097	88271	88444	88617	88791	88964	89137	89310	89482	89655
8.4	2.89828	90000	90172	90345	90517	90689	90861	91033	91204	91376
8.5	2.91548	91719	91890	92062	92233	92404	92575	92746	92916	93087
8.6	2.93258	93428	93598	93769	93939	94109	94279	94449	94618	94788
8.7	2.94958	95127	95296	95466	95635	95804	95973	96142	96311	96479
8.8	2.96648	96816	96985	97153	97321	97489	97658	97825	97993	98161
8.9	2.98329	98496	98664	98831	98998	99166	99333	99500	99666	99833
9.0	3.00000	00167	00333	00500	00666	00832	00998	01164	01330	01496
9.1	3.01662	01828	01993	02159	02324	02490	02655	02820	02985	03150
9.2	3.03315	03480	03645	03809	03974	04138	04302	04467	04631	04795
9.3	3.04959	05123	05287	05450	05614	05778	05941	06105	06268	06431
9.4	3.06594	06757	06920	07083	07246	07409	07571	07734	07896	08058
9.5	3.08221	08383	08545	08707	08869	09031	09192	09354	09516	09677
9.6	3.09839	10000	10161	10322	10483	10644	10805	10966	11127	11288
9.7	3.11448	11609	11769	11929	12090	12250	12410	12570	12730	12890
9.8	3.13050	13209	13369	13528	13688	13847	14006	14166	14325	14484
9.9	3.14643	14802	14960	15119	15278	15436	15595	15753	15911	16070
10.0	3.16228	16386	16544	16702	16860	17017	17175	17333	17490	17648
10.1	3.17805	17962	18119	18277	18434	18591	18748	18904	19061	19218
	0	1	2	3	4	5	6	7	8	9

	246	245	244	243	242	241	240	239	238	237	236	235	234
1	25	25	24	24	24	24	24	24	24	24	24	24	23
2	49	49	49	49	48	48	48	48	48	47	47	47	47
3	74	74	73	73	73	72	72	72	71	71	71	71	70
4	98	98	98	97	97	96	96	96	95	95	94	94	94
5	123	123	122	122	121	121	120	120	119	119	118	118	117
6	148	147	146	146	145	145	144	143	143	142	142	141	140
7	172	172	171	170	169	169	168	167	167	166	165	165	164
8	197	196	195	194	194	193	192	191	190	190	189	188	187
9	221	221	220	219	218	217	216	215	214	213	212	212	211

	233	232	231	230	229	228	227	226	225	224	223	222	221
1	23	23	23	23	23	23	23	23	23	22	22	22	22
2	47	46	46	46	46	46	45	45	45	45	45	44	44
3	70	70	69	69	69	68	68	68	68	67	67	67	66
4	93	93	92	92	92	91	91	90	90	90	89	89	88
5	117	116	116	115	115	114	114	113	113	112	112	111	111
6	140	139	139	138	137	137	136	136	135	134	134	133	133
7	163	162	162	161	160	160	159	158	158	157	156	155	155
8	186	186	185	184	183	182	182	181	180	179	178	178	177
9	210	209	208	207	206	205	204	203	203	202	201	200	199

	220	219	218	217	216	215	214	213	212	211	210	209	208
1	22	22	22	22	22	22	21	21	21	21	21	21	21
2	44	44	44	43	43	43	43	43	42	42	42	42	42
3	66	66	65	65	65	65	64	64	64	63	63	63	62
4	88	88	87	87	86	86	86	85	85	84	84	84	83
5	110	110	109	109	108	108	107	107	106	106	105	105	104
6	132	131	131	130	130	129	128	128	127	127	126	125	125
7	154	153	153	152	151	151	150	149	148	148	147	146	146
8	176	175	174	174	173	172	171	170	170	169	168	167	166
9	198	197	196	195	194	194	193	192	191	190	189	188	187

	207	206	205	204	203	202	201	200	199	198	197	196	195
1	21	21	21	20	20	20	20	20	20	20	20	20	20
2	41	41	41	41	41	40	40	40	40	40	39	39	39
3	62	62	62	61	61	61	60	60	60	59	59	59	59
4	83	82	82	82	81	81	80	80	80	79	79	78	78
5	104	103	103	102	102	101	101	100	100	99	99	98	98
6	124	124	123	122	122	121	121	120	119	119	118	118	117
7	145	144	144	143	142	141	141	140	139	139	138	137	137
8	166	165	164	163	162	162	161	160	159	158	158	157	156
9	186	185	185	184	183	182	181	180	179	178	177	176	176

	194	193	192	191	190	189	188	187	186	185	184	183	182
1	19	19	19	19	19	19	19	19	19	19	18	18	18
2	39	39	38	38	38	38	38	37	37	37	37	37	36
3	58	58	58	57	57	57	56	56	56	56	55	55	55
4	78	77	77	76	76	76	75	75	74	74	74	73	73
5	97	97	96	96	95	95	94	94	93	93	92	92	91
6	116	116	115	115	114	113	113	112	112	111	110	110	109
7	136	135	134	134	133	132	132	131	130	130	129	128	127
8	155	154	154	153	152	151	150	150	149	148	147	146	146
9	175	174	173	172	171	170	169	168	167	167	166	165	164

	181	180	179	178	177	176	175	174	173	172	171	170	169
1	18	18	18	18	18	18	18	17	17	17	17	17	17
2	36	36	36	36	35	35	35	35	35	34	34	34	34
3	54	54	54	53	53	53	53	52	52	52	51	51	51
4	72	72	72	71	71	70	70	70	69	69	68	68	68
5	91	90	90	89	89	88	88	87	87	86	86	85	85
6	109	108	107	107	106	106	105	104	104	103	103	102	101
7	127	126	125	125	124	123	123	122	121	120	120	119	118
8	145	144	143	142	142	141	140	139	138	138	137	136	135
9	163	162	161	160	159	158	158	157	156	155	154	153	152

	168	167	166	165	164	163	162	161	160	159	158	157	156
1	17	17	17	17	16	16	16	16	16	16	16	16	16
2	34	33	33	33	33	33	32	32	32	32	32	31	31
3	50	50	50	50	49	49	49	48	48	48	47	47	47
4	67	67	66	66	66	65	65	64	64	64	63	63	62
5	84	84	83	83	82	82	81	81	80	80	79	79	78
6	101	100	100	99	98	98	97	97	96	95	95	94	94
7	118	117	116	116	115	114	113	113	112	111	111	110	109
8	134	134	133	132	131	130	130	129	128	127	126	126	125
9	151	150	149	149	148	147	146	145	144	143	142	141	140

TABLE II.

This gives the time T of describing a parabolic arc by a comet, the sum of the extreme radii $r+r'$ being found at the top, and the chord c at the left side of the page.

Chord c.	Sum of the radii $r+r''$.																						
	0,01		0,02		0,03		0,04		0,05		0,06		0,07		0,08		0,09		0,10		0,11		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,027	14	0,041	9	0,050	8	0,058	7	0,065	6	0,071	6	0,077	5	0,082	5	0,087	5	0,092	4	0,096	5	0,0001
0,02			0,078	21	0,099	16	0,115	14	0,129	13	0,142	11	0,153	11	0,164	10	0,174	10	0,184	9	0,193	8	0,000[illegible]
0,03					0,142	28	0,170	22	0,192	19	0,211	18	0,229	16	0,245	15	0,260	15	0,275	13	0,288	13	0,000[illegible]
0,04							0,219	33	0,252	27	0,279	24	0,303	22	0,325	21	0,346	19	0,365	18	0,383	18	0,001[illegible]
0,05									0,306	38	0,344	31	0,375	29	0,404	26	0,430	25	0,455	23	0,478	22	0,002[illegible]
0,06											0,403	41	0,444	36	0,480	33	0,513	30	0,543	28	0,571	27	0,003[illegible]
0,07													0,508	45	0,553	40	0,593	36	0,629	33	0,662	32	0,004[illegible]
0,08															0,620	49	0,669	44	0,713	39	0,752	37	0,006[illegible]
0,09																	0,740	53	0,793	46	0,839	43	0,008[illegible]
0,10																			0,867	56	0,923	49	0,010[illegible]
0,11																					1,000	59	0,0121
	0,0001		0,0002		0,0004		0,0008		0,0012		0,0018		0,0024		0,0032		0,0040		0,0050		0,0061		c^2
	$\frac{1}{2}\cdot(r+r'')^2$ or $r^2+r''^2$ nearly.																						

TABLES FOR COMPUTING THE ORBIT OF A COMET.

Table I. This is a table of square roots, adapted to the calculation of the orbit of a Comet, by methods similar to that proposed by Dr. Olbers. We have by inspection, in this table, the root of any number, from 0,001 to 10,19; and by using the small tables of proportional parts, given in the margin, *the root may be obtained from the number, or the number from the root, to five places of decimals.* This requires no particular explanation, since the arrangement is the same as that of a common table of logarithms. We may also observe that when the quantity x, whose root is to be found, is less than 0,102, it is convenient to find the root of $100x$, and then divide the result by 10; which is done by merely transposing the decimal point. Thus if $x=0{,}0961$, we may find the root of $9{,}61=3{,}1$, and transpose the point one figure, and we shall obtain $\sqrt{0{,}0961}=0{,}31$. In like manner, if we have $c^2=0{,}080087$, we get by the table $\sqrt{8{,}0087}=2{,}82996$, whence $c=0{,}282996$; by this means the proportional parts are more easily obtained.

Table II. The argument at the top of the table is the sum of the two radii vectores of the comet r, r''; the mean distance of the earth from the sun being taken for unity. On the left side column of the table, is the length c of the chord, connecting the extreme parts of these radii. The corresponding number represents the time T, given by Lambert's formula [750, 750′]; supposing the comet to move in a parabolic orbit,

$$T=9^{\text{days}},688724.\left\{(r+r''+c)^{\frac{3}{2}}-(r+r''-c)^{\frac{3}{2}}\right\}.$$

Thus if $r+r''=2{,}20$, and $c=0{,}20$, we shall have $T=8^{\text{days}},619$. The proportional parts for the fractions of $r+r''$ beyond two places of decimals, are placed at the right hand side of the page, those for c, in the column at the bottom of the table, nearly below the corresponding tabular time T. In using Table II. we must enter it with the values of $r+r''$ and c; taking them to two places of decimals; and find, by inspection, the corresponding *chief term of T.* The variation of T, corresponding to the successive tabular values of $r+r''$, is given in the same horizontal line with the chief term of T; and we must find also the variation, corresponding to the successive tabular values of c, in the vertical column, immediately below the chief term of T. The increments of T, corresponding to the fractional parts of $r+r''$ and c, beyond the second decimal place, are to be found and *added* to the chief term T, to obtain the true value of T.

In general it will be sufficiently accurate to use for the argument of the proportional parts in the table in the side column, the tabular number in the column of differences corresponding to the chief term of T; but when very great accuracy is required, we may find it for the exact value of c; by taking a proportional part of the difference of the two nearest numbers in the table.

To show, by an example, the use of this table, we shall suppose $r+r''=1{,}96280$, $c=0{,}24573$. Then we shall have for the chief term of T, corresponding to 1,96 and 0,24 the value $9^{\text{days}},760$; the differences between this and the next numbers being 25, and 406, respectively. The proportional parts corresponding to the decimals ,00280 and ,00573 are 7 and 233; the sum of these three quantities is $9{,}760+0{,}007+0{,}233=10^{\text{days}}$, the value of T required.

In the right hand column of the table is given the value of c^2. At the bottom of the table is given the values of $\frac{1}{2}.(r+r'')^2$, which may be used, instead of $r^2+r''^2$, in the first approximation to the value of ϱ. In this case the calculation is made merely *by inspection;* using the nearest numbers in the table, and taking them to one or two places of decimals; without using the tables of *proportional parts, which are exclusively adapted to the values of $r+r''$* and c.

These two tables are designed to facilitate the computation of the value of ϱ, from the three equations (A), (B), (C), which are similar to these in the following system; in which r, r'' *represent the radii vectores at the first and third observations; c the intercepted chord; ϱ the curtate distance of the comet from the earth; the interval between the observations, expressed in days, being given and represented by T.* The equation (D) which is the sum of the equation (A), (B), may be used in the first approximation to the value of ϱ. It is not absolutely necessary, to use the equation (D), but it will frequently be found to have a tendency to abridge the calculations.

TABLE II.

This gives the time T of describing a parabolic arc by a comet, the sum of the extreme radii $r + r''$ being found at the top, and the chord c at the left side of the page.

Sum of the Radii $r + r''$.																							
Chord c.	0,12		0,13		0,14		0,15		0,16		0,17		0,18		0,19		0,20		0,21		0,22		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,101	4	0,105	4	0,109	4	0,113	3	0,116	4	0,120	3	0,123	4	0,127	3	0,130	3	0,133	3	0,13[illegible]	3	0,0001
0,02	0,201	8	0,209	8	0,217	8	0,225	7	0,232	8	0,240	7	0,247	6	0,253	7	0,260	6	0,266	7	[illegible]	6	0,0004
0,03	0,301	13	0,314	12	0,326	11	0,337	11	0,348	11	0,359	11	0,370	10	0,380	10	0,390	9	0,399	10	[illegible]	[illegible]	0,0009
0,04	0,401	17	0,418	16	0,434	15	0,449	15	0,464	14	0,478	14	0,492	14	0,506	13	0,519	13	0,532	13	0,545	12	0,0016
0,05	0,500	21	0,521	20	0,541	19	0,560	19	0,579	18	0,597	18	0,615	17	0,632	16	0,648	16	0,664	16	0,680	[illegible]	0,0025
0,06	0,598	25	0,623	24	0,647	24	0,671	22	0,693	22	0,715	21	0,736	21	0,757	20	0,777	19	0,796	19	0,815	19	0,0036
0,07	0,694	30	0,724	29	0,753	28	0,781	26	0,807	26	0,833	25	0,858	24	0,882	23	0,905	23	0,928	22	[illegible]	[illegible]	0,0049
0,08	0,789	33	0,824	33	0,857	32	0,889	31	0,920	[illegible]	0,949	[illegible]	[illegible]	[illegible]	1,006	27	1,033	26	1,059	25	[illegible]	[illegible]	0,0064
0,09	0,883	[illegible]	0,922	38	0,960	37	0,997	35	1,032	33	1,065	33	1,098	31	1,129	31	1,160	29	1,189	29	1,21[illegible]	[illegible]	0,0081
0,10	0,972	[illegible]	1,018	44	1,062	41	1,103	39	1,142	38	1,180	36	1,216	35	1,251	35	1,286	33	1,319	32	1,351	[illegible]	0,0100
0,11	1,050	[illegible]	1,112	[illegible]	1,161	46	1,207	44	1,251	42	1,293	41	1,334	39	1,373	38	1,411	36	1,447	36	[illegible]	35	0,0121
0,12	1,139	[illegible]	[illegible]	56	1,257	52	[illegible]	[illegible]	1,358	47	1,405	45	1,450	43	1,493	42	1,535	40	1,575	39	1,61[illegible]	39	0,0144
0,13			1,[illegible]84	66	1,350	58	[illegible]	[illegible]	1,463	52	1,515	49	1,564	47	1,611	46	1,657	45	1,702	43	1,745	[illegible]	0,0169
0,14					1,435	[illegible]	1,50[illegible]	[illegible]	1,565	57	1,622	54	1,676	52	1,728	50	1,778	49	1,827	47	1,874	[illegible]	0,0196
0,15							1,59[illegible]	71	1,663	64	1,722	59	1,780	57	1,843	55	1,895	[illegible]	1,950	51	[illegible]	[illegible]	0,0225
0,16									1,734	73	1,827	66	1,893	[illegible]	1,956	59	2,015	57	2,072	55	[illegible]	[illegible]	0,0256
0,17											1,921	75	1,996	69	2,065	65	2,130	62	2,192	59	[illegible]	[illegible]	0,0289
0,18													2,093	78	2,171	71	2,242	67	2,309	65	[illegible]	[illegible]	0,0324
0,19															2,270	80	2,350	74	2,424	69	[illegible]	67	0,0361
0,20																	2,451	83	2,534	76	[illegible]	[illegible]	[illegible]
0,21																			[illegible]	[illegible]	[illegible]	[illegible]	0,0441
0,22																					[illegible]	87	0,0484
	.0072		.0085		.0098		.0113		.0128		.0145		.0162		.0181		.0200		.0221		.0242		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

Proportional parts for the Chord.

	80	83	86	89	92	95	98	101	104	107	110	113	116	119	122	125	128	131	134	137
1	8	8	9	9	9	10	10	10	10	11	11	11	12	12	12	13	13	13	13	14
2	16	17	17	18	18	19	20	20	21	21	22	23	23	24	24	25	26	26	27	27
3	24	25	26	27	28	29	29	30	31	32	33	34	35	36	37	38	38	39	40	41
4	32	33	34	36	37	38	39	40	42	43	44	45	46	48	49	50	51	52	54	55
5	40	42	43	45	46	48	49	51	52	54	55	57	58	60	61	63	64	66	67	69
6	48	50	52	53	55	57	59	61	62	64	66	68	70	71	73	75	77	79	80	82
7	56	58	60	62	64	67	69	71	73	75	77	79	81	83	85	88	90	92	94	96
8	64	66	69	71	74	76	78	81	83	86	88	90	93	95	98	100	102	105	107	110
9	72	75	77	80	83	86	88	91	94	96	99	102	104	107	110	113	115	118	121	123

EXAMPLE I.

To show the use of these tables we shall apply them to the determination of the value of ϱ from the three following equations, corresponding to observations of the Comet of 1779; as in page xiii. of Dr. Olbers' *Abhandlung*, &c.

$$r^2 = 0,98240 + 0,87363.\varrho + 2,33263.\varrho^2 ; \qquad (A)$$
$$r''^2 = 0,98861 + 2,11869.\varrho + 2,88041.\varrho^2 ; \qquad (B)$$
$$c^2 = 0,04188 + 0,006845.\varrho + 0,208501.\varrho^2 ; \qquad (C)$$
$$r^2 + r''^2 = 1,97101 + 2,99232.\varrho + 5,21304.\varrho^2 . \qquad (D)$$

Time $T = 11^{\text{days}},834$.

COMPUTATION OF ϱ FROM THE ABOVE EQUATIONS, (A), (B), (C).

	r^2.	r''^2.	c^2.	r, r'', c.	T.
Hypothesis I.	0,98240	0,98861	0,04188	$r = 1,20599$	11,645
	0,26209	0,63560	,00205	$r'' = 1,37229$	19
$\varrho = 0,3$	0,20994	0,25924	,01876	$r + r'' = 2,57838$	18
	1,45443	1,88345	0,06269	$c = 0,25038$	11,682
Hypothesis II.	0,98240	0,98861	0,041880	$r = 1,21311$	11,690
Add $\frac{1}{40}$ makes	,26864	,65149	,002104	$r'' = 1,38291$	13
	,22057	,27236	,019716	$r + r'' = 2,59602$	111
$\varrho = 0,3075$	1,47161	1,91246	0,063700	$c = 0,25239$	11,814
Hypothesis III.	0,98240	0,98861	0,041880	$r = 1,21427$	11,690
Add $\frac{1}{250}$ or ,00123	0,26971	,65410	,002112	$r'' = 1,38465$	20
	,22233	,27454	,019874	$r + r'' = 2,59892$	127
$\varrho = 0,30873$	1,47444	1,91725	0,063866	$c = 0,25271$	11,837
Hypothesis IV.	0,98240	0,98861	0,041880	$r = 1,21412$	11,690
Less $\frac{1}{2000}$ or ,00015	,26958	,65377	,002111	$r'' = 1,38443$	19
	,22211	,27427	,019854	$r + r'' = 2,59855$	125
$\varrho = 0,30858$	1,47409	1,91665	0,063845	$c = 0,25267$	11,834

Coefficients of ϱ.			
	0,87363	2,11869	0,006845
	0,26209	0,63560	,002053
$\frac{1}{40}$	655	1589	51
	0,26864	0,65149	,002104
$\frac{1}{250}$	107	261	8
	0,26971	0,65410	,002112
$-\frac{1}{2000}$	−13	−33	−1
	0,26958	0,65377	,002111

Coefficients of ϱ^2.			
	2,33263	2,88041	0,208501
	0,20994	0,25924	0,018766
$\frac{2}{40}$	1050	1296	938
$\frac{1}{80}$	13	16	12
	0,22057	0,27236	0,019716
$\frac{2}{250}$	176	218	158
$\frac{1}{500}$			
	0,22233	0,27454	0,019874
$-\frac{1}{1000}$	−22	−27	−20
	0,22211	0,27427	,019854

TABLE II.

This gives the time T of describing a parabolic arc by a comet, the sum of the extreme radii $r+r''$ being found at the top, and the chord c at the left side of the page.

Chord c.	0,23 Days	dif.	0,24 Days	dif.	0,25 Days	dif.	0,26 Days	dif.	0,27 Days	dif.	0,28 Days	dif.	0,29 Days	dif.	
	Sum of the Radii $r+r''$.														
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,139	3	0,142	3	0,145	3	0,148	3	0,151	3	0,154	3	0,157	2	0,0001
0,02	0,279	6	0,285	6	0,291	5	0,296	6	0,302	6	0,308	5	0,313	5	0,0004
0,03	0,418	9	0,427	9	0,436	8	0,444	9	0,453	8	0,461	8	0,469	8	0,0009
0,04	0,557	12	0,569	12	0,581	11	0,592	12	0,604	11	0,615	11	0,626	10	0,0016
0,05	0,696	15	0,711	14	0,725	15	0,740	14	0,754	14	0,768	14	0,782	13	0,0025
0,06	0,834	18	0,852	18	0,870	17	0,887	17	0,904	17	0,921	16	0,937	17	0,0036
0,07	0,972	21	0,993	21	1,014	20	1,034	20	1,054	20	1,074	19	1,093	19	0,0049
0,08	1,109	25	1,134	24	1,158	23	1,181	23	1,204	22	1,226	22	1,248	22	0,0064
0,09	1,240	28	1,274	27	1,301	26	1,327	26	1,353	25	1,378	25	1,403	24	0,0081
0,10	1,383	30	1,413	30	1,443	30	1,473	28	1,501	29	1,530	27	1,557	27	0,0100
0,11	1,518	34	1,552	33	1,585	33	1,618	31	1,649	32	1,681	30	1,711	30	0,0121
0,12	1,653	37	1,690	36	1,726	36	1,762	35	1,797	34	1,831	33	1,864	33	0,0144
0,13	1,786	41	1,827	40	1,867	39	1,906	38	1,944	37	1,981	36	2,017	36	0,0169
0,14	1,919	44	1,963	43	2,006	42	2,048	41	2,089	41	2,130	39	2,169	39	0,0196
0,15	2,050	48	2,098	47	2,145	45	2,190	44	2,234	44	2,278	42	2,320	42	0,0225
0,16	2,180	52	2,232	50	2,282	49	2,331	47	2,378	47	2,425	46	2,471	44	0,0256
0,17	2,309	55	2,364	54	2,418	52	2,470	51	2,521	50	2,571	49	2,620	48	0,0289
0,18	2,435	60	2,495	57	2,552	57	2,609	54	2,663	53	2,716	52	2,768	51	0,0324
0,19	2,560	64	2,624	61	2,685	60	2,745	59	2,804	56	2,860	56	2,916	54	0,0361
0,20	2,682	68	2,750	66	2,816	64	2,880	62	2,942	61	3,003	59	3,062	57	0,0400
0,21	2,800	74	2,874	71	2,945	69	3,014	66	3,080	64	3,144	62	3,206	61	0,0441
0,22	2,915	80	2,995	77	3,072	73	3,145	70	3,215	68	3,283	66	3,349	65	0,0484
0,23	3,023	89	3,112	83	3,195	78	3,273	75	3,348	72	3,420	71	3,491	68	0,0529
0,24			3,222	92	3,314	84	3,398	80	3,478	78	3,556	74	3,630	72	0,0576
0,25					3,425	94	3,519	87	3,606	82	3,688	79	3,767	77	0,0625
0,26							3,633	96	3,729	88	3,817	85	3,902	81	0,0676
0,27									3,845	97	3,942	91	4,033	86	0,0729
0,28											4,060	100	4,160	92	0,0784
0,29													4,280	101	0,0841
	,0265		,0288		,0313		,0338		,0365		,0392		,0421		c^2

$\frac{1}{2} \cdot (r+r'')^2$ or $r^2+r''^2$ nearly.

Prop. parts for the sum of the Radii

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	[illegible]
4	0	1	1	2	2	2	[illegible]	3	[illegible]
5	1	1	2	2	3	3	4	4	[illegible]
6	1	1	2	2	3	4	4	5	5
7	1	1	2	[illegible]	[illegible]	4	[illegible]	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	[illegible]
11	1	2	3	4	6	7	[illegible]	[illegible]	[illegible]
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

Proportional parts for the Chord.

	112	116	120	124	128	132	136	140	144	148	152	156
1	11	12	12	12	13	13	14	14	14	15	15	16
2	22	23	24	25	26	26	27	28	29	30	30	31
3	34	35	36	37	38	40	41	42	43	44	46	47
4	45	46	48	50	51	53	54	56	58	59	61	62
5	56	58	60	62	64	66	68	70	72	74	76	78
6	67	70	72	74	77	79	82	84	86	89	91	94
7	78	81	84	87	90	92	95	98	101	104	106	109
8	90	93	96	99	102	106	109	112	115	118	122	125
9	101	104	108	112	115	119	122	126	130	133	137	140

In the first approximation we shall use the equations (C) (D), computing the numbers to one or two places of decimals. Now if we suppose $\varrho=1$, these equations become $c^2=0{,}26$, $r^2+r''^2=10{,}2$. The first of these numbers is to be found on the right hand side of Table II., and the second at the bottom, the corresponding value of T is nearly 31 days. This being nearly *three* times the actual value of T, we may take for ϱ one third part of the value first assumed, or $\varrho=\frac{1}{3}$; then repeating the preceding calculation, with one more decimal in c^2, we get $c^2=0{,}067$; $r^2+r''^2=3{,}55$, and the corresponding value of T is about $12\frac{1}{2}$ days; so that we must decrease ϱ a little more. We shall therefore take for the first hypothesis of the preceding table $\varrho=0{,}3$, and use the equations (A), (B), (C), making the calculation to a greater degree of accuracy, and we get $T=11{,}682$. This time being rather too small, the value of ϱ is increased $\frac{1}{40}$, in Hypothesis II., and the resulting value of T becomes 11,814. Increasing ϱ by $\frac{1}{250}$, we obtain in a third hypothesis $T=11{,}837$, which is rather too large. Finally decreasing this last value of ϱ by $\frac{1}{2000}$ part, we obtain $\varrho=0{,}30858$, and $T=11^{\text{days}},834$, and this value of T agrees with that by observation. We may use the values obtained by this last operation, as being very near the true values so that we shall have $\varrho=0{,}30858$, $r=1{,}21412$, $r''=1{,}38413$, $c=0{,}25267$; which are almost identically the same with those obtained in the above mentioned work of Dr. Olbers.

TABLE II.

This gives the time T of describing a parabolic arc by a comet, the sum of the extreme radii $r+r''$ being found at the top, and the chord c at the left side of the page.

Sum of the Radii $r+r''$.

Chord c.	0,30		0,31		0,32		0,33		0,34		0,35		0,36		0,37		0,38		0,39		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,159	3	0,162	2	0,164	3	0,167	2	0,169	3	0,172	2	0,174	3	0,177	2	0,179	3	0,182	2	0,0001
0,02	0,318	6	0,324	5	0,329	5	0,334	5	0,339	5	0,344	5	0,349	5	0,354	4	0,358	5	0,363	5	0,0004
0,03	0,477	8	0,485	8	0,493	8	0,501	7	0,508	8	0,516	7	0,523	7	0,530	7	0,537	7	0,544	7	0,0009
0,04	0,636	11	0,647	10	0,657	10	0,667	11	0,678	9	0,687	10	0,697	10	0,707	9	0,716	10	0,726	9	0,0016
0,05	0,795	13	0,808	13	0,821	13	0,834	13	0,847	12	0,859	12	0,871	12	0,883	12	0,895	12	0,907	12	0,0025
0,06	0,954	15	0,969	16	0,985	15	1,000	16	1,016	14	1,030	15	1,045	15	1,060	14	1,074	14	1,088	14	0,0036
0,07	1,112	18	1,130	19	1,149	18	1,167	17	1,184	18	1,202	17	1,219	17	1,236	16	1,252	17	1,269	16	0,0049
0,08	1,270	21	1,291	21	1,312	20	1,332	21	1,353	20	1,373	19	1,392	20	1,412	19	1,431	19	1,450	18	0,0064
0,09	1,427	24	1,451	24	1,475	23	1,498	23	1,521	22	1,543	22	1,565	22	1,587	22	1,609	21	1,630	21	0,0081
0,10	1,584	27	1,611	26	1,637	26	1,663	26	1,689	25	1,714	24	1,738	25	1,763	24	1,787	23	1,810	23	0,0100
0,11	1,741	30	1,771	29	1,800	28	1,828	28	1,856	26	1,884	27	1,911	27	1,938	26	1,964	26	1,990	26	0,0121
0,12	1,897	33	1,930	31	1,961	31	1,992	31	2,023	30	2,053	30	2,083	29	2,112	29	2,141	28	2,169	29	0,0144
0,13	2,053	35	2,088	34	2,122	34	2,156	33	2,189	33	2,222	33	2,255	31	2,286	32	2,318	31	2,349	30	0,0169
0,14	2,208	38	2,246	37	2,283	36	2,319	36	2,355	36	2,391	35	2,426	34	2,460	34	2,494	33	2,527	33	0,0196
0,15	2,362	41	2,403	40	2,443	39	2,482	39	2,521	38	2,559	37	2,596	37	2,633	37	2,670	35	2,705	36	0,0225
0,16	2,515	44	2,559	43	2,602	42	2,644	42	2,686	40	2,726	40	2,766	40	2,806	39	2,845	38	2,883	38	0,0256
0,17	2,668	46	2,714	46	2,760	45	2,805	45	2,850	43	2,893	43	2,936	42	2,978	42	3,020	40	3,060	41	0,0289
0,18	2,819	50	2,869	49	2,918	48	2,966	47	3,013	46	3,059	46	3,105	45	3,150	44	3,194	43	3,237	43	0,0324
0,19	2,970	53	3,023	52	3,075	51	3,126	49	3,175	50	3,225	48	3,273	47	3,320	47	3,367	46	3,413	45	0,0361
0,20	3,119	56	3,175	55	3,230	54	3,284	53	3,337	52	3,389	51	3,440	50	3,490	50	3,540	48	3,588	48	0,0400
0,21	3,267	60	3,327	58	3,385	57	3,442	56	3,498	55	3,553	54	3,607	53	3,660	52	3,712	51	3,763	51	0,0441
0,22	3,414	63	3,477	61	3,538	60	3,598	59	3,657	58	3,715	57	3,772	56	3,828	55	3,883	54	3,937	53	0,0484
0,23	3,559	66	3,625	65	3,690	64	3,754	62	3,816	61	3,877	60	3,937	58	3,995	58	4,053	57	4,110	56	0,0529
0,24	3,702	70	3,772	69	3,841	67	3,908	65	3,973	64	4,037	63	4,100	62	4,162	60	4,222	60	4,282	59	0,0576
0,25	3,844	74	3,918	72	3,990	70	4,060	69	4,129	68	4,197	66	4,263	64	4,327	64	4,391	62	4,453	61	0,0625
0,26	3,983	78	4,061	76	4,137	74	4,211	73	4,284	70	4,354	70	4,424	67	4,491	67	4,558	65	4,623	64	0,0676
0,27	4,119	83	4,202	80	4,282	79	4,361	76	4,437	74	4,511	72	4,583	71	4,654	70	4,724	68	4,792	67	0,0729
0,28	4,252	88	4,340	85	4,425	83	4,508	80	4,588	77	4,665	76	4,741	75	4,816	73	4,889	71	4,960	70	0,0784
0,29	4,381	95	4,476	90	4,566	86	4,652	84	4,736	82	4,818	80	4,898	78	4,976	76	5,052	75	5,127	73	0,0841
0,30	4,503	103	4,606	97	4,703	91	4,794	89	4,883	86	4,969	84	5,053	81	5,134	80	5,214	78	5,292	76	0,0900
0,31			4,730	105	4,835	98	4,933	94	5,027	90	5,117	88	5,205	85	5,290	84	5,374	81	5,455	80	0,0961
0,32					4,961	107	5,068	100	5,168	95	5,263	92	5,355	90	5,445	87	5,532	85	5,617	83	0,1024
0,33							5,195	109	5,304	102	5,406	97	5,503	94	5,597	91	5,688	89	5,777	87	0,1089
0,34									5,433	110	5,543	104	5,647	99	5,740	96	5,842	93	5,935	90	0,1156
0,35											5,674	113	5,787	105	5,892	101	5,993	97	6,090	95	0,1225
0,36													5,919	114	6,033	107	6,140	103	6,243	99	0,1296
0,37															6,168	115	6,283	109	6,392	104	0,1369
0,38																	6,419	118	6,537	110	0,1444
0,39																			6,674	119	0,1521
	,0450		,0481		,0512		,0545		,0578		,0613		,0648		,0685		,0722		,0761		c^2

$\frac{1}{2}\cdot(r+r'')^2$ or $r^2+r''^2$ nearly.

Proportional parts for the Chord.

	122	125	128	131	134	137	140	143	146	149	152	155	158	161	164	167	170	173	176	179	182
1	12	13	13	13	13	14	14	14	15	15	15	16	16	16	16	17	17	17	18	18	18
2	24	25	26	26	27	27	28	29	29	30	30	31	32	32	33	33	34	35	35	36	36
3	37	38	38	39	40	41	42	43	44	45	46	47	47	48	49	50	51	52	53	54	55
4	49	50	51	52	54	55	56	57	58	60	61	62	63	64	66	67	68	69	70	72	73
5	61	63	64	66	67	69	70	72	73	75	76	78	79	81	82	84	85	87	88	90	91
6	73	75	77	79	80	82	84	86	88	89	91	93	95	97	98	100	102	104	106	107	109
7	85	88	90	92	94	96	98	100	102	104	106	109	111	113	115	117	119	121	123	125	127
8	98	100	102	105	107	110	112	114	117	119	122	124	126	129	131	134	136	138	141	143	146
9	110	113	115	118	121	123	126	129	131	134	137	140	142	145	148	150	153	156	158	161	164

In making these successive operations, it is convenient to vary ϱ, by some aliquot part of its value, represented by $\frac{1}{p}.\varrho$; p being an integral number; since by this means we are enabled to deduce any one of the coefficients of ϱ, in the successive operations, from that which immediately precedes it; as in the small tables of the preceding example. Thus if we represent by A_1, A_2, A_3, A_4, the successive values of the term $0{,}87363.\varrho$ of the equation (A), we shall have, when $\varrho=0{,}3$, $A_1=0{,}87363.\varrho=0{,}26209$, in the first operation. In the second hypothesis, this is to be increased by $\frac{1}{40}\cdot A_1=0{,}00655$; by which means it becomes $A_2=0{,}26864$. In the third hypothesis this is increased $\frac{1}{250}\cdot A_2=0{,}00107$; making $A_3=0{,}26971$. In the fourth hypothesis, it is decreased $\frac{1}{2000}\cdot A_3=0{,}00013$, making the final value $A_4=0{,}26958$; as in the preceding table. In like manner if the coefficient of ϱ^2, in any operation be represented by A, and we increase ϱ in the next operation by the quantity $\frac{1}{p}.\varrho$, the value of A will become $A.(1+\frac{2}{p}+\frac{1}{p^2})=A+\frac{2}{p}.A+\frac{1}{2p}.B$; using for brevity $B=\frac{2}{p}.A$. From this formula we obtain the successive values of A, as in the third table of the preceding example. In this way we obtain the values, in the successive operations, with very little additional labor.

TABLE II.

This gives the time T of describing a parabolic arc by a comet, the sum of the extreme radii $r + r''$ being found at the top, and the chord c at the left side of the page.

	Sum of the Radii $r + r'$.														
Chord	0,40		0,41		0,42		0,43		0,44		0,45		0,46		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,184	2	0,186	2	0,188	3	0,191	2	0,193	2	0,195	2	0,197	2	0,0001
0,02	0,368	4	0,372	5	0,377	4	0,381	5	0,386	4	0,390	4	0,394	4	0,0004
0,03	0,551	7	0,558	7	0,565	7	0,572	6	0,578	7	0,585	6	0,591	7	0,0009
0,04	0,735	9	0,744	9	0,753	9	0,762	9	0,771	9	0,780	8	0,788	9	0,0016
0,05	0,919	11	0,930	11	0,941	11	0,952	11	0,963	11	0,974	11	0,985	11	0,0025
0,06	1,102	14	1,116	13	1,129	14	1,143	13	1,156	13	1,169	13	1,182	13	0,0036
0,07	1,285	16	1,301	16	1,317	16	1,333	15	1,348	15	1,363	16	1,379	15	0,0049
0,08	1,468	19	1,487	18	1,505	18	1,523	17	1,540	18	1,558	17	1,575	17	0,0064
0,09	1,651	21	1,672	20	1,692	20	1,712	20	1,732	20	1,752	19	1,771	20	0,0081
0,10	1,833	23	1,856	23	1,879	23	1,902	22	1,924	22	1,946	21	1,967	22	0,0100
0,11	2,016	25	2,041	25	2,066	25	2,091	24	2,115	24	2,139	24	2,163	24	0,0121
0,12	2,198	27	2,225	28	2,253	27	2,280	26	2,306	27	2,333	26	2,359	26	0,0144
0,13	2,379	30	2,409	30	2,439	29	2,468	29	2,497	29	2,526	28	2,554	28	0,0169
0,14	2,560	33	2,593	32	2,625	31	2,656	32	2,688	31	2,719	30	2,749	30	0,0196
0,15	2,741	35	2,776	34	2,810	34	2,844	34	2,878	33	2,911	33	2,944	32	0,0225
0,16	2,921	37	3,958	37	3,995	37	3,032	35	3,067	36	3,103	35	3,138	35	0,0256
0,17	3,101	39	3,140	40	3,180	38	3,218	39	3,257	37	3,294	38	3,332	37	0,0289
0,18	3,280	42	3,322	42	3,364	41	3,405	40	3,445	41	3,486	39	3,525	39	0,0324
0,19	3,458	45	3,503	44	3,547	44	3,591	43	3,634	42	3,676	42	3,718	41	0,0361
0,20	3,636	48	3,684	46	3,730	46	3,776	45	3,821	45	3,866	44	3,910	44	0,0400
0,21	3,814	49	3,863	49	3,912	49	3,961	48	4,009	47	4,056	46	4,102	46	0,0441
0,22	3,990	52	4,042	52	4,094	51	4,145	50	4,195	50	4,245	49	4,294	48	0,0484
0,23	4,166	55	4,221	54	4,275	53	4,328	53	4,381	52	4,433	51	4,484	51	0,0529
0,24	4,341	57	4,398	57	4,455	56	4,511	55	4,566	55	4,621	54	4,675	53	0,0576
0,25	4,514	61	4,575	59	4,634	59	4,693	58	4,751	57	4,808	56	4,864	55	0,0625
0,26	4,687	64	4,751	62	4,813	61	4,874	60	4,934	60	4,994	59	5,053	58	0,0676
0,27	4,859	66	4,925	65	4,990	64	5,054	63	5,117	62	5,179	62	5,241	60	0,0729
0,28	5,030	69	5,099	68	5,167	66	5,233	66	5,299	65	5,364	64	5,428	63	0,0784
0,29	5,200	72	5,272	70	5,342	70	5,412	68	5,480	67	5,547	67	5,614	65	0,0841
0,30	5,368	75	5,443	73	5,516	73	5,589	71	5,660	70	5,730	69	5,799	68	0,0900
0,31	5,535	78	5,613	77	5,690	75	5,765	74	5,839	73	5,912	72	5,984	70	0,0961
0,32	5,700	81	5,781	80	5,861	79	5,940	77	6,017	75	6,092	75	6,167	73	0,1024
0,33	5,864	84	5,948	83	6,031	82	6,113	80	6,193	79	6,272	77	6,349	76	0,1089
0,34	6,025	89	6,114	86	6,200	85	6,285	83	6,368	82	6,450	80	6,530	79	0,1156
0,35	6,185	92	6,277	90	6,367	88	6,455	87	6,542	84	6,626	84	6,710	82	0,1225
0,36	6,342	96	6,438	94	6,532	92	6,624	89	6,713	88	6,801	87	6,888	85	0,1296
0,37	6,496	101	6,597	98	6,695	95	6,790	94	6,884	91	6,975	90	7,065	88	0,1369
0,38	6,647	106	6,753	102	6,855	100	6,955	97	7,052	95	7,147	93	7,240	91	0,1444
0,39	6,793	112	6,905	108	7,013	104	7,117	101	7,218	99	7,317	96	7,413	95	0,1521
0,40	6,933	120	7,053	114	7,167	109	7,276	106	7,382	102	7,484	101	7,585	98	0,1600
0,41			7,194	123	7,317	115	7,432	110	7,542	108	7,650	104	7,754	102	0,1681
0,42					7,459	124	7,583	117	7,700	112	7,812	109	7,921	106	0,1764
0,43							7,727	126	7,853	118	7,971	114	8,085	110	0,1849
0,44									7,998	127	8,125	120	8,245	115	0,1936
0,45											8,272	129	8,401	121	0,2025
	,0800		,0841		,0882		,0925		,0968		,1013		,1058		c^2
	$\frac{1}{2} \cdot (r + r')^2$ or $r^2 + r'^2$ nearly.														

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

Proportional parts for the Chord.

	184	185	186	187	188	189	190	191	192	193	194	195	196	197
1	18	19	19	19	19	19	19	19	19	19	19	20	20	20
2	37	37	37	37	38	38	38	38	38	39	39	39	39	39
3	55	56	56	56	56	57	57	57	58	58	58	59	59	59
4	74	74	74	75	75	76	76	76	77	77	78	78	78	79
5	92	93	93	94	94	95	95	96	96	97	97	98	98	99
6	110	111	112	112	113	113	114	115	115	116	116	117	118	118
7	129	130	130	131	132	132	133	134	134	135	136	137	137	138
8	147	148	149	150	150	151	152	153	154	154	155	156	157	158
9	166	167	167	168	169	170	171	172	173	174	175	176	176	177

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord	Sum of the Radii $r + r''$.																				
	0,47		0,48		0,49		0,50		0,51		0,52		0,53		0,54		0,55		0,56		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,199	2	0,201	2	0,203	3	0,206	2	0,208	2	0,210	2	0,212	2	0,214	2	0,216	2	0,218	1	0,0001
0,02	0,398	5	0,403	4	0,407	4	0,411	4	0,415	4	0,419	4	0,423	4	0,427	4	0,431	4	0,435	4	0,0004
0,03	0,598	6	0,604	6	0,610	6	0,616	7	0,623	6	0,629	6	0,635	6	0,641	6	0,647	5	0,652	6	0,0009
0,04	0,797	8	0,805	9	0,814	8	0,822	8	0,830	8	0,838	8	0,846	8	0,854	8	0,862	8	0,870	8	0,0016
0,05	0,996	10	1,006	11	1,017	10	1,027	10	1,037	11	1,048	10	1,058	10	1,068	9	1,077	10	1,087	10	0,0025
0,06	1,195	12	1,207	13	1,220	12	1,232	13	1,245	12	1,257	12	1,269	12	1,281	12	1,293	11	1,304	12	0,0036
0,07	1,394	14	1,408	15	1,423	15	1,438	14	1,452	14	1,466	14	1,480	14	1,494	14	1,508	14	1,522	13	0,0049
0,08	1,592	17	1,609	17	1,626	16	1,642	17	1,659	16	1,675	16	1,691	16	1,707	16	1,723	15	1,738	16	0,0064
0,09	1,791	19	1,810	19	1,829	18	1,847	19	1,866	18	1,884	18	1,902	18	1,920	18	1,938	17	1,955	18	0,0081
0,10	1,989	21	2,010	21	2,031	21	2,052	20	2,072	21	2,093	20	2,113	20	2,133	20	2,153	19	2,172	20	0,0100
0,11	2,187	23	2,210	23	2,233	23	2,256	23	2,279	22	2,301	22	2,323	22	2,345	22	2,367	22	2,389	21	0,0121
0,12	2,385	25	2,410	25	2,435	25	2,460	25	2,485	25	2,510	24	2,534	24	2,558	24	2,582	23	2,605	23	0,0144
0,13	2,582	28	2,610	27	2,637	27	2,664	27	2,691	27	2,718	26	2,744	26	2,770	26	2,796	25	2,821	26	0,0169
0,14	2,779	30	2,809	30	2,839	29	2,868	29	2,897	28	2,925	29	2,954	28	2,982	28	3,010	27	3,037	27	0,0196
0,15	2,976	32	3,008	32	3,040	31	3,071	31	3,102	31	3,133	30	3,163	30	3,193	30	3,223	30	3,253	29	0,0225
0,16	3,173	34	3,207	34	3,241	33	3,274	33	3,307	33	3,340	33	3,373	32	3,405	32	3,437	31	3,468	31	0,0256
0,17	3,369	36	3,405	36	3,441	36	3,477	35	3,512	35	3,547	35	3,582	34	3,616	34	3,650	33	3,683	33	0,0289
0,18	3,564	39	3,603	38	3,641	38	3,679	37	3,716	38	3,754	36	3,790	36	3,826	36	3,862	36	3,898	37	0,0324
0,19	3,759	41	3,800	41	3,841	40	3,881	39	3,920	40	3,960	38	3,998	39	4,037	38	4,075	37	4,112	38	0,0361
0,20	3,954	43	3,997	43	4,040	42	4,082	42	4,124	41	4,165	41	4,206	41	4,247	40	4,287	40	4,327	39	0,0400
0,21	4,148	46	4,194	45	4,239	44	4,283	44	4,327	44	4,371	43	4,414	42	4,456	42	4,498	42	4,540	42	0,0441
0,22	4,342	48	4,390	47	4,437	47	4,484	46	4,530	46	4,576	45	4,621	44	4,665	45	4,710	44	4,754	43	0,0484
0,23	4,535	50	4,585	50	4,635	49	4,684	48	4,732	48	4,780	47	4,827	47	4,874	46	4,920	46	4,966	46	0,0529
0,24	4,728	52	4,780	52	4,832	51	4,883	51	4,934	50	4,984	49	5,033	49	5,082	49	5,131	48	5,179	47	0,0576
0,25	4,919	55	4,974	54	5,028	54	5,082	53	5,135	52	5,187	52	5,239	51	5,290	51	5,341	50	5,391	49	0,0625
0,26	5,111	57	5,168	56	5,224	56	5,280	55	5,335	55	5,390	54	5,444	53	5,497	53	5,550	52	5,602	52	0,0676
0,27	5,301	60	5,361	59	5,420	58	5,478	57	5,535	57	5,592	56	5,648	56	5,704	55	5,759	54	5,813	54	0,0729
0,28	5,491	62	5,553	61	5,614	61	5,675	59	5,734	60	5,794	58	5,852	58	5,910	57	5,967	57	6,024	56	0,0784
0,29	5,679	65	5,744	64	5,808	63	5,871	62	5,933	61	5,994	61	6,055	60	6,115	60	6,175	58	6,233	59	0,0841
0,30	5,867	67	5,934	67	6,001	65	6,066	65	6,131	64	6,195	63	6,258	62	6,320	62	6,382	61	6,443	60	0,0900
0,31	6,054	70	6,124	69	6,193	68	6,261	67	6,328	66	6,394	65	6,459	65	6,524	64	6,588	63	6,651	63	0,0961
0,32	6,240	73	6,313	71	6,384	70	6,454	70	6,524	69	6,593	67	6,660	67	6,727	67	6,794	65	6,859	65	0,1024
0,33	6,425	75	6,500	74	6,574	73	6,647	72	6,719	71	6,790	70	6,860	70	6,930	68	6,998	68	7,066	67	0,1089
0,34	6,609	78	6,687	76	6,763	76	6,839	75	6,914	73	6,987	73	7,060	72	7,132	71	7,203	70	7,273	69	0,1156
0,35	6,792	80	6,872	80	6,952	78	7,030	77	7,107	76	7,183	75	7,258	74	7,332	74	7,406	72	7,478	72	0,1225
0,36	6,973	83	7,056	83	7,139	81	7,220	79	7,299	79	7,378	78	7,456	76	7,532	76	7,608	75	7,683	74	0,1296
0,37	7,153	86	7,239	85	7,324	84	7,408	83	7,491	81	7,572	80	7,652	80	7,732	78	7,810	77	7,887	76	0,1369
0,38	7,331	90	7,421	88	7,509	86	7,595	86	7,681	84	7,765	83	7,848	82	7,930	80	8,010	80	8,090	79	0,1444
0,39	7,508	93	7,601	91	7,692	89	7,781	89	7,870	87	7,957	85	8,042	85	8,127	83	8,210	82	8,292	81	0,1521
0,40	7,683	96	7,779	94	7,873	93	7,966	91	8,057	90	8,147	88	8,235	87	8,322	86	8,408	85	8,493	84	0,1600
0,41	7,856	99	7,955	98	8,053	96	8,149	94	8,243	93	8,336	91	8,427	90	8,517	89	8,606	87	8,693	86	0,1681
0,42	8,027	103	8,130	101	8,231	99	8,330	98	8,428	96	8,524	94	8,618	93	8,711	91	8,802	90	8,892	89	0,1764
0,43	8,195	107	8,302	105	8,407	103	8,510	101	8,611	99	8,710	97	8,807	96	8,903	94	8,997	93	9,090	91	0,1849
0,44	8,360	112	8,472	109	8,581	107	8,688	104	8,792	102	8,894	100	8,994	99	9,093	97	9,190	96	9,286	94	0,1936
0,45	8,522	117	8,639	113	8,752	111	8,863	108	8,971	106	9,077	103	9,180	102	9,282	100	9,382	99	9,481	97	0,2025
0,50							9,689	136	9,825	129	9,954	124	10,078	120	10,198	118	10,316	115	10,431	113	0,2500
0,55																	11,178	143	11,321	136	0,3025
	,1105		,1152		,1201		,1250		,1301		,1352		,1405		,1458		,1513		,1568		c^2
	$\frac{1}{2} \cdot (r + r')^2$ or $r^2 + r'^2$ nearly.																				

	162	164	166	168	170	172	174	176	178	180	182	184	186	188	190	192	194	196	198	200	202
1	16	16	17	17	17	17	17	18	18	18	18	18	19	19	19	19	19	20	20	20	20
2	32	33	33	34	34	34	35	35	36	36	36	37	37	38	38	38	39	39	40	40	40
3	49	49	50	50	51	52	52	53	53	54	55	55	56	56	57	58	58	59	59	60	61
4	65	66	66	67	68	69	70	70	71	72	73	74	74	75	76	77	78	78	79	80	81
5	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101
6	97	98	100	101	102	103	104	106	107	108	109	110	112	113	114	115	116	118	119	120	121
7	113	115	116	118	119	120	122	123	125	126	127	129	130	132	133	134	136	137	139	140	141
8	130	131	133	134	136	138	139	141	142	144	146	147	149	150	152	154	155	157	158	160	162
9	146	148	149	151	153	155	157	158	160	162	164	166	167	169	171	173	175	176	178	180	182

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	0,57 Days	dif.	0,58 Days	dif.	0,59 Days	dif.	0,60 Days	dif.	0,61 Days	dif.	0,62 Days	dif.	
	Sum of the Radii $r + r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,219	2	0,221	2	0,223	2	0,225	2	0,227	2	0,229	2	0,0001
0,02	0,439	4	0,443	4	0,447	3	0,450	4	0,454	4	0,458	3	0,0004
0,03	0,658	6	0,664	6	0,670	5	0,675	6	0,681	6	0,687	5	0,0009
0,04	0,878	7	0,885	8	0,893	7	0,900	8	0,908	7	0,915	8	0,0016
0,05	1,097	9	1,106	10	1,116	9	1,125	10	1,135	9	1,144	9	0,0025
0,06	1,316	12	1,328	11	1,339	11	1,350	12	1,362	11	1,373	11	0,0036
0,07	1,535	14	1,549	13	1,562	13	1,575	13	1,588	13	1,601	13	0,0049
0,08	1,754	15	1,769	16	1,785	15	1,800	15	1,815	15	1,830	14	0,0064
0,09	1,973	17	1,990	17	2,007	17	2,024	17	2,041	17	2,058	17	0,0081
0,10	2,192	19	2,211	19	2,230	19	2,249	19	2,268	18	2,286	19	0,0100
0,11	2,410	21	2,431	21	2,452	21	2,473	21	2,494	20	2,514	21	0,0121
0,12	2,628	24	2,652	22	2,674	23	2,697	23	2,720	22	2,742	22	0,0144
0,13	2,847	25	2,872	24	2,896	25	2,921	25	2,946	24	2,970	24	0,0169
0,14	3,064	27	3,091	27	3,118	27	3,145	26	3,171	26	3,197	26	0,0196
0,15	3,282	29	3,311	29	3,340	28	3,368	29	3,397	28	3,425	27	0,0225
0,16	3,499	31	3,530	31	3,561	31	3,592	30	3,622	30	3,652	29	0,0256
0,17	3,716	33	3,749	33	3,782	32	3,814	33	3,847	31	3,878	32	0,0289
0,18	3,933	35	3,968	35	4,003	34	4,037	34	4,071	34	4,105	33	0,0324
0,19	4,150	37	4,187	36	4,223	37	4,260	36	4,296	35	4,331	36	0,0361
0,20	4,366	39	4,405	38	4,443	39	4,482	38	4,520	37	4,557	37	0,0400
0,21	4,582	41	4,623	40	4,663	40	4,703	40	4,743	40	4,783	39	0,0441
0,22	4,797	43	4,840	43	4,883	42	4,925	42	4,967	41	5,008	41	0,0484
0,23	5,012	45	5,057	45	5,102	44	5,146	44	5,190	43	5,233	43	0,0529
0,24	5,226	47	5,273	47	5,320	46	5,366	46	5,412	46	5,458	44	0,0576
0,25	5,440	50	5,490	48	5,538	49	5,587	47	5,634	48	5,682	47	0,0625
0,26	5,654	51	5,705	51	5,756	50	5,806	50	5,856	49	5,905	49	0,0676
0,27	5,867	53	5,920	53	5,973	53	6,026	51	6,077	52	6,129	51	0,0729
0,28	6,080	55	6,135	55	6,190	54	6,244	54	6,298	54	6,352	52	0,0784
0,29	6,292	57	6,349	57	6,406	57	6,463	56	6,519	55	6,574	55	0,0841
0,30	6,503	60	6,563	59	6,622	58	6,680	58	6,738	58	6,796	57	0,0900
0,31	6,714	62	6,776	61	6,837	61	6,898	60	6,958	59	7,017	59	0,0961
0,32	6,924	64	6,988	63	7,051	63	7,114	62	7,176	62	7,238	61	0,1024
0,33	7,133	67	7,200	65	7,265	65	7,330	64	7,394	64	7,458	63	0,1089
0,34	7,342	68	7,410	68	7,478	67	7,545	67	7,612	66	7,678	65	0,1156
0,35	7,550	71	7,621	70	7,691	69	7,760	69	7,829	68	7,897	67	0,1225
0,36	7,757	73	7,830	73	7,903	71	7,974	71	8,045	70	8,115	69	0,1296
0,37	7,963	76	8,039	74	8,113	74	8,187	73	8,260	73	8,333	71	0,1369
0,38	8,169	78	8,247	77	8,324	76	8,400	75	8,475	75	8,550	73	0,1444
0,39	8,373	81	8,454	79	8,533	78	8,611	78	8,689	77	8,766	76	0,1521
0,40	8,577	83	8,660	81	8,741	81	8,822	80	8,902	79	8,981	78	0,1600
0,41	8,779	86	8,865	84	8,949	83	9,032	82	9,114	82	9,196	80	0,1681
0,42	8,981	88	9,069	86	9,155	86	9,241	85	9,326	83	9,409	83	0,1764
0,43	9,181	91	9,272	89	9,361	88	9,449	87	9,536	86	9,622	85	0,1849
0,44	9,380	93	9,473	92	9,565	91	9,656	89	9,745	89	9,834	87	0,1936
0,45	9,578	96	9,674	94	9,768	94	9,862	92	9,954	91	10,045	89	0,2025
0,50	10,544	111	10,655	109	10,764	107	10,871	106	10,977	104	11,081	103	0,2500
0,55	11,457	131	11,588	127	11,715	125	11,840	122	11,962	120	12,082	118	0,3025
0,60							12,736	150	12,886	143	13,029	137	0,3600
	,1625		,1682		,1741		,1800		,1861		,1922		c^2

$\frac{1}{2} . (r + r'')^2$ or $r^2 + r''^2$ nearly.

	204	206	208	210	212	214	216	218	220	222	224	226	228	230
1	20	21	21	21	21	21	22	22	22	22	22	23	23	23
2	41	41	42	42	42	43	43	44	44	44	45	45	46	46
3	61	62	62	63	64	64	65	65	66	67	67	68	68	69
4	82	82	83	84	85	86	86	87	88	89	90	90	91	92
5	102	103	104	105	106	107	108	109	110	111	112	113	114	115
6	122	124	125	126	127	128	130	131	132	133	134	136	137	138
7	143	144	146	147	148	150	151	153	154	155	157	158	160	161
8	163	165	166	168	170	171	173	174	176	178	179	181	182	184
9	184	185	187	189	191	193	194	196	198	200	202	203	205	207

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	0,63		0,64		0,65		0,66		0,67		0,68		0,69		0,70		0,71		0,72		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Dasy	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,231	2	0,233	1	0,234	2	0,236	2	0,238	2	0,240	1	0,241	2	0,243	2	0,245	2	0,247	1	0,0001
0,02	0,461	4	0,465	4	0,469	3	0,472	4	0,476	3	0,479	4	0,483	3	0,486	4	0,490	3	0,493	4	0,0004
0,03	0,692	6	0,698	5	0,703	5	0,708	6	0,714	5	0,719	5	0,724	6	0,730	5	0,735	5	0,740	5	0,0009
0,04	0,923	7	0,930	7	0,937	7	0,944	8	0,952	7	0,959	7	0,966	7	0,973	7	0,980	6	0,986	7	0,0016
0,05	1,153	9	1,162	9	1,171	9	1,180	9	1,189	9	1,198	9	1,207	9	1,216	8	1,224	9	1,233	8	0,0025
0,06	1,384	11	1,395	11	1,406	10	1,416	11	1,427	11	1,438	10	1,448	11	1,459	10	1,469	10	1,479	11	0,0036
0,07	1,614	13	1,627	13	1,640	12	1,652	13	1,665	12	1,677	12	1,689	13	1,702	12	1,714	12	1,726	12	0,0049
0,08	1,844	15	1,859	15	1,874	14	1,888	14	1,902	14	1,916	14	1,930	14	1,944	14	1,958	14	1,972	14	0,0064
0,09	2,075	16	2,091	16	2,107	17	2,124	16	2,140	16	2,156	15	2,171	16	2,187	16	2,203	15	2,218	16	0,0081
0,10	2,305	18	2,323	18	2,341	18	2,359	18	2,377	18	2,395	17	2,412	18	2,430	17	2,447	17	2,464	17	0,0100
0,11	2,535	20	2,555	20	2,575	19	2,594	20	2,614	20	2,634	19	2,653	19	2,672	19	2,691	19	2,710	19	0,0121
0,12	2,764	22	2,786	22	2,808	22	2,830	21	2,851	21	2,872	22	2,894	21	2,915	20	2,935	21	2,956	21	0,0144
0,13	2,994	24	3,018	23	3,041	24	3,065	23	3,088	23	3,111	23	3,134	23	3,157	22	3,179	23	3,202	22	0,0169
0,14	3,223	26	3,249	25	3,274	26	3,300	25	3,325	25	3,350	24	3,374	25	3,399	24	3,423	24	3,447	24	0,0196
0,15	3,452	28	3,480	27	3,507	27	3,534	27	3,561	27	3,588	26	3,614	27	3,641	26	3,667	26	3,693	26	0,0225
0,16	3,681	30	3,711	29	3,740	29	3,769	29	3,798	28	3,826	28	3,854	28	3,882	28	3,910	28	3,938	27	0,0256
0,17	3,910	31	3,941	31	3,972	31	4,003	31	4,034	30	4,064	30	4,094	30	4,124	30	4,154	29	4,183	29	0,0289
0,18	4,138	34	4,172	32	4,204	33	4,237	32	4,269	33	4,302	31	4,333	32	4,365	32	4,397	31	4,428	31	0,0324
0,19	4,367	35	4,402	34	4,436	35	4,471	34	4,505	34	4,539	34	4,573	33	4,606	33	4,639	33	4,672	33	0,0361
0,20	4,594	37	4,631	37	4,668	36	4,704	36	4,740	36	4,776	36	4,812	35	4,847	35	4,882	35	4,917	34	0,0400
0,21	4,822	39	4,861	38	4,899	39	4,938	37	4,975	38	5,013	37	5,050	37	5,087	37	5,124	37	5,161	36	0,0441
0,22	5,049	41	5,090	40	5,130	40	5,170	40	5,210	40	5,250	39	5,289	39	5,328	38	5,366	38	5,404	38	0,0484
0,23	5,276	43	5,319	42	5,361	42	5,403	42	5,445	41	5,486	41	5,527	41	5,568	40	5,608	40	5,648	40	0,0529
0,24	5,502	45	5,547	44	5,591	44	5,635	44	5,679	43	5,722	43	5,765	42	5,807	42	5,849	42	5,891	42	0,0576
0,25	5,729	46	5,775	46	5,821	46	5,867	45	5,912	45	5,957	45	6,002	45	6,047	43	6,090	44	6,134	44	0,0625
0,26	5,954	49	6,003	48	6,051	48	6,099	47	6,146	47	6,193	46	6,239	46	6,285	46	6,331	46	6,377	45	0,0676
0,27	6,180	50	6,230	50	6,280	50	6,330	49	6,379	49	6,428	48	6,476	48	6,524	48	6,572	47	6,619	47	0,0729
0,28	6,404	53	6,457	52	6,509	51	6,560	51	6,611	51	6,662	50	6,712	50	6,762	50	6,812	49	6,861	49	0,0784
0,29	6,629	54	6,683	54	6,737	54	6,791	53	6,844	52	6,896	52	6,948	52	7,000	52	7,052	50	7,102	51	0,0841
0,30	6,853	56	6,909	56	6,965	55	7,020	55	7,075	55	7,130	54	7,184	54	7,238	53	7,291	53	7,344	52	0,0900
0,31	7,076	59	7,135	57	7,192	58	7,250	57	7,307	56	7,363	56	7,419	56	7,475	55	7,530	54	7,584	55	0,0961
0,32	7,299	60	7,359	60	7,419	60	7,479	59	7,538	58	7,596	58	7,654	57	7,711	57	7,768	57	7,825	56	0,1024
0,33	7,521	63	7,584	62	7,646	61	7,707	61	7,768	60	7,828	60	7,888	59	7,947	59	8,006	59	8,065	58	0,1089
0,34	7,743	64	7,807	64	7,871	64	7,935	63	7,998	62	8,060	62	8,122	61	8,183	61	8,244	60	8,304	60	0,1156
0,35	7,964	67	8,031	66	8,097	65	8,162	65	8,227	64	8,291	64	8,355	63	8,418	63	8,481	62	8,543	62	0,1225
0,36	8,184	69	8,253	68	8,321	68	8,389	67	8,456	66	8,522	66	8,588	65	8,653	64	8,717	65	8,782	63	0,1296
0,37	8,404	71	8,475	70	8,545	70	8,615	69	8,684	68	8,752	68	8,820	67	8,887	67	8,954	66	9,020	65	0,1369
0,38	8,623	73	8,696	73	8,769	71	8,840	71	8,911	71	8,982	69	9,051	69	9,120	69	9,189	68	9,257	67	0,1444
0,39	8,842	75	8,917	74	8,991	74	9,065	73	9,138	73	9,211	71	9,282	71	9,353	71	9,424	70	9,494	69	0,1521
0,40	9,059	78	9,137	76	9,213	76	9,289	75	9,364	75	9,439	74	9,513	73	9,586	72	9,658	72	9,730	71	0,1600
0,41	9,276	80	9,356	79	9,435	78	9,513	77	9,590	76	9,666	76	9,742	75	9,817	75	9,892	74	9,966	73	0,1681
0,42	9,492	82	9,574	81	9,655	80	9,735	80	9,815	78	9,893	78	9,971	78	10,049	76	10,125	76	10,201	75	0,1764
0,43	9,707	84	9,791	84	9,875	82	9,957	82	10,039	80	10,119	81	10,200	79	10,279	78	10,357	78	10,435	78	0,1849
0,44	9,921	87	10,008	85	10,093	85	10,178	84	10,262	83	10,345	82	10,427	82	10,509	80	10,589	80	10,669	79	0,1936
0,45	10,134	89	10,223	88	10,311	87	10,398	86	10,484	85	10,569	85	10,654	83	10,737	83	10,820	82	10,902	82	0,2025
0,50	11,184	101	11,285	101	11,386	99	11,485	97	11,582	97	11,679	96	11,775	95	11,870	93	11,963	93	12,056	92	0,2500
0,55	12,200	116	12,316	114	12,430	112	12,542	111	12,653	110	12,763	108	12,871	107	12,978	105	13,083	105	13,188	103	0,3025
0,60	13,166	135	13,301	131	13,432	129	13,561	126	13,687	125	13,812	122	13,934	121	14,055	118	14,173	118	14,291	116	0,3600
0,65					14,361	156	14,517	149	14,666	145	14,811	140	14,951	138	15,089	135	15,224	133	15,357	130	0,4225
0,70															16,049	163	16,212	155	16,367	151	0,4900
	,1985		,2048		,2113		,2178		,2245		,2312		,2381		,2450		,2521		,2592		c^2

Sum of the radii $r + r''$.

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	213	215	217	219	221	223	225	227	229	231	232	233	234	235	236	237	238	239	240	241	242	243
1	21	22	22	22	22	22	23	23	23	23	23	23	23	24	24	24	24	24	24	24	24	24
2	43	43	43	44	44	45	45	45	46	46	46	47	47	47	47	47	48	48	48	48	48	49
3	64	65	65	66	66	67	68	68	69	69	70	70	70	71	71	71	71	72	72	72	73	73
4	85	86	87	88	88	89	90	91	92	92	93	93	94	94	94	95	95	96	96	96	97	97
5	107	108	109	110	111	112	113	114	115	116	116	117	117	118	118	119	119	120	120	121	121	122
6	128	129	130	131	133	134	135	136	137	139	139	140	140	141	142	142	143	143	144	145	145	146
7	149	151	152	153	155	156	158	159	160	162	162	163	164	165	165	166	167	167	168	169	169	170
8	170	172	174	175	177	178	180	182	183	185	186	186	187	188	189	190	190	191	192	193	194	194
9	192	194	195	197	199	201	203	204	206	208	209	210	211	212	212	213	214	215	216	217	218	219

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord c.	0,73 Days	dif.	0,74 Days	dif.	0,75 Days	dif.	0,76 Days	dif.	0,77 Days	dif.	0,78 Days	dif.	
	Sum of the Radii $r+r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,248	2	0,250	2	0,252	1	0,253	2	0,255	2	0,257	1	0,0001
0,02	0,497	3	0,500	3	0,503	4	0,507	3	0,510	3	0,513	4	0,0004
0,03	0,745	5	0,750	5	0,755	5	0,760	5	0,765	5	0,770	5	0,0009
0,04	0,993	7	1,000	7	1,007	6	1,013	7	1,020	7	1,027	6	0,0016
0,05	1,241	9	1,250	8	1,258	9	1,267	8	1,275	8	1,283	9	0,0025
0,06	1,490	10	1,500	10	1,510	10	1,520	10	1,530	10	1,540	10	0,0036
0,07	1,738	12	1,750	11	1,761	12	1,773	12	1,785	11	1,796	12	0,0049
0,08	1,986	13	1,999	14	2,013	13	2,026	14	2,040	13	2,053	13	0,0064
0,09	2,234	15	2,249	15	2,264	15	2,279	15	2,294	15	2,309	15	0,0081
0,10	2,481	17	2,498	17	2,515	17	2,532	17	2,549	16	2,565	17	0,0100
0,11	2,729	19	2,748	18	2,766	19	2,785	18	2,803	18	2,821	19	0,0121
0,12	2,977	20	2,997	20	3,017	21	3,038	20	3,058	19	3,077	20	0,0144
0,13	3,224	22	3,246	22	3,268	22	3,290	22	3,312	21	3,333	22	0,0169
0,14	3,471	24	3,495	24	3,519	23	3,542	24	3,566	23	3,589	23	0,0196
0,15	3,719	25	3,744	25	3,769	26	3,795	25	3,820	25	3,845	24	0,0225
0,16	3,965	28	3,993	27	4,020	27	4,047	26	4,073	27	4,100	26	0,0256
0,17	4,212	29	4,241	29	4,270	29	4,299	28	4,327	28	4,355	28	0,0289
0,18	4,459	30	4,489	31	4,520	30	4,550	30	4,580	30	4,610	30	0,0324
0,19	4,705	32	4,737	33	4,770	32	4,802	32	4,834	31	4,865	32	0,0361
0,20	4,951	34	4,985	34	5,019	34	5,053	34	5,087	33	5,120	33	0,0400
0,21	5,197	36	5,233	36	5,269	35	5,304	35	5,339	35	5,374	35	0,0441
0,22	5,442	38	5,480	38	5,518	37	5,555	37	5,592	37	5,629	36	0,0484
0,23	5,688	39	5,727	39	5,766	39	5,805	39	5,844	38	5,882	39	0,0529
0,24	5,933	41	5,974	41	6,015	41	6,056	40	6,096	40	6,136	40	0,0576
0,25	6,178	43	6,221	42	6,263	43	6,306	42	6,348	42	6,390	41	0,0625
0,26	6,422	45	6,467	44	6,511	44	6,555	44	6,599	44	6,643	43	0,0676
0,27	6,666	47	6,713	46	6,759	46	6,805	45	6,850	46	6,896	45	0,0729
0,28	6,910	48	6,958	48	7,006	48	7,054	47	7,101	47	7,148	47	0,0784
0,29	7,153	50	7,203	50	7,253	50	7,303	49	7,352	48	7,400	49	0,0841
0,30	7,396	52	7,448	52	7,500	51	7,551	51	7,602	50	7,652	50	0,0900
0,31	7,639	53	7,692	54	7,746	53	7,799	53	7,852	52	7,904	52	0,0961
0,32	7,881	55	7,936	56	7,992	55	8,047	54	8,101	54	8,155	54	0,1024
0,33	8,123	57	8,180	57	8,237	57	8,294	56	8,350	56	8,406	55	0,1089
0,34	8,364	59	8,423	59	8,482	59	8,541	58	8,599	57	8,656	57	0,1156
0,35	8,605	61	8,666	61	8,727	60	8,787	60	8,847	59	8,906	59	0,1225
0,36	8,845	63	8,908	63	8,971	62	9,033	62	9,095	61	9,156	61	0,1296
0,37	9,085	65	9,150	64	9,214	64	9,278	64	9,342	63	9,405	63	0,1369
0,38	9,324	67	9,391	67	9,458	65	9,523	66	9,589	65	9,654	64	0,1444
0,39	9,563	69	9,632	68	9,700	68	9,768	67	9,835	67	9,902	66	0,1521
0,40	9,801	71	9,872	70	9,942	70	10,012	69	10,081	69	10,150	68	0,1600
0,41	10,039	73	10,112	72	10,184	71	10,255	71	10,326	71	10,397	70	0,1681
0,42	10,276	75	10,351	74	10,425	73	10,498	73	10,571	72	10,643	72	0,1764
0,43	10,513	76	10,589	76	10,665	76	10,741	74	10,815	75	10,890	73	0,1849
0,44	10,748	79	10,827	78	10,905	77	10,982	77	11,059	76	11,135	76	0,1936
0,45	10,984	80	11,064	80	11,144	79	11,223	79	11,302	78	11,380	77	0,2025
0,50	12,148	91	12,239	90	12,329	90	12,419	88	12,507	88	12,595	87	0,2500
0,55	13,291	102	13,393	101	13,494	101	13,595	99	13,694	98	13,792	98	0,3025
0,60	14,407	114	14,521	113	14,634	112	14,746	111	14,857	110	14,967	108	0,3600
0,65	15,487	129	15,616	127	15,743	125	15,868	124	15,992	122	16,114	121	0,4225
0,70	16,518	147	16,665	144	16,809	141	16,950	139	17,089	136	17,225	135	0,4900
0,75					17,799	169	17,968	161	18,129	157	18,286	153	0,5625
	,2665		,2738		,2813		,2888		,2965		,3042		c^2

$\frac{1}{2}\,.\,(r + r'')^2$ or $r^2 + r''^2$ nearly.

	244	245	246	247	248	249	250	251	252	253	254	255	256	257
1	24	25	25	25	25	25	25	25	25	25	25	26	26	26
2	49	49	49	49	50	50	50	50	50	51	51	51	51	51
3	73	74	74	74	74	75	75	75	76	76	76	77	77	77
4	98	98	98	99	99	100	100	100	101	101	102	102	102	103
5	122	123	123	124	124	125	125	126	126	127	127	128	128	129
6	146	147	148	148	149	149	150	151	151	152	152	153	154	154
7	171	172	172	173	174	174	175	176	176	177	178	179	179	180
8	195	196	197	198	198	199	200	201	202	202	203	204	205	206
9	220	221	221	222	223	224	225	226	227	228	229	230	230	231

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord	Sum of the radii $r+r''$.																				
	0,79		0,80		0,81		0,82		0,83		0,84		0,85		0,86		0,87		0,88		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,258	2	0,260	2	0,262	1	0,263	2	0,265	1	0,266	2	0,268	2	0,270	1	0,271	2	0,273	1	0,0001
0,02	0,517	3	0,520	3	0,523	3	0,526	4	0,530	3	0,533	3	0,536	3	0,539	3	0,542	3	0,545	3	0,0004
0,03	0,775	5	0,780	5	0,785	5	0,790	4	0,794	5	0,799	5	0,804	5	0,809	4	0,813	5	0,818	5	0,0009
0,04	1,033	7	1,040	6	1,046	7	1,053	6	1,059	6	1,065	7	1,072	6	1,078	6	1,084	7	1,091	6	0,0016
0,05	1,292	8	1,300	8	1,308	8	1,316	8	1,324	8	1,332	8	1,340	8	1,348	7	1,355	8	1,363	8	0,0025
0,06	1,550	9	1,559	10	1,569	10	1,579	9	1,588	10	1,598	10	1,608	9	1,617	9	1,626	10	1,636	9	0,0036
0,07	1,808	11	1,819	12	1,831	11	1,842	11	1,853	11	1,864	11	1,875	11	1,886	11	1,897	11	1,908	11	0,0049
0,08	2,066	13	2,079	13	2,092	13	2,105	13	2,118	12	2,130	13	2,143	13	2,156	12	2,168	13	2,181	12	0,0064
0,09	2,324	15	2,339	14	2,353	15	2,368	14	2,382	14	2,396	15	2,411	14	2,425	14	2,439	14	2,453	14	0,0081
0,10	2,582	16	2,598	16	2,614	16	2,630	16	2,646	16	2,662	16	2,678	16	2,694	16	2,710	15	2,725	16	0,0100
0,11	2,840	17	2,857	18	2,875	18	2,893	18	2,911	17	2,928	18	2,946	17	2,963	17	2,980	17	2,997	17	0,0121
0,12	3,097	20	3,117	19	3,136	20	3,156	19	3,175	19	3,194	19	3,213	19	3,232	19	3,251	18	3,269	19	0,0144
0,13	3,355	21	3,376	21	3,397	21	3,418	21	3,439	21	3,460	20	3,480	21	3,501	20	3,521	20	3,541	21	0,0169
0,14	3,612	23	3,635	23	3,658	22	3,680	23	3,703	22	3,725	22	3,747	23	3,770	21	3,791	22	3,813	22	0,0196
0,15	3,869	25	3,894	24	3,918	25	3,943	24	3,967	24	3,991	23	4,014	24	4,038	24	4,062	23	4,085	23	0,0225
0,16	4,126	27	4,153	26	4,179	26	4,205	25	4,230	26	4,256	25	4,281	26	4,307	25	4,332	25	4,357	24	0,0256
0,17	4,383	28	4,411	28	4,439	27	4,466	28	4,494	27	4,521	27	4,548	27	4,575	27	4,602	26	4,628	26	0,0289
0,18	4,640	30	4,670	29	4,699	29	4,728	29	4,757	29	4,786	28	4,814	29	4,843	28	4,871	28	4,899	28	0,0324
0,19	4,897	31	4,928	31	4,959	31	4,990	30	5,020	31	5,051	30	5,081	30	5,111	30	5,141	29	5,170	30	0,0361
0,20	5,153	33	5,186	32	5,218	33	5,251	32	5,283	32	5,315	32	5,347	32	5,379	31	5,410	31	5,441	32	0,0400
0,21	5,409	35	5,444	34	5,478	34	5,512	34	5,546	34	5,580	33	5,613	33	5,646	33	5,679	33	5,712	33	0,0441
0,22	5,665	36	5,701	36	5,737	36	5,773	35	5,808	36	5,844	35	5,879	35	5,914	34	5,948	35	5,983	34	0,0484
0,23	5,921	38	5,959	37	5,996	38	6,034	37	6,071	37	6,108	36	6,144	37	6,181	36	6,217	36	6,253	36	0,0529
0,24	6,176	40	6,216	39	6,255	39	6,294	39	6,333	38	6,371	39	6,410	38	6,448	38	6,486	37	6,523	38	0,0576
0,25	6,431	41	6,472	41	6,513	41	6,554	41	6,595	40	6,635	40	6,675	40	6,715	39	6,754	39	6,793	39	0,0625
0,26	6,686	43	6,729	43	6,772	42	6,814	42	6,856	42	6,898	42	6,940	41	6,981	41	7,022	41	7,063	41	0,0676
0,27	6,941	44	6,985	45	7,030	44	7,074	44	7,118	43	7,161	43	7,204	43	7,247	43	7,290	43	7,333	42	0,0729
0,28	7,195	46	7,241	46	7,287	46	7,333	46	7,379	45	7,424	45	7,469	44	7,513	45	7,558	44	7,602	44	0,0784
0,29	7,449	48	7,497	48	7,545	47	7,592	47	7,639	47	7,686	47	7,733	46	7,779	46	7,825	46	7,871	45	0,0841
0,30	7,702	50	7,752	50	7,802	49	7,851	49	7,900	48	7,948	49	7,997	47	8,044	48	8,092	47	8,139	47	0,0900
0,31	7,956	51	8,007	52	8,059	50	8,109	51	8,160	50	8,210	50	8,260	50	8,310	49	8,359	49	8,408	49	0,0961
0,32	8,209	53	8,262	53	8,315	53	8,368	52	8,420	52	8,472	51	8,523	51	8,574	51	8,625	51	8,676	50	0,1024
0,33	8,461	55	8,516	55	8,571	54	8,625	54	8,679	54	8,733	53	8,786	53	8,839	53	8,892	52	8,944	52	0,1089
0,34	8,713	57	8,770	57	8,827	56	8,883	55	8,938	56	8,994	55	9,049	54	9,103	54	9,157	54	9,211	54	0,1156
0,35	8,965	59	9,024	58	9,082	58	9,140	57	9,197	57	9,254	57	9,311	56	9,367	56	9,423	55	9,478	56	0,1225
0,36	9,217	60	9,277	60	9,337	59	9,396	59	9,455	59	9,514	58	9,572	58	9,630	58	9,688	57	9,745	57	0,1296
0,37	9,468	62	9,530	61	9,591	62	9,653	60	9,713	61	9,774	60	9,834	60	9,894	59	9,953	59	10,012	58	0,1369
0,38	9,718	64	9,782	63	9,845	63	9,908	63	9,971	62	10,033	62	10,095	61	10,156	61	10,217	61	10,278	60	0,1444
0,39	9,968	66	10,034	65	10,099	65	10,164	64	10,228	64	10,292	63	10,355	64	10,419	62	10,481	62	10,543	62	0,1521
0,40	10,218	67	10,285	67	10,352	67	10,419	66	10,485	65	10,550	66	10,616	64	10,680	65	10,745	64	10,809	63	0,1600
0,41	10,467	69	10,536	69	10,605	68	10,673	68	10,741	67	10,808	67	10,875	67	10,942	66	11,008	66	11,074	65	0,1681
0,42	10,715	71	10,786	71	10,857	70	10,927	70	10,997	69	11,066	69	11,135	68	11,203	68	11,271	67	11,338	67	0,1764
0,43	10,963	73	11,036	73	11,109	72	11,181	71	11,252	71	11,323	71	11,394	70	11,464	69	11,533	69	11,602	69	0,1849
0,44	11,211	74	11,285	75	11,360	74	11,434	73	11,507	73	11,580	72	11,652	72	11,724	71	11,795	71	11,866	70	0,1936
0,45	11,457	77	11,534	76	11,610	76	11,686	75	11,761	75	11,836	74	11,910	73	11,983	73	12,056	73	12,129	72	0,2025
0,50	12,682	87	12,769	86	12,855	85	12,940	84	13,024	84	13,108	83	13,191	83	13,274	82	13,356	81	13,437	81	0,2500
0,55	13,890	96	13,986	96	14,082	95	14,177	94	14,271	94	14,365	92	14,457	92	14,549	92	14,641	90	14,731	90	0,3025
0,60	15,075	108	15,183	106	15,289	106	15,395	104	15,499	104	15,603	103	15,706	102	15,808	101	15,909	100	16,009	100	0,3600
0,65	16,235	119	16,354	118	16,472	117	16,589	116	16,705	114	16,819	114	16,933	112	17,045	112	17,157	110	17,267	110	0,4225
0,70	17,360	133	17,493	131	17,624	130	17,754	128	17,882	126	18,008	126	18,134	124	18,258	123	18,381	121	18,502	121	0,4900
0,75	18,439	149	18,588	148	18,736	144	18,880	143	19,023	140	19,163	139	19,302	137	19,439	136	19,575	134	19,709	132	0,5625
0,80			19,609	174	19,783	167	19,950	162	20,112	159	20,271	156	20,427	152	20,579	151	20,730	148	20,878	147	0,6400
0,85													21,475	180	21,655	173	21,828	168	21,996	164	0,7225
	,3121		,3200		,3281		,3362		,3445		,3528		,3613		,3698		,3785		,3872		c^2
	$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.																				

	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260	261	262	263	264	265	266	267	268
1	25	25	25	25	25	25	25	25	25	26	26	26	26	26	26	26	26	26	26	27	27	27	27
2	49	49	50	50	50	50	50	51	51	51	51	51	52	52	52	52	52	53	53	53	53	53	54
3	74	74	74	75	75	75	76	76	76	77	77	77	77	78	78	78	79	79	79	80	80	80	80
4	98	99	99	100	100	100	101	101	102	102	102	103	103	104	104	104	105	105	106	106	106	107	107
5	123	124	124	125	125	126	126	127	127	128	128	129	129	130	130	131	131	132	132	133	133	134	134
6	148	148	149	149	150	151	151	152	152	153	154	154	155	155	156	157	157	158	158	159	160	160	161
7	172	173	174	174	175	176	176	177	178	179	179	180	181	181	182	183	183	184	185	186	186	187	188
8	197	198	198	199	200	201	202	202	203	204	205	206	206	207	208	209	210	210	211	212	213	214	214
9	221	222	223	224	225	226	227	228	229	230	230	231	232	233	234	235	236	237	238	239	239	240	241

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r + r''$. 0,89 Days	dif.	0,90 Days	dif.	0,91 Days	dif.	0,92 Days	dif.	0,93 Days	dif.	0,94 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,274	2	0,276	1	0,277	2	0,279	1	0,280	2	0,282	1	0,0001
0,02	0,548	3	0,551	4	0,555	3	0,558	3	0,561	3	0,564	3	0,0004
0,03	0,825	4	0,827	5	0,832	4	0,836	5	0,841	4	0,845	5	0,0009
0,04	1,097	6	1,103	6	1,109	6	1,115	6	1,121	6	1,127	6	0,0016
0,05	1,371	8	1,379	7	1,386	8	1,394	7	1,401	8	1,409	7	0,0025
0,06	1,645	9	1,654	9	1,663	9	1,672	10	1,682	9	1,691	9	0,0036
0,07	1,919	11	1,930	10	1,940	11	1,951	11	1,962	10	1,972	11	0,0049
0,08	2,193	12	2,205	12	2,217	13	2,230	12	2,242	12	2,254	12	0,0064
0,09	2,467	14	2,481	13	2,494	14	2,508	14	2,522	13	2,535	14	0,0081
0,10	2,741	15	2,756	15	2,771	16	2,787	15	2,802	15	2,817	15	0,0100
0,11	3,014	17	3,031	17	3,048	17	3,065	17	3,082	16	3,098	17	0,0121
0,12	3,288	19	3,307	18	3,325	18	3,343	18	3,361	18	3,379	18	0,0144
0,13	3,562	20	3,582	19	3,601	20	3,621	20	3,641	20	3,661	19	0,0169
0,14	3,835	22	3,857	21	3,878	21	3,899	22	3,921	21	3,942	21	0,0196
0,15	4,108	23	4,131	23	4,154	23	4,177	23	4,200	23	4,223	22	0,0225
0,16	4,381	25	4,406	25	4,431	24	4,455	24	4,479	24	4,503	24	0,0256
0,17	4,654	27	4,681	26	4,707	26	4,733	25	4,758	26	4,784	26	0,0289
0,18	4,927	28	4,955	28	4,983	27	5,010	28	5,038	27	5,065	27	0,0324
0,19	5,200	29	5,229	30	5,259	29	5,288	28	5,316	29	5,345	29	0,0361
0,20	5,473	30	5,503	31	5,534	31	5,565	30	5,595	30	5,625	30	0,0400
0,21	5,745	32	5,777	33	5,810	32	5,842	32	5,874	32	5,906	31	0,0441
0,22	6,017	34	6,051	34	6,085	34	6,119	33	6,152	33	6,185	34	0,0484
0,23	6,289	36	6,325	35	6,360	35	6,395	35	6,430	35	6,465	35	0,0529
0,24	6,561	37	6,598	37	6,635	37	6,672	36	6,708	37	6,745	36	0,0576
0,25	6,832	39	6,871	39	6,910	38	6,948	38	6,986	38	7,024	38	0,0625
0,26	7,104	40	7,144	40	7,184	40	7,224	40	7,264	39	7,303	40	0,0676
0,27	7,375	42	7,417	41	7,458	42	7,500	41	7,541	41	7,582	41	0,0729
0,28	7,646	43	7,689	43	7,732	44	7,776	42	7,818	43	7,861	42	0,0784
0,29	7,916	45	7,961	45	8,006	45	8,051	44	8,095	44	8,139	44	0,0841
0,30	8,186	47	8,233	47	8,280	46	8,326	46	8,372	46	8,418	45	0,0900
0,31	8,457	48	8,505	48	8,553	48	8,601	47	8,648	48	8,696	47	0,0961
0,32	8,726	50	8,776	50	8,826	49	8,875	49	8,924	49	8,973	49	0,1024
0,33	8,996	51	9,047	52	9,099	51	9,150	50	9,200	51	9,251	50	0,1089
0,34	9,265	53	9,318	53	9,371	53	9,424	52	9,476	52	9,528	52	0,1156
0,35	9,534	54	9,588	55	9,643	54	9,697	54	9,751	54	9,805	53	0,1225
0,36	9,802	57	9,859	56	9,915	56	9,971	55	10,026	55	10,081	55	0,1296
0,37	10,070	58	10,128	58	10,186	58	10,244	57	10,301	56	10,357	57	0,1369
0,38	10,338	60	10,398	59	10,457	59	10,516	59	10,575	58	10,633	58	0,1444
0,39	10,605	62	10,667	61	10,728	61	10,789	60	10,849	60	10,909	60	0,1521
0,40	10,872	63	10,935	63	10,998	63	11,061	62	11,123	61	11,184	61	0,1600
0,41	11,139	65	11,204	64	11,268	64	11,332	64	11,396	63	11,459	63	0,1681
0,42	11,405	67	11,472	66	11,538	65	11,603	66	11,669	64	11,733	65	0,1764
0,43	11,671	68	11,739	68	11,807	67	11,874	67	11,941	67	12,008	66	0,1849
0,44	11,936	70	12,006	69	12,075	69	12,144	69	12,213	68	12,281	68	0,1936
0,45	12,201	72	12,273	71	12,344	70	12,414	71	12,485	70	12,555	69	0,2025
0,50	13,518	80	13,598	80	13,678	79	13,757	79	13,836	78	13,914	78	0,2500
0,55	14,821	90	14,911	88	14,999	88	15,087	88	15,175	87	15,262	86	0,3025
0,60	16,109	98	16,207	98	16,305	98	16,403	96	16,499	96	16,595	96	0,3600
0,65	17,377	109	17,486	107	17,593	107	17,700	107	17,807	105	17,912	105	0,4225
0,70	18,623	119	18,742	118	18,860	118	18,978	116	19,094	115	19,209	115	0,4900
0,75	19,841	131	19,972	130	20,102	128	20,230	128	20,358	126	20,484	125	0,5625
0,80	21,025	144	21,169	143	21,312	141	21,453	139	21,592	138	21,730	137	0,6400
0,85	22,160	161	22,321	159	22,480	156	22,636	154	22,790	152	22,942	149	0,7225
0,90			23,398	185	23,583	179	23,762	173	23,935	170	24,105	166	0,8100
	,3961		,4050		,4141		,4232		,4325		,4418		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	269	270	271	272	273	274	275	276	277	278	279	280	281	282
1	27	27	27	27	27	27	28	28	28	28	28	28	28	28
2	54	54	54	54	55	55	55	55	55	56	56	56	56	56
3	81	81	81	82	82	82	83	83	83	83	84	84	84	85
4	108	108	108	109	109	110	110	110	111	111	112	112	112	113
5	135	135	136	136	137	137	138	138	139	139	140	140	141	141
6	161	162	163	163	164	164	165	166	166	167	167	168	169	169
7	188	189	190	190	191	192	193	193	194	195	195	196	197	197
8	215	216	217	218	218	219	220	221	222	222	223	224	225	226
9	242	243	244	245	246	247	248	248	249	250	251	252	253	254

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r + r''$. 0,95 Days	dif.	0,96 Days	dif.	0,97 Days	dif.	0,98 Days	dif.	0,99 Days	dif.	1,00 Days	dif.	1,01 Days	dif.	1,02 Days	dif.	1,03 Days	dif.	1,04 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,283	2	0,285	1	0,286	2	0,288	1	0,289	2	0,291	1	0,292	2	0,294	1	0,295	1	0,296	2	0,0001
0,02	0,567	3	0,570	3	0,573	2	0,575	3	0,578	3	0,581	3	0,584	3	0,587	3	0,590	3	0,593	3	0,0004
0,03	0,850	4	0,854	5	0,859	4	0,863	5	0,868	4	0,872	4	0,876	5	0,881	4	0,885	4	0,889	4	0,0009
0,04	1,133	6	1,139	6	1,145	6	1,151	6	1,157	6	1,163	5	1,168	6	1,174	6	1,180	6	1,186	5	0,0016
0,05	1,416	8	1,424	7	1,431	8	1,439	7	1,446	7	1,453	7	1,460	8	1,468	7	1,475	7	1,482	7	0,0025
0,06	1,700	8	1,708	9	1,717	9	1,726	9	1,735	9	1,744	8	1,752	9	1,761	9	1,770	8	1,778	9	0,0036
0,07	1,983	10	1,993	10	2,003	11	2,014	10	2,024	10	2,034	10	2,044	10	2,054	11	2,065	10	2,075	9	0,0049
0,08	2,266	12	2,278	12	2,290	11	2,301	12	2,313	12	2,325	11	2,336	12	2,348	11	2,359	12	2,371	11	0,0064
0,09	2,549	13	2,562	13	2,575	14	2,589	13	2,602	13	2,615	13	2,628	13	2,641	13	2,654	13	2,667	13	0,0081
0,10	2,832	15	2,847	14	2,861	15	2,876	15	2,891	14	2,905	15	2,920	14	2,934	15	2,949	14	2,963	14	0,0100
0,11	3,115	16	3,131	16	3,147	16	3,163	17	3,180	16	3,196	16	3,212	16	3,228	15	3,243	16	3,259	16	0,0121
0,12	3,397	18	3,415	18	3,433	18	3,451	17	3,468	18	3,486	17	3,503	18	3,521	17	3,538	17	3,555	17	0,0144
0,13	3,680	19	3,699	20	3,719	19	3,738	19	3,757	19	3,776	19	3,795	19	3,814	18	3,832	19	3,851	18	0,0169
0,14	3,963	21	3,984	20	4,004	21	4,025	20	4,045	21	4,066	20	4,086	21	4,107	20	4,127	20	4,147	20	0,0196
0,15	4,245	22	4,267	23	4,290	22	4,312	22	4,334	22	4,356	22	4,378	21	4,399	22	4,421	21	4,442	22	0,0225
0,16	4,527	24	4,551	24	4,575	24	4,599	23	4,622	24	4,646	23	4,669	23	4,692	23	4,715	23	4,738	23	0,0256
0,17	4,810	25	4,835	25	4,860	25	4,885	25	4,910	25	4,935	25	4,960	25	4,985	24	5,009	24	5,033	25	0,0289
0,18	5,092	27	5,119	26	5,145	27	5,172	26	5,198	27	5,225	26	5,251	26	5,277	26	5,303	26	5,329	26	0,0324
0,19	5,374	28	5,402	28	5,430	28	5,458	28	5,486	28	5,514	28	5,542	27	5,569	28	5,597	27	5,624	27	0,0361
0,20	5,655	30	5,685	30	5,715	30	5,745	29	5,774	29	5,803	30	5,833	29	5,862	28	5,890	29	5,919	29	0,0400
0,21	5,937	32	5,969	31	6,000	31	6,031	31	6,062	31	6,093	30	6,123	31	6,154	30	6,184	30	6,214	30	0,0441
0,22	6,219	33	6,252	32	6,284	33	6,317	32	6,349	33	6,382	32	6,414	32	6,446	31	6,477	32	6,509	31	0,0484
0,23	6,500	34	6,534	35	6,569	34	6,603	34	6,637	33	6,670	34	6,704	33	6,737	34	6,771	33	6,804	33	0,0529
0,24	6,781	36	6,817	36	6,853	35	6,888	36	6,924	35	6,959	35	6,994	35	7,029	35	7,064	34	7,098	34	0,0576
0,25	7,062	37	7,099	38	7,137	37	7,174	37	7,211	36	7,247	37	7,284	36	7,320	36	7,356	36	7,392	36	0,0625
0,26	7,343	39	7,382	38	7,420	39	7,459	38	7,497	39	7,536	38	7,574	37	7,611	38	7,649	38	7,687	37	0,0676
0,27	7,623	41	7,664	40	7,704	40	7,744	40	7,784	40	7,824	39	7,863	40	7,903	39	7,942	39	7,981	38	0,0729
0,28	7,903	42	7,945	42	7,987	42	8,029	41	8,070	42	8,112	41	8,153	40	8,193	41	8,234	40	8,274	40	0,0784
0,29	8,183	44	8,227	43	8,270	44	8,314	42	8,356	43	8,399	43	8,442	42	8,484	42	8,526	42	8,568	42	0,0841
0,30	8,463	45	8,508	45	8,553	45	8,598	44	8,642	45	8,687	44	8,731	43	8,774	44	8,818	43	8,861	43	0,0900
0,31	8,743	46	8,789	47	8,836	46	8,882	46	8,928	46	8,974	45	9,019	46	9,065	45	9,110	44	9,154	45	0,0961
0,32	9,022	48	9,070	48	9,118	48	9,166	47	9,213	48	9,261	47	9,308	46	9,354	47	9,401	46	9,447	46	0,1024
0,33	9,301	50	9,351	49	9,400	50	9,450	49	9,499	48	9,547	49	9,596	48	9,644	48	9,692	48	9,740	47	0,1089
0,34	9,580	51	9,631	51	9,682	51	9,733	51	9,784	50	9,834	50	9,884	50	9,934	49	9,983	49	10,032	49	0,1156
0,35	9,858	53	9,911	53	9,964	52	10,016	52	10,068	52	10,120	52	10,172	51	10,223	51	10,274	51	10,325	50	0,1225
0,36	10,136	55	10,191	54	10,245	54	10,299	54	10,353	53	10,406	53	10,459	53	10,512	52	10,564	53	10,617	52	0,1296
0,37	10,414	56	10,470	56	10,526	55	10,581	56	10,637	54	10,691	55	10,746	54	10,800	54	10,854	54	10,908	54	0,1369
0,38	10,691	58	10,749	58	10,807	57	10,864	56	10,920	57	10,977	56	11,033	56	11,089	55	11,144	56	11,200	55	0,1444
0,39	10,969	59	11,028	59	11,087	58	11,145	59	11,204	58	11,262	57	11,319	58	11,377	57	11,434	57	11,491	56	0,1521
0,40	11,245	61	11,306	61	11,367	60	11,427	60	11,487	59	11,546	60	11,606	59	11,665	58	11,723	58	11,781	58	0,1600
0,41	11,522	62	11,584	63	11,647	61	11,708	62	11,770	61	11,831	61	11,892	60	11,952	60	12,012	60	12,072	59	0,1681
0,42	11,798	64	11,862	64	11,926	63	11,989	63	12,052	63	12,115	62	12,177	62	12,239	62	12,301	61	12,362	61	0,1764
0,43	12,074	65	12,139	66	12,205	65	12,270	64	12,334	65	12,399	63	12,462	64	12,526	63	12,589	63	12,652	63	0,1849
0,44	12,349	67	12,416	67	12,483	67	12,550	66	12,616	66	12,682	65	12,747	65	12,812	65	12,877	65	12,942	64	0,1936
0,45	12,624	69	12,693	68	12,761	69	12,830	67	12,897	68	12,965	67	13,032	67	13,099	66	13,165	66	13,231	65	0,2025
0,50	13,992	77	14,069	77	14,146	77	14,223	75	14,298	76	14,374	75	14,449	74	14,523	75	14,598	73	14,671	74	0,2500
0,55	15,348	86	15,434	85	15,519	85	15,604	84	15,688	84	15,772	83	15,855	83	15,938	82	16,020	82	16,102	81	0,3025
0,60	16,691	94	16,785	94	16,879	94	16,973	92	17,065	93	17,158	91	17,249	91	17,340	91	17,431	90	17,521	89	0,3600
0,65	18,017	103	18,120	104	18,224	102	18,326	102	18,428	101	18,529	100	18,629	100	18,729	99	18,828	98	18,926	98	0,4225
0,70	19,324	113	19,437	113	19,550	112	19,662	111	19,773	110	19,883	110	19,993	109	20,102	108	20,210	107	20,317	107	0,4900
0,75	20,609	124	20,733	123	20,856	122	20,978	121	21,099	120	21,219	119	21,338	118	21,456	117	21,573	117	21,690	116	0,5625
0,80	21,867	135	22,002	134	22,136	133	22,269	132	22,401	130	22,531	130	22,661	128	22,789	127	22,916	127	23,043	125	0,6400
0,85	23,091	149	23,240	146	23,386	145	23,531	144	23,675	142	23,817	140	23,957	140	24,097	138	24,235	137	24,372	136	0,7225
0,90	24,271	164	24,435	161	24,596	160	24,756	157	24,913	155	25,068	154	25,222	151	25,373	151	25,524	148	25,672	148	0,8100
0,95	25,374	191	25,565	184	25,749	178	25,927	175	26,102	172	26,274	169	26,443	167	26,610	165	26,775	162	26,937	160	0,9025
1,00											27,404	196	27,600	189	27,789	183	27,972	181	28,153	177	1,0000
	,4513		,4608		,4705		,4802		,4901		,5000		,5101		,5202		,5305		,5108		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	291	292	293
1	28	28	28	28	28	28	28	28	28	28	29	29	29	29	29	29	29	29	29
2	55	55	55	56	56	56	56	56	57	57	57	57	57	58	58	58	58	58	59
3	83	83	83	83	84	84	84	85	85	85	86	86	86	86	87	87	87	88	88
4	110	110	111	111	112	112	112	113	113	114	114	114	115	115	116	116	116	117	117
5	138	138	139	139	140	140	141	141	142	142	143	143	144	144	145	145	146	146	147
6	165	166	166	167	167	168	169	169	170	170	171	172	172	173	173	174	175	175	176
7	193	193	194	195	195	196	197	197	198	199	200	200	201	202	202	203	204	204	205
8	220	221	222	222	223	224	225	226	226	227	228	229	230	230	231	232	233	234	234
9	248	248	249	250	251	252	253	254	255	256	257	257	258	259	260	261	262	263	264

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	1,05 Days	dif.	1,06 Days	dif.	1,07 Days	dif.	1,08 Days	dif.	1,09 Days	dif.	1,10 Days	dif.	
	Sum of the Radii $r + r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,298	1	0,299	2	0,301	1	0,302	1	0,303	2	0,305	1	0,0001
0,02	0,596	3	0,599	2	0,601	3	0,604	3	0,607	3	0,610	2	0,0004
0,03	0,893	5	0,898	4	0,902	4	0,906	4	0,910	5	0,915	4	0,0009
0,04	1,191	6	1,197	6	1,203	5	1,208	6	1,214	5	1,219	6	0,0016
0,05	1,489	7	1,496	7	1,503	7	1,510	7	1,517	7	1,524	7	0,0025
0,06	1,787	8	1,795	9	1,804	8	1,812	9	1,821	8	1,829	8	0,0036
0,07	2,084	10	2,094	10	2,104	10	2,114	10	2,124	10	2,134	9	0,0049
0,08	2,382	11	2,393	12	2,405	11	2,416	11	2,427	11	2,438	11	0,0064
0,09	2,680	12	2,692	13	2,705	13	2,718	12	2,730	13	2,743	12	0,0081
0,10	2,977	14	2,991	15	3,006	14	3,020	14	3,034	13	3,047	14	0,0100
0,11	3,275	15	3,290	16	3,306	15	3,321	16	3,337	15	3,352	15	0,0121
0,12	3,572	17	3,589	17	3,606	17	3,623	17	3,640	16	3,656	17	0,0144
0,13	3,869	19	3,888	18	3,906	18	3,924	19	3,943	18	3,961	18	0,0169
0,14	4,167	20	4,187	19	4,206	20	4,226	20	4,246	19	4,265	19	0,0196
0,15	4,464	21	4,485	21	4,506	21	4,527	21	4,548	21	4,569	21	0,0225
0,16	4,761	23	4,784	22	4,806	23	4,829	22	4,851	22	4,873	22	0,0256
0,17	5,058	24	5,082	24	5,106	24	5,130	24	5,154	23	5,177	24	0,0289
0,18	5,355	25	5,380	26	5,406	25	5,431	25	5,456	25	5,481	25	0,0324
0,19	5,651	27	5,678	27	5,705	27	5,732	26	5,758	27	5,785	26	0,0361
0,20	5,948	28	5,976	28	6,004	29	6,033	28	6,061	28	6,089	27	0,0400
0,21	6,244	30	6,274	30	6,304	29	6,333	30	6,363	29	6,392	29	0,0441
0,22	6,540	32	6,572	31	6,603	31	6,634	31	6,665	30	6,695	31	0,0484
0,23	6,837	32	6,869	33	6,902	32	6,934	33	6,967	32	6,999	33	0,0529
0,24	7,132	35	7,167	34	7,201	33	7,234	34	7,268	34	7,302	33	0,0576
0,25	7,428	36	7,464	35	7,499	36	7,535	35	7,570	35	7,605	34	0,0625
0,26	7,724	37	7,761	37	7,798	37	7,835	36	7,871	36	7,907	37	0,0676
0,27	8,019	39	8,058	38	8,096	38	8,134	38	8,172	38	8,210	38	0,0729
0,28	8,314	40	8,354	40	8,394	40	8,434	39	8,473	39	8,512	39	0,0784
0,29	8,610	41	8,651	41	8,692	41	8,733	41	8,774	41	8,815	40	0,0841
0,30	8,904	43	8,947	43	8,990	42	9,032	43	9,075	42	9,117	42	0,0900
0,31	9,199	44	9,243	44	9,287	44	9,331	44	9,375	44	9,419	43	0,0961
0,32	9,493	46	9,539	46	9,585	45	9,630	45	9,675	45	9,720	45	0,1024
0,33	9,787	48	9,835	47	9,882	47	9,929	46	9,975	47	10,022	46	0,1089
0,34	10,081	49	10,130	49	10,179	48	10,227	48	10,275	48	10,323	47	0,1156
0,35	10,375	50	10,425	50	10,475	50	10,525	50	10,575	49	10,624	49	0,1225
0,36	10,669	51	10,720	52	10,772	51	10,823	51	10,874	51	10,925	50	0,1296
0,37	10,962	53	11,015	53	11,068	52	11,120	53	11,173	52	11,225	52	0,1369
0,38	11,255	54	11,309	55	11,364	54	11,418	54	11,472	53	11,525	54	0,1444
0,39	11,547	56	11,603	56	11,659	56	11,715	55	11,770	55	11,825	55	0,1521
0,40	11,839	58	11,897	58	11,955	57	12,012	56	12,068	57	12,125	56	0,1600
0,41	12,131	60	12,191	59	12,250	58	12,308	58	12,366	58	12,424	58	0,1681
0,42	12,423	61	12,484	60	12,544	60	12,604	60	12,664	60	12,724	59	0,1764
0,43	12,715	62	12,777	62	12,839	61	12,900	62	12,962	60	13,022	61	0,1849
0,44	13,006	63	13,069	64	13,133	63	13,196	63	13,259	62	13,321	62	0,1936
0,45	13,296	66	13,362	65	13,427	64	13,491	64	13,555	64	13,619	64	0,2025
0,50	14,745	73	14,818	72	14,890	72	14,962	72	15,034	72	15,106	71	0,2500
0,55	16,183	81	16,264	80	16,344	80	16,424	80	16,504	79	16,583	79	0,3025
0,60	17,610	89	17,699	89	17,788	87	17,875	88	17,963	87	18,050	86	0,3600
0,65	19,024	98	19,122	96	19,218	96	19,314	96	19,410	95	19,505	94	0,4225
0,70	20,424	105	20,529	106	20,635	104	20,739	104	20,843	104	20,947	103	0,4900
0,75	21,806	115	21,921	114	22,035	114	22,149	112	22,261	112	22,373	112	0,5625
0,80	23,168	125	23,293	124	23,417	122	23,539	122	23,661	121	23,782	121	0,6400
0,85	24,508	135	24,643	133	24,776	133	24,909	132	25,041	131	25,172	129	0,7225
0,90	25,820	140	25,960	144	26,110	144	26,254	142	26,396	141	26,537	140	0,8100
0,95	27,097	159	27,256	157	27,413	156	27,569	154	27,723	152	27,875	151	0,9025
1,00	28,330	174	28,504	172	28,676	169	28,845	168	29,013	165	29,178	164	1,0000
	.5513		,5618		.5725		.5832		.5941		,6050		c^2

$\frac{1}{2}\cdot(r + r'^2)$ or $r^2 + r'^2$ nearly.

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

	294	295	296	297	298	299	300	301	302	303	304	305
1	29	30	30	30	30	30	30	30	30	30	30	31
2	59	59	59	59	60	60	60	60	60	61	61	61
3	88	89	89	89	89	90	90	90	91	91	91	92
4	118	118	118	119	119	120	120	120	121	121	122	122
5	147	148	148	149	149	150	150	151	151	152	152	153
6	176	177	178	178	179	179	180	181	181	182	182	183
7	206	207	207	208	209	209	210	211	211	212	213	214
8	235	236	237	238	238	239	240	241	242	242	243	244
9	265	266	266	267	268	269	270	271	272	273	274	275

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord	Sum of the Radii $r+r''$.																				
c.	1,11		1,12		1,13		1,14		1,15		1,16		1,17		1,18		1,19		1,20		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,306	2	0,308	1	0,309	1	0,310	2	0,312	1	0,313	1	0,314	2	0,316	1	0,317	1	0,318	2	0,0001
0,02	0,612	3	0,615	3	0,618	3	0,621	2	0,623	3	0,626	3	0,629	2	0,631	3	0,634	3	0,637	2	0,0004
0,03	0,919	4	0,923	4	0,927	4	0,931	4	0,935	4	0,939	4	0,943	4	0,947	4	0,951	4	0,955	4	0,0009
0,04	1,225	5	1,230	6	1,236	5	1,241	6	1,247	5	1,252	6	1,258	5	1,263	5	1,268	6	1,274	5	0,0016
0,05	1,531	7	1,538	7	1,545	7	1,552	6	1,558	7	1,565	7	1,572	7	1,579	6	1,585	7	1,592	7	0,0025
0,06	1,837	8	1,845	9	1,854	8	1,862	8	1,870	8	1,878	8	1,886	8	1,894	8	1,902	8	1,910	8	0,0036
0,07	2,143	10	2,153	10	2,163	9	2,172	10	2,182	9	2,191	9	2,200	10	2,210	9	2,219	10	2,229	9	0,0049
0,08	2,449	11	2,460	11	2,471	11	2,482	11	2,493	11	2,504	11	2,515	10	2,525	11	2,536	11	2,547	10	0,0064
0,09	2,755	13	2,768	12	2,780	12	2,792	13	2,805	12	2,817	12	2,829	12	2,841	12	2,853	12	2,865	12	0,0081
0,10	3,061	14	3,075	14	3,089	13	3,102	14	3,116	14	3,130	13	3,143	13	3,156	14	3,170	13	3,183	13	0,0100
0,11	3,367	15	3,382	15	3,397	15	3,412	15	3,427	15	3,442	15	3,457	15	3,472	15	3,487	14	3,501	15	0,0121
0,12	3,673	17	3,690	16	3,706	16	3,722	17	3,739	16	3,755	16	3,771	16	3,787	16	3,803	16	3,819	16	0,0144
0,13	3,979	18	3,997	17	4,014	18	4,032	18	4,050	18	4,068	17	4,085	18	4,103	17	4,120	17	4,137	17	0,0169
0,14	4,284	20	4,304	19	4,323	19	4,342	19	4,361	19	4,380	19	4,399	19	4,418	18	4,436	19	4,455	19	0,0196
0,15	4,590	21	4,611	20	4,631	21	4,652	20	4,672	21	4,693	20	4,713	20	4,733	20	4,753	20	4,773	20	0,0225
0,16	4,895	23	4,918	22	4,940	21	4,961	22	4,983	22	5,005	21	5,026	22	5,048	21	5,069	22	5,091	21	0,0256
0,17	5,201	23	5,224	24	5,248	23	5,271	23	5,294	23	5,317	23	5,340	23	5,363	23	5,386	22	5,408	23	0,0289
0,18	5,506	25	5,531	25	5,556	24	5,580	25	5,605	24	5,629	25	5,654	24	5,678	24	5,702	24	5,726	24	0,0324
0,19	5,811	27	5,838	26	5,864	26	5,890	26	5,916	25	5,941	26	5,967	26	5,993	25	6,018	25	6,043	26	0,0361
0,20	6,116	28	6,144	27	6,171	28	6,199	27	6,226	27	6,253	27	6,280	27	6,307	27	6,334	27	6,361	26	0,0400
0,21	6,421	29	6,450	29	6,479	29	6,508	29	6,537	28	6,565	28	6,593	29	6,622	28	6,650	28	6,678	28	0,0441
0,22	6,726	30	6,756	31	6,787	30	6,817	30	6,847	30	6,877	30	6,907	29	6,936	30	6,966	29	6,995	29	0,0484
0,23	7,031	31	7,062	32	7,094	32	7,126	31	7,157	31	7,188	31	7,219	31	7,250	31	7,281	31	7,312	31	0,0529
0,24	7,335	33	7,368	33	7,401	33	7,434	33	7,467	33	7,500	32	7,532	33	7,565	32	7,597	32	7,629	32	0,0576
0,25	7,639	35	7,674	35	7,709	34	7,743	34	7,777	34	7,811	34	7,845	34	7,879	33	7,912	34	7,946	33	0,0625
0,26	7,944	36	7,980	36	8,016	35	8,051	36	8,087	35	8,122	35	8,157	35	8,192	35	8,227	35	8,262	35	0,0676
0,27	8,248	37	8,285	37	8,322	37	8,359	37	8,396	37	8,433	37	8,470	36	8,506	36	8,542	37	8,579	36	0,0729
0,28	8,551	39	8,590	39	8,629	38	8,667	39	8,706	38	8,744	38	8,782	38	8,820	37	8,857	38	8,895	37	0,0784
0,29	8,855	40	8,895	40	8,935	40	8,975	40	9,015	40	9,055	39	9,094	39	9,133	39	9,172	39	9,211	39	0,0841
0,30	9,159	41	9,200	42	9,242	41	9,283	41	9,324	41	9,365	41	9,406	40	9,446	41	9,487	40	9,527	40	0,0900
0,31	9,462	43	9,505	43	9,548	43	9,591	42	9,633	42	9,675	42	9,717	42	9,759	42	9,801	42	9,843	41	0,0961
0,32	9,765	44	9,809	45	9,854	44	9,898	44	9,942	43	9,985	44	10,029	43	10,072	43	10,115	43	10,158	43	0,1024
0,33	10,068	46	10,114	45	10,159	46	10,205	45	10,250	45	10,295	45	10,340	45	10,385	44	10,429	45	10,474	44	0,1089
0,34	10,370	48	10,418	47	10,465	47	10,512	47	10,559	46	10,605	46	10,651	46	10,697	46	10,743	46	10,789	45	0,1156
0,35	10,673	49	10,722	48	10,770	49	10,819	48	10,867	48	10,915	47	10,962	48	11,010	47	11,057	47	11,104	47	0,1225
0,36	10,975	50	11,025	50	11,075	50	11,125	50	11,175	49	11,224	49	11,273	49	11,322	48	11,370	49	11,419	48	0,1296
0,37	11,277	52	11,329	51	11,380	51	11,431	51	11,482	51	11,533	50	11,583	51	11,634	50	11,684	49	11,733	50	0,1369
0,38	11,579	53	11,632	53	11,685	52	11,737	53	11,790	52	11,842	52	11,894	51	11,945	52	11,997	51	12,048	51	0,1444
0,39	11,880	55	11,935	54	11,989	54	12,043	54	12,097	53	12,150	54	12,204	53	12,257	52	12,309	53	12,362	52	0,1521
0,40	12,181	56	12,237	56	12,293	55	12,348	56	12,404	55	12,459	54	12,513	55	12,568	54	12,622	54	12,676	54	0,1600
0,41	12,482	58	12,540	57	12,597	57	12,654	56	12,710	57	12,767	56	12,823	56	12,879	55	12,934	56	12,990	55	0,1681
0,42	12,783	59	12,842	58	12,900	59	12,959	58	13,017	58	13,075	57	13,132	57	13,189	57	13,246	57	13,303	57	0,1764
0,43	13,083	61	13,144	60	13,204	59	13,263	60	13,323	59	13,382	59	13,441	59	13,500	58	13,558	58	13,616	58	0,1849
0,44	13,383	62	13,445	62	13,507	61	13,568	61	13,629	60	13,689	61	13,750	60	13,810	60	13,870	59	13,929	59	0,1936
0,45	13,683	63	13,746	63	13,809	63	13,872	62	13,934	62	13,996	62	14,058	62	14,120	61	14,181	61	14,242	61	0,2025
0,50	15,177	70	15,247	71	15,318	70	15,388	70	15,458	69	15,527	69	15,596	69	15,665	68	15,733	68	15,801	68	0,2500
0,55	16,662	78	16,740	78	16,818	77	16,895	77	16,972	77	17,049	77	17,126	75	17,201	76	17,277	75	17,352	75	0,3025
0,60	18,136	86	18,222	86	18,308	85	18,393	85	18,478	84	18,562	84	18,646	83	18,729	83	18,812	87	18,895	82	0,3600
0,65	19,599	94	19,693	94	19,787	93	19,880	92	19,972	92	20,064	92	20,156	91	20,247	90	20,337	91	20,428	89	0,4225
0,70	21,050	102	21,152	101	21,253	101	21,354	101	21,455	100	21,555	99	21,654	99	21,753	98	21,851	98	21,949	97	0,4900
0,75	22,485	110	22,595	110	22,705	110	22,815	108	22,923	109	23,032	107	23,139	107	23,246	106	23,352	106	23,458	105	0,5625
0,80	23,903	119	24,022	119	24,141	118	24,259	117	24,376	117	24,493	116	24,609	115	24,724	115	24,839	114	24,953	113	0,6400
0,85	25,301	129	25,430	128	25,558	127	25,685	127	25,812	125	25,937	125	26,062	124	26,186	123	26,309	123	26,432	121	0,7225
0,90	26,677	139	26,816	138	26,954	137	27,091	136	27,227	135	27,362	134	27,496	133	27,629	132	27,761	132	27,893	130	0,8100
0,95	28,026	150	28,176	149	28,325	147	28,472	146	28,618	145	28,763	144	28,907	143	29,050	142	29,192	141	29,333	140	0,9025
1,00	29,342	162	29,504	161	29,665	159	29,824	157	29,981	156	30,137	155	30,292	154	30,446	152	30,598	151	30,749	150	1,0000
	,6161		,6272		,6385		,6498		,6613		,6728		,6845		,6962		,7081		,7200		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	300	301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316
1	30	30	30	30	30	31	31	31	31	31	31	31	31	31	31	32	32
2	60	60	60	61	61	61	61	61	62	62	62	62	62	63	63	63	63
3	90	90	91	91	91	92	92	92	92	93	93	93	94	94	94	95	95
4	120	120	121	121	122	122	122	123	123	124	124	124	125	125	126	126	126
5	150	151	151	152	152	153	153	154	154	155	155	156	156	157	157	158	158
6	180	181	181	182	182	183	184	184	185	185	186	187	187	188	188	189	190
7	210	211	211	212	213	214	214	215	216	216	217	218	218	219	220	221	221
8	240	241	242	242	243	244	245	246	246	247	248	249	250	250	251	252	253
9	270	271	272	273	274	275	275	276	277	278	279	280	281	282	283	284	284

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	1,21 Days	dif.	1,22 Days	dif.	1,23 Days	dif.	1,24 Days	dif.	1,25 Days	dif.	1,26 Days	dif.	
	Sum of the Radii $r + r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,320	1	0,321	1	0,322	2	0,324	1	0,325	1	0,326	2	0,0001
0,02	0,639	3	0,642	3	0,645	2	0,647	3	0,650	3	0,653	2	0,0004
0,03	0,959	4	0,963	4	0,967	4	0,971	4	0,975	4	0,979	4	0,0009
0,04	1,279	5	1,284	5	1,289	6	1,295	5	1,300	5	1,305	5	0,0016
0,05	1,599	6	1,605	7	1,612	6	1,618	7	1,625	6	1,631	7	0,0025
0,06	1,918	8	1,926	8	1,934	8	1,942	8	1,950	7	1,957	8	0,0036
0,07	2,238	9	2,247	9	2,256	9	2,265	9	2,274	10	2,284	9	0,0049
0,08	2,557	11	2,568	10	2,578	11	2,589	10	2,599	11	2,610	10	0,0064
0,09	2,877	12	2,889	12	2,901	11	2,912	12	2,924	12	2,936	11	0,0081
0,10	3,196	14	3,210	13	3,223	13	3,236	13	3,249	13	3,262	13	0,0100
0,11	3,516	14	3,530	15	3,545	14	3,559	15	3,574	14	3,588	14	0,0121
0,12	3,835	16	3,851	16	3,867	15	3,882	16	3,898	16	3,914	15	0,0144
0,13	4,154	18	4,172	17	4,189	17	4,206	17	4,223	17	4,240	16	0,0169
0,14	4,474	18	4,492	19	4,511	18	4,529	18	4,547	18	4,565	19	0,0196
0,15	4,793	20	4,813	19	4,832	20	4,852	20	4,872	19	4,891	20	0,0225
0,16	5,112	21	5,133	21	5,154	21	5,175	21	5,196	21	5,217	20	0,0256
0,17	5,431	22	5,453	23	5,476	22	5,498	22	5,520	22	5,542	22	0,0289
0,18	5,750	24	5,774	23	5,797	24	5,821	23	5,844	24	5,868	23	0,0324
0,19	6,069	25	6,094	25	6,119	25	6,144	24	6,168	25	6,193	25	0,0361
0,20	6,387	27	6,414	26	6,440	26	6,466	26	6,492	26	6,518	26	0,0400
0,21	6,706	28	6,734	27	6,761	28	6,789	27	6,816	28	6,844	27	0,0441
0,22	7,024	29	7,053	29	7,082	29	7,111	29	7,140	29	7,169	28	0,0484
0,23	7,343	30	7,373	30	7,403	31	7,434	30	7,464	30	7,494	30	0,0529
0,24	7,661	32	7,693	31	7,724	32	7,756	31	7,787	31	7,818	32	0,0576
0,25	7,979	33	8,012	33	8,045	33	8,078	33	8,111	32	8,143	33	0,0625
0,26	8,297	34	8,331	35	8,366	34	8,400	34	8,434	34	8,468	34	0,0676
0,27	8,615	35	8,650	36	8,686	36	8,722	35	8,757	35	8,792	35	0,0729
0,28	8,932	37	8,969	37	9,006	37	9,043	37	9,080	37	9,117	36	0,0784
0,29	9,250	38	9,288	39	9,327	38	9,365	38	9,403	38	9,441	37	0,0841
0,30	9,567	40	9,607	40	9,647	39	9,686	39	9,725	40	9,765	39	0,0900
0,31	9,884	41	9,925	41	9,966	41	10,007	41	10,048	40	10,088	41	0,0961
0,32	10,201	43	10,244	42	10,286	42	10,328	42	10,370	42	10,412	42	0,1024
0,33	10,518	44	10,562	44	10,606	43	10,649	43	10,692	44	10,736	43	0,1089
0,34	10,834	46	10,880	45	10,925	45	10,970	44	11,014	45	11,059	44	0,1156
0,35	11,151	46	11,197	47	11,244	46	11,290	46	11,336	46	11,382	46	0,1225
0,36	11,467	48	11,515	48	11,563	47	11,610	48	11,658	47	11,705	47	0,1296
0,37	11,783	49	11,832	50	11,882	48	11,930	49	11,979	49	12,028	48	0,1369
0,38	12,099	51	12,150	50	12,200	50	12,250	50	12,300	50	12,350	50	0,1444
0,39	12,414	52	12,466	52	12,518	52	12,570	51	12,621	52	12,673	51	0,1521
0,40	12,730	53	12,783	53	12,836	53	12,889	53	12,942	53	12,995	52	0,1600
0,41	13,045	55	13,100	54	13,154	55	13,209	54	13,263	54	13,317	53	0,1681
0,42	13,360	56	13,416	56	13,472	56	13,528	55	13,583	55	13,638	55	0,1764
0,43	13,674	58	13,732	57	13,789	57	13,846	57	13,903	57	13,960	56	0,1849
0,44	13,988	60	14,048	58	14,106	59	14,165	58	14,223	58	14,281	58	0,1936
0,45	14,303	60	14,363	60	14,423	60	14,483	60	14,543	59	14,602	59	0,2025
0,50	15,869	67	15,936	67	16,003	67	16,070	67	16,137	66	16,203	66	0,2500
0,55	17,427	75	17,502	74	17,576	74	17,650	74	17,724	73	17,797	73	0,3025
0,60	18,977	82	19,059	81	19,140	81	19,221	81	19,302	80	19,382	80	0,3600
0,65	20,517	89	20,606	89	20,695	89	20,784	88	20,872	87	20,959	87	0,4225
0,70	22,046	97	22,143	96	22,239	96	22,335	96	22,431	95	22,526	94	0,4900
0,75	23,563	105	23,668	104	23,772	103	23,875	103	23,978	103	24,081	102	0,5625
0,80	25,066	113	25,179	112	25,291	111	25,402	111	25,513	110	25,623	110	0,6400
0,85	26,553	122	26,675	120	26,795	120	26,915	119	27,034	118	27,152	118	0,7225
0,90	28,023	130	28,153	129	28,282	128	28,410	128	28,538	126	28,664	127	0,8100
0,95	29,473	139	29,612	138	29,750	137	29,887	137	30,024	135	30,159	135	0,9025
1,00	30,899	149	31,048	148	31,196	147	31,343	145	31,488	145	31,633	144	1,0000
	,7321		,7442		,7565		,7688		,7813		,7938		c^2

$\frac{1}{2}\,.\,(r + r'^2)$ or $r^2 + r''^2$ nearly.

	317	318	319	320	321	322	323	324	325	326	327
1	32	32	32	32	32	32	32	32	33	33	33
2	63	64	64	64	64	64	65	65	65	65	65
3	95	95	96	96	96	97	97	97	98	98	98
4	127	127	128	128	128	129	129	130	130	130	131
5	159	159	160	160	161	161	162	162	163	163	164
6	190	191	191	192	193	193	194	194	195	196	196
7	222	223	223	224	225	225	226	227	228	228	229
8	254	254	255	256	257	258	258	259	260	261	262
9	285	286	287	288	289	290	291	292	293	293	294

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord	Sum of the radii $r+r'$.																				
	1,27		1,28		1,29		1,30		1,31		1,32		1,33		1,34		1,35		1,36		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,328	1	0,329	1	0,330	1	0,331	2	0,333	1	0,334	1	0,335	1	0,336	2	0,338	1	0,339	1	0,0001
0,02	0,655	3	0,658	2	0,660	3	0,663	2	0,665	3	0,668	2	0,670	3	0,673	2	0,675	3	0,678	2	0,0004
0,03	0,983	4	0,987	3	0,990	4	0,994	4	0,998	4	1,002	4	1,006	3	1,009	4	1,013	4	1,017	4	0,0009
0,04	1,310	5	1,315	5	1,320	6	1,326	5	1,331	5	1,336	5	1,341	5	1,346	5	1,351	5	1,356	5	0,0016
0,05	1,638	6	1,644	7	1,651	6	1,657	6	1,663	7	1,670	6	1,676	6	1,682	7	1,689	6	1,695	6	0,0025
0,06	1,965	8	1,973	8	1,981	7	1,988	8	1,996	8	2,004	7	2,011	8	2,019	7	2,026	8	2,034	7	0,0036
0,07	2,293	9	2,302	9	2,311	9	2,320	8	2,328	9	2,337	9	2,346	9	2,355	9	2,364	9	2,373	8	0,0049
0,08	2,620	10	2,630	11	2,641	10	2,651	10	2,661	10	2,671	10	2,681	10	2,691	10	2,701	10	2,711	10	0,0064
0,09	2,947	12	2,959	12	2,971	11	2,982	12	2,994	11	3,005	11	3,016	12	3,028	11	3,039	11	3,050	11	0,0081
0,10	3,275	13	3,288	12	3,300	13	3,313	13	3,326	13	3,339	12	3,351	13	3,364	12	3,376	13	3,389	12	0,0100
0,11	3,602	14	3,616	14	3,630	14	3,644	14	3,658	14	3,672	14	3,686	14	3,700	14	3,714	14	3,728	13	0,0121
0,12	3,929	16	3,945	15	3,960	15	3,975	16	3,991	15	4,006	15	4,021	15	4,036	15	4,051	15	4,066	15	0,0144
0,13	4,256	17	4,273	17	4,290	16	4,306	17	4,323	17	4,340	16	4,356	16	4,372	17	4,389	16	4,405	16	0,0169
0,14	4,584	18	4,602	18	4,620	17	4,637	18	4,655	18	4,673	18	4,691	17	4,708	18	4,726	17	4,743	18	0,0196
0,15	4,911	19	4,930	19	4,949	19	4,968	19	4,987	19	5,006	19	5,025	19	5,044	19	5,063	19	5,082	19	0,0225
0,16	5,237	21	5,258	21	5,279	20	5,299	21	5,320	20	5,340	20	5,360	20	5,380	20	5,400	20	5,420	20	0,0256
0,17	5,564	22	5,586	22	5,608	22	5,630	22	5,652	21	5,673	22	5,695	21	5,716	21	5,737	22	5,759	21	0,0289
0,18	5,891	23	5,914	23	5,937	24	5,961	22	5,983	23	6,006	23	6,029	23	6,052	22	6,074	23	6,097	[illegible]	0,0324
0,19	6,218	24	6,242	25	6,267	24	6,291	24	6,315	24	6,339	25	6,364	23	6,387	24	6,411	24	6,435	24	0,0361
0,20	6,544	26	6,570	26	6,596	26	6,622	25	6,647	25	6,672	26	6,698	25	6,723	25	6,748	25	6,773	25	0,0400
0,21	6,871	27	6,898	27	6,925	27	6,952	27	6,979	26	7,005	27	7,032	27	7,059	26	7,085	26	7,111	26	0,0441
0,22	7,197	29	7,226	28	7,254	28	7,282	28	7,310	28	7,338	28	7,366	28	7,394	28	7,422	27	7,449	28	0,0484
0,23	7,524	29	7,553	30	7,583	29	7,612	30	7,642	29	7,671	29	7,700	29	7,729	29	7,758	29	7,787	29	0,0529
0,24	7,850	31	7,881	31	7,912	30	7,942	31	7,973	31	8,004	30	8,034	30	8,064	31	8,095	30	8,125	[illegible]	0,0576
0,25	8,176	32	8,208	32	8,240	32	8,272	32	8,304	32	8,336	32	8,368	31	8,399	32	8,431	31	8,462	31	0,0625
0,26	8,502	33	8,535	34	8,569	33	8,602	33	8,635	33	8,668	33	8,701	33	8,734	33	8,767	33	8,800	32	0,0676
0,27	8,827	35	8,862	35	8,897	35	8,932	34	8,966	35	9,001	34	9,035	34	9,069	34	9,103	34	9,137	34	0,0729
0,28	9,153	36	9,189	36	9,225	36	9,261	36	9,297	36	9,333	35	9,368	36	9,404	35	9,439	35	9,474	35	0,0784
0,29	9,478	38	9,516	37	9,553	38	9,591	37	9,628	37	9,665	37	9,702	36	9,738	37	9,775	36	9,811	37	0,0841
0,30	9,804	39	9,843	38	9,881	39	9,920	38	9,958	39	9,997	38	10,035	38	10,073	38	10,111	37	10,148	38	0,0900
0,31	10,129	40	10,169	40	10,209	40	10,249	40	10,289	39	10,328	40	10,368	39	10,407	39	10,446	39	10,485	39	0,0961
0,32	10,454	41	10,495	42	10,537	41	10,578	41	10,619	41	10,660	40	10,700	41	10,741	40	10,781	41	10,822	40	0,1024
0,33	10,779	42	10,821	43	10,864	43	10,907	42	10,949	42	10,991	42	11,033	42	11,075	42	11,117	41	11,158	42	0,1089
0,34	11,103	44	11,147	44	11,191	44	11,235	44	11,279	43	11,322	44	11,366	43	11,409	43	11,452	42	11,494	43	0,1156
0,35	11,428	45	11,473	46	11,519	45	11,564	45	11,609	44	11,653	45	11,698	44	11,742	45	11,787	44	11,831	44	0,1225
0,36	11,752	47	11,799	46	11,845	47	11,892	46	11,938	46	11,984	46	12,030	46	12,076	45	12,121	46	12,167	45	0,1296
0,37	12,076	48	12,124	48	12,172	48	12,220	48	12,268	47	12,315	47	12,362	47	12,409	47	12,456	47	12,503	46	0,1369
0,38	12,400	49	12,449	50	12,499	49	12,548	49	12,597	48	12,645	49	12,694	48	12,742	48	12,790	48	12,838	48	0,1444
0,39	12,724	51	12,775	50	12,825	50	12,875	51	12,926	50	12,976	49	13,025	50	13,075	49	13,124	50	13,174	49	0,1521
0,40	13,047	52	13,099	52	13,151	52	13,203	51	13,254	52	13,306	51	13,357	51	13,408	50	13,458	51	13,509	50	0,1600
0,41	13,370	54	13,424	53	13,477	53	13,530	53	13,583	53	13,636	52	13,688	52	13,740	52	13,792	52	13,844	52	0,1681
0,42	13,693	55	13,748	55	13,803	54	13,857	54	13,911	54	13,965	54	14,019	54	14,073	53	14,126	53	14,179	53	0,1764
0,43	14,016	56	14,072	56	14,128	56	14,184	56	14,240	55	14,295	55	14,350	55	14,405	54	14,459	55	14,514	54	0,1849
0,44	14,339	57	14,396	58	14,454	57	14,511	56	14,567	57	14,624	56	14,680	57	14,737	55	14,792	56	14,848	56	0,1936
0,45	14,661	59	14,720	59	14,779	58	14,837	58	14,895	58	14,953	58	15,011	57	15,068	57	15,125	57	15,182	57	0,2025
0,50	16,269	66	16,335	65	16,400	65	16,465	65	16,530	65	16,595	64	16,659	64	16,723	64	16,787	63	16,850	64	0,2500
0,55	17,870	72	17,942	73	18,015	71	18,086	72	18,158	71	18,229	72	18,301	70	18,371	71	18,442	70	18,512	70	0,3025
0,60	19,462	80	19,542	79	19,621	79	19,700	79	19,779	78	19,857	78	19,935	77	20,012	78	20,090	77	20,167	76	0,3600
0,65	21,046	87	21,133	86	21,219	86	21,305	86	21,391	85	21,476	85	21,561	84	21,645	85	21,730	83	21,813	84	0,4225
0,70	22,620	94	22,714	94	22,808	93	22,901	93	22,994	92	23,086	92	23,178	91	23,269	92	23,361	90	23,451	91	0,4900
0,75	24,183	101	24,284	101	24,385	101	24,486	100	24,586	100	24,686	99	24,785	98	24,883	99	24,982	98	25,080	97	0,5625
0,80	25,733	109	25,842	109	25,951	108	26,059	108	26,167	107	26,274	106	26,380	106	26,486	106	26,592	105	26,697	105	0,6400
0,85	27,270	117	27,387	116	27,503	116	27,619	115	27,734	115	27,849	114	27,963	114	28,077	113	28,190	113	28,303	112	0,7225
0,90	28,791	125	28,916	124	29,040	124	29,164	124	29,288	122	29,410	123	29,533	121	29,654	121	29,775	120	29,895	120	0,8100
0,95	30,294	134	30,428	133	30,561	132	30,693	132	30,825	131	30,956	130	31,086	130	31,216	128	31,344	129	31,473	127	0,9025
1,00	31,777	143	31,920	142	32,062	141	32,203	141	32,344	139	32,483	139	32,622	138	32,760	137	32,897	137	33,034	135	1,0000
	,8065		,8192		,8321		,8450		,8581		,8712		,8845		,8978		,9113		,9248		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	322	323	324	325	326	327	328	329	330	331	332	333	334	335	336	337	
1	32	32	32	33	33	33	33	33	33	33	33	33	33	34	34	34	1
2	64	65	65	65	65	65	66	66	66	66	66	67	67	67	67	67	2
3	97	97	97	98	98	98	98	99	99	99	100	100	100	101	101	101	3
4	129	129	130	130	130	131	131	132	132	132	133	133	134	134	134	135	4
5	161	162	162	163	163	164	164	165	165	166	166	167	167	168	168	169	5
6	193	194	194	195	196	196	197	197	198	199	199	200	200	201	202	202	6
7	225	226	227	228	228	229	230	230	231	232	232	233	234	235	235	236	7
8	258	258	259	260	261	262	262	263	264	265	266	266	267	268	269	270	8
9	290	291	292	293	293	294	295	296	297	298	299	300	301	302	302	303	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	1,37 Days	dif.	1,38 Days	dif.	1,39 Days	dif.	1,40 Days	dif.	1,41 Days	dif.	1,42 Days	dif.	
	Sum of the Radii $r + r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,340	1	0,341	2	0,343	1	0,344	1	0,345	1	0,346	2	0,0001
0,02	0,680	3	0,683	2	0,685	3	0,688	2	0,690	3	0,693	2	0,0004
0,03	1,021	3	1,024	4	1,028	4	1,032	3	1,035	4	1,039	4	0,0009
0,04	1,361	5	1,366	5	1,371	5	1,376	5	1,381	4	1,385	5	0,0016
0,05	1,701	6	1,707	6	1,713	6	1,719	7	1,726	6	1,732	6	0,0025
0,06	2,041	8	2,049	7	2,056	7	2,063	8	2,071	7	2,078	7	0,0036
0,07	2,381	9	2,390	9	2,399	8	2,407	9	2,416	8	2,424	9	0,0049
0,08	2,721	10	2,731	10	2,741	10	2,751	10	2,761	10	2,771	9	0,0064
0,09	3,061	12	3,073	11	3,084	11	3,095	11	3,106	11	3,117	11	0,0081
0,10	3,401	13	3,414	12	3,426	12	3,438	13	3,451	12	3,463	12	0,0100
0,11	3,741	14	3,755	14	3,769	13	3,782	14	3,796	13	3,809	13	0,0121
0,12	4,081	15	4,096	15	4,111	15	4,126	14	4,140	15	4,155	15	0,0144
0,13	4,421	16	4,437	16	4,453	16	4,469	16	4,485	16	4,501	16	0,0169
0,14	4,761	17	4,778	18	4,796	17	4,813	17	4,830	17	4,847	17	0,0196
0,15	5,101	18	5,119	19	5,138	18	5,156	19	5,175	18	5,193	18	0,0225
0,16	5,440	20	5,460	20	5,480	20	5,500	19	5,519	20	5,539	19	0,0256
0,17	5,780	21	5,801	21	5,822	21	5,843	21	5,864	21	5,885	20	0,0289
0,18	6,119	23	6,142	22	6,164	22	6,186	22	6,208	22	6,230	22	0,0324
0,19	6,459	23	6,482	24	6,506	23	6,529	24	6,553	23	6,576	23	0,0361
0,20	6,798	25	6,823	25	6,848	24	6,872	25	6,897	25	6,922	24	0,0400
0,21	7,137	27	7,164	26	7,190	25	7,215	26	7,241	26	7,267	26	0,0441
0,22	7,477	27	7,504	27	7,531	27	7,558	27	7,585	27	7,612	27	0,0484
0,23	7,816	28	7,844	29	7,873	28	7,901	28	7,929	28	7,957	29	0,0529
0,24	8,155	29	8,184	30	8,214	30	8,244	29	8,273	30	8,303	29	0,0576
0,25	8,493	32	8,525	31	8,556	30	8,586	31	8,617	31	8,648	30	0,0625
0,26	8,832	32	8,864	33	8,897	32	8,929	32	8,961	32	8,993	33	0,0676
0,27	9,171	33	9,204	34	9,238	33	9,271	33	9,304	34	9,338	33	0,0729
0,28	9,509	35	9,544	35	9,579	34	9,613	35	9,648	34	9,682	35	0,0784
0,29	9,848	36	9,884	36	9,920	36	9,956	35	9,991	36	10,027	35	0,0841
0,30	10,186	37	10,223	37	10,260	38	10,298	37	10,335	36	10,371	37	0,0900
0,31	10,524	38	10,562	39	10,601	38	10,639	39	10,678	38	10,716	38	0,0961
0,32	10,862	40	10,902	39	10,941	40	10,981	40	11,021	39	11,060	39	0,1024
0,33	11,200	41	11,241	41	11,282	41	11,323	40	11,363	41	11,404	40	0,1089
0,34	11,537	43	11,580	42	11,622	42	11,664	42	11,706	42	11,748	42	0,1156
0,35	11,875	43	11,918	44	11,962	43	12,005	44	12,049	43	12,092	43	0,1225
0,36	12,212	45	12,257	45	12,302	44	12,346	45	12,391	44	12,435	44	0,1296
0,37	12,549	46	12,595	46	12,641	46	12,687	46	12,733	46	12,779	45	0,1369
0,38	12,886	48	12,934	47	12,981	47	13,028	47	13,075	47	13,122	47	0,1444
0,39	13,223	49	13,272	48	13,320	49	13,369	48	13,417	48	13,465	48	0,1521
0,40	13,559	50	13,609	50	13,659	50	13,709	50	13,759	49	13,808	49	0,1600
0,41	13,896	51	13,947	51	13,998	51	14,049	51	14,100	51	14,151	50	0,1681
0,42	14,232	53	14,285	52	14,337	52	14,389	52	14,441	52	14,493	52	0,1764
0,43	14,568	54	14,622	54	14,676	53	14,729	54	14,783	53	14,836	53	0,1849
0,44	14,904	55	14,959	55	15,014	55	15,069	54	15,123	55	15,178	54	0,1936
0,45	15,239	57	15,296	56	15,352	56	15,408	56	15,464	56	15,520	55	0,2025
0,50	16,914	63	16,977	63	17,040	62	17,102	62	17,164	63	17,227	61	0,2500
0,55	18,582	70	18,652	69	18,721	69	18,790	69	18,859	68	18,927	69	0,3025
0,60	20,243	76	20,319	76	20,395	76	20,471	76	20,547	75	20,622	74	0,3600
0,65	21,897	83	21,980	83	22,063	82	22,145	82	22,227	82	22,309	81	0,4225
0,70	23,542	90	23,632	89	23,721	89	23,810	89	23,899	89	23,988	88	0,4900
0,75	25,177	97	25,274	96	25,370	97	25,467	95	25,562	96	25,658	95	0,5625
0,80	26,802	104	26,906	103	27,009	104	27,113	102	27,215	103	27,318	102	0,6400
0,85	28,415	111	28,526	111	28,637	111	28,748	109	28,857	110	28,967	109	0,7225
0,90	30,015	119	30,134	118	30,252	118	30,370	117	30,487	117	30,604	116	0,8100
0,95	31,600	127	31,727	126	31,853	126	31,979	125	32,104	124	32,228	124	0,9025
1,00	33,169	135	33,304	135	33,439	133	33,572	133	33,705	132	33,837	132	1,0000
	,9385		,9522		,9661		,9800		,9941		1,0082		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r'^2$ nearly.

	338	339	340	341	342	343	344	345	346	347	
1	34	34	34	34	34	34	34	35	35	35	1
2	68	68	68	68	68	69	69	69	69	69	2
3	101	102	102	102	103	103	103	104	104	104	3
4	135	136	136	136	137	137	138	138	138	139	4
5	169	170	170	171	171	172	172	173	173	174	5
6	203	203	204	205	205	206	206	207	208	208	6
7	237	237	238	239	239	240	241	242	242	243	7
8	270	271	272	273	274	274	275	276	277	278	8
9	304	305	306	307	308	309	310	311	311	312	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	1,43 Days	dif.	1,44 Days	dif.	1,45 Days	dif.	1,46 Days	dif.	1,47 Days	dif.	1,48 Days	dif.	1,49 Days	dif.	1,50 Days	dif.	1,51 Days	dif.	1,52 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,348	1	0,349	1	0,350	1	0,351	1	0,352	2	0,354	1	0,355	1	0,356	1	0,357	1	0,358	2	0,0001
0,02	0,695	3	0,698	2	0,700	2	0,702	3	0,705	2	0,707	3	0,710	2	0,712	2	0,714	3	0,717	2	0,0004
0,03	1,043	3	1,046	4	1,050	4	1,054	3	1,057	4	1,061	3	1,064	4	1,068	4	1,072	3	1,075	4	0,0009
0,04	1,390	5	1,395	5	1,400	5	1,405	5	1,410	4	1,414	5	1,419	5	1,424	5	1,429	4	1,433	5	0,0016
0,05	1,738	6	1,744	6	1,750	6	1,756	6	1,762	6	1,768	6	1,774	6	1,780	6	1,786	6	1,792	6	0,0025
0,06	2,085	8	2,093	7	2,100	7	2,107	7	2,114	7	2,121	8	2,129	7	2,136	7	2,143	7	2,150	7	0,0036
0,07	2,433	8	2,441	9	2,450	8	2,458	9	2,467	8	2,475	8	2,483	9	2,492	8	2,500	8	2,508	8	0,0049
0,08	2,780	10	2,790	10	2,800	9	2,809	10	2,819	10	2,829	9	2,838	10	2,848	9	2,857	9	2,866	10	0,0064
0,09	3,128	11	3,139	11	3,150	10	3,160	11	3,171	11	3,182	11	3,193	10	3,203	11	3,214	11	3,225	10	0,0081
0,10	3,475	12	3,487	12	3,499	12	3,511	12	3,523	12	3,535	12	3,547	12	3,559	12	3,571	12	3,583	12	0,0100
0,11	3,822	14	3,836	13	3,849	13	3,862	14	3,876	13	3,889	13	3,902	13	3,915	13	3,928	13	3,941	13	0,0121
0,12	4,170	14	4,184	15	4,199	14	4,213	15	4,228	14	4,242	14	4,256	15	4,271	14	4,285	14	4,299	14	0,0144
0,13	4,517	16	4,533	16	4,549	15	4,564	16	4,580	15	4,595	16	4,611	15	4,626	16	4,642	15	4,657	15	0,0169
0,14	4,864	17	4,881	17	4,898	17	4,915	17	4,932	17	4,949	16	4,965	17	4,982	17	4,999	16	5,015	17	0,0196
0,15	5,211	19	5,230	18	5,248	18	5,266	18	5,284	18	5,302	18	5,320	18	5,338	17	5,355	18	5,373	18	0,0225
0,16	5,558	20	5,578	19	5,597	20	5,617	19	5,636	19	5,655	19	5,674	19	5,693	19	5,712	19	5,731	19	0,0256
0,17	5,905	21	5,926	21	5,947	20	5,967	21	5,988	20	6,008	20	6,028	21	6,049	20	6,069	20	6,089	20	0,0289
0,18	6,252	22	6,274	22	6,296	22	6,318	21	6,339	22	6,361	21	6,382	22	6,404	21	6,425	22	6,447	21	0,0324
0,19	6,599	23	6,622	23	6,645	23	6,668	23	6,691	23	6,714	23	6,737	22	6,759	23	6,782	22	6,804	23	0,0361
0,20	6,946	24	6,970	24	6,994	25	7,019	24	7,043	24	7,067	24	7,091	23	7,114	24	7,138	24	7,162	23	0,0400
0,21	7,293	25	7,318	26	7,344	25	7,369	25	7,394	25	7,419	26	7,445	25	7,470	25	7,495	24	7,519	25	0,0441
0,22	7,639	27	7,666	27	7,693	26	7,719	27	7,746	26	7,772	26	7,798	27	7,825	26	7,851	26	7,877	26	0,0484
0,23	7,986	28	8,014	28	8,042	27	8,069	28	8,097	28	8,125	27	8,152	28	8,180	27	8,207	27	8,234	27	0,0529
0,24	8,332	29	8,361	29	8,390	29	8,419	29	8,448	29	8,477	29	8,506	29	8,535	28	8,563	28	8,591	29	0,0576
0,25	8,678	31	8,709	30	8,739	30	8,769	31	8,800	30	8,830	29	8,859	30	8,889	30	8,919	30	8,949	29	0,0625
0,26	9,025	31	9,056	32	9,088	31	9,119	32	9,151	31	9,182	31	9,213	31	9,244	31	9,275	31	9,306	30	0,0676
0,27	9,371	33	9,404	32	9,436	33	9,469	33	9,502	32	9,534	32	9,566	33	9,599	32	9,631	32	9,663	32	0,0729
0,28	9,717	34	9,751	34	9,785	34	9,819	33	9,852	34	9,886	34	9,920	33	9,953	33	9,986	34	10,020	33	0,0784
0,29	10,062	36	10,098	35	10,133	35	10,168	35	10,203	35	10,238	35	10,273	34	10,307	35	10,342	34	10,376	35	0,0841
0,30	10,408	37	10,445	36	10,481	37	10,518	36	10,554	36	10,590	36	10,626	36	10,662	35	10,697	36	10,733	35	0,0900
0,31	10,754	38	10,792	37	10,829	38	10,867	37	10,904	38	10,942	37	10,979	37	11,016	37	11,053	37	11,090	36	0,0961
0,32	11,099	39	11,138	39	11,177	39	11,216	39	11,255	38	11,293	39	11,332	38	11,370	38	11,408	38	11,446	38	0,1024
0,33	11,444	41	11,485	40	11,525	40	11,565	40	11,605	40	11,645	39	11,684	40	11,724	39	11,763	39	11,802	39	0,1089
0,34	11,790	41	11,831	42	11,873	41	11,914	41	11,955	41	11,996	41	12,037	40	12,077	41	12,118	40	12,158	41	0,1156
0,35	12,135	42	12,177	43	12,220	43	12,263	42	12,305	42	12,347	42	12,389	42	12,431	42	12,473	41	12,514	42	0,1225
0,36	12,479	44	12,523	44	12,567	44	12,611	44	12,655	43	12,698	43	12,741	43	12,784	43	12,827	43	12,870	43	0,1296
0,37	12,824	45	12,869	46	12,915	44	12,959	45	13,004	45	13,049	44	13,093	45	13,138	44	13,182	44	13,226	44	0,1369
0,38	13,169	46	13,215	47	13,262	46	13,308	46	13,354	46	13,400	45	13,445	46	13,491	45	13,536	45	13,581	46	0,1444
0,39	13,513	48	13,561	47	13,608	48	13,656	47	13,703	47	13,750	47	13,797	47	13,844	46	13,890	47	13,937	46	0,1521
0,40	13,857	49	13,906	49	13,955	49	14,004	48	14,052	49	14,101	48	14,149	48	14,197	48	14,245	47	14,292	48	0,1600
0,41	14,201	51	14,252	50	14,302	49	14,351	50	14,401	50	14,451	49	14,500	49	14,549	49	14,598	49	14,647	49	0,1681
0,42	14,545	52	14,597	51	14,648	51	14,699	51	14,750	51	14,801	50	14,851	51	14,902	50	14,952	50	15,002	50	0,1764
0,43	14,889	52	14,941	53	14,994	52	15,046	53	15,099	52	15,151	52	15,203	51	15,254	52	15,306	51	15,357	51	0,1849
0,44	15,232	54	15,286	54	15,340	54	15,394	53	15,447	53	15,500	53	15,553	53	15,606	53	15,659	53	15,712	52	0,1936
0,45	15,575	56	15,631	55	15,686	55	15,741	54	15,795	55	15,850	54	15,904	54	15,958	54	16,012	54	16,066	54	0,2025
0,50	17,288	62	17,350	61	17,411	62	17,473	61	17,534	60	17,594	61	17,655	60	17,715	60	17,775	60	17,835	60	0,2500
0,55	18,996	68	19,064	67	19,131	68	19,199	67	19,266	67	19,333	67	19,400	67	19,467	66	19,533	66	19,599	66	0,3025
0,60	20,696	75	20,771	74	20,845	74	20,919	74	20,993	73	21,066	73	21,139	73	21,212	73	21,285	72	21,357	72	0,3600
0,65	22,390	81	22,471	81	22,552	80	22,632	81	22,713	79	22,792	80	22,872	79	22,951	79	23,030	79	23,109	78	0,4225
0,70	24,076	87	24,163	88	24,251	87	24,338	87	24,425	86	24,511	86	24,597	86	24,683	85	24,768	85	24,853	85	0,4900
0,75	25,753	94	25,847	94	25,941	94	26,035	93	26,128	93	26,221	93	26,314	92	26,406	92	26,498	92	26,590	91	0,5625
0,80	27,420	101	27,521	101	27,622	101	27,723	100	27,823	100	27,923	99	28,022	99	28,121	99	28,220	98	28,318	98	0,6400
0,85	29,076	108	29,184	109	29,293	107	29,400	107	29,507	107	29,614	106	29,720	106	29,826	106	29,932	105	30,037	104	0,7225
0,90	30,720	116	30,836	116	30,952	114	31,066	115	31,181	113	31,294	114	31,408	112	31,520	113	31,633	112	31,745	111	0,8100
0,95	32,352	123	32,475	123	32,598	122	32,720	121	32,841	121	32,962	121	33,083	120	33,203	119	33,322	119	33,441	119	0,9025
1,00	33,969	131	34,100	130	34,230	130	34,360	129	34,489	128	34,617	128	34,745	128	34,873	126	34,999	126	35,125	126	1,0000
	1,0225		1,0368		1,0513		1,0658		1,0805		1,0952		1,1101		1,1250		1,1401		1,1552		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	
1	34	34	35	35	35	35	35	35	35	35	35	35	36	36	36	36	36	1
2	69	69	69	69	69	70	70	70	70	70	71	71	71	71	71	72	72	2
3	103	103	104	104	104	104	105	105	105	106	106	106	107	107	107	107	108	3
4	137	138	138	138	139	139	140	140	140	141	141	142	142	142	143	143	144	4
5	172	172	173	173	174	174	175	175	176	176	177	177	178	178	179	179	180	5
6	206	206	207	208	208	209	209	210	211	211	212	212	213	214	214	215	215	6
7	240	241	242	242	243	244	244	245	246	246	247	248	249	249	250	251	251	7
8	274	275	276	277	278	278	279	280	281	282	282	283	284	285	286	286	287	8
9	309	310	311	311	312	313	314	315	316	317	318	319	320	320	321	322	323	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r+r''$. 1,53 Days	dif.	1,54 Days	dif.	1,55 Days	dif.	1,56 Days	dif.	1,57 Days	dif.	1,58 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,360	1	0,361	1	0,362	1	0,363	1	0,364	1	0,365	2	0,0001
0,02	0,719	2	0,721	3	0,724	2	0,726	2	0,728	3	0,731	2	0,0004
0,03	1,079	3	1,082	4	1,086	3	1,089	4	1,093	3	1,096	4	0,0009
0,04	1,438	5	1,443	4	1,447	5	1,452	5	1,457	4	1,461	5	0,0016
0,05	1,798	5	1,803	6	1,809	6	1,815	6	1,821	6	1,827	5	0,0025
0,06	2,157	7	2,164	7	2,171	7	2,178	7	2,185	7	2,192	7	0,0036
0,07	2,516	9	2,525	8	2,533	8	2,541	8	2,549	8	2,557	8	0,0049
0,08	2,876	9	2,885	10	2,895	9	2,904	9	2,913	10	2,923	9	0,0064
0,09	3,235	11	3,246	10	3,256	11	3,267	10	3,277	11	3,288	10	0,0081
0,10	3,595	11	3,606	12	3,618	12	3,630	11	3,641	12	3,653	12	0,0100
0,11	3,954	13	3,967	13	3,980	13	3,993	12	4,005	13	4,018	13	0,0121
0,12	4,313	14	4,327	14	4,341	14	4,355	14	4,369	14	4,383	14	0,0144
0,13	4,672	16	4,688	15	4,703	15	4,718	15	4,733	15	4,748	15	0,0169
0,14	5,032	16	5,048	16	5,064	17	5,081	16	5,097	16	5,113	16	0,0196
0,15	5,391	17	5,408	18	5,426	17	5,443	18	5,461	17	5,478	18	0,0225
0,16	5,750	19	5,769	18	5,787	19	5,806	19	5,825	18	5,843	19	0,0256
0,17	6,109	20	6,129	20	6,149	20	6,169	19	6,188	20	6,208	20	0,0289
0,18	6,468	21	6,489	21	6,510	21	6,531	21	6,552	21	6,573	21	0,0324
0,19	6,827	22	6,849	22	6,871	22	6,893	23	6,916	22	6,938	22	0,0361
0,20	7,185	24	7,209	23	7,232	24	7,256	23	7,279	23	7,302	23	0,0400
0,21	7,544	25	7,569	24	7,593	25	7,618	24	7,642	25	7,667	24	0,0441
0,22	7,903	26	7,929	25	7,954	26	7,980	26	8,006	25	8,031	26	0,0484
0,23	8,261	27	8,288	27	8,315	27	8,342	27	8,369	27	8,396	26	0,0529
0,24	8,620	28	8,648	28	8,676	28	8,704	28	8,732	28	8,760	28	0,0576
0,25	8,978	30	9,008	29	9,037	29	9,066	29	9,095	29	9,124	29	0,0625
0,26	9,336	31	9,367	31	9,398	30	9,428	30	9,458	30	9,488	31	0,0676
0,27	9,695	31	9,726	32	9,758	32	9,790	31	9,821	32	9,853	31	0,0729
0,28	10,053	33	10,086	33	10,119	32	10,151	33	10,184	33	10,217	33	0,0784
0,29	10,411	34	10,445	34	10,479	34	10,513	34	10,547	33	10,580	34	0,0841
0,30	10,768	36	10,804	35	10,839	35	10,874	35	10,909	35	10,944	35	0,0900
0,31	11,126	37	11,163	36	11,199	36	11,235	37	11,272	36	11,308	36	0,0961
0,32	11,484	38	11,522	37	11,559	38	11,597	37	11,634	37	11,671	37	0,1024
0,33	11,841	39	11,880	39	11,919	39	11,958	38	11,996	39	12,035	38	0,1089
0,34	12,199	40	12,239	40	12,279	40	12,319	39	12,358	40	12,398	39	0,1156
0,35	12,556	41	12,597	41	12,638	41	12,679	41	12,720	41	12,761	41	0,1225
0,36	12,913	42	12,955	43	12,998	42	13,040	42	13,082	42	13,124	42	0,1296
0,37	13,270	44	13,314	43	13,357	44	13,401	43	13,444	43	13,487	43	0,1369
0,38	13,627	44	13,671	45	13,716	45	13,761	44	13,805	45	13,850	44	0,1444
0,39	13,983	46	14,029	46	14,075	46	14,121	46	14,167	45	14,212	46	0,1521
0,40	14,340	47	14,387	47	14,434	47	14,481	47	14,528	47	14,575	46	0,1600
0,41	14,696	49	14,745	48	14,793	48	14,841	48	14,889	48	14,937	48	0,1681
0,42	15,052	50	15,102	49	15,151	50	15,201	49	15,250	49	15,299	49	0,1764
0,43	15,408	51	15,459	51	15,510	50	15,560	51	15,611	50	15,661	50	0,1849
0,44	15,764	52	15,816	52	15,868	52	15,920	51	15,971	52	16,023	51	0,1936
0,45	16,120	53	16,173	53	16,226	53	16,279	53	16,332	53	16,385	52	0,2025
0,50	17,895	59	17,954	60	18,014	59	18,073	58	18,131	59	18,190	59	0,2500
0,55	19,665	66	19,731	65	19,796	65	19,861	65	19,926	65	19,991	64	0,3025
0,60	21,429	72	21,501	72	21,573	71	21,644	71	21,715	71	21,786	70	0,3600
0,65	23,187	78	23,265	78	23,343	78	23,421	77	23,498	77	23,575	77	0,4225
0,70	24,938	85	25,023	84	25,107	84	25,191	83	25,274	83	25,357	83	0,4900
0,75	26,681	91	26,772	91	26,863	90	26,953	90	27,043	90	27,133	89	0,5625
0,80	28,416	97	28,513	97	28,610	97	28,707	97	28,804	96	28,900	95	0,6400
0,85	30,141	104	30,245	104	30,349	103	30,452	103	30,555	103	30,658	102	0,7225
0,90	31,856	111	31,967	111	32,078	110	32,188	109	32,297	110	32,407	108	0,8100
0,95	33,560	118	33,678	117	33,795	117	33,912	117	34,029	116	34,145	115	0,9025
1,00	35,251	125	35,376	125	35,501	124	35,625	123	35,748	123	35,871	123	10,000
	1,1705		1,1858		1,2013		1,2168		1,2325		1,2482		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	356	357	358	359	360	361	362	363	364	365	366	
1	36	36	36	36	36	36	36	36	36	37	37	1
2	71	71	72	72	72	72	72	73	73	73	73	2
3	107	107	107	108	108	108	109	109	109	110	110	3
4	142	143	143	144	144	144	145	145	146	146	146	4
5	178	179	179	180	180	181	181	182	182	183	183	5
6	214	214	215	215	216	217	217	218	218	219	220	6
7	249	250	251	251	252	253	253	254	255	256	256	7
8	285	286	286	287	288	289	290	290	291	292	293	8
9	320	321	322	323	324	325	326	327	328	329	329	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	1,59 Days	dif.	1,60 Days	dif.	1,61 Days	dif.	1,62 Days	dif.	1,63 Days	dif.	1,64 Days	dif.	1,65 Days	dif.	1,66 Days	dif.	1,67 Days	dif.	1,68 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,367	1	0,368	1	0,369	1	0,370	1	0,371	1	0,372	1	0,373	2	0,375	1	0,376	1	0,377	1	0,0001
0,02	0,733	2	0,735	3	0,738	3	0,740	2	0,742	2	0,744	3	0,747	2	0,749	2	0,751	2	0,753	2	0,0004
0,03	1,100	3	1,103	3	1,106	4	1,110	3	1,113	4	1,117	3	1,120	3	1,123	4	1,127	3	1,130	4	0,0009
0,04	1,466	5	1,471	4	1,475	5	1,480	4	1,484	5	1,489	4	1,493	5	1,498	4	1,502	5	1,507	4	0,0016
0,05	1,832	6	1,838	6	1,844	6	1,850	5	1,855	6	1,861	6	1,867	5	1,872	6	1,878	6	1,884	5	0,0025
0,06	2,199	7	2,206	7	2,213	7	2,220	6	2,226	7	2,233	7	2,240	7	2,247	7	2,254	6	2,260	7	0,0036
0,07	2,565	8	2,573	8	2,581	8	2,589	8	2,597	8	2,605	8	2,613	8	2,621	8	2,629	8	2,637	8	0,0049
0,08	2,932	9	2,941	9	2,950	9	2,959	9	2,968	10	2,978	9	2,987	9	2,996	9	3,005	9	3,014	9	0,0064
0,09	3,298	11	3,309	10	3,319	10	3,329	10	3,339	11	3,350	10	3,360	10	3,370	10	3,380	10	3,390	10	0,0081
0,10	3,665	11	3,676	12	3,688	11	3,699	11	3,710	12	3,722	11	3,733	11	3,744	12	3,756	11	3,767	11	0,0100
0,11	4,031	12	4,043	13	4,056	13	4,069	12	4,081	13	4,094	12	4,106	13	4,119	12	4,131	12	4,143	13	0,0121
0,12	4,397	14	4,411	14	4,425	13	4,438	14	4,452	14	4,466	13	4,479	14	4,493	13	4,506	14	4,520	13	0,0144
0,13	4,763	15	4,778	15	4,793	15	4,808	15	4,823	15	4,838	14	4,852	15	4,867	15	4,882	14	4,896	15	0,0169
0,14	5,129	17	5,146	16	5,162	16	5,178	16	5,194	16	5,210	16	5,226	15	5,241	16	5,257	16	5,273	16	0,0196
0,15	5,496	17	5,513	17	5,530	17	5,547	17	5,564	17	5,581	17	5,598	17	5,615	17	5,632	17	5,649	17	0,0225
0,16	5,862	18	5,880	19	5,899	18	5,917	18	5,935	18	5,953	18	5,971	19	5,990	18	6,008	18	6,026	18	0,0256
0,17	6,228	19	6,247	20	6,267	19	6,286	20	6,306	19	6,325	19	6,344	20	6,364	19	6,383	19	6,402	19	0,0289
0,18	6,594	20	6,614	21	6,635	21	6,656	20	6,676	21	6,697	20	6,717	21	6,738	20	6,758	20	6,778	20	0,0324
0,19	6,960	21	6,981	22	7,003	22	7,025	22	7,047	21	7,068	22	7,090	21	7,111	22	7,133	21	7,154	22	0,0361
0,20	7,325	23	7,348	23	7,371	23	7,394	23	7,417	23	7,440	23	7,463	22	7,485	23	7,508	22	7,530	23	0,0400
0,21	7,691	24	7,715	24	7,739	25	7,764	24	7,788	23	7,811	24	7,835	24	7,859	24	7,883	23	7,906	24	0,0441
0,22	8,057	25	8,082	25	8,107	26	8,133	25	8,158	25	8,183	25	8,208	25	8,233	25	8,258	24	8,282	25	0,0484
0,23	8,422	27	8,449	26	8,475	27	8,502	26	8,528	26	8,554	26	8,580	26	8,606	26	8,632	26	8,658	26	0,0529
0,24	8,788	28	8,816	27	8,843	28	8,871	27	8,898	27	8,925	28	8,953	27	8,980	27	9,007	27	9,034	27	0,0576
0,25	9,153	29	9,182	29	9,211	29	9,240	28	9,268	29	9,297	28	9,325	28	9,353	29	9,382	28	9,410	28	0,0625
0,26	9,519	30	9,549	30	9,579	29	9,608	30	9,638	30	9,668	29	9,697	30	9,727	29	9,756	29	9,785	30	0,0676
0,27	9,884	31	9,915	31	9,946	31	9,977	31	10,008	31	10,039	30	10,069	31	10,100	31	10,131	30	10,161	30	0,0729
0,28	10,249	32	10,281	33	10,314	32	10,346	32	10,378	32	10,410	32	10,442	31	10,473	32	10,505	31	10,536	32	0,0784
0,29	10,614	34	10,648	33	10,681	33	10,714	33	10,747	33	10,780	33	10,813	33	10,846	33	10,879	33	10,912	32	0,0841
0,30	10,979	35	11,014	34	11,048	35	11,083	34	11,117	34	11,151	34	11,185	34	11,219	34	11,253	34	11,287	34	0,0900
0,31	11,344	36	11,380	35	11,415	36	11,451	35	11,486	36	11,522	35	11,557	35	11,592	35	11,627	35	11,662	35	0,0961
0,32	11,708	37	11,745	37	11,782	37	11,819	37	11,856	36	11,892	37	11,929	36	11,965	36	12,001	36	12,037	36	0,1024
0,33	12,073	38	12,111	38	12,149	38	12,187	38	12,225	38	12,263	37	12,300	38	12,338	37	12,375	37	12,412	37	0,1089
0,34	12,437	40	12,477	39	12,516	39	12,555	39	12,594	39	12,633	39	12,672	38	12,710	39	12,749	38	12,787	38	0,1156
0,35	12,802	40	12,842	41	12,883	40	12,923	40	12,963	40	13,003	40	13,043	40	13,083	39	13,122	40	13,162	39	0,1225
0,36	13,166	42	13,208	41	13,249	42	13,291	41	13,332	41	13,373	41	13,414	41	13,455	41	13,496	40	13,536	41	0,1296
0,37	13,530	43	13,573	43	13,616	42	13,658	43	13,701	42	13,743	42	13,785	42	13,827	42	13,869	42	13,911	42	0,1369
0,38	13,894	44	13,938	44	13,982	44	14,026	43	14,069	44	14,113	43	14,156	43	14,199	43	14,242	43	14,285	43	0,1444
0,39	14,258	45	14,303	45	14,348	45	14,393	45	14,438	44	14,482	45	14,527	44	14,571	44	14,615	45	14,660	44	0,1521
0,40	14,621	47	14,668	46	14,714	46	14,760	46	14,806	46	14,852	46	14,898	45	14,943	45	14,988	46	15,034	45	0,1600
0,41	14,985	47	15,032	48	15,080	47	15,127	47	15,174	47	15,221	47	15,268	47	15,315	46	15,361	47	15,408	46	0,1681
0,42	15,348	49	15,397	48	15,445	49	15,494	48	15,542	48	15,590	48	15,638	48	15,686	48	15,734	47	15,781	48	0,1764
0,43	15,711	50	15,761	50	15,811	50	15,861	49	15,910	49	15,959	50	16,009	49	16,058	48	16,106	49	16,155	49	0,1849
0,44	16,074	51	16,125	51	16,176	51	16,227	51	16,278	50	16,328	51	16,379	50	16,429	50	16,479	50	16,529	49	0,1936
0,45	16,437	52	16,489	53	16,542	52	16,594	51	16,645	52	16,697	51	16,748	52	16,800	51	16,851	51	16,902	51	0,2025
0,50	18,249	58	18,307	58	18,365	58	18,423	58	18,481	57	18,538	57	18,595	58	18,653	57	18,710	56	18,766	57	0,2500
0,55	20,055	65	20,120	64	20,184	63	20,247	64	20,311	64	20,375	63	20,438	63	20,501	63	20,564	62	20,626	61	0,3025
0,60	21,856	71	21,927	70	21,997	70	22,067	69	22,136	70	22,206	69	22,275	69	22,344	69	22,413	68	22,481	68	0,3600
0,65	23,652	76	23,728	76	23,804	76	23,880	76	23,956	76	24,032	75	24,107	75	24,182	74	24,256	75	24,331	7	0,4225
0,70	25,440	83	25,523	82	25,605	83	25,688	81	25,769	82	25,851	81	25,932	81	26,013	81	26,094	80	26,174	81	0,4900
0,75	27,222	89	27,311	88	27,399	89	27,488	88	27,576	88	27,664	87	27,751	87	27,838	87	27,925	86	28,011	87	0,5625
0,80	28,995	96	29,091	95	29,186	94	29,280	95	29,375	94	29,469	93	29,562	94	29,656	93	29,749	92	29,841	9	0,6400
0,85	30,760	102	30,862	101	30,963	101	31,064	101	31,165	100	31,265	100	31,365	100	31,465	99	31,564	99	31,663	99	0,7225
0,90	32,515	109	32,624	108	32,732	107	32,839	108	32,947	106	33,053	107	33,160	106	33,266	105	33,371	106	33,477	105	0,8100
0,95	34,260	115	34,375	115	34,490	114	34,604	114	34,718	114	34,832	112	34,944	113	35,057	112	35,169	112	35,281	111	0,9025
1,00	35,994	122	36,116	121	36,237	121	36,358	121	36,479	120	36,599	120	36,719	119	36,838	119	36,957	118	37,075	118	1,0000
	1,2611		1,2800		1,2961		1,3122		1,3285		1,3148		1,3613		1,3778		1,3945		1,4112		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	
1	36	36	37	37	37	37	37	37	37	37	37	37	38	38	38	1
2	73	73	73	73	73	74	74	74	74	74	75	75	75	75	75	2
3	109	109	110	110	110	110	111	111	111	112	112	112	113	113	113	3
4	145	146	146	146	147	147	148	148	148	149	149	150	150	150	151	4
5	182	182	183	183	184	184	185	185	186	186	187	187	188	188	189	5
6	218	218	219	220	220	221	221	222	223	223	224	224	225	226	226	6
7	254	255	256	256	257	258	258	259	260	260	261	262	263	263	264	7
8	290	291	292	293	294	294	295	296	297	298	298	299	300	301	302	8
9	327	328	329	329	330	331	332	333	334	335	336	337	338	338	339	9

Sum of the Radii $r+r''$.

Chord c.	1,69 Days	dif.	1,70 Days	dif.	1,71 Days	dif.	1,72 Days	dif.	1,73 Days	dif.	1,74 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,378	1	0,379	1	0,380	1	0,381	1	0,382	1	0,383	2	0,0001
0,02	0,756	2	0,758	2	0,760	2	0,762	3	0,765	2	0,767	2	0,0004
0,03	1,134	3	1,137	3	1,140	4	1,144	3	1,147	3	1,150	4	0,0009
0,04	1,511	5	1,516	4	1,520	5	1,525	4	1,529	5	1,534	4	0,0016
0,05	1,889	6	1,895	5	1,900	6	1,906	5	1,911	6	1,917	5	0,0025
0,06	2,267	7	2,274	6	2,280	7	2,287	7	2,294	6	2,300	7	0,0036
0,07	2,645	8	2,653	7	2,660	8	2,668	8	2,676	8	2,684	7	0,0049
0,08	3,023	9	3,032	8	3,040	9	3,049	9	3,058	9	3,067	9	0,0064
0,09	3,400	10	3,410	10	3,420	10	3,430	10	3,440	10	3,450	10	0,0081
0,10	3,778	11	3,789	11	3,800	11	3,811	12	3,823	11	3,834	11	0,0100
0,11	4,156	12	4,168	12	4,180	12	4,192	13	4,205	12	4,217	12	0,0121
0,12	4,533	14	4,547	13	4,560	13	4,573	14	4,587	13	4,600	13	0,0144
0,13	4,911	14	4,925	15	4,940	14	4,954	15	4,969	14	4,983	14	0,0169
0,14	5,289	15	5,304	16	5,320	15	5,335	16	5,351	15	5,366	16	0,0196
0,15	5,666	17	5,683	17	5,700	16	5,716	17	5,733	16	5,749	17	0,0225
0,16	6,044	17	6,061	18	6,079	18	6,097	18	6,115	17	6,132	18	0,0256
0,17	6,421	19	6,440	19	6,459	19	6,478	19	6,497	18	6,515	19	0,0289
0,18	6,798	20	6,818	20	6,838	20	6,858	20	6,878	20	6,898	20	0,0324
0,19	7,176	21	7,197	21	7,218	21	7,239	21	7,260	21	7,281	21	0,0361
0,20	7,553	22	7,575	22	7,597	23	7,620	22	7,642	22	7,664	22	0,0400
0,21	7,930	23	7,953	24	7,977	23	8,000	23	8,023	24	8,047	23	0,0441
0,22	8,307	25	8,332	24	8,356	25	8,381	24	8,405	24	8,429	25	0,0481
0,23	8,684	26	8,710	25	8,735	26	8,761	26	8,787	25	8,812	25	0,0529
0,24	9,061	27	9,088	27	9,115	26	9,141	27	9,168	27	9,195	26	0,0576
0,25	9,438	28	9,466	28	9,494	28	9,522	27	9,549	28	9,577	28	0,0625
0,26	9,815	29	9,844	29	9,873	29	9,902	29	9,931	28	9,959	29	0,0676
0,27	10,191	31	10,222	30	10,252	30	10,282	30	10,312	30	10,342	29	0,0729
0,28	10,568	31	10,599	32	10,631	31	10,662	31	10,693	31	10,724	31	0,0784
0,29	10,944	33	10,977	32	11,009	33	11,042	32	11,074	32	11,106	32	0,0841
0,30	11,321	33	11,354	34	11,388	33	11,421	34	11,455	33	11,488	33	0,0900
0,31	11,697	35	11,732	35	11,767	34	11,801	35	11,836	34	11,870	34	0,0961
0,32	12,073	36	12,109	36	12,145	36	12,181	35	12,216	36	12,252	35	0,1024
0,33	12,449	37	12,486	37	12,523	37	12,560	37	12,597	36	12,633	37	0,1089
0,34	12,825	39	12,864	38	12,902	38	12,940	37	12,977	38	13,015	38	0,1156
0,35	13,201	40	13,241	39	13,280	39	13,319	39	13,358	39	13,397	38	0,1225
0,36	13,577	40	13,617	41	13,658	40	13,698	40	13,738	40	13,778	40	0,1296
0,37	13,953	41	13,994	42	14,036	41	14,077	41	14,118	41	14,159	41	0,1369
0,38	14,328	43	14,371	42	14,413	43	14,456	42	14,498	42	14,540	42	0,1444
0,39	14,704	43	14,747	44	14,791	44	14,835	43	14,878	43	14,921	44	0,1521
0,40	15,079	45	15,124	45	15,169	44	15,213	45	15,258	44	15,302	45	0,1600
0,41	15,454	46	15,500	46	15,546	46	15,592	45	15,637	46	15,683	45	0,1681
0,42	15,829	47	15,876	47	15,923	47	15,970	47	16,017	47	16,064	46	0,1764
0,43	16,204	48	16,252	48	16,300	48	16,348	48	16,396	48	16,444	48	0,1849
0,44	16,578	50	16,628	49	16,677	49	16,726	50	16,776	49	16,825	48	0,1936
0,45	16,953	51	17,004	50	17,054	50	17,104	51	17,155	50	17,205	50	0,2025
0,50	18,823	56	18,879	57	18,936	56	18,992	56	19,048	55	19,103	56	0,2500
0,55	20,689	62	20,751	62	20,813	62	20,875	62	20,937	61	20,998	61	0,3025
0,60	22,550	68	22,618	68	22,686	67	22,753	68	22,821	67	22,888	67	0,3600
0,65	24,405	74	24,479	74	24,553	73	24,626	74	24,700	73	24,773	73	0,4225
0,70	26,255	80	26,335	79	26,414	80	26,494	79	26,573	79	26,652	78	0,4900
0,75	28,098	86	28,184	85	28,269	86	28,355	85	28,440	85	28,525	84	0,5625
0,80	29,934	92	30,026	91	30,117	92	30,209	91	30,300	91	30,391	91	0,6400
0,85	31,762	98	31,860	98	31,958	98	32,056	97	32,153	97	32,250	96	0,7225
0,90	33,582	104	33,686	104	33,790	104	33,894	104	33,998	103	34,101	102	0,8100
0,95	35,392	111	35,503	111	35,614	110	35,724	109	35,833	110	35,943	109	0,9025
1,00	37,193	117	37,310	117	37,427	117	37,544	116	37,660	116	37,776	115	1,0000
	1,4281		1,4450		1,4621		1,4792		1,4965		1,5138		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

	374	375	376	377	378	379	380	381	382	383	384	
1	37	38	38	38	38	38	38	38	38	38	38	1
2	75	75	75	75	76	76	76	76	76	77	77	2
3	112	113	113	113	113	114	114	114	115	115	115	3
4	150	150	150	151	151	152	152	152	153	153	154	4
5	187	188	188	189	189	190	190	191	191	192	192	5
6	224	225	226	226	227	227	228	229	229	230	230	6
7	262	263	263	264	265	265	266	267	267	268	269	7
8	299	300	301	302	302	303	304	305	306	306	307	8
9	337	338	338	339	340	341	342	343	344	345	346	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

	Sum of the Radii $r + r''$.																				
Chord c.	1,75		1,76		1,77		1,78		1,79		1,80		1,81		1,82		1,83		1,84		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,385	1	0,386	1	0,387	1	0,388	1	0,389	1	0,390	1	0,391	1	0,392	1	0,393	1	0,394	1	0,0001
0,02	0,769	2	0,771	2	0,773	3	0,776	2	0,778	2	0,780	2	0,782	2	0,784	2	0,786	3	0,789	2	0,0004
0,03	1,154	3	1,157	3	1,160	3	1,163	4	1,167	3	1,170	3	1,173	3	1,176	4	1,180	3	1,183	3	0,0009
0,04	1,538	4	1,542	5	1,547	4	1,551	4	1,555	5	1,560	4	1,564	4	1,568	5	1,573	4	1,577	4	0,0016
0,05	1,922	6	1,928	5	1,933	6	1,939	5	1,944	6	1,950	5	1,955	6	1,961	5	1,966	5	1,971	6	0,0025
0,06	2,307	7	2,314	6	2,320	7	2,327	6	2,333	7	2,340	6	2,346	7	2,353	6	2,359	7	2,366	6	0,0036
0,07	2,691	8	2,699	8	2,707	7	2,714	8	2,722	8	2,730	7	2,737	8	2,745	7	2,752	8	2,760	7	0,0049
0,08	3,076	9	3,085	8	3,093	9	3,102	9	3,111	8	3,119	9	3,128	9	3,137	8	3,145	9	3,154	8	0,0064
0,09	3,460	10	3,470	10	3,480	10	3,490	10	3,500	9	3,509	10	3,519	10	3,529	9	3,538	10	3,548	10	0,0081
0,10	3,845	11	3,856	10	3,866	11	3,877	11	3,888	11	3,899	11	3,910	11	3,921	11	3,932	10	3,942	11	0,0100
0,11	4,229	12	4,241	12	4,253	12	4,265	12	4,277	12	4,289	12	4,301	12	4,313	12	4,325	11	4,336	12	0,0121
0,12	4,613	13	4,626	14	4,640	13	4,653	13	4,666	13	4,679	13	4,692	13	4,705	13	4,718	12	4,730	13	0,0144
0,13	4,997	15	5,012	14	5,026	14	5,040	14	5,054	14	5,068	14	5,082	15	5,097	14	5,111	13	5,124	14	0,0169
0,14	5,382	15	5,397	15	5,412	16	5,428	15	5,443	15	5,458	15	5,473	15	5,488	15	5,503	16	5,519	14	0,0196
0,15	5,766	16	5,782	17	5,799	16	5,815	16	5,831	17	5,848	16	5,864	16	5,880	16	5,896	16	5,912	17	0,0225
0,16	6,150	18	6,168	17	6,185	18	6,203	17	6,220	17	6,237	18	6,255	17	6,272	17	6,289	17	6,306	18	0,0256
0,17	6,534	19	6,553	18	6,571	19	6,590	18	6,608	19	6,627	18	6,645	19	6,664	18	6,682	18	6,700	18	0,0289
0,18	6,918	20	6,938	20	6,958	19	6,977	20	6,997	19	7,016	20	7,036	19	7,055	20	7,075	19	7,094	19	0,0324
0,19	7,302	21	7,323	21	7,344	21	7,365	20	7,385	21	7,406	20	7,426	21	7,447	20	7,467	21	7,488	20	0,0361
0,20	7,686	22	7,708	22	7,730	22	7,752	22	7,774	21	7,795	22	7,817	22	7,839	21	7,860	22	7,882	21	0,0400
0,21	8,070	23	8,093	23	8,116	23	8,139	23	8,162	23	8,185	22	8,207	23	8,230	23	8,253	22	8,275	23	0,0441
0,22	8,454	24	8,478	24	8,502	24	8,526	24	8,550	24	8,574	24	8,598	23	8,621	24	8,645	24	8,669	23	0,0484
0,23	8,837	26	8,863	25	8,888	25	8,913	25	8,938	25	8,963	25	8,988	25	9,013	25	9,038	24	9,062	25	0,0529
0,24	9,221	26	9,247	27	9,274	26	9,300	26	9,326	26	9,352	26	9,378	26	9,404	26	9,430	26	9,456	26	0,0576
0,25	9,605	27	9,632	27	9,659	28	9,687	27	9,714	27	9,741	27	9,768	27	9,795	27	9,822	27	9,849	27	0,0625
0,26	9,988	29	10,017	28	10,045	29	10,074	28	10,102	28	10,130	28	10,158	29	10,187	28	10,215	28	10,243	27	0,0676
0,27	10,371	30	10,401	30	10,431	29	10,460	30	10,490	29	10,519	29	10,548	30	10,578	29	10,607	29	10,636	29	0,0729
0,28	10,755	31	10,786	30	10,816	31	10,847	30	10,877	31	10,908	30	10,938	31	10,969	30	10,999	30	11,029	30	0,0784
0,29	11,138	32	11,170	32	11,202	31	11,233	32	11,265	32	11,297	31	11,328	32	11,360	31	11,391	31	11,422	31	0,0841
0,30	11,521	33	11,554	33	11,587	33	11,620	33	11,653	32	11,685	33	11,718	32	11,750	33	11,783	32	11,815	32	0,0900
0,31	11,904	34	11,938	34	11,972	34	12,006	34	12,040	34	12,074	34	12,108	33	12,141	34	12,175	33	12,208	33	0,0961
0,32	12,287	35	12,322	35	12,357	36	12,393	34	12,427	35	12,462	35	12,497	35	12,532	34	12,566	35	12,601	34	0,1024
0,33	12,670	36	12,706	37	12,743	36	12,779	36	12,815	36	12,851	36	12,887	35	12,922	36	12,958	35	12,993	36	0,1089
0,34	13,053	37	13,090	37	13,127	38	13,165	37	13,202	37	13,239	37	13,276	37	13,313	36	13,349	37	13,386	37	0,1156
0,35	13,435	39	13,474	38	13,512	39	13,551	38	13,589	39	13,627	38	13,665	38	13,703	38	13,741	38	13,779	37	0,1225
0,36	13,818	39	13,857	40	13,897	40	13,937	39	13,976	39	14,015	39	14,054	39	14,093	39	14,132	39	14,171	39	0,1296
0,37	14,200	41	14,241	41	14,282	40	14,322	41	14,363	40	14,403	40	14,443	40	14,483	40	14,523	40	14,563	40	0,1369
0,38	14,582	42	14,624	42	14,666	42	14,708	41	14,749	42	14,791	41	14,832	41	14,873	42	14,915	41	14,956	40	0,1444
0,39	14,965	43	15,008	43	15,051	42	15,093	43	15,136	43	15,179	42	15,221	42	15,263	43	15,306	42	15,348	42	0,1521
0,40	15,347	44	15,391	44	15,435	44	15,479	43	15,522	44	15,566	44	15,610	43	15,653	43	15,696	44	15,740	43	0,1600
0,41	15,728	46	15,774	45	15,819	45	15,864	45	15,909	45	15,954	44	15,998	45	16,043	44	16,087	44	16,131	45	0,1681
0,42	16,110	47	16,157	46	16,203	46	16,249	46	16,295	46	16,341	46	16,387	45	16,432	46	16,478	45	16,523	45	0,1764
0,43	16,492	47	16,539	48	16,587	47	16,634	47	16,681	47	16,728	47	16,775	47	16,822	46	16,868	47	16,915	46	0,1849
0,44	16,873	49	16,922	48	16,970	49	17,019	48	17,067	48	17,115	48	17,163	48	17,211	48	17,259	47	17,306	48	0,1936
0,45	17,255	49	17,304	50	17,354	50	17,404	49	17,453	49	17,502	49	17,551	49	17,600	49	17,649	49	17,698	48	0,2025
0,50	19,159	55	19,214	56	19,270	55	19,325	55	19,380	55	19,435	54	19,489	55	19,544	54	19,598	54	19,652	54	0,2500
0,55	21,059	61	21,120	61	21,181	61	21,242	61	21,303	60	21,363	60	21,423	60	21,483	60	21,543	60	21,603	59	0,3025
0,60	22,955	67	23,022	66	23,088	67	23,155	66	23,221	66	23,287	66	23,353	66	23,419	65	23,484	65	23,549	65	0,3600
0,65	24,846	72	24,918	73	24,991	72	25,063	72	25,135	71	25,206	72	25,278	71	25,349	71	25,420	71	25,491	71	0,4225
0,70	26,730	79	26,809	78	26,887	78	26,965	78	27,043	77	27,120	78	27,198	77	27,275	76	27,351	77	27,428	76	0,4900
0,75	28,609	85	28,694	84	28,778	84	28,862	83	28,945	83	29,028	83	29,111	83	29,194	83	29,277	82	29,359	82	0,5625
0,80	30,482	90	30,572	90	30,662	89	30,751	90	30,841	89	30,930	89	31,019	89	31,108	88	31,196	88	31,284	88	0,6400
0,85	32,346	97	32,443	96	32,539	95	32,634	96	32,730	95	32,825	95	32,920	94	33,014	94	33,108	94	33,202	94	0,7225
0,90	34,203	103	34,306	102	34,408	102	34,510	101	34,611	101	34,712	101	34,813	100	34,913	101	35,014	99	35,113	100	0,8100
0,95	36,052	108	36,160	109	36,269	108	36,377	107	36,484	107	36,591	107	36,698	107	36,805	106	36,911	106	37,017	105	0,9025
1,00	37,891	115	38,006	115	38,121	114	38,235	113	38,348	114	38,462	113	38,575	113	38,688	112	38,800	112	38,912	111	1,0000
	1,5313		1,5488		1,5665		1,5842		1,6021		1,6200		1,6381		1,6562		1,6745		1,6928		c^2
	$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.																				

	381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	
1	38	38	38	38	39	39	39	39	39	39	39	39	39	39	40	1
2	76	76	77	77	77	77	77	78	78	78	78	78	79	79	79	2
3	114	115	115	115	116	116	116	116	117	117	117	118	118	118	119	3
4	152	153	153	154	154	154	155	155	156	156	156	157	157	158	158	4
5	191	191	192	192	193	193	194	194	195	195	196	196	197	197	198	5
6	229	229	230	230	231	232	232	233	233	234	235	235	236	236	237	6
7	267	267	268	269	270	270	271	272	272	273	274	274	275	276	277	7
8	305	306	306	307	308	309	310	310	311	312	313	314	314	315	316	8
9	343	344	345	346	347	347	348	349	350	351	352	353	354	355	356	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	1,85 Days	dif.	1,86 Days	dif.	1,87 Days	dif.	1,88 Days	dif.	1,89 Days	dif.	1,90 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,395	1	0,396	1	0,397	2	0,399	1	0,400	1	0,401	1	0,0001
0,02	0,791	2	0,793	2	0,795	2	0,797	2	0,799	2	0,801	2	0,0004
0,03	1,186	3	1,189	3	1,192	4	1,196	3	1,199	3	1,202	3	0,0009
0,04	1,581	5	1,586	4	1,590	4	1,594	4	1,598	5	1,603	4	0,0016
0,05	1,977	5	1,982	5	1,987	6	1,993	5	1,998	5	2,003	5	0,0025
0,06	2,372	6	2,378	7	2,385	6	2,391	6	2,397	7	2,404	6	0,0036
0,07	2,767	8	2,775	7	2,782	8	2,790	7	2,797	7	2,804	8	0,0049
0,08	3,162	9	3,171	9	3,180	8	3,188	9	3,197	8	3,205	8	0,0064
0,09	3,558	9	3,567	10	3,577	9	3,586	10	3,596	10	3,606	9	0,0081
0,10	3,953	11	3,964	10	3,974	11	3,985	10	3,995	11	4,006	11	0,0100
0,11	4,348	12	4,360	12	4,372	11	4,383	12	4,395	12	4,407	11	0,0121
0,12	4,743	13	4,756	13	4,769	13	4,782	12	4,794	13	4,807	13	0,0144
0,13	5,138	14	5,152	14	5,166	14	5,180	14	5,194	13	5,207	14	0,0169
0,14	5,533	15	5,548	15	5,563	15	5,578	15	5,593	15	5,608	15	0,0196
0,15	5,929	16	5,945	16	5,961	15	5,976	16	5,992	16	6,008	16	0,0225
0,16	6,324	17	6,341	17	6,358	17	6,375	17	6,392	17	6,409	16	0,0256
0,17	6,718	19	6,737	18	6,755	18	6,773	18	6,791	18	6,809	18	0,0289
0,18	7,113	20	7,133	19	7,152	19	7,171	19	7,190	19	7,209	19	0,0324
0,19	7,508	21	7,529	20	7,549	20	7,569	20	7,589	20	7,609	20	0,0361
0,20	7,903	21	7,924	22	7,946	21	7,967	21	7,988	21	8,009	21	0,0400
0,21	8,298	22	8,320	23	8,343	22	8,365	22	8,387	22	8,409	23	0,0441
0,22	8,692	24	8,716	23	8,739	24	8,763	23	8,786	23	8,809	24	0,0484
0,23	9,087	25	9,112	24	9,136	25	9,161	24	9,185	24	9,209	25	0,0529
0,24	9,482	25	9,507	26	9,533	25	9,558	26	9,584	25	9,609	26	0,0576
0,25	9,876	27	9,903	26	9,929	27	9,956	27	9,983	26	10,009	26	0,0625
0,26	10,270	28	10,298	28	10,326	28	10,354	27	10,381	28	10,409	27	0,0676
0,27	10,665	29	10,694	28	10,722	29	10,751	29	10,780	28	10,808	29	0,0729
0,28	11,059	30	11,089	30	11,119	30	11,149	29	11,178	30	11,208	30	0,0784
0,29	11,453	31	11,484	31	11,515	31	11,546	31	11,577	31	11,608	30	0,0841
0,30	11,847	32	11,879	32	11,911	32	11,943	32	11,975	32	12,007	32	0,0900
0,31	12,241	33	12,274	34	12,308	33	12,341	32	12,373	33	12,406	33	0,0961
0,32	12,635	34	12,669	35	12,704	34	12,738	34	12,772	34	12,806	33	0,1024
0,33	13,029	35	13,064	36	13,100	35	13,135	35	13,170	35	13,205	35	0,1089
0,34	13,423	36	13,459	36	13,495	37	13,532	36	13,568	36	13,604	36	0,1156
0,35	13,816	38	13,854	37	13,891	37	13,928	38	13,966	37	14,003	37	0,1225
0,36	14,210	38	14,248	39	14,287	38	14,325	38	14,363	39	14,402	38	0,1296
0,37	14,603	40	14,643	39	14,682	40	14,722	39	14,761	39	14,800	40	0,1369
0,38	14,996	41	15,037	41	15,078	40	15,118	41	15,159	40	15,199	40	0,1444
0,39	15,390	41	15,431	42	15,473	42	15,515	41	15,556	42	15,598	41	0,1521
0,40	15,783	43	15,826	42	15,868	43	15,911	43	15,954	42	15,996	43	0,1600
0,41	16,176	44	16,220	43	16,263	44	16,307	44	16,351	43	16,394	44	0,1681
0,42	16,568	45	16,613	45	16,658	45	16,703	45	16,748	45	16,793	44	0,1764
0,43	16,961	46	17,007	46	17,053	46	17,099	46	17,145	46	17,191	45	0,1849
0,44	17,354	47	17,401	47	17,448	47	17,495	47	17,542	47	17,589	46	0,1936
0,45	17,746	48	17,794	49	17,843	48	17,891	48	17,939	48	17,987	47	0,2025
0,50	19,706	54	19,760	54	19,814	53	19,867	54	19,921	53	19,974	53	0,2500
0,55	21,662	60	21,722	59	21,781	59	21,840	59	21,899	59	21,958	58	0,3025
0,60	23,614	65	23,679	65	23,744	65	23,809	64	23,873	64	23,937	64	0,3600
0,65	25,562	70	25,632	71	25,703	70	25,773	70	25,843	69	25,912	70	0,4225
0,70	27,504	76	27,580	76	27,656	76	27,732	75	27,807	76	27,883	75	0,4900
0,75	29,441	82	29,523	81	29,604	82	29,686	81	29,767	81	29,848	80	0,5625
0,80	31,372	87	31,459	88	31,547	86	31,633	87	31,720	87	31,807	86	0,6400
0,85	33,296	93	33,389	93	33,482	93	33,575	92	33,667	93	33,760	92	0,7225
0,90	35,213	99	35,312	99	35,411	99	35,510	98	35,608	98	35,706	98	0,8100
0,95	37,122	105	37,227	105	37,332	105	37,437	104	37,541	104	37,645	103	0,9025
1,00	39,023	111	39,134	111	39,245	111	39,356	110	39,466	110	39,576	109	1,0000
	1,7113		1,7298		1,7485		1,7672		1,7861		1,8050		c^2

Sum of the Radii $r + r''$.

$\frac{1}{2}\cdot(r + r'')^2$ or $r^2 + r''^2$ nearly.

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

	392	393	394	395	396	397	398	399	400	401	
1	39	39	39	40	40	40	40	40	40	40	1
2	78	79	79	79	79	79	80	80	80	80	2
3	118	118	118	119	119	119	119	120	120	120	3
4	157	157	158	158	158	159	159	160	160	160	4
5	196	197	197	198	198	199	199	200	200	201	5
6	235	236	236	237	238	238	239	239	240	241	6
7	274	275	276	277	277	278	279	279	280	281	7
8	314	314	315	316	317	318	318	319	320	321	8
9	353	354	355	356	356	357	358	359	360	361	9

TABLE II.—To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord c.	1,91 Days	dif.	1,92 Days	dif.	1,93 Days	dif.	1,94 Days	dif.	1,95 Days	dif.	1,96 Days	dif.	1,97 Days	dif.	1,98 Days	dif.	1,99 Days	dif.	2,00 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,402	1	0,403	1	0,404	1	0,405	1	0,406	1	0,407	1	0,408	1	0,409	1	0,410	1	0,411	1	0,0001
0,02	0,803	2	0,805	3	0,808	2	0,810	2	0,812	2	0,814	2	0,816	2	0,818	2	0,820	2	0,822	2	0,0004
0,03	1,205	3	1,208	3	1,211	4	1,215	3	1,218	3	1,221	3	1,224	3	1,227	3	1,230	3	1,233	3	0,0009
0,04	1,607	4	1,611	4	1,615	4	1,619	5	1,624	4	1,628	4	1,632	4	1,636	4	1,640	4	1,644	4	0,0016
0,05	2,008	6	2,014	5	2,019	5	2,024	5	2,029	6	2,035	5	2,040	5	2,045	5	2,050	5	2,055	5	0,0025
0,06	2,410	6	2,416	7	2,423	6	2,429	6	2,435	6	2,441	7	2,448	6	2,454	6	2,460	6	2,466	6	0,0036
0,07	2,812	7	2,819	7	2,826	8	2,834	7	2,841	7	2,848	8	2,856	7	2,863	7	2,870	7	2,877	7	0,0049
0,08	3,213	9	3,222	8	3,230	9	3,239	8	3,247	8	3,255	8	3,263	9	3,272	8	3,280	8	3,288	8	0,0064
0,09	3,615	9	3,624	10	3,634	9	3,643	10	3,653	9	3,662	9	3,671	10	3,681	9	3,690	9	3,699	9	0,0081
0,10	4,017	10	4,027	11	4,038	10	4,048	10	4,058	11	4,069	10	4,079	11	4,090	10	4,100	10	4,110	10	0,0100
0,11	4,418	12	4,430	11	4,441	12	4,453	11	4,464	12	4,476	11	4,487	11	4,498	12	4,510	11	4,521	11	0,0121
0,12	4,820	12	4,832	13	4,845	12	4,857	13	4,870	12	4,882	13	4,895	12	4,907	13	4,920	12	4,932	12	0,0144
0,13	5,221	14	5,235	13	5,248	14	5,262	14	5,276	13	5,289	14	5,303	13	5,316	13	5,329	14	5,343	13	0,0169
0,14	5,623	14	5,637	15	5,652	15	5,667	14	5,681	15	5,696	14	5,710	15	5,725	14	5,739	15	5,754	14	0,0196
0,15	6,024	16	6,040	15	6,055	16	6,071	16	6,087	15	6,102	16	6,118	15	6,133	16	6,149	15	6,164	16	0,0225
0,16	6,425	17	6,442	17	6,459	17	6,476	16	6,492	17	6,509	17	6,526	16	6,542	17	6,559	16	6,575	17	0,0256
0,17	6,827	18	6,845	17	6,862	18	6,880	18	6,898	18	6,916	17	6,933	18	6,951	17	6,968	18	6,986	17	0,0289
0,18	7,228	19	7,247	19	7,266	19	7,285	18	7,303	19	7,322	19	7,341	18	7,359	19	7,378	19	7,397	18	0,0324
0,19	7,629	20	7,649	20	7,669	20	7,689	20	7,709	20	7,729	19	7,748	20	7,768	20	7,788	19	7,807	20	0,0361
0,20	8,030	21	8,051	21	8,072	21	8,093	21	8,114	21	8,135	21	8,156	20	8,176	21	8,197	21	8,218	20	0,0400
0,21	8,432	22	8,454	22	8,476	22	8,498	22	8,520	21	8,541	22	8,563	22	8,585	22	8,607	21	8,628	22	0,0441
0,22	8,833	23	8,856	23	8,879	23	8,902	23	8,925	23	8,948	23	8,971	22	8,993	23	9,016	23	9,039	22	0,0484
0,23	9,234	24	9,258	24	9,282	24	9,306	24	9,330	24	9,354	24	9,378	24	9,402	23	9,425	24	9,449	24	0,0529
0,24	9,635	25	9,660	25	9,685	25	9,710	25	9,735	25	9,760	25	9,785	25	9,810	25	9,835	24	9,859	25	0,0576
0,25	10,035	27	10,062	26	10,088	26	10,114	26	10,140	26	10,166	26	10,192	26	10,218	26	10,244	26	10,270	25	0,0625
0,26	10,436	28	10,464	27	10,491	27	10,518	27	10,545	27	10,572	27	10,599	27	10,626	27	10,653	27	10,680	27	0,0676
0,27	10,837	28	10,865	29	10,894	28	10,922	28	10,950	28	10,978	28	11,006	28	11,034	28	11,062	28	11,090	28	0,0729
0,28	11,238	29	11,267	29	11,296	30	11,326	29	11,355	29	11,384	29	11,413	29	11,442	29	11,471	29	11,500	29	0,0784
0,29	11,638	31	11,669	30	11,699	31	11,730	30	11,760	30	11,790	30	11,820	30	11,850	30	11,880	30	11,910	30	0,0841
0,30	12,039	31	12,070	32	12,102	31	12,133	32	12,165	31	12,196	31	12,227	31	12,258	31	12,289	31	12,320	31	0,0900
0,31	12,439	33	12,472	32	12,504	33	12,537	32	12,569	33	12,602	32	12,634	32	12,666	32	12,698	32	12,730	32	0,0961
0,32	12,839	34	12,873	34	12,907	33	12,940	34	12,974	33	13,007	33	13,040	34	13,074	33	13,107	33	13,140	33	0,1024
0,33	13,240	34	13,274	35	13,309	35	13,344	34	13,378	35	13,413	34	13,447	34	13,481	34	13,515	34	13,549	34	0,1089
0,34	13,640	36	13,676	35	13,711	36	13,747	36	13,783	35	13,818	35	13,853	36	13,889	35	13,924	35	13,959	35	0,1156
0,35	14,040	37	14,077	37	14,114	36	14,150	37	14,187	36	14,223	37	14,260	36	14,296	36	14,332	37	14,369	36	0,1225
0,36	14,440	38	14,478	38	14,516	37	14,553	38	14,591	38	14,629	37	14,666	38	14,704	37	14,741	37	14,778	37	0,1296
0,37	14,840	39	14,879	39	14,918	38	14,956	39	14,995	39	15,034	38	15,072	39	15,111	38	15,149	38	15,187	38	0,1369
0,38	15,239	40	15,279	40	15,319	40	15,359	40	15,399	40	15,439	39	15,478	40	15,518	39	15,557	40	15,597	39	0,1444
0,39	15,639	41	15,680	41	15,721	41	15,762	41	15,803	41	15,844	40	15,884	41	15,925	40	15,965	41	16,006	40	0,1521
0,40	16,039	42	16,081	42	16,123	42	16,165	42	16,207	42	16,249	41	16,290	42	16,332	41	16,373	42	16,415	41	0,1600
0,41	16,438	43	16,481	43	16,524	43	16,567	43	16,610	43	16,653	43	16,696	43	16,739	42	16,781	43	16,824	42	0,1681
0,42	16,837	45	16,882	44	16,926	44	16,970	44	17,014	44	17,058	44	17,102	43	17,145	44	17,189	43	17,232	44	0,1764
0,43	17,236	46	17,282	45	17,327	45	17,372	45	17,417	45	17,462	45	17,507	45	17,552	45	17,597	44	17,641	45	0,1849
0,44	17,635	47	17,682	46	17,728	47	17,775	46	17,821	46	17,867	46	17,913	46	17,959	45	18,004	46	18,050	45	0,1936
0,45	18,034	48	18,082	47	18,129	48	18,177	47	18,224	47	18,271	47	18,318	47	18,365	47	18,412	46	18,458	47	0,2025
0,50	20,027	53	20,080	53	20,133	53	20,186	52	20,238	52	20,290	53	20,343	52	20,395	52	20,447	52	20,499	51	0,2500
0,55	22,016	58	22,074	59	22,133	58	22,191	58	22,249	57	22,306	58	22,364	57	22,421	58	22,479	57	22,536	57	0,3025
0,60	24,001	64	24,065	64	24,129	63	24,192	63	24,255	64	24,319	62	24,381	63	24,444	63	24,507	62	24,569	63	0,3600
0,65	25,982	69	26,051	69	26,120	69	26,189	69	26,258	68	26,326	69	26,395	68	26,463	68	26,531	68	26,599	67	0,4225
0,70	27,958	74	28,032	75	28,107	74	28,181	75	28,256	74	28,330	73	28,403	74	28,477	73	28,550	74	28,624	73	0,4900
0,75	29,928	81	30,009	80	30,089	80	30,169	79	30,248	80	30,328	79	30,407	79	30,486	79	30,565	79	30,644	78	0,5625
0,80	31,893	86	31,979	86	32,065	85	32,150	86	32,236	85	32,321	84	32,405	85	32,490	84	32,574	84	32,658	84	0,6400
0,85	33,852	91	33,943	92	34,035	91	34,126	91	34,217	90	34,307	91	34,398	90	34,488	90	34,578	89	34,667	90	0,7225
0,90	35,804	97	35,901	97	35,998	97	36,095	97	36,192	96	36,288	96	36,384	96	36,480	95	36,575	95	36,670	95	0,8100
0,95	37,748	103	37,851	103	37,954	103	38,057	102	38,159	102	38,261	102	38,363	101	38,464	102	38,566	100	38,666	101	0,9025
1,00	39,685	109	39,794	109	39,903	108	40,011	109	40,120	107	40,227	108	40,335	107	40,442	107	40,549	106	40,655	107	1,0000
	1,8241		1,8432		1,8625		1,8818		1,9013		1,9208		1,9405		1,9602		1,9801		2,0000		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	399	400	401	402	403	404	405	406	407	408	409	410	411	
1	40	40	40	40	40	40	41	41	41	41	41	41	41	1
2	80	80	80	80	81	81	81	81	81	82	82	82	82	2
3	120	120	120	121	121	121	122	122	122	122	123	123	123	3
4	160	160	160	161	161	162	162	162	163	163	164	164	164	4
5	200	200	201	201	202	202	203	203	204	204	205	205	206	5
6	239	240	241	241	242	242	243	244	244	245	245	246	247	6
7	279	280	281	281	282	283	284	284	285	286	286	287	288	7
8	319	320	321	322	322	323	324	325	326	326	327	328	329	8
9	359	360	361	362	363	364	365	365	366	367	368	369	370	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord c.	Sum of the Radii $r+r''$. 2,01 Days	dif.	2,02 Days	dif.	2,03 Days	dif.	2,04 Days	dif.	2,05 Days	dif.	2,06 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,412	1	0,413	1	0,414	1	0,415	1	0,416	1	0,417	1	0,0001
0,02	0,824	2	0,826	2	0,828	2	0,830	2	0,832	2	0,834	2	0,0004
0,03	1,236	3	1,239	3	1,242	3	1,245	3	1,248	4	1,252	3	0,0009
0,04	1,648	4	1,652	4	1,656	5	1,661	4	1,665	4	1,669	4	0,0016
0,05	2,060	5	2,065	6	2,071	5	2,076	5	2,081	5	2,086	5	0,0025
0,06	2,472	7	2,479	6	2,485	6	2,491	6	2,497	6	2,503	6	0,0036
0,07	2,884	8	2,892	7	2,899	7	2,906	7	2,913	7	2,920	7	0,0049
0,08	3,296	9	3,305	8	3,313	8	3,321	8	3,329	8	3,337	8	0,0064
0,09	3,708	10	3,718	9	3,727	9	3,736	9	3,745	9	3,754	9	0,0081
0,10	4,120	11	4,131	10	4,141	10	4,151	10	4,161	10	4,171	10	0,0100
0,11	4,532	12	4,544	11	4,555	11	4,566	11	4,577	11	4,588	12	0,0121
0,12	4,944	13	4,957	12	4,969	12	4,981	12	4,993	12	5,005	13	0,0144
0,13	5,356	13	5,369	14	5,383	13	5,396	13	5,409	13	5,422	14	0,0169
0,14	5,768	14	5,782	15	5,797	14	5,811	14	5,825	14	5,839	15	0,0196
0,15	6,180	15	6,195	16	6,211	15	6,226	15	6,241	15	6,256	15	0,0225
0,16	6,592	16	6,608	16	6,624	17	6,641	16	6,657	16	6,673	16	0,0256
0,17	7,003	18	7,021	17	7,038	17	7,055	18	7,073	17	7,090	17	0,0289
0,18	7,415	18	7,433	19	7,452	18	7,470	19	7,489	18	7,507	18	0,0324
0,19	7,827	19	7,846	20	7,866	19	7,885	19	7,904	20	7,924	19	0,0361
0,20	8,238	21	8,259	20	8,279	21	8,300	20	8,320	20	8,340	21	0,0400
0,21	8,650	21	8,671	22	8,693	21	8,714	22	8,736	21	8,757	21	0,0441
0,22	9,061	23	9,084	22	9,106	23	9,129	22	9,151	23	9,174	22	0,0484
0,23	9,473	23	9,496	24	9,520	23	9,543	24	9,567	23	9,590	23	0,0529
0,24	9,884	25	9,909	24	9,933	25	9,958	24	9,982	25	10,007	24	0,0576
0,25	10,295	26	10,321	26	10,347	25	10,372	26	10,398	25	10,423	25	0,0625
0,26	10,707	26	10,733	27	10,760	27	10,787	26	10,813	26	10,839	27	0,0676
0,27	11,118	28	11,146	27	11,173	28	11,201	27	11,228	28	11,256	27	0,0729
0,28	11,529	29	11,558	28	11,586	29	11,615	29	11,644	28	11,672	28	0,0784
0,29	11,940	30	11,970	30	12,000	29	12,029	30	12,059	29	12,088	30	0,0841
0,30	12,351	31	12,382	31	12,413	30	12,443	31	12,474	30	12,504	31	0,0900
0,31	12,762	32	12,794	31	12,825	32	12,857	32	12,889	31	12,920	32	0,0961
0,32	13,173	33	13,206	32	13,238	33	13,271	33	13,304	32	13,336	33	0,1024
0,33	13,583	34	13,617	34	13,651	34	13,685	34	13,719	33	13,752	34	0,1089
0,34	13,994	35	14,029	35	14,064	35	14,099	34	14,133	35	14,168	34	0,1156
0,35	14,405	36	14,441	35	14,476	36	14,512	36	14,548	36	14,584	35	0,1225
0,36	14,815	37	14,852	37	14,889	37	14,926	37	14,963	36	14,999	37	0,1296
0,37	15,225	39	15,264	37	15,301	38	15,339	38	15,377	38	15,415	37	0,1369
0,38	15,636	39	15,675	39	15,714	39	15,753	38	15,791	39	15,830	39	0,1444
0,39	16,046	40	16,086	40	16,126	40	16,166	40	16,206	39	16,245	40	0,1521
0,40	16,456	41	16,497	41	16,538	41	16,579	41	16,620	41	16,661	40	0,1600
0,41	16,866	42	16,908	42	16,950	42	16,992	42	17,034	42	17,076	42	0,1681
0,42	17,276	43	17,319	43	17,362	43	17,405	43	17,448	43	17,491	43	0,1764
0,43	17,686	44	17,730	44	17,774	44	17,818	44	17,862	44	17,906	44	0,1849
0,44	18,095	45	18,140	46	18,186	45	18,231	45	18,276	45	18,321	44	0,1936
0,45	18,505	46	18,551	46	18,597	46	18,643	46	18,689	46	18,735	46	0,2025
0,50	20,550	52	20,602	52	20,654	51	20,705	51	20,756	51	20,807	51	0,2500
0,55	22,593	57	22,650	56	22,706	57	22,763	56	22,819	57	22,876	56	0,3025
0,60	24,632	62	24,694	62	24,756	62	24,818	61	24,879	62	24,941	61	0,3600
0,65	26,666	68	26,734	67	26,801	67	26,868	67	26,935	67	27,002	67	0,4225
0,70	28,697	72	28,769	73	28,842	73	28,915	72	28,987	72	29,059	72	0,4900
0,75	30,722	78	30,800	78	30,878	78	30,956	78	31,034	77	31,111	77	0,5625
0,80	32,742	84	32,826	83	32,909	84	32,993	82	33,075	83	33,158	83	0,6400
0,85	34,757	89	34,846	89	34,935	88	35,023	89	35,112	88	35,200	88	0,7225
0,90	36,765	95	36,860	94	36,954	94	37,048	94	37,142	94	37,236	93	0,8100
0,95	38,767	100	38,867	100	38,967	100	39,067	99	39,166	99	39,265	99	0,9025
1,00	40,762	106	40,868	105	40,973	106	41,079	105	41,184	104	41,288	105	1,0000
	2,0201		2,0402		2,0605		2,0808		2,1013		2,1218		c^2

$\frac{1}{2} \cdot (r+r'')^2$ or $r^2 + r''^2$ nearly.

Prop. parts for the sum of the Radii.	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

	409	410	411	412	413	414	415	416	417	418	
1	41	41	41	41	41	41	42	42	42	42	1
2	82	82	82	82	83	83	83	83	83	84	2
3	123	123	123	124	124	124	125	125	125	125	3
4	164	164	164	165	165	166	166	166	167	167	4
5	205	205	206	206	207	207	208	208	209	209	5
6	245	246	247	247	248	248	249	250	250	251	6
7	286	287	288	288	289	290	291	291	292	293	7
8	327	328	329	330	330	331	332	333	334	334	8
9	368	369	370	371	372	373	374	374	375	376	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r + r''$. 2,07 Days	dif.	2,08 Days	dif.	2,09 Days	dif.	2,10 Days	dif.	2,11 Days	dif.	2,12 Days	dif.	2,13 Days	dif.	2,14 Days	dif.	2,15 Days	dif.	2,16 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,418	1	0,419	1	0,420	1	0,421	1	0,422	1	0,423	1	0,424	1	0,425	1	0,426	1	0,427	1	0,0001
0,02	0,836	2	0,838	2	0,840	2	0,842	2	0,844	2	0,846	2	0,848	2	0,850	2	0,852	2	0,854	2	0,0004
0,03	1,255	3	1,258	3	1,261	3	1,264	3	1,267	3	1,270	3	1,273	3	1,276	3	1,279	3	1,282	2	0,0009
0,04	1,673	4	1,677	4	1,681	4	1,685	4	1,689	4	1,693	4	1,697	4	1,701	4	1,705	4	1,709	4	0,0016
0,05	2,091	5	2,096	5	2,101	5	2,106	5	2,111	5	2,116	5	2,121	5	2,126	5	2,131	5	2,136	5	0,0025
0,06	2,509	6	2,515	6	2,521	6	2,527	6	2,533	6	2,539	6	2,545	6	2,551	6	2,557	6	2,563	6	0,0036
0,07	2,927	7	2,934	7	2,941	7	2,948	7	2,955	7	2,962	7	2,969	7	2,976	7	2,983	7	2,990	7	0,0049
0,08	3,345	8	3,353	8	3,361	8	3,369	8	3,377	8	3,385	8	3,393	8	3,401	8	3,409	8	3,417	8	0,0064
0,09	3,763	9	3,772	10	3,782	9	3,791	9	3,800	9	3,809	9	3,818	9	3,827	8	3,835	9	3,844	9	0,0081
0,10	4,181	11	4,192	10	4,202	10	4,212	10	4,222	10	4,232	10	4,242	10	4,252	10	4,262	9	4,271	10	0,0100
0,11	4,600	11	4,611	11	4,622	11	4,633	11	4,644	11	4,655	11	4,666	11	4,677	11	4,688	11	4,699	10	0,0121
0,12	5,018	12	5,030	12	5,042	12	5,054	12	5,066	12	5,078	12	5,090	12	5,102	12	5,114	12	5,126	11	0,0144
0,13	5,436	13	5,449	13	5,462	13	5,475	13	5,488	13	5,501	13	5,514	13	5,527	13	5,540	13	5,553	12	0,0169
0,14	5,854	14	5,868	14	5,882	14	5,896	14	5,910	14	5,924	14	5,938	14	5,952	14	5,966	14	5,980	13	0,0196
0,15	6,271	16	6,287	15	6,302	15	6,317	15	6,332	15	6,347	15	6,362	15	6,377	15	6,392	14	6,406	15	0,0225
0,16	6,689	17	6,706	16	6,722	16	6,738	16	6,754	16	6,770	16	6,786	16	6,802	16	6,818	15	6,833	16	0,0256
0,17	7,107	17	7,124	18	7,142	17	7,159	17	7,176	17	7,193	17	7,210	17	7,227	16	7,243	17	7,260	17	0,0289
0,18	7,525	18	7,543	18	7,561	18	7,579	18	7,597	18	7,615	18	7,633	18	7,651	18	7,669	18	7,687	18	0,0324
0,19	7,943	19	7,962	19	7,981	19	8,000	19	8,019	19	8,038	19	8,057	19	8,076	19	8,095	19	8,114	19	0,0361
0,20	8,361	20	8,381	20	8,401	20	8,421	20	8,441	20	8,461	20	8,481	20	8,501	20	8,521	20	8,541	19	0,0400
0,21	8,778	21	8,799	22	8,821	21	8,842	21	8,863	21	8,884	21	8,905	21	8,926	21	8,947	20	8,967	21	0,0441
0,22	9,196	22	9,218	22	9,240	22	9,262	22	9,284	22	9,306	22	9,328	22	9,350	22	9,372	22	9,394	22	0,0484
0,23	9,613	24	9,637	23	9,660	23	9,683	23	9,706	23	9,729	23	9,752	23	9,775	23	9,798	23	9,821	22	0,0529
0,24	10,031	24	10,055	24	10,079	25	10,104	24	10,128	24	10,152	24	10,176	23	10,199	24	10,223	24	10,247	24	0,0576
0,25	10,448	26	10,474	25	10,499	25	10,524	25	10,549	25	10,574	25	10,599	25	10,624	25	10,649	25	10,674	24	0,0625
0,26	10,866	26	10,892	26	10,918	26	10,944	27	10,971	26	10,997	26	11,023	25	11,048	26	11,074	26	11,100	26	0,0676
0,27	11,283	27	11,310	28	11,338	27	11,365	27	11,392	27	11,419	27	11,446	27	11,473	27	11,500	26	11,526	27	0,0729
0,28	11,700	29	11,729	28	11,757	28	11,785	28	11,813	28	11,841	28	11,869	28	11,897	28	11,925	28	11,953	27	0,0784
0,29	12,118	29	12,147	29	12,176	29	12,205	29	12,234	29	12,263	29	12,292	29	12,321	29	12,350	29	12,379	29	0,0841
0,30	12,535	30	12,565	30	12,595	30	12,625	31	12,656	30	12,686	30	12,716	30	12,746	29	12,775	30	12,805	30	0,0900
0,31	12,952	31	12,983	31	13,014	32	13,046	31	13,077	31	13,108	31	13,139	31	13,170	31	13,201	30	13,231	31	0,0961
0,32	13,369	32	13,401	32	13,433	33	13,466	32	13,498	32	13,530	32	13,562	32	13,594	32	13,626	31	13,657	32	0,1024
0,33	13,786	33	13,819	33	13,852	34	13,886	33	13,919	33	13,952	33	13,985	33	14,018	33	14,051	32	14,083	33	0,1089
0,34	14,202	35	14,237	34	14,271	34	14,305	35	14,340	34	14,374	34	14,408	34	14,442	33	14,475	34	14,509	34	0,1156
0,35	14,619	36	14,655	35	14,690	35	14,725	35	14,760	35	14,795	35	14,830	35	14,865	35	14,900	35	14,935	35	0,1225
0,36	15,036	36	15,072	37	15,109	36	15,145	36	15,181	36	15,217	36	15,253	36	15,289	36	15,325	36	15,361	35	0,1296
0,37	15,452	38	15,490	37	15,527	37	15,564	38	15,602	37	15,639	37	15,676	37	15,713	37	15,750	36	15,786	37	0,1369
0,38	15,869	38	15,907	39	15,946	38	15,984	38	16,022	38	16,060	38	16,098	38	16,136	38	16,174	38	16,212	38	0,1444
0,39	16,285	40	16,325	39	16,364	39	16,403	40	16,443	39	16,482	39	16,521	39	16,560	39	16,599	38	16,637	39	0,1521
0,40	16,701	41	16,742	40	16,782	41	16,823	40	16,863	40	16,903	40	16,943	40	16,983	40	17,023	40	17,063	39	0,1600
0,41	17,118	41	17,159	42	17,201	41	17,242	41	17,283	41	17,324	41	17,365	41	17,406	41	17,447	41	17,488	41	0,1681
0,42	17,534	42	17,576	43	17,619	42	17,661	42	17,703	43	17,746	42	17,788	42	17,830	41	17,871	42	17,913	42	0,1764
0,43	17,950	43	17,993	44	18,037	43	18,080	43	18,123	44	18,167	43	18,210	43	18,253	43	18,296	42	18,338	43	0,1849
0,44	18,365	45	18,410	45	18,455	44	18,499	44	18,543	45	18,588	44	18,632	44	18,676	44	18,720	43	18,763	44	0,1936
0,45	18,781	40	18,827	45	18,872	46	18,918	45	18,963	45	19,008	46	19,054	45	19,099	44	19,143	45	19,188	45	0,2025
0,50	20,858	51	20,909	51	20,960	50	21,010	51	21,061	50	21,111	50	21,161	50	21,211	50	21,261	50	21,311	50	0,2500
0,55	22,932	56	22,988	56	23,044	55	23,099	56	23,155	55	23,210	56	23,266	55	23,321	55	23,376	55	23,431	55	0,3025
0,60	25,002	61	25,063	61	25,124	61	25,185	61	25,246	61	25,307	60	25,367	60	25,427	60	25,487	60	25,547	60	0,3600
0,65	27,069	66	27,135	66	27,201	66	27,267	66	27,333	66	27,399	66	27,465	65	27,530	65	27,595	65	27,660	65	0,4225
0,70	29,131	71	29,202	72	29,274	71	29,345	71	29,416	71	29,487	71	29,558	71	29,629	70	29,699	70	29,769	70	0,4900
0,75	31,188	77	31,265	77	31,342	77	31,419	76	31,495	76	31,571	76	31,647	76	31,723	76	31,799	75	31,874	75	0,5625
0,80	33,241	82	33,323	82	33,405	82	33,487	82	33,569	81	33,650	82	33,732	81	33,813	80	33,893	81	33,974	81	0,6400
0,85	35,288	88	35,376	87	35,463	87	35,550	87	35,637	87	35,724	87	35,811	86	35,897	86	35,983	86	36,069	86	0,7225
0,90	37,329	93	37,422	93	37,515	93	37,608	92	37,700	92	37,792	92	37,884	92	37,976	92	38,068	91	38,159	91	0,8100
0,95	39,364	99	39,463	98	39,561	98	39,659	98	39,757	98	39,855	97	39,952	97	40,049	97	40,146	97	40,243	96	0,9025
1,00	41,393	104	41,497	104	41,601	103	41,704	104	41,808	103	41,911	102	42,013	103	42,116	102	42,218	102	42,320	102	1,0000
	2,1425		2,1632		2,1841		2,2050		2,2261		2,2472		2,2685		2,2898		2,3113		2,3328		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	415	416	417	418	419	420	421	422	423	424	425	426	427	428	
1	42	42	42	42	42	42	42	42	42	42	43	43	43	43	1
2	83	83	83	84	84	84	84	84	85	85	85	85	85	86	2
3	125	125	125	125	126	126	126	127	127	127	128	128	128	128	3
4	166	166	167	167	168	168	168	169	169	170	170	170	171	171	4
5	208	208	209	209	210	210	211	211	212	212	213	213	214	214	5
6	249	250	250	251	251	252	253	253	254	254	255	256	256	257	6
7	291	291	292	293	293	294	295	295	296	297	298	298	299	300	7
8	332	333	334	334	335	336	337	338	338	339	340	341	342	342	8
9	374	374	375	376	377	378	379	380	381	382	383	383	384	385	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r + r''$. 2,17 Days	dif.	2,18 Days	dif.	2,19 Days	dif.	2,20 Days	dif.	2,21 Days	dif.	2,22 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,428	1	0,429	1	0,430	1	0,431	1	0,432	1	0,433	1	0,0001
0,02	0,856	2	0,858	2	0,860	2	0,862	2	0,864	2	0,866	2	0,0004
0,03	1,284	3	1,287	3	1,290	3	1,293	3	1,296	3	1,299	3	0,0009
0,04	1,713	4	1,717	4	1,721	3	1,724	4	1,728	4	1,732	4	0,0016
0,05	2,141	5	2,146	5	2,151	5	2,156	4	2,160	5	2,165	5	0,0025
0,06	2,569	6	2,575	6	2,581	6	2,587	6	2,593	5	2,598	6	0,0036
0,07	2,997	7	3,004	7	3,011	7	3,018	7	3,025	6	3,031	7	0,0049
0,08	3,425	8	3,433	8	3,441	8	3,449	8	3,457	7	3,464	8	0,0064
0,09	3,853	9	3,862	9	3,871	9	3,880	9	3,889	8	3,897	9	0,0081
0,10	4,281	10	4,291	10	4,301	10	4,311	10	4,321	9	4,330	10	0,0100
0,11	4,709	11	4,720	11	4,731	11	4,742	11	4,753	10	4,763	11	0,0121
0,12	5,137	12	5,149	12	5,161	12	5,173	12	5,185	11	5,196	12	0,0144
0,13	5,565	13	5,578	13	5,591	13	5,604	13	5,617	12	5,629	13	0,0169
0,14	5,993	14	6,007	14	6,021	14	6,035	13	6,048	14	6,062	14	0,0196
0,15	6,421	15	6,436	15	6,451	15	6,466	14	6,480	15	6,495	15	0,0225
0,16	6,849	16	6,865	16	6,881	15	6,896	16	6,912	16	6,928	15	0,0256
0,17	7,277	17	7,294	17	7,311	16	7,327	17	7,344	16	7,360	17	0,0289
0,18	7,705	18	7,723	17	7,740	18	7,758	18	7,776	17	7,793	18	0,0324
0,19	8,133	18	8,151	19	8,170	19	8,189	18	8,207	19	8,226	18	0,0361
0,20	8,560	20	8,580	20	8,600	19	8,619	20	8,639	20	8,659	19	0,0400
0,21	8,988	21	9,009	20	9,029	21	9,050	21	9,071	20	9,091	21	0,0441
0,22	9,416	21	9,437	22	9,459	22	9,481	21	9,502	22	9,524	21	0,0484
0,23	9,843	23	9,866	23	9,889	22	9,911	23	9,934	22	9,956	23	0,0529
0,24	10,271	24	10,295	23	10,318	24	10,342	23	10,365	24	10,389	23	0,0576
0,25	10,698	25	10,723	25	10,748	24	10,772	25	10,797	24	10,821	25	0,0625
0,26	11,126	25	11,151	26	11,177	26	11,203	25	11,228	26	11,254	25	0,0676
0,27	11,553	27	11,580	26	11,606	27	11,633	26	11,659	27	11,686	26	0,0729
0,28	11,980	28	12,008	28	12,036	27	12,063	28	12,091	27	12,118	27	0,0784
0,29	12,408	28	12,436	29	12,465	28	12,493	29	12,522	28	12,550	29	0,0841
0,30	12,835	30	12,865	29	12,894	30	12,924	29	12,953	29	12,982	30	0,0900
0,31	13,262	31	13,293	30	13,323	31	13,354	30	13,384	30	13,414	31	0,0961
0,32	13,689	32	13,721	31	13,752	32	13,784	31	13,815	31	13,846	32	0,1024
0,33	14,116	33	14,149	32	14,181	33	14,214	32	14,246	32	14,278	33	0,1089
0,34	14,543	33	14,576	34	14,610	33	14,643	34	14,677	33	14,710	33	0,1156
0,35	14,970	34	15,004	35	15,039	34	15,073	35	15,108	34	15,142	34	0,1225
0,36	15,396	36	15,432	36	15,468	35	15,503	35	15,538	36	15,574	35	0,1296
0,37	15,823	37	15,860	36	15,896	37	15,933	36	15,969	36	16,005	36	0,1369
0,38	16,250	37	16,287	38	16,325	37	16,362	37	16,399	38	16,437	37	0,1444
0,39	16,676	39	16,715	38	16,753	39	16,792	38	16,830	38	16,868	38	0,1521
0,40	17,102	40	17,142	40	17,182	39	17,221	39	17,260	39	17,299	40	0,1600
0,41	17,529	40	17,569	41	17,610	40	17,650	41	17,691	40	17,731	40	0,1681
0,42	17,955	42	17,997	41	18,038	41	18,079	42	18,121	41	18,162	41	0,1764
0,43	18,381	43	18,424	42	18,466	42	18,508	43	18,551	42	18,593	42	0,1849
0,44	18,807	44	18,851	43	18,894	43	18,937	44	18,981	43	19,024	43	0,1936
0,45	19,233	44	19,277	45	19,322	44	19,366	45	19,411	44	19,455	44	0,2025
0,50	21,361	49	21,410	50	21,460	49	21,509	49	21,558	50	21,608	49	0,2500
0,55	23,486	54	23,540	55	23,595	54	23,649	54	23,703	55	23,758	54	0,3025
0,60	25,607	60	25,667	60	25,727	59	25,786	59	25,845	59	25,904	59	0,3600
0,65	27,725	65	27,790	65	27,855	64	27,919	65	27,984	64	28,048	64	0,4225
0,70	29,839	70	29,909	70	29,979	70	30,049	69	30,118	69	30,187	69	0,4900
0,75	31,949	75	32,024	75	32,099	75	32,174	74	32,248	75	32,323	74	0,5625
0,80	34,055	80	34,135	80	34,215	80	34,295	79	34,374	80	34,454	79	0,6400
0,85	36,155	85	36,240	86	36,326	85	36,411	84	36,495	85	36,580	84	0,7225
0,90	38,250	90	38,340	91	38,431	90	38,521	90	38,611	90	38,701	90	0,8100
0,95	40,339	96	40,435	96	40,531	95	40,626	96	40,722	95	40,817	95	0,9025
1,00	42,422	101	42,523	101	42,624	101	42,725	101	42,826	100	42,926	101	1,0000
	2,3515		2,3762		2,3981		2,4200		2,4421		2,4642		c^2

$\frac{1}{2} \cdot (r + r')^2$ or $r^2 + r''^2$ nearly.

	426	427	428	429	430	431	432	433	434	
1	43	43	43	43	43	43	43	43	43	1
2	85	85	86	86	86	86	86	87	87	2
3	128	128	128	129	129	129	130	130	130	3
4	170	171	171	172	172	172	173	173	174	4
5	213	214	214	215	215	216	216	217	217	5
6	256	256	257	257	258	259	259	260	260	6
7	298	299	300	300	301	302	302	303	304	7
8	341	342	342	343	344	345	346	346	347	8
9	383	384	385	386	387	388	389	390	391	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	2,23 Days	dif.	2,24 Days	dif.	2,25 Days	dif.	2,26 Days	dif.	2,27 Days	dif.	2,28 Days	dif.	2,29 Days	dif.	2,30 Days	dif.	2,31 Days	dif.	2,32 Days	dif.	
	Sum of the Radii $r + r''$.																				
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,434	1	0,435	1	0,436	1	0,437	1	0,438	1	0,439	1	0,440	1	0,441	1	0,442	1	0,443	1	0,0001
0,02	0,868	2	0,870	2	0,872	2	0,874	2	0,876	2	0,878	2	0,880	2	0,882	2	0,884	1	0,885	2	0,0004
0,03	1,302	3	1,305	3	1,308	3	1,311	3	1,314	3	1,317	3	1,320	2	1,322	3	1,325	3	1,328	3	0,0009
0,04	1,736	4	1,740	4	1,744	4	1,748	4	1,752	4	1,756	3	1,759	4	1,763	4	1,767	4	1,771	4	0,0016
0,05	2,170	5	2,175	5	2,180	5	2,185	5	2,190	4	2,194	5	2,199	5	2,204	5	2,209	5	2,214	4	0,0025
0,06	2,604	6	2,610	6	2,616	6	2,622	5	2,627	6	2,633	6	2,639	6	2,645	6	2,651	5	2,656	6	0,0036
0,07	3,038	7	3,045	7	3,052	7	3,059	6	3,065	7	3,072	7	3,079	7	3,086	6	3,092	7	3,099	7	0,0049
0,08	3,472	8	3,480	8	3,488	8	3,496	7	3,503	8	3,511	8	3,519	7	3,526	8	3,534	8	3,542	7	0,0064
0,09	3,906	9	3,915	9	3,924	8	3,932	9	3,941	9	3,950	8	3,958	9	3,967	9	3,976	8	3,984	9	0,0081
0,10	4,340	10	4,350	10	4,360	9	4,369	10	4,379	10	4,389	9	4,398	10	4,408	9	4,417	10	4,427	9	0,0100
0,11	4,774	11	4,785	10	4,795	11	4,806	11	4,817	10	4,827	11	4,838	10	4,848	11	4,859	10	4,869	11	0,0121
0,12	5,208	12	5,220	11	5,231	12	5,243	11	5,254	12	5,266	12	5,278	11	5,289	12	5,301	11	5,312	12	0,0144
0,13	5,642	13	5,655	12	5,667	13	5,680	12	5,692	13	5,705	12	5,717	13	5,730	12	5,742	13	5,755	12	0,0169
0,14	6,076	13	6,089	14	6,103	13	6,116	14	6,130	13	6,143	14	6,157	13	6,170	14	6,184	13	6,197	14	0,0196
0,15	6,510	14	6,524	15	6,539	14	6,553	15	6,568	14	6,582	15	6,597	14	6,611	14	6,625	15	6,640	14	0,0225
0,16	6,943	16	6,959	15	6,974	16	6,990	15	7,005	16	7,021	15	7,036	16	7,052	15	7,067	15	7,082	15	0,0256
0,17	7,377	17	7,394	16	7,410	17	7,427	16	7,443	16	7,459	17	7,476	16	7,492	16	7,508	17	7,525	16	0,0289
0,18	7,811	17	7,828	18	7,846	17	7,863	18	7,881	17	7,898	17	7,915	18	7,933	17	7,950	17	7,967	17	0,0324
0,19	8,244	19	8,263	18	8,281	19	8,300	18	8,318	19	8,337	18	8,355	18	8,373	18	8,391	18	8,409	18	0,0361
0,20	8,678	20	8,698	19	8,717	19	8,736	20	8,756	19	8,775	19	8,794	19	8,813	20	8,833	19	8,852	19	0,0400
0,21	9,112	20	9,132	21	9,153	20	9,173	20	9,193	20	9,213	21	9,234	20	9,254	20	9,274	20	9,294	20	0,0441
0,22	9,545	22	9,567	21	9,588	21	9,609	22	9,631	21	9,652	21	9,673	21	9,694	21	9,715	21	9,736	21	0,0484
0,23	9,979	22	10,001	22	10,023	23	10,046	22	10,068	22	10,090	22	10,112	22	10,134	22	10,156	22	10,178	22	0,0529
0,24	10,412	24	10,436	23	10,459	23	10,482	23	10,505	23	10,528	24	10,552	23	10,575	23	10,598	23	10,621	23	0,0576
0,25	10,846	24	10,870	24	10,894	24	10,918	25	10,943	24	10,967	24	10,991	24	11,015	24	11,039	24	11,063	24	0,0625
0,26	11,279	25	11,304	25	11,329	26	11,355	25	11,380	25	11,405	25	11,430	25	11,455	25	11,480	25	11,505	25	0,0676
0,27	11,712	26	11,738	27	11,765	26	11,791	26	11,817	26	11,843	26	11,869	26	11,895	26	11,921	26	11,947	26	0,0729
0,28	12,145	28	12,173	27	12,200	27	12,227	27	12,254	27	12,281	27	12,308	27	12,335	27	12,362	27	12,389	26	0,0784
0,29	12,579	28	12,607	28	12,635	28	12,663	28	12,691	28	12,719	28	12,747	28	12,775	28	12,803	28	12,831	27	0,0841
0,30	13,012	29	13,041	29	13,070	29	13,099	29	13,128	29	13,157	29	13,186	29	13,215	29	13,244	28	13,272	29	0,0900
0,31	13,445	30	13,475	30	13,505	30	13,535	30	13,565	30	13,595	30	13,625	30	13,655	29	13,684	30	13,714	30	0,0961
0,32	13,878	31	13,909	31	13,940	31	13,971	31	14,002	31	14,033	31	14,064	31	14,095	30	14,125	31	14,156	30	0,1024
0,33	14,311	32	14,343	32	14,375	32	14,407	32	14,439	32	14,471	31	14,502	32	14,534	32	14,566	32	14,598	31	0,1089
0,34	14,743	34	14,777	33	14,810	33	14,843	33	14,876	32	14,908	33	14,941	33	14,974	33	15,007	32	15,039	33	0,1156
0,35	15,176	34	15,210	34	15,244	34	15,278	34	15,312	34	15,346	34	15,380	33	15,413	34	15,447	34	15,481	33	0,1225
0,36	15,609	35	15,644	35	15,679	35	15,714	35	15,749	35	15,784	34	15,818	35	15,853	34	15,887	35	15,922	34	0,1296
0,37	16,041	36	16,077	36	16,113	36	16,149	36	16,185	36	16,221	36	16,257	35	16,292	36	16,328	35	16,363	36	0,1369
0,38	16,474	37	16,511	37	16,548	37	16,585	37	16,622	36	16,658	37	16,695	37	16,732	36	16,768	37	16,805	36	0,1444
0,39	16,906	38	16,944	38	16,982	38	17,020	38	17,058	38	17,096	37	17,133	38	17,171	37	17,208	38	17,246	37	0,1521
0,40	17,339	39	17,378	39	17,417	38	17,455	39	17,494	39	17,533	39	17,572	38	17,610	38	17,648	39	17,687	38	0,1600
0,41	17,771	40	17,811	40	17,851	40	17,891	39	17,930	40	17,970	40	18,010	39	18,049	40	18,089	39	18,128	39	0,1681
0,42	18,203	41	18,244	41	18,285	41	18,326	40	18,366	41	18,407	41	18,448	40	18,488	41	18,529	40	18,569	40	0,1764
0,43	18,635	42	18,677	42	18,719	42	18,761	41	18,802	42	18,844	42	18,886	41	18,927	41	18,968	42	19,010	41	0,1849
0,44	19,067	43	19,110	43	19,153	43	19,196	42	19,238	43	19,281	42	19,323	43	19,366	42	19,408	42	19,450	43	0,1936
0,45	19,499	44	19,543	44	19,587	43	19,630	44	19,674	44	19,718	43	19,761	44	19,805	43	19,848	43	19,891	43	0,2025
0,50	21,657	49	21,706	48	21,754	49	21,803	49	21,852	48	21,900	48	21,948	49	21,997	48	22,045	48	22,093	48	0,2500
0,55	23,812	53	23,865	54	23,919	54	23,973	53	24,026	54	24,080	53	24,133	53	24,186	53	24,239	53	24,292	53	0,3025
0,60	25,963	59	26,022	59	26,081	59	26,140	58	26,198	58	26,256	59	26,315	58	26,373	58	26,431	57	26,488	58	0,3600
0,65	28,112	64	28,176	63	28,239	64	28,303	63	28,366	64	28,430	63	28,493	63	28,556	63	28,619	62	28,681	63	0,4225
0,70	30,256	69	30,325	69	30,394	69	30,463	68	30,531	68	30,599	69	30,668	67	30,735	68	30,803	68	30,871	67	0,4900
0,75	32,397	74	32,471	74	32,545	73	32,618	74	32,692	73	32,765	73	32,838	73	32,911	73	32,984	73	33,057	72	0,5625
0,80	34,533	79	34,612	79	34,691	79	34,770	78	34,848	79	34,927	78	35,005	78	35,083	78	35,161	77	35,238	78	0,6400
0,85	36,664	85	36,749	84	36,833	84	36,917	83	37,000	84	37,084	83	37,167	83	37,250	83	37,333	82	37,415	83	0,7225
0,90	38,791	89	38,880	89	38,969	89	39,058	89	39,147	89	39,236	88	39,324	88	39,412	88	39,500	88	39,588	87	0,8100
0,95	40,912	94	41,006	95	41,101	94	41,195	94	41,289	93	41,382	94	41,476	93	41,569	93	41,662	93	41,755	93	0,9025
1,00	43,027	99	43,126	100	43,226	99	43,325	100	43,425	98	43,523	99	43,622	99	43,721	98	43,819	98	43,917	97	1,0000
	2,4865		2,5088		2,5313		2,5538		2,5765		2,5992		2,6221		2,6450		2,6681		2,6912		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	432	433	434	435	436	437	438	439	440	441	442	443	
1	43	43	43	44	44	44	44	44	44	44	44	44	1
2	86	87	87	87	87	87	88	88	88	88	88	89	2
3	130	130	130	131	131	131	131	132	132	132	133	133	3
4	173	173	174	174	174	175	175	176	176	176	177	177	4
5	216	217	217	218	218	219	219	220	220	221	221	222	5
6	259	260	260	261	262	262	263	263	264	265	265	266	6
7	302	303	304	305	305	306	307	307	308	309	309	310	7
8	346	346	347	348	349	350	350	351	352	353	354	354	8
9	389	390	391	392	392	393	394	395	396	397	398	399	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord	Sum of the Radii $r+r''$.												
c.	2,33		2,34		2,35		2,36		2,37		2,38		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,444	1	0,445	1	0,446	1	0,447	1	0,447	1	0,448	1	0,0001
0,02	0,887	2	0,889	2	0,891	2	0,893	2	0,895	2	0,897	2	0,0004
0,03	1,331	3	1,334	3	1,337	3	1,340	2	1,342	3	1,345	3	0,0009
0,04	1,775	4	1,779	3	1,782	4	1,786	4	1,790	4	1,794	3	0,0016
0,05	2,218	5	2,223	5	2,228	5	2,233	4	2,237	5	2,242	5	0,0025
0,06	2,662	6	2,668	5	2,673	6	2,679	6	2,685	5	2,690	6	0,0036
0,07	3,106	6	3,112	7	3,119	7	3,126	6	3,132	7	3,139	6	0,0049
0,08	3,549	8	3,557	7	3,564	8	3,572	8	3,580	7	3,587	8	0,0064
0,09	3,993	8	4,001	9	4,010	8	4,018	9	4,027	8	4,035	9	0,0081
0,10	4,436	10	4,446	9	4,455	10	4,465	9	4,474	10	4,484	9	0,0100
0,11	4,880	10	4,890	11	4,901	10	4,911	11	4,922	10	4,932	10	0,0121
0,12	5,324	11	5,335	11	5,346	12	5,358	11	5,369	11	5,380	12	0,0144
0,13	5,767	12	5,779	13	5,792	12	5,804	12	5,816	13	5,829	12	0,0169
0,14	6,211	13	6,224	13	6,237	13	6,250	14	6,264	13	6,277	13	0,0196
0,15	6,654	14	6,668	15	6,683	14	6,697	14	6,711	14	6,725	14	0,0225
0,16	7,097	16	7,113	15	7,128	15	7,143	15	7,158	15	7,173	15	0,0256
0,17	7,541	16	7,557	16	7,573	16	7,589	16	7,605	16	7,621	16	0,0289
0,18	7,984	17	8,001	17	8,018	17	8,035	18	8,053	16	8,069	17	0,0324
0,19	8,427	19	8,446	18	8,464	18	8,482	18	8,500	18	8,518	17	0,0361
0,20	8,871	19	8,890	19	8,909	19	8,928	19	8,947	19	8,966	18	0,0400
0,21	9,314	20	9,334	20	9,354	20	9,374	20	9,394	20	9,414	19	0,0441
0,22	9,757	21	9,778	21	9,799	21	9,820	21	9,841	21	9,862	20	0,0484
0,23	10,200	22	10,222	22	10,244	22	10,266	22	10,288	21	10,309	22	0,0529
0,24	10,644	22	10,666	23	10,689	23	10,712	23	10,735	22	10,757	23	0,0576
0,25	11,087	23	11,110	24	11,134	24	11,158	24	11,182	23	11,205	24	0,0625
0,26	11,530	24	11,554	25	11,579	25	11,604	24	11,628	25	11,653	24	0,0676
0,27	11,973	25	11,998	26	12,024	26	12,050	25	12,075	26	12,101	25	0,0729
0,28	12,415	27	12,442	27	12,469	26	12,495	27	12,522	26	12,548	27	0,0784
0,29	12,858	28	12,886	27	12,913	28	12,941	27	12,968	28	12,996	27	0,0841
0,30	13,301	29	13,330	28	13,358	29	13,387	28	13,415	28	13,443	29	0,0900
0,31	13,744	29	13,773	30	13,803	29	13,832	30	13,862	29	13,891	29	0,0961
0,32	14,186	31	14,217	30	14,247	31	14,278	30	14,308	30	14,338	30	0,1024
0,33	14,629	32	14,661	31	14,692	31	14,723	31	14,754	32	14,786	31	0,1089
0,34	15,072	32	15,104	32	15,136	33	15,169	32	15,201	32	15,233	32	0,1156
0,35	15,514	33	15,547	34	15,581	33	15,614	33	15,647	33	15,680	33	0,1225
0,36	15,956	35	15,991	34	16,025	34	16,059	34	16,093	34	16,127	34	0,1296
0,37	16,399	35	16,434	35	16,469	35	16,504	35	16,539	35	16,574	35	0,1369
0,38	16,841	36	16,877	36	16,913	36	16,949	36	16,985	36	17,021	36	0,1444
0,39	17,283	37	17,320	37	17,357	37	17,394	37	17,431	37	17,468	37	0,1521
0,40	17,725	38	17,763	38	17,801	38	17,839	38	17,877	38	17,915	38	0,1600
0,41	18,167	39	18,206	39	18,245	39	18,284	39	18,323	39	18,362	39	0,1681
0,42	18,609	40	18,649	40	18,689	40	18,729	40	18,769	40	18,809	39	0,1764
0,43	19,051	41	19,092	41	19,133	41	19,174	41	19,215	40	19,255	41	0,1849
0,44	19,493	42	19,535	42	19,577	41	19,618	42	19,660	42	19,702	41	0,1936
0,45	19,934	43	19,977	43	20,020	43	20,063	43	20,106	42	20,148	43	0,2025
0,50	22,141	48	22,189	47	22,236	48	22,284	48	22,332	47	22,379	47	0,2500
0,55	24,345	53	24,398	52	24,450	53	24,503	52	24,555	52	24,607	52	0,3025
0,60	26,546	58	26,604	57	26,661	57	26,718	58	26,776	57	26,833	56	0,3600
0,65	28,744	62	28,806	63	28,869	62	28,931	62	28,993	62	29,055	62	0,4225
0,70	30,938	68	31,006	67	31,073	67	31,140	67	31,207	67	31,274	66	0,4900
0,75	33,129	72	33,201	73	33,274	72	33,346	71	33,417	72	33,489	71	0,5625
0,80	35,316	77	35,393	77	35,470	77	35,547	77	35,624	76	35,700	77	0,6400
0,85	37,498	82	37,580	82	37,662	82	37,744	82	37,826	81	37,907	82	0,7225
0,90	39,675	88	39,763	87	39,850	87	39,937	86	40,023	87	40,110	86	0,8100
0,95	41,848	92	41,940	92	42,032	92	42,124	92	42,216	91	42,307	92	0,9025
1,00	44,014	98	44,112	97	44,209	97	44,306	97	44,403	97	44,500	96	1,0000
	2,7145		2,7378		2,7613		2,7848		2,8085		2,8322		c^2
	$\frac{1}{2} \cdot (r+r'')^2$ or $r^2 + r''^2$ nearly.												

	441	442	443	444	445	446	447	448	449	
1	44	44	44	44	45	45	45	45	45	1
2	88	88	89	89	89	89	89	90	90	2
3	132	133	133	133	134	134	134	134	135	3
4	176	177	177	178	178	178	179	179	180	4
5	221	221	222	222	223	223	224	224	225	5
6	265	265	266	266	267	268	268	269	269	6
7	309	309	310	311	312	312	313	314	314	7
8	353	354	354	355	356	357	358	358	359	8
9	397	398	399	400	401	401	402	403	404	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r + r''$. 2,39 Days	dif.	2,40 Days	dif.	2,41 Days	dif.	2,42 Days	dif.	2,43 Days	dif.	2,44 Days	dif.	2,45 Days	dif.	2,46 Days	dif.	2,47 Days	dif.	2,48 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,449	1	0,450	1	0,451	1	0,452	1	0,453	1	0,454	1	0,455	1	0,456	1	0,457	1	0,458	1	0,0001
0,02	0,899	2	0,901	1	0,902	2	0,904	2	0,906	2	0,908	2	0,910	2	0,912	2	0,914	1	0,915	2	0,0004
0,03	1,348	3	1,351	3	1,354	2	1,356	3	1,359	3	1,362	3	1,365	3	1,368	2	1,370	3	1,373	3	0,0009
0,04	1,797	4	1,801	4	1,805	4	1,809	3	1,812	4	1,816	4	1,820	4	1,824	3	1,827	4	1,831	4	0,0016
0,05	2,247	4	2,251	5	2,256	5	2,261	4	2,265	5	2,270	5	2,275	4	2,279	5	2,284	5	2,289	4	0,0025
0,06	2,696	6	2,702	5	2,707	6	2,713	6	2,719	5	2,724	6	2,730	5	2,735	6	2,741	5	2,746	6	0,0036
0,07	3,145	7	3,152	6	3,158	7	3,165	7	3,172	6	3,178	7	3,185	6	3,191	7	3,198	6	3,204	6	0,0049
0,08	3,595	7	3,602	8	3,610	7	3,617	8	3,625	7	3,632	8	3,640	7	3,647	7	3,654	8	3,662	7	0,0064
0,09	4,044	8	4,052	9	4,061	8	4,069	9	4,078	8	4,086	8	4,094	9	4,103	8	4,111	8	4,119	9	0,0081
0,10	4,493	10	4,503	9	4,512	9	4,521	10	4,531	9	4,540	9	4,549	10	4,559	9	4,568	9	4,577	9	0,0100
0,11	4,942	11	4,953	10	4,963	10	4,973	11	4,984	10	4,994	10	5,004	10	5,014	11	5,025	10	5,035	10	0,0121
0,12	5,392	11	5,403	11	5,414	11	5,425	12	5,437	11	5,448	11	5,459	11	5,470	11	5,481	11	5,492	11	0,0144
0,13	5,841	12	5,853	12	5,865	12	5,877	13	5,890	12	5,902	12	5,914	12	5,926	12	5,938	12	5,950	12	0,0169
0,14	6,290	13	6,303	13	6,316	13	6,329	13	6,342	14	6,356	13	6,369	13	6,382	13	6,395	12	6,407	13	0,0196
0,15	6,739	14	6,753	14	6,767	14	6,781	14	6,795	14	6,809	14	6,823	14	6,837	14	6,851	14	6,865	14	0,0225
0,16	7,188	15	7,203	15	7,218	15	7,233	15	7,248	15	7,263	15	7,278	15	7,293	15	7,308	14	7,322	15	0,0256
0,17	7,637	16	7,653	16	7,669	16	7,685	16	7,701	16	7,717	16	7,733	15	7,748	16	7,764	16	7,780	16	0,0289
0,18	8,086	17	8,103	17	8,120	17	8,137	17	8,154	17	8,171	16	8,187	17	8,204	17	8,221	16	8,237	17	0,0324
0,19	8,535	18	8,553	18	8,571	18	8,589	18	8,607	17	8,624	18	8,642	18	8,660	17	8,677	18	8,695	17	0,0361
0,20	8,984	19	9,003	19	9,022	19	9,041	18	9,059	19	9,078	19	9,097	18	9,115	19	9,134	18	9,152	19	0,0400
0,21	9,433	20	9,453	20	9,473	19	9,492	20	9,512	20	9,532	19	9,551	20	9,571	19	9,590	20	9,610	19	0,0441
0,22	9,882	21	9,903	21	9,924	20	9,944	21	9,965	20	9,985	21	10,006	20	10,026	21	10,047	20	10,067	20	0,0484
0,23	10,331	22	10,353	21	10,374	22	10,396	21	10,417	22	10,439	21	10,460	22	10,482	21	10,503	21	10,524	21	0,0529
0,24	10,780	22	10,802	23	10,825	22	10,847	23	10,870	22	10,892	23	10,915	22	10,937	22	10,959	22	10,981	22	0,0576
0,25	11,229	23	11,252	24	11,276	23	11,299	23	11,322	24	11,346	23	11,369	23	11,392	23	11,415	24	11,439	23	0,0625
0,26	11,677	25	11,702	24	11,726	25	11,751	24	11,775	24	11,799	24	11,823	25	11,848	24	11,872	24	11,896	24	0,0676
0,27	12,126	25	12,151	26	12,177	25	12,202	25	12,227	25	12,252	26	12,278	25	12,303	25	12,328	25	12,353	25	0,0729
0,28	12,575	26	12,601	26	12,627	27	12,654	26	12,680	26	12,706	26	12,732	26	12,758	26	12,784	26	12,810	26	0,0784
0,29	13,023	27	13,050	28	13,078	27	13,105	27	13,132	27	13,159	27	13,186	27	13,213	27	13,240	27	13,267	27	0,0841
0,30	13,472	28	13,500	28	13,528	28	13,556	28	13,584	28	13,612	28	13,640	28	13,668	28	13,696	28	13,724	27	0,0900
0,31	13,920	29	13,949	29	13,978	29	14,007	29	14,036	29	14,065	29	14,094	29	14,123	29	14,152	29	14,181	28	0,0961
0,32	14,368	31	14,399	30	14,429	30	14,459	30	14,489	29	14,518	30	14,548	30	14,578	30	14,608	29	14,637	30	0,1024
0,33	14,817	31	14,848	31	14,879	31	14,910	31	14,941	30	14,971	31	15,002	31	15,033	31	15,064	30	15,094	31	0,1089
0,34	15,265	32	15,297	32	15,329	32	15,361	32	15,393	31	15,424	32	15,456	32	15,488	31	15,519	32	15,551	31	0,1156
0,35	15,713	33	15,746	33	15,779	33	15,812	33	15,845	32	15,877	33	15,910	32	15,942	33	15,975	32	16,007	33	0,1225
0,36	16,161	34	16,195	34	16,229	34	16,263	34	16,297	33	16,330	34	16,364	33	16,397	34	16,431	33	16,464	33	0,1296
0,37	16,609	35	16,644	35	16,679	35	16,714	34	16,748	35	16,783	34	16,817	35	16,852	34	16,886	34	16,920	35	0,1369
0,38	17,057	36	17,093	36	17,129	35	17,164	36	17,200	36	17,236	35	17,271	35	17,306	36	17,342	35	17,377	35	0,1444
0,39	17,505	37	17,542	37	17,579	36	17,615	37	17,652	36	17,688	37	17,725	36	17,761	36	17,797	36	17,833	36	0,1521
0,40	17,953	38	17,991	37	18,028	38	18,066	37	18,103	38	18,141	37	18,178	37	18,215	37	18,252	37	18,289	37	0,1600
0,41	18,401	38	18,439	39	18,478	38	18,516	39	18,555	38	18,593	38	18,631	39	18,670	38	18,708	38	18,746	38	0,1681
0,42	18,848	40	18,888	39	18,927	40	18,967	39	19,006	40	19,046	39	19,085	39	19,124	39	19,163	39	19,202	39	0,1764
0,43	19,296	40	19,336	41	19,377	40	19,417	41	19,458	40	19,498	40	19,538	40	19,578	40	19,618	40	19,658	40	0,1849
0,44	19,743	42	19,785	41	19,826	42	19,868	41	19,909	41	19,950	41	19,991	41	20,032	41	20,073	41	20,114	40	0,1936
0,45	20,191	42	20,233	43	20,276	42	20,318	42	20,360	42	20,402	42	20,444	42	20,486	42	20,528	42	20,570	41	0,2025
0,50	22,426	48	22,474	47	22,521	47	22,568	47	22,615	46	22,661	47	22,708	47	22,755	46	22,801	47	22,848	46	0,2500
0,55	24,659	52	24,711	52	24,763	52	24,815	52	24,867	51	24,918	52	24,970	51	25,021	51	25,072	51	25,123	51	0,3025
0,60	26,889	57	26,946	57	27,003	56	27,059	57	27,116	56	27,172	56	27,228	57	27,285	56	27,341	55	27,396	56	0,3600
0,65	29,117	61	29,178	62	29,240	61	29,301	61	29,362	61	29,423	61	29,484	61	29,545	61	29,606	60	29,606	61	0,4225
0,70	31,340	67	31,407	66	31,473	66	31,539	66	31,605	66	31,671	66	31,737	66	31,803	65	31,868	65	31,933	66	0,4900
0,75	33,560	72	33,632	71	33,703	71	33,774	71	33,845	71	33,916	70	33,986	71	34,057	70	34,127	70	34,197	70	0,5625
0,80	35,777	76	35,853	76	35,929	76	36,005	76	36,081	75	36,156	76	36,232	75	36,307	75	36,382	75	36,457	75	0,6400
0,85	37,989	81	38,070	81	38,151	81	38,232	80	38,312	81	38,393	80	38,473	80	38,553	80	38,633	80	38,713	79	0,7225
0,90	40,196	86	40,282	86	40,368	86	40,454	85	40,539	86	40,625	85	40,710	85	40,795	85	40,880	84	40,964	85	0,8100
0,95	42,399	91	42,490	91	42,581	90	42,671	91	42,762	90	42,852	90	42,942	90	43,032	90	43,122	89	43,211	89	0,9025
1,00	44,596	96	44,692	96	44,788	96	44,884	95	44,979	95	45,074	95	45,169	95	45,264	95	45,359	94	45,453	94	1,0000
	2,8561		2,8800		2,9041		2,9282		2,9525		2,9768		3,0013		3,0258		3,0505		3,0752		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	447	448	449	450	451	452	453	454	455	456	457	458	
1	45	45	45	45	45	45	45	45	46	46	46	46	1
2	89	90	90	90	90	90	91	91	91	91	91	92	2
3	134	134	135	135	135	136	136	136	137	137	137	137	3
4	179	179	180	180	180	181	181	182	182	182	183	183	4
5	224	224	225	225	226	226	227	227	228	228	229	229	5
6	268	269	269	270	271	271	272	272	273	274	274	275	6
7	313	314	314	315	316	316	317	318	319	319	320	321	7
8	358	358	359	360	361	362	362	363	364	365	366	366	8
9	402	403	404	405	406	407	408	409	410	410	411	412	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord c.	2,49 Days	dif.	2,50 Days	dif.	2,51 Days	dif.	2,52 Days	dif.	2,53 Days	dif.	2,54 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,459	1	0,460	0	0,460	1	0,461	1	0,462	1	0,463	1	0,0001
0,02	0,917	2	0,919	2	0,921	2	0,923	2	0,925	1	0,926	2	0,0004
0,03	1,376	3	1,379	2	1,381	3	1,384	3	1,387	3	1,390	2	0,0009
0,04	1,835	3	1,838	4	1,842	4	1,846	3	1,849	4	1,853	4	0,0016
0,05	2,293	5	2,298	4	2,302	5	2,307	5	2,312	4	2,316	5	0,0025
0,06	2,752	5	2,757	6	2,763	5	2,768	6	2,774	5	2,779	6	0,0036
0,07	3,210	7	3,217	6	3,223	7	3,230	6	3,236	7	3,243	6	0,0049
0,08	3,669	7	3,676	8	3,684	7	3,691	7	3,698	8	3,706	7	0,0064
0,09	4,128	8	4,136	8	4,144	8	4,152	9	4,161	8	4,169	8	0,0081
0,10	4,586	9	4,595	10	4,605	9	4,614	9	4,623	9	4,632	9	0,0100
0,11	5,045	10	5,055	10	5,065	10	5,075	10	5,085	10	5,095	10	0,0121
0,12	5,503	11	5,514	11	5,525	11	5,536	11	5,547	11	5,558	11	0,0144
0,13	5,962	12	5,974	12	5,986	12	5,998	12	6,010	11	6,021	12	0,0169
0,14	6,420	13	6,433	13	6,446	13	6,459	13	6,472	13	6,485	12	0,0196
0,15	6,879	14	6,893	13	6,906	14	6,920	14	6,934	14	6,948	13	0,0225
0,16	7,337	15	7,352	15	7,367	14	7,381	15	7,396	15	7,411	14	0,0256
0,17	7,796	15	7,811	16	7,827	16	7,843	15	7,858	16	7,874	15	0,0289
0,18	8,254	17	8,271	16	8,287	17	8,304	16	8,320	17	8,337	16	0,0324
0,19	8,712	18	8,730	17	8,747	18	8,765	17	8,782	17	8,799	18	0,0361
0,20	9,171	18	9,189	18	9,207	19	9,226	18	9,244	18	9,262	19	0,0400
0,21	9,629	19	9,648	20	9,668	19	9,687	19	9,706	19	9,725	19	0,0441
0,22	10,087	20	10,107	21	10,128	20	10,148	20	10,168	20	10,188	20	0,0484
0,23	10,545	22	10,567	21	10,588	21	10,609	21	10,630	21	10,651	21	0,0529
0,24	11,003	23	11,026	22	11,048	22	11,070	22	11,092	22	11,114	21	0,0576
0,25	11,462	23	11,485	23	11,508	23	11,531	22	11,553	23	11,576	23	0,0625
0,26	11,920	24	11,944	24	11,968	23	11,991	24	12,015	24	12,039	24	0,0676
0,27	12,378	25	12,403	24	12,427	25	12,452	25	12,477	25	12,502	24	0,0729
0,28	12,836	25	12,861	26	12,887	26	12,913	26	12,939	25	12,964	26	0,0784
0,29	13,294	26	13,320	27	13,347	27	13,374	26	13,400	27	13,427	26	0,0841
0,30	13,751	28	13,779	28	13,807	27	13,834	28	13,862	27	13,889	27	0,0900
0,31	14,209	29	14,238	28	14,266	29	14,295	28	14,323	28	14,351	29	0,0961
0,32	14,667	29	14,696	30	14,726	29	14,755	30	14,785	29	14,814	29	0,1024
0,33	15,125	30	15,155	30	15,185	31	15,216	30	15,246	30	15,276	30	0,1089
0,34	15,582	32	15,614	31	15,645	31	15,676	31	15,707	31	15,738	31	0,1156
0,35	16,040	32	16,072	32	16,104	32	16,136	32	16,168	32	16,200	32	0,1225
0,36	16,497	33	16,530	34	16,564	33	16,597	33	16,630	33	16,663	32	0,1296
0,37	16,955	34	16,989	34	17,023	34	17,057	34	17,091	34	17,125	33	0,1369
0,38	17,412	35	17,447	35	17,482	35	17,517	35	17,552	35	17,587	34	0,1444
0,39	17,869	36	17,905	36	17,941	36	17,977	36	18,013	35	18,048	36	0,1521
0,40	18,326	37	18,363	37	18,400	37	18,437	37	18,474	36	18,510	37	0,1600
0,41	18,784	37	18,821	38	18,859	38	18,897	38	18,935	37	18,972	38	0,1681
0,42	19,241	38	19,279	39	19,318	39	19,357	38	19,395	39	19,434	38	0,1764
0,43	19,698	39	19,737	40	19,777	39	19,816	40	19,856	39	19,895	40	0,1849
0,44	20,154	41	20,195	41	20,236	40	20,276	41	20,317	40	20,357	40	0,1936
0,45	20,611	42	20,653	41	20,694	42	20,736	41	20,777	41	20,818	41	0,2025
0,50	22,894	46	22,940	46	22,986	46	23,032	46	23,078	46	23,124	46	0,2500
0,55	25,174	51	25,225	51	25,276	51	25,327	50	25,377	51	25,428	50	0,3025
0,60	27,452	56	27,508	55	27,563	56	27,619	55	27,674	55	27,729	55	0,3600
0,65	29,727	60	29,787	60	29,847	61	29,908	59	29,967	60	30,027	60	0,4225
0,70	31,999	65	32,064	65	32,129	64	32,193	65	32,258	65	32,323	64	0,4900
0,75	34,267	70	34,337	69	34,406	70	34,476	69	34,545	70	34,615	69	0,5625
0,80	36,532	74	36,606	75	36,681	74	36,755	74	36,829	74	36,903	74	0,6400
0,85	38,792	80	38,872	79	38,951	79	39,030	79	39,109	79	39,188	78	0,7225
0,90	41,049	84	41,133	84	41,217	84	41,301	83	41,384	84	41,468	83	0,8100
0,95	43,300	90	43,390	88	43,478	89	43,567	89	43,656	88	43,744	88	0,9025
1,00	45,547	94	45,641	94	45,735	94	45,829	93	45,922	93	46,015	93	1,0000
	3,1001		3,1250		3,1501		3,1752		3,2005		3,2258		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	456	457	458	459	460	461	462	463	464	
1	46	46	46	46	46	46	46	46	46	1
2	91	91	92	92	92	92	92	93	93	2
3	137	137	137	138	138	138	139	139	139	3
4	182	183	183	184	184	184	185	185	186	4
5	228	229	229	230	230	231	231	232	232	5
6	274	274	275	275	276	277	277	278	278	6
7	319	320	321	321	322	323	323	324	325	7
8	365	366	366	367	368	369	370	370	371	8
9	410	411	412	413	414	415	416	417	418	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r + r''$. 2,55		2,56		2,57		2,58		2,59		2,60		2,61		2,62		2,63		2,64		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,464	1	0,465	1	0,466	1	0,467	1	0,468	1	0,469	1	0,470	0	0,470	1	0,471	1	0,472	1	0,0001
0,02	0,928	2	0,930	2	0,932	2	0,934	2	0,936	1	0,937	2	0,939	2	0,941	2	0,943	2	0,945	1	0,0004
0,03	1,392	3	1,395	3	1,398	3	1,401	2	1,403	3	1,406	3	1,409	2	1,411	3	1,414	3	1,417	2	0,0009
0,04	1,857	3	1,860	4	1,864	3	1,867	4	1,871	4	1,875	3	1,878	4	1,882	3	1,885	4	1,889	4	0,0016
0,05	2,321	4	2,325	5	2,330	4	2,334	5	2,339	4	2,343	5	2,348	4	2,352	5	2,357	4	2,361	5	0,0025
0,06	2,785	5	2,790	6	2,796	5	2,801	6	2,807	5	2,812	5	2,817	6	2,823	5	2,828	6	2,834	5	0,0036
0,07	3,249	6	3,255	7	3,262	6	3,268	6	3,274	7	3,281	6	3,287	6	3,293	7	3,300	6	3,306	6	0,0049
0,08	3,713	7	3,720	8	3,728	7	3,735	7	3,742	7	3,749	7	3,756	8	3,764	7	3,771	7	3,778	7	0,0064
0,09	4,177	8	4,185	8	4,193	9	4,202	8	4,210	8	4,218	8	4,226	8	4,234	8	4,242	8	4,250	8	0,0081
0,10	4,641	9	4,650	9	4,659	9	4,668	9	4,677	10	4,687	9	4,696	9	4,705	8	4,713	9	4,722	9	0,0100
0,11	5,105	10	5,115	10	5,125	10	5,135	10	5,145	10	5,155	10	5,165	10	5,175	10	5,185	10	5,195	9	0,0121
0,12	5,569	11	5,580	11	5,591	11	5,602	11	5,613	11	5,624	10	5,634	11	5,645	11	5,656	11	5,667	10	0,0144
0,13	6,033	12	6,045	12	6,057	12	6,069	11	6,080	12	6,092	12	6,104	12	6,116	11	6,127	12	6,139	12	0,0169
0,14	6,497	13	6,510	13	6,523	12	6,535	13	6,548	13	6,561	12	6,573	13	6,586	12	6,598	13	6,611	13	0,0196
0,15	6,961	14	6,975	14	6,989	13	7,002	14	7,016	13	7,029	14	7,043	13	7,056	14	7,070	13	7,083	14	0,0225
0,16	7,425	15	7,440	14	7,454	15	7,469	14	7,483	15	7,498	14	7,512	14	7,526	15	7,541	14	7,555	14	0,0256
0,17	7,889	16	7,905	15	7,920	15	7,935	16	7,951	15	7,966	15	7,981	16	7,997	15	8,012	15	8,027	15	0,0289
0,18	8,353	16	8,369	17	8,386	16	8,402	16	8,418	17	8,435	16	8,451	16	8,467	16	8,483	16	8,499	16	0,0324
0,19	8,817	17	8,834	17	8,851	18	8,869	17	8,886	17	8,903	17	8,920	17	8,937	17	8,954	17	8,971	17	0,0361
0,20	9,281	18	9,299	18	9,317	18	9,335	18	9,353	18	9,371	18	9,389	18	9,407	18	9,425	18	9,443	18	0,0400
0,21	9,744	20	9,764	19	9,783	19	9,802	19	9,821	19	9,840	18	9,858	19	9,877	19	9,896	19	9,915	19	0,0441
0,22	10,208	20	10,228	20	10,248	20	10,268	20	10,288	20	10,308	20	10,328	19	10,347	20	10,367	20	10,387	20	0,0484
0,23	10,672	21	10,693	21	10,714	20	10,734	21	10,755	21	10,776	21	10,797	21	10,818	20	10,838	21	10,859	20	0,0529
0,24	11,135	22	11,157	22	11,179	22	11,201	22	11,223	21	11,244	22	11,266	21	11,287	22	11,309	22	11,331	21	0,0576
0,25	11,599	23	11,622	23	11,645	22	11,667	23	11,690	22	11,712	23	11,735	22	11,757	23	11,780	22	11,802	23	0,0625
0,26	12,063	23	12,086	24	12,110	23	12,133	24	12,157	24	12,181	23	12,204	23	12,227	24	12,251	23	12,274	23	0,0676
0,27	12,526	25	12,551	24	12,575	25	12,600	24	12,624	25	12,649	24	12,673	24	12,697	25	12,722	24	12,746	24	0,0729
0,28	12,990	25	13,015	26	13,041	25	13,066	25	13,091	26	13,117	25	13,142	25	13,167	25	13,192	25	13,217	25	0,0784
0,29	13,453	26	13,479	27	13,506	26	13,532	26	13,558	27	13,585	26	13,611	26	13,637	26	13,663	26	13,689	26	0,0841
0,30	13,916	28	13,944	27	13,971	27	13,998	27	14,025	28	14,053	27	14,080	27	14,107	27	14,134	26	14,160	27	0,0900
0,31	14,380	28	14,408	28	14,436	28	14,464	28	14,492	28	14,520	28	14,548	28	14,576	28	14,604	28	14,632	28	0,0961
0,32	14,843	29	14,872	29	14,901	29	14,930	29	14,959	29	14,988	29	15,017	29	15,046	29	15,075	28	15,103	29	0,1024
0,33	15,306	30	15,336	30	15,366	30	15,396	30	15,426	30	15,456	30	15,486	29	15,515	30	15,545	30	15,575	29	0,1089
0,34	15,769	31	15,800	31	15,831	31	15,862	31	15,893	31	15,924	30	15,954	31	15,985	31	16,016	30	16,046	30	0,1156
0,35	16,232	32	16,264	32	16,296	32	16,328	32	16,360	31	16,391	32	16,423	31	16,454	32	16,486	31	16,517	32	0,1225
0,36	16,695	33	16,728	33	16,761	33	16,794	32	16,826	33	16,859	32	16,891	33	16,924	32	16,956	32	16,988	33	0,1296
0,37	17,158	34	17,192	34	17,226	33	17,259	34	17,293	33	17,326	34	17,360	33	17,393	33	17,426	34	17,460	33	0,1369
0,38	17,621	35	17,656	35	17,691	34	17,725	34	17,759	35	17,794	34	17,828	34	17,862	35	17,897	34	17,931	34	0,1444
0,39	18,084	36	18,120	35	18,155	36	18,191	35	18,226	35	18,261	35	18,296	36	18,332	35	18,367	35	18,402	35	0,1521
0,40	18,547	36	18,583	37	18,620	36	18,656	36	18,692	37	18,729	36	18,765	36	18,801	36	18,837	36	18,873	35	0,1600
0,41	19,010	37	19,047	37	19,084	38	19,122	37	19,159	37	19,196	37	19,233	37	19,270	37	19,307	36	19,343	37	0,1681
0,42	19,472	38	19,510	39	19,549	38	19,587	38	19,625	38	19,663	38	19,701	38	19,739	38	19,777	37	19,814	38	0,1764
0,43	19,935	39	19,974	39	20,013	39	20,052	39	20,091	39	20,130	39	20,169	39	20,208	38	20,246	39	20,285	39	0,1849
0,44	20,397	40	20,437	40	20,477	40	20,517	40	20,557	40	20,597	40	20,637	40	20,677	39	20,716	40	20,756	39	0,1936
0,45	20,859	42	20,901	41	20,942	40	20,982	41	21,023	41	21,064	41	21,105	40	21,145	41	21,186	40	21,226	41	0,2025
0,50	23,170	46	23,216	45	23,261	46	23,307	45	23,352	46	23,398	45	23,443	45	23,488	45	23,533	45	23,578	45	0,2500
0,55	25,478	51	25,529	50	25,579	50	25,629	50	25,679	50	25,729	50	25,779	49	25,828	50	25,878	49	25,927	50	0,3025
0,60	27,784	55	27,839	55	27,894	55	27,949	54	28,003	55	28,058	54	28,112	54	28,166	55	28,221	54	28,275	54	0,3600
0,65	30,087	60	30,147	59	30,206	59	30,265	60	30,325	59	30,384	59	30,443	59	30,502	58	30,560	59	30,619	59	0,4225
0,70	32,387	64	32,451	64	32,515	64	32,579	64	32,643	64	32,707	64	32,771	63	32,834	63	32,897	64	32,961	63	0,4900
0,75	34,684	69	34,753	68	34,821	69	34,890	69	34,959	68	35,027	68	35,095	68	35,163	68	35,231	68	35,299	68	0,5625
0,80	36,977	73	37,050	74	37,124	73	37,197	74	37,271	73	37,344	73	37,417	72	37,489	73	37,562	72	37,634	73	0,6400
0,85	39,266	79	39,345	78	39,423	78	39,501	78	39,579	78	39,657	77	39,734	78	39,812	77	39,889	77	39,966	7	0,7225
0,90	41,551	84	41,635	83	41,718	82	41,800	83	41,883	82	41,965	83	42,048	82	42,130	82	42,212	82	42,294	81	0,8100
0,95	43,832	88	43,920	88	44,008	88	44,096	87	44,183	87	44,270	87	44,357	87	44,444	87	44,531	86	44,617	87	0,9025
1,00	46,108	93	46,201	93	46,294	92	46,386	92	46,478	92	46,570	92	46,662	92	46,754	91	46,845	92	46,937	91	1,0000
	3,2513		3,2768		3,3025		3,3282		3,3541		3,3800		3,4061		3,4322		3,4585		3,4848		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	462	463	464	465	466	467	468	469	470	471	472	473	
1	46	46	46	47	47	47	47	47	47	47	47	47	1
2	92	93	93	93	93	93	94	94	94	94	94	95	2
3	139	139	139	140	140	140	140	141	141	141	142	142	3
4	185	185	186	186	186	187	187	188	188	188	189	189	4
5	231	232	232	233	233	234	234	235	235	236	236	237	5
6	277	278	278	279	280	280	281	281	282	283	283	284	6
7	323	324	325	326	326	327	328	328	329	330	330	331	7
8	370	370	371	372	373	374	374	375	376	377	378	378	8
9	416	417	418	419	419	420	421	422	423	424	425	426	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord c.	2,65 Days	dif.	2,66 Days	dif.	2,67 Days	dif.	2,68 Days	dif.	2,69 Days	dif.	2,70 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,473	1	0,474	1	0,475	1	0,476	1	0,477	1	0,478	0	0,0001
0,02	0,946	2	0,948	2	0,950	2	0,952	1	0,953	2	0,955	2	0,0004
0,03	1,419	3	1,422	3	1,425	2	1,427	3	1,430	3	1,433	2	0,0009
0,04	1,893	3	1,896	4	1,900	3	1,903	4	1,907	3	1,910	4	0,0016
0,05	2,366	4	2,370	5	2,375	4	2,379	5	2,384	4	2,388	4	0,0025
0,06	2,839	5	2,844	6	2,850	5	2,855	5	2,860	6	2,866	5	0,0036
0,07	3,312	6	3,318	7	3,325	6	3,331	6	3,337	6	3,343	6	0,0049
0,08	3,785	7	3,792	7	3,799	8	3,807	7	3,814	7	3,821	7	0,0064
0,09	4,258	8	4,266	8	4,274	8	4,282	8	4,290	8	4,298	8	0,0081
0,10	4,731	9	4,740	9	4,749	9	4,758	9	4,767	9	4,776	9	0,0100
0,11	5,204	10	5,214	10	5,224	10	5,234	10	5,244	9	5,253	10	0,0121
0,12	5,677	11	5,688	11	5,699	11	5,710	10	5,720	11	5,731	10	0,0144
0,13	6,151	11	6,162	12	6,174	11	6,185	12	6,197	11	6,208	12	0,0169
0,14	6,624	12	6,636	12	6,648	13	6,661	12	6,673	13	6,686	12	0,0196
0,15	7,097	13	7,110	13	7,123	14	7,137	13	7,150	13	7,163	13	0,0225
0,16	7,569	15	7,584	14	7,598	14	7,612	14	7,626	15	7,641	14	0,0256
0,17	8,042	16	8,058	15	8,073	15	8,088	15	8,103	15	8,118	15	0,0289
0,18	8,515	16	8,531	16	8,547	16	8,563	16	8,579	16	8,595	16	0,0324
0,19	8,988	17	9,005	17	9,022	17	9,039	17	9,056	17	9,073	16	0,0361
0,20	9,461	18	9,479	18	9,497	17	9,514	18	9,532	18	9,550	18	0,0400
0,21	9,934	19	9,953	18	9,971	19	9,990	19	10,009	18	10,027	19	0,0441
0,22	10,407	19	10,426	20	10,446	19	10,465	20	10,485	19	10,504	20	0,0484
0,23	10,879	21	10,900	20	10,920	21	10,941	20	10,961	21	10,982	20	0,0529
0,24	11,352	21	11,373	22	11,395	21	11,416	22	11,438	21	11,459	21	0,0576
0,25	11,825	22	11,847	22	11,869	23	11,892	22	11,914	22	11,936	22	0,0625
0,26	12,297	24	12,321	23	12,344	23	12,367	23	12,390	23	12,413	23	0,0676
0,27	12,770	24	12,794	24	12,818	24	12,842	24	12,866	24	12,890	24	0,0729
0,28	13,242	25	13,267	25	13,292	25	13,317	25	13,342	25	13,367	25	0,0784
0,29	13,715	26	13,741	26	13,767	25	13,792	26	13,818	26	13,844	26	0,0841
0,30	14,187	27	14,214	27	14,241	27	14,268	26	14,294	27	14,321	26	0,0900
0,31	14,660	27	14,687	28	14,715	28	14,743	27	14,770	28	14,798	27	0,0961
0,32	15,132	29	15,161	28	15,189	29	15,218	28	15,246	28	15,274	29	0,1024
0,33	15,604	30	15,634	29	15,663	30	15,693	29	15,722	29	15,751	29	0,1089
0,34	16,076	31	16,107	30	16,137	30	16,167	31	16,198	30	16,228	30	0,1156
0,35	16,549	31	16,580	31	16,611	31	16,642	31	16,673	31	16,704	31	0,1225
0,36	17,021	32	17,053	32	17,085	32	17,117	32	17,149	32	17,181	32	0,1296
0,37	17,493	33	17,526	33	17,559	33	17,592	33	17,625	33	17,658	32	0,1369
0,38	17,965	34	17,999	34	18,033	33	18,066	34	18,100	34	18,134	34	0,1444
0,39	18,437	35	18,472	34	18,506	35	18,541	35	18,576	34	18,610	35	0,1521
0,40	18,908	36	18,944	36	18,980	36	19,016	35	19,051	36	19,087	35	0,1600
0,41	19,380	37	19,417	37	19,454	36	19,490	37	19,527	36	19,563	36	0,1681
0,42	19,852	38	19,890	37	19,927	37	19,964	38	20,002	37	20,039	37	0,1764
0,43	20,324	38	20,362	38	20,400	39	20,439	38	20,477	38	20,515	38	0,1849
0,44	20,795	40	20,835	39	20,874	39	20,913	39	20,952	39	20,991	39	0,1936
0,45	21,267	40	21,307	40	21,347	40	21,387	40	21,427	40	21,467	40	0,2025
0,50	23,623	45	23,668	44	23,712	45	23,757	45	23,802	44	23,846	44	0,2500
0,55	25,977	49	26,026	49	26,075	50	26,125	49	26,174	49	26,223	48	0,3025
0,60	28,329	53	28,382	54	28,436	54	28,490	53	28,543	54	28,597	53	0,3600
0,65	30,678	58	30,736	58	30,794	59	30,853	58	30,911	58	30,969	57	0,4225
0,70	33,024	63	33,087	63	33,150	62	33,212	63	33,275	63	33,338	62	0,4900
0,75	35,367	68	35,435	67	35,502	67	35,569	68	35,637	67	35,704	67	0,5625
0,80	37,707	72	37,779	72	37,851	72	37,923	72	37,995	71	38,066	72	0,6400
0,85	40,043	77	40,120	77	40,197	76	40,273	76	40,349	77	40,426	76	0,7225
0,90	42,375	82	42,457	81	42,538	81	42,619	81	42,700	81	42,781	81	0,8100
0,95	44,704	86	44,790	86	44,876	86	44,962	85	45,047	86	45,133	85	0,9025
1,00	47,028	91	47,119	90	47,209	91	47,300	90	47,390	90	47,480	90	1,0000
	3,5113		3,5378		3,5645		3,5912		3,6181		3,6450		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r'^2$ nearly.

	471	472	473	474	475	476	477	478	
1	47	47	47	47	48	48	48	48	1
2	94	94	95	95	95	95	95	96	2
3	141	142	142	142	143	143	143	143	3
4	188	189	189	190	190	190	191	191	4
5	236	236	237	237	238	238	239	239	5
6	283	283	284	284	285	286	286	287	6
7	330	330	331	332	333	333	334	335	7
8	377	378	378	379	380	381	382	382	8
9	424	425	426	427	428	428	429	430	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	2,71		2,72		2,73		2,74		2,75		2,76		2,77		2,78		2,79		2,80		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,478	1	0,479	1	0,480	1	0,481	1	0,482	1	0,483	1	0,484	1	0,485	1	0,486	0	0,486	1	0,0001
0,02	0,957	2	0,959	2	0,961	1	0,962	2	0,964	2	0,966	2	0,968	1	0,969	2	0,971	2	0,973	1	0,0004
0,03	1,435	3	1,438	3	1,441	2	1,443	3	1,446	3	1,449	2	1,451	3	1,454	2	1,456	3	1,459	3	0,0009
0,04	1,914	3	1,917	4	1,921	4	1,925	3	1,928	4	1,932	3	1,935	3	1,938	4	1,942	3	1,945	4	0,0016
0,05	2,392	5	2,397	4	2,401	5	2,406	4	2,410	4	2,414	5	2,419	4	2,423	4	2,427	5	2,432	4	0,0025
0,06	2,871	5	2,876	5	2,881	6	2,887	5	2,892	5	2,897	5	2,902	6	2,908	5	2,913	5	2,918	5	0,0036
0,07	3,349	7	3,356	6	3,362	6	3,368	6	3,374	6	3,380	6	3,386	6	3,392	6	3,398	7	3,405	6	0,0049
0,08	3,828	7	3,835	7	3,842	7	3,849	7	3,856	7	3,863	7	3,870	7	3,877	7	3,884	7	3,891	7	0,0064
0,09	4,306	8	4,314	8	4,322	8	4,330	8	4,338	8	4,346	8	4,354	7	4,361	8	4,369	8	4,377	8	0,0081
0,10	4,785	8	4,793	9	4,802	9	4,811	9	4,820	9	4,829	8	4,837	9	4,846	9	4,855	8	4,863	9	0,0100
0,11	5,263	10	5,273	9	5,282	10	5,292	10	5,302	9	5,311	10	5,321	10	5,331	9	5,340	10	5,350	9	0,0121
0,12	5,741	11	5,752	11	5,763	10	5,773	11	5,784	10	5,794	11	5,805	10	5,815	11	5,826	10	5,836	10	0,0144
0,13	6,220	11	6,231	12	6,243	11	6,254	12	6,266	11	6,277	11	6,288	12	6,300	11	6,311	11	6,322	12	0,0169
0,14	6,698	12	6,710	13	6,723	12	6,735	12	6,747	13	6,760	12	6,772	12	6,784	12	6,796	12	6,808	13	0,0196
0,15	7,176	14	7,190	13	7,203	13	7,216	13	7,229	13	7,242	13	7,255	14	7,269	13	7,282	13	7,295	13	0,0225
0,16	7,655	14	7,669	14	7,683	14	7,697	14	7,711	14	7,725	14	7,739	14	7,753	14	7,767	14	7,781	14	0,0256
0,17	8,133	15	8,148	15	8,163	15	8,178	15	8,193	15	8,208	15	8,223	14	8,237	15	8,252	15	8,267	15	0,0289
0,18	8,611	16	8,627	16	8,643	16	8,659	16	8,675	15	8,690	16	8,706	16	8,722	16	8,738	15	8,753	16	0,0324
0,19	9,089	17	9,106	17	9,123	17	9,140	16	9,156	17	9,173	17	9,190	16	9,206	17	9,223	16	9,239	17	0,0361
0,20	9,568	17	9,585	18	9,603	17	9,620	18	9,638	18	9,656	17	9,673	18	9,691	17	9,708	17	9,725	18	0,0400
0,21	10,046	18	10,064	19	10,083	18	10,101	19	10,120	18	10,138	18	10,156	19	10,175	18	10,193	18	10,211	19	0,0441
0,22	10,524	19	10,543	20	10,563	19	10,582	19	10,601	20	10,621	19	10,640	19	11,659	19	10,678	19	10,697	19	0,0484
0,23	11,002	20	11,022	21	11,043	20	11,063	20	11,083	20	11,103	20	11,123	20	11,143	20	11,163	20	11,183	20	0,0529
0,24	11,480	21	11,501	21	11,522	21	11,543	22	11,565	21	11,586	21	11,607	21	11,628	20	11,648	21	11,669	21	0,0576
0,25	11,958	22	11,980	22	12,002	22	12,024	22	12,046	22	12,068	22	12,090	22	12,112	21	12,133	22	12,155	22	0,0625
0,26	12,436	23	12,459	23	12,482	23	12,505	23	12,528	22	12,550	23	12,573	23	12,596	22	12,618	23	12,641	23	0,0676
0,27	12,914	24	12,938	24	12,962	23	12,985	24	13,009	24	13,033	23	13,056	24	13,080	23	13,103	24	13,127	23	0,0729
0,28	13,392	24	13,416	25	13,441	25	13,466	24	13,490	25	13,515	24	13,539	25	13,564	24	13,588	25	13,613	24	0,0784
0,29	13,870	25	13,895	26	13,921	25	13,946	26	13,972	25	13,997	26	14,023	25	14,048	25	14,073	25	14,098	26	0,0841
0,30	14,347	27	14,374	26	14,400	27	14,427	26	14,453	26	14,479	27	14,506	26	14,532	26	14,558	26	14,584	26	0,0900
0,31	14,825	27	14,852	28	14,880	27	14,907	27	14,934	28	14,962	27	14,989	27	15,016	27	15,043	27	15,070	27	0,0961
0,32	15,303	28	15,331	28	15,359	28	15,387	29	15,416	28	15,444	28	15,472	28	15,500	27	15,527	28	15,555	28	0,1024
0,33	15,780	30	15,810	29	15,839	29	15,868	29	15,897	29	15,926	29	15,955	28	15,983	29	16,012	29	16,041	29	0,1089
0,34	16,258	30	16,288	30	16,318	30	16,348	30	16,378	30	16,408	29	16,437	30	16,467	30	16,497	29	16,526	30	0,1156
0,35	16,735	31	16,766	31	16,797	31	16,828	31	16,859	31	16,890	30	16,920	31	16,951	30	16,981	31	17,012	30	0,1225
0,36	17,213	32	17,245	32	17,277	31	17,308	32	17,340	31	17,371	32	17,403	31	17,434	32	17,466	31	17,497	32	0,1296
0,37	17,690	33	17,723	33	17,756	32	17,788	33	17,821	32	17,853	33	17,886	32	17,918	32	17,950	33	17,983	32	0,1369
0,38	18,168	33	18,201	34	18,235	33	18,268	34	18,302	33	18,335	33	18,368	34	18,402	33	18,435	33	18,468	33	0,1444
0,39	18,645	34	18,679	35	18,714	34	18,748	34	18,782	35	18,817	34	18,851	34	18,885	34	18,919	34	18,953	34	0,1521
0,40	19,122	36	19,158	35	19,193	35	19,228	35	19,263	35	19,298	35	19,333	35	19,368	35	19,403	35	19,438	35	0,1600
0,41	19,599	37	19,636	36	19,672	36	19,708	36	19,744	36	19,780	36	19,816	36	19,852	36	19,888	35	19,923	36	0,1681
0,42	20,076	38	20,114	37	20,151	37	20,188	37	20,225	36	20,261	37	20,298	37	20,335	37	20,372	36	20,408	37	0,1764
0,43	20,553	38	20,591	38	20,629	38	20,667	38	20,705	38	20,743	38	20,781	37	20,818	38	20,856	37	20,893	38	0,1849
0,44	21,030	39	21,069	39	21,108	39	21,147	39	21,186	38	21,224	39	21,263	38	21,301	39	21,340	38	21,378	38	0,1936
0,45	21,507	40	21,547	40	21,587	39	21,626	40	21,666	40	21,706	39	21,745	39	21,784	40	21,824	39	21,863	39	0,2025
0,50	23,890	45	23,935	44	23,979	44	24,023	44	24,067	44	24,111	44	24,155	44	24,199	43	24,242	44	24,286	44	0,2500
0,55	26,271	49	26,320	49	26,369	48	26,417	49	26,466	48	26,514	49	26,563	48	26,611	48	26,659	48	26,707	48	0,3025
0,60	28,650	53	28,703	54	28,757	53	28,810	53	28,863	52	28,915	53	28,968	53	29,021	52	29,073	53	29,126	52	0,3600
0,65	31,026	58	31,084	58	31,142	57	31,199	58	31,257	57	31,314	57	31,371	57	31,428	57	31,485	57	31,542	57	0,4225
0,70	33,400	62	33,462	62	33,524	62	33,586	62	33,648	62	33,710	62	33,772	61	33,833	62	33,895	61	33,956	61	0,4900
0,75	35,771	66	35,837	67	35,904	67	35,971	66	36,037	66	36,103	66	36,169	66	36,235	66	36,301	66	36,367	66	0,5625
0,80	38,138	71	38,209	71	38,280	72	38,352	70	38,422	71	38,493	71	38,564	70	38,634	71	38,705	70	38,775	70	0,6400
0,85	40,502	76	40,578	75	40,653	76	40,729	76	40,805	75	40,880	75	40,955	75	41,030	75	41,105	75	41,180	75	0,7225
0,90	42,862	80	42,942	81	43,023	80	43,103	80	43,183	80	43,263	80	43,343	79	43,422	80	43,502	79	43,581	79	0,8100
0,95	45,218	85	45,303	85	45,388	85	45,473	85	45,558	84	45,642	85	45,727	84	45,811	84	45,895	84	45,979	83	0,9025
1,00	47,570	90	47,660	90	47,750	89	47,839	89	47,928	89	48,017	89	48,106	89	48,195	89	48,284	88	48,372	88	1,0000
	3,6721		3,6992		3,7265		3,7538		3,7813		3,8088		3,8365		3,8642		3,8921		3,9200		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	477	478	479	480	481	482	483	484	485	486	487	
1	48	48	48	48	48	48	48	48	49	49	49	1
2	95	96	96	96	96	96	97	97	97	97	97	2
3	143	143	144	144	144	145	145	145	146	146	146	3
4	191	191	192	192	192	193	193	194	194	194	195	4
5	239	239	240	240	241	241	242	242	243	243	244	5
6	286	287	287	288	289	289	290	290	291	292	292	6
7	334	335	335	336	337	337	338	339	340	340	341	7
8	382	382	383	384	385	386	386	387	388	389	390	8
9	429	430	431	432	433	434	435	436	437	437	438	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r+r''$. 2,81 Days	dif.	2,82 Days	dif.	2,83 Days	dif.	2,84 Days	dif.	2,85 Days	dif.	2,86 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,487	1	0,488	1	0,489	1	0,490	1	0,491	1	0,492	0	0,0001
0,02	0,974	2	0,976	2	0,978	2	0,980	1	0,981	2	0,983	2	0,0004
0,03	1,462	2	1,464	3	1,467	3	1,470	2	1,472	3	1,475	2	0,0009
0,04	1,949	3	1,952	4	1,956	3	1,959	4	1,963	3	1,966	4	0,0016
0,05	2,436	4	2,440	5	2,445	4	2,449	4	2,453	5	2,458	4	0,0025
0,06	2,923	6	2,929	5	2,934	5	2,939	5	2,944	5	2,949	5	0,0036
0,07	3,411	6	3,417	6	3,423	6	3,429	6	3,435	6	3,441	6	0,0049
0,08	3,898	7	3,905	7	3,912	7	3,919	6	3,925	7	3,932	7	0,0064
0,09	4,385	8	4,393	8	4,401	7	4,408	8	4,416	8	4,424	8	0,0081
0,10	4,872	9	4,881	8	4,889	9	4,898	9	4,907	8	4,915	9	0,0100
0,11	5,359	10	5,369	9	5,378	10	5,388	9	5,397	10	5,407	9	0,0121
0,12	5,846	11	5,857	10	5,867	11	5,878	10	5,888	10	5,898	11	0,0144
0,13	6,334	11	6,345	11	6,356	11	6,367	11	6,378	12	6,390	11	0,0169
0,14	6,821	12	6,833	12	6,845	12	6,857	12	6,869	12	6,881	12	0,0196
0,15	7,308	13	7,321	13	7,334	13	7,347	13	7,360	12	7,372	13	0,0225
0,16	7,795	14	7,809	13	7,822	14	7,836	14	7,850	14	7,864	14	0,0256
0,17	8,282	15	8,297	14	8,311	15	8,326	15	8,341	14	8,355	15	0,0289
0,18	8,769	15	8,784	16	8,800	15	8,815	16	8,831	16	8,847	15	0,0324
0,19	9,256	16	9,272	17	9,289	16	9,305	16	9,321	17	9,338	16	0,0361
0,20	9,743	17	9,760	17	9,777	18	9,795	17	9,812	17	9,829	17	0,0400
0,21	10,230	18	10,248	18	10,266	18	10,284	18	10,302	18	10,320	18	0,0441
0,22	10,716	20	10,736	19	10,755	19	10,774	19	10,793	19	10,812	18	0,0484
0,23	11,203	20	11,223	20	11,243	20	11,263	20	11,283	20	11,303	19	0,0529
0,24	11,690	21	11,711	21	11,732	20	11,752	21	11,773	21	11,794	20	0,0576
0,25	12,177	22	12,199	21	12,220	22	12,242	21	12,263	22	12,285	21	0,0625
0,26	12,664	22	12,686	23	12,709	22	12,731	23	12,754	22	12,776	22	0,0676
0,27	13,150	24	13,174	23	13,197	23	13,220	24	13,244	23	13,267	23	0,0729
0,28	13,637	24	13,661	25	13,686	24	13,710	24	13,734	24	13,758	24	0,0784
0,29	14,124	25	14,149	25	14,174	25	14,199	25	14,224	25	14,249	25	0,0841
0,30	14,610	26	14,636	26	14,662	26	14,688	26	14,714	26	14,740	26	0,0900
0,31	15,097	27	15,124	26	15,150	27	15,177	27	15,204	27	15,231	26	0,0961
0,32	15,583	28	15,611	28	15,639	27	15,666	28	15,694	27	15,721	28	0,1024
0,33	16,070	28	16,098	29	16,127	28	16,155	29	16,184	28	16,212	29	0,1089
0,34	16,556	29	16,585	30	16,615	29	16,644	30	16,674	29	16,703	29	0,1156
0,35	17,042	31	17,073	30	17,103	30	17,133	30	17,163	31	17,194	30	0,1225
0,36	17,529	31	17,560	31	17,591	31	17,622	31	17,653	31	17,684	31	0,1296
0,37	18,015	32	18,047	32	18,079	32	18,111	32	18,143	32	18,175	32	0,1369
0,38	18,501	33	18,534	33	18,567	33	18,600	32	18,632	33	18,665	33	0,1444
0,39	18,987	34	19,021	34	19,055	33	19,088	34	19,122	34	19,156	33	0,1521
0,40	19,473	35	19,508	34	19,542	35	19,577	35	19,612	34	19,646	34	0,1600
0,41	19,959	36	19,995	35	20,030	36	20,066	35	20,101	35	20,136	36	0,1681
0,42	20,445	36	20,481	37	20,518	36	20,554	36	20,590	37	20,627	36	0,1764
0,43	20,931	37	20,968	37	21,005	38	21,043	37	21,080	37	21,117	37	0,1849
0,44	21,416	39	21,455	38	21,493	38	21,531	38	21,569	38	21,607	38	0,1936
0,45	21,902	39	21,941	39	21,980	39	22,019	39	22,058	39	22,097	39	0,2025
0,50	24,330	43	24,373	43	24,416	44	24,460	43	24,503	43	24,546	43	0,2500
0,55	26,755	48	26,803	48	26,851	47	26,898	48	26,946	48	26,994	47	0,3025
0,60	29,178	53	29,231	52	29,283	52	29,335	52	29,387	52	29,439	51	0,3600
0,65	31,599	57	31,656	56	31,712	57	31,769	56	31,825	57	31,882	56	0,4225
0,70	34,017	62	34,079	61	34,140	60	34,200	61	34,261	61	34,322	60	0,4900
0,75	36,433	65	36,498	66	36,564	65	36,629	65	36,694	65	36,759	65	0,5625
0,80	38,845	70	38,915	70	38,985	70	39,055	70	39,125	69	39,194	70	0,6400
0,85	41,255	74	41,329	74	41,403	75	41,478	74	41,552	74	41,626	73	0,7225
0,90	43,660	79	43,739	79	43,818	79	43,897	78	43,975	79	44,054	78	0,8100
0,95	46,062	84	46,146	83	46,229	83	46,312	83	46,395	83	46,478	83	0,9025
1,00	48,460	88	48,548	88	48,636	88	48,724	88	48,812	87	48,899	87	1,0000
	3,9481		3,9762		4,0045		4,0328		4,0613		4,0898		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	485	486	487	488	489	490	491	492	
1	49	49	49	49	49	49	49	49	1
2	97	97	97	98	98	98	98	98	2
3	146	146	146	146	147	147	147	148	3
4	194	194	195	195	196	196	196	197	4
5	243	243	244	244	245	245	246	246	5
6	291	292	292	293	293	294	295	295	6
7	340	340	341	342	342	343	344	344	7
8	388	389	390	390	391	392	393	394	8
9	437	437	438	439	440	441	442	443	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	2,87 Days	dif.	2,88 Days	dif.	2,89 Days	dif.	2,90 Days	dif.	2,91 Days	dif.	2,92 Days	dif.	2,93 Days	dif.	2,94 Days	dif.	2,95 Days	dif.	2,96 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,492	1	0,493	1	0,494	1	0,495	1	0,496	1	0,497	1	0,498	0	0,498	1	0,499	1	0,500	1	0,0001
0,02	0,985	2	0,987	1	0,988	2	0,990	2	0,992	1	0,993	2	0,995	2	0,997	1	0,998	2	1,000	2	0,0004
0,03	1,477	3	1,480	2	1,482	3	1,485	2	1,487	3	1,490	3	1,493	2	1,495	3	1,498	2	1,500	3	0,0009
0,04	1,970	3	1,973	3	1,976	4	1,980	3	1,983	4	1,987	3	1,990	4	1,994	3	1,997	3	2,000	4	0,0016
0,05	2,462	4	2,466	5	2,471	4	2,475	4	2,479	4	2,483	5	2,488	4	2,492	4	2,496	4	2,500	5	0,0025
0,06	2,954	6	2,960	5	2,965	5	2,970	5	2,975	5	2,980	5	2,985	5	2,990	5	2,995	5	3,000	5	0,0036
0,07	3,447	6	3,453	6	3,459	6	3,465	6	3,471	6	3,477	6	3,483	6	3,489	5	3,494	6	3,500	6	0,0049
0,08	3,939	7	3,946	7	3,953	7	3,960	7	3,967	6	3,973	7	3,980	7	3,987	7	3,994	6	4,000	7	0,0064
0,09	4,432	7	4,439	8	4,447	8	4,455	7	4,462	8	4,470	8	4,478	7	4,485	8	4,493	7	4,500	8	0,0081
0,10	4,924	8	4,932	9	4,941	9	4,950	8	4,958	9	4,967	8	4,975	9	4,984	8	4,992	8	5,000	9	0,0100
0,11	5,416	10	5,426	9	5,435	9	5,444	10	5,454	9	5,463	10	5,473	9	5,482	9	5,491	9	5,500	10	0,0121
0,12	5,909	10	5,919	10	5,929	10	5,939	11	5,950	10	5,960	10	5,970	10	5,980	10	5,990	10	6,000	11	0,0144
0,13	6,401	11	6,412	11	6,423	11	6,434	11	6,445	11	6,456	11	6,467	11	6,478	11	6,489	11	6,500	11	0,0169
0,14	6,893	12	6,905	12	6,917	12	6,929	12	6,941	12	6,953	12	6,965	12	6,977	12	6,989	11	7,000	12	0,0196
0,15	7,385	13	7,398	13	7,411	13	7,424	13	7,437	12	7,449	13	7,462	13	7,475	13	7,488	12	7,500	13	0,0225
0,16	7,878	13	7,891	14	7,905	14	7,919	13	7,932	14	7,946	14	7,960	13	7,973	14	7,987	13	8,000	14	0,0256
0,17	8,370	14	8,384	15	8,399	14	8,413	15	8,428	14	8,442	15	8,457	14	8,471	15	8,486	14	8,500	14	0,0289
0,18	8,862	15	8,877	16	8,893	15	8,908	16	8,924	15	8,939	15	8,954	15	8,969	16	8,985	15	9,000	15	0,0324
0,19	9,354	16	9,370	17	9,387	16	9,403	16	9,419	16	9,435	16	9,451	17	9,468	16	9,484	16	9,500	16	0,0361
0,20	9,846	17	9,863	18	9,881	17	9,898	17	9,915	17	9,932	17	9,949	17	9,966	17	9,983	17	10,000	16	0,0400
0,21	10,338	18	10,356	18	10,374	18	10,392	18	10,410	18	10,428	18	10,446	18	10,464	18	10,482	17	10,499	18	0,0441
0,22	10,830	19	10,849	19	10,868	19	10,887	19	10,906	18	10,924	19	10,943	19	10,962	18	10,980	19	10,999	19	0,0484
0,23	11,322	20	11,342	20	11,362	20	11,382	19	11,401	20	11,421	19	11,440	20	11,460	19	11,479	20	11,499	19	0,0529
0,24	11,814	21	11,835	21	11,856	20	11,876	21	11,897	20	11,917	20	11,937	21	11,958	20	11,978	20	11,998	21	0,0576
0,25	12,306	22	12,328	21	12,349	22	12,371	21	12,392	21	12,413	22	12,435	21	12,456	21	12,477	21	12,498	21	0,0625
0,26	12,798	23	12,821	22	12,843	22	12,865	22	12,887	22	12,909	23	12,932	22	12,954	22	12,976	22	12,998	22	0,0676
0,27	13,290	23	13,313	24	13,337	23	13,360	23	13,383	23	13,406	23	13,429	23	13,452	22	13,474	23	13,497	23	0,0729
0,28	13,782	24	13,806	24	13,830	24	13,854	24	13,878	24	13,902	24	13,926	23	13,949	24	13,973	24	13,997	24	0,0784
0,29	14,274	25	14,299	25	14,324	24	14,348	25	14,373	25	14,398	25	14,423	24	14,447	25	14,472	24	14,496	25	0,0841
0,30	14,766	25	14,791	26	14,817	26	14,843	25	14,868	26	14,894	25	14,919	26	14,945	25	14,970	26	14,996	25	0,0900
0,31	15,257	27	15,284	27	15,311	26	15,337	27	15,364	26	15,390	26	15,416	27	15,443	26	15,469	26	15,495	26	0,0961
0,32	15,749	27	15,776	28	15,804	27	15,831	28	15,859	27	15,886	27	15,913	27	15,940	27	15,967	28	15,995	27	0,1024
0,33	16,241	28	16,269	28	16,297	28	16,325	29	16,354	28	16,382	28	16,410	28	16,438	28	16,466	28	16,494	28	0,1089
0,34	16,732	29	16,761	30	16,791	29	16,820	29	16,849	29	16,878	29	16,907	28	16,935	29	16,964	29	16,993	29	0,1156
0,35	17,224	30	17,254	30	17,284	30	17,314	30	17,344	29	17,373	30	17,403	30	17,433	30	17,463	29	17,492	30	0,1225
0,36	17,715	31	17,746	31	17,777	31	17,808	31	17,839	30	17,869	31	17,900	30	17,930	31	17,961	31	17,992	30	0,1296
0,37	18,207	31	18,238	32	18,270	32	18,302	31	18,333	32	18,365	31	18,396	32	18,428	31	18,459	32	18,491	31	0,1369
0,38	18,698	33	18,731	32	18,763	33	18,796	32	18,828	33	18,861	32	18,893	32	18,925	33	18,958	32	18,990	32	0,1444
0,39	19,189	34	19,223	33	19,256	34	19,290	33	19,323	33	19,356	33	19,389	34	19,423	33	19,456	33	19,489	33	0,1521
0,40	19,680	35	19,715	34	19,749	34	19,783	35	19,818	34	19,852	34	19,886	34	19,920	34	19,954	34	19,988	34	0,1600
0,41	20,172	35	20,207	35	20,242	35	20,277	35	20,312	35	20,347	35	20,382	35	20,417	35	20,452	35	20,487	34	0,1681
0,42	20,663	36	20,699	36	20,735	36	20,771	36	20,807	36	20,843	35	20,878	36	20,914	36	20,950	35	20,985	36	0,1764
0,43	21,154	37	21,191	37	21,228	37	21,265	36	21,301	37	21,338	37	21,375	36	21,411	37	21,448	36	21,484	37	0,1849
0,44	21,645	38	21,683	37	21,720	38	21,758	38	21,796	37	21,833	38	21,871	37	21,908	38	21,946	37	21,983	37	0,1936
0,45	22,136	38	22,174	39	22,213	39	22,252	38	22,290	39	22,329	38	22,307	38	22,405	38	22,443	39	22,482	38	0,2025
0,50	24,589	43	24,632	43	24,675	43	24,718	43	24,761	43	24,804	42	24,846	43	24,889	42	24,931	43	24,974	42	0,2500
0,55	27,041	47	27,088	48	27,136	47	27,183	47	27,230	47	27,277	47	27,324	47	27,371	47	27,418	46	27,464	47	0,3025
0,60	29,490	52	29,542	52	29,594	51	29,645	52	29,697	51	29,748	51	29,799	52	29,851	51	29,902	51	29,953	51	0,3600
0,65	31,938	56	31,994	56	32,050	56	32,106	55	32,161	56	32,217	56	32,273	55	32,328	56	32,384	55	32,439	55	0,4225
0,70	34,382	61	34,443	60	34,503	60	34,563	61	34,624	60	34,684	60	34,744	59	34,803	60	34,863	60	34,923	59	0,4900
0,75	36,824	65	36,889	65	36,954	65	37,019	64	37,083	65	37,148	64	37,212	64	37,276	64	37,340	64	37,404	64	0,5625
0,80	39,264	69	39,333	69	39,402	69	39,471	69	39,540	69	39,609	68	39,677	69	39,746	68	39,814	68	39,882	69	0,6400
0,85	41,699	74	41,773	74	41,847	73	41,920	73	41,993	74	42,067	73	42,140	72	42,212	73	42,285	73	42,358	72	0,7225
0,90	44,132	78	44,210	78	44,288	78	44,366	78	44,444	77	44,521	78	44,599	77	44,676	77	44,753	77	44,830	77	0,8100
0,95	46,561	83	46,644	82	46,726	82	46,808	82	46,890	83	46,973	81	47,054	82	47,136	82	47,218	81	47,299	81	0,9025
1,00	48,986	87	49,073	87	49,160	87	49,247	87	49,334	86	49,420	86	49,506	86	49,592	86	49,678	86	49,764	86	1,0000
	4,1185		4,1472		4,1761		4,2050		4,2341		4,2632		4,2925		4,3218		4,3513		4,3808		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	491	492	493	494	495	496	497	498	499	500	
1	49	49	49	49	50	50	50	50	50	50	1
2	98	98	99	99	99	99	99	100	100	100	2
3	147	148	148	148	149	149	149	149	150	150	3
4	196	197	197	198	198	198	199	199	200	200	4
5	246	246	247	247	248	248	249	249	250	250	5
6	295	295	296	296	297	298	298	299	299	300	6
7	344	344	345	346	347	347	348	349	349	350	7
8	393	394	394	395	396	397	398	398	399	400	8
9	442	443	444	445	446	446	447	448	449	450	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r'$.

Chord c.	2,97 Days	dif.	2,98 Days	dif.	2,99 Days	dif.	3,00 Days	dif.	3,01 Days	dif.	3,02 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,501	1	0,502	1	0,503	0	0,503	1	0,504	1	0,505	1	0,0001
0,02	1,002	2	1,004	1	1,005	2	1,007	2	1,009	1	1,010	2	0,0004
0,03	1,503	2	1,505	3	1,508	2	1,510	3	1,513	2	1,515	3	0,0009
0,04	2,004	3	2,007	3	2,010	4	2,014	3	2,017	3	2,020	4	0,0016
0,05	2,505	4	2,509	4	2,513	4	2,517	4	2,521	5	2,526	4	0,0025
0,06	3,005	6	3,011	5	3,016	5	3,021	5	3,026	5	3,031	5	0,0036
0,07	3,506	6	3,512	6	3,518	6	3,524	6	3,530	6	3,536	6	0,0049
0,08	4,007	7	4,014	7	4,021	6	4,027	7	4,034	7	4,041	7	0,0064
0,09	4,508	8	4,516	7	4,523	8	4,531	7	4,538	8	4,546	7	0,0081
0,10	5,009	8	5,017	9	5,026	8	5,034	9	5,043	8	5,051	8	0,0100
0,11	5,510	9	5,519	9	5,528	10	5,538	9	5,547	9	5,556	9	0,0121
0,12	6,011	10	6,021	10	6,031	10	6,041	10	6,051	10	6,061	10	0,0144
0,13	6,511	11	6,522	11	6,533	11	6,544	11	6,555	11	6,566	11	0,0169
0,14	7,012	12	7,024	12	7,036	12	7,048	11	7,059	12	7,071	12	0,0196
0,15	7,513	13	7,526	12	7,538	13	7,551	12	7,563	13	7,576	12	0,0225
0,16	8,014	13	8,027	14	8,041	13	8,054	14	8,068	13	8,081	13	0,0256
0,17	8,514	15	8,529	14	8,543	14	8,557	15	8,572	14	8,586	14	0,0289
0,18	9,015	15	9,030	15	9,045	16	9,061	15	9,076	15	9,091	15	0,0324
0,19	9,516	16	9,532	16	9,548	16	9,564	16	9,580	16	9,596	16	0,0361
0,20	10,016	17	10,033	17	10,050	17	10,067	17	10,084	16	10,100	17	0,0400
0,21	10,517	18	10,535	17	10,552	18	10,570	18	10,588	17	10,605	18	0,0441
0,22	11,018	18	11,036	19	11,055	18	11,073	19	11,092	18	11,110	19	0,0484
0,23	11,518	20	11,538	19	11,557	19	11,576	20	11,596	19	11,615	19	0,0529
0,24	12,019	20	12,039	20	12,059	20	12,079	21	12,100	20	12,120	20	0,0576
0,25	12,519	21	12,540	21	12,561	21	12,582	21	12,603	21	12,624	21	0,0625
0,26	13,020	22	13,042	22	13,064	21	13,085	22	13,107	22	13,129	22	0,0676
0,27	13,520	23	13,543	23	13,566	22	13,588	23	13,611	23	13,634	22	0,0729
0,28	14,021	23	14,044	24	14,068	23	14,091	24	14,115	23	14,138	24	0,0784
0,29	14,521	24	14,545	25	14,570	24	14,594	24	14,618	25	14,643	24	0,0841
0,30	15,021	25	15,046	26	15,072	25	15,097	25	15,122	25	15,147	25	0,0900
0,31	15,521	27	15,548	26	15,574	26	15,600	26	15,626	26	15,652	26	0,0961
0,32	16,022	27	16,049	27	16,076	26	16,102	27	16,129	27	16,156	27	0,1024
0,33	16,522	28	16,550	27	16,577	28	16,605	28	16,633	28	16,661	27	0,1089
0,34	17,022	29	17,051	28	17,079	29	17,108	28	17,136	29	17,165	28	0,1156
0,35	17,522	29	17,551	30	17,581	29	17,610	30	17,640	29	17,669	29	0,1225
0,36	18,022	30	18,052	31	18,083	30	18,113	30	18,143	30	18,173	31	0,1296
0,37	18,522	31	18,553	31	18,584	31	18,615	32	18,647	31	18,678	31	0,1369
0,38	19,022	32	19,054	32	19,086	32	19,118	32	19,150	32	19,182	32	0,1444
0,39	19,522	33	19,555	33	19,588	32	19,620	33	19,653	33	19,686	32	0,1521
0,40	20,022	33	20,055	34	20,089	34	20,123	33	20,156	34	20,190	33	0,1600
0,41	20,521	35	20,556	34	20,590	35	20,625	34	20,659	35	20,694	34	0,1681
0,42	21,021	35	21,056	36	21,092	35	21,127	35	21,162	36	21,198	35	0,1764
0,43	21,521	36	21,557	36	21,593	36	21,629	37	21,666	36	21,702	36	0,1849
0,44	22,020	37	22,057	37	22,094	37	22,131	37	22,168	37	22,205	37	0,1936
0,45	22,520	38	22,558	38	22,596	38	22,634	37	22,671	38	22,709	38	0,2025
0,50	25,016	42	25,058	43	25,101	42	25,143	42	25,185	42	25,227	42	0,2500
0,55	27,511	46	27,557	47	27,604	46	27,650	47	27,697	46	27,743	46	0,3025
0,60	30,004	50	30,054	51	30,105	51	30,156	50	30,206	51	30,257	50	0,3600
0,65	32,494	55	32,549	55	32,604	55	32,659	55	32,714	55	32,769	54	0,4225
0,70	34,982	60	35,042	59	35,101	59	35,160	59	35,219	59	35,278	59	0,4900
0,75	37,468	63	37,531	64	37,595	64	37,659	63	37,722	63	37,785	63	0,5625
0,80	39,951	68	40,019	68	40,087	67	40,154	68	40,222	68	40,290	67	0,6400
0,85	42,430	73	42,503	72	42,575	72	42,647	72	42,719	72	42,791	72	0,7225
0,90	44,907	77	44,984	76	45,060	77	45,137	76	45,213	76	45,289	76	0,8100
0,95	47,380	81	47,461	81	47,542	81	47,623	81	47,704	80	47,784	81	0,9025
1,00	49,850	85	49,935	86	50,021	85	50,106	85	50,191	85	50,276	85	1,0000
	4,4105		4,4102		4,4701		4,5000		4,5301		4,5602		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r'^2$ nearly.

	499	500	501	502	503	504	505	506	
1	50	50	50	50	50	50	51	51	1
2	100	100	100	100	101	101	101	101	2
3	150	150	150	151	151	151	152	152	3
4	200	200	200	201	201	202	202	202	4
5	250	250	251	251	252	252	253	253	5
6	299	300	301	301	302	302	303	304	6
7	349	350	351	351	352	353	354	354	7
8	399	400	401	402	402	403	404	405	8
9	449	450	451	452	453	454	455	455	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	3,03 Days	dif.	3,04 Days	dif.	3,05 Days	dif.	3,06 Days	dif.	3,07 Days	dif.	3,08 Days	dif.	3,09 Days	dif.	3,10 Days	dif.	3,11 Days	dif.	3,12 Days	dif.	
	Sum of the Radii $r + r''$.																				
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,506	1	0,507	1	0,508	1	0,508	1	0,509	1	0,510	1	0,511	1	0,512	1	0,513	0	0,513	1	0,0001
0,02	1,012	2	1,014	1	1,015	2	1,017	2	1,019	1	1,020	2	1,022	2	1,024	1	1,025	2	1,027	1	0,0004
0,03	1,518	2	1,520	3	1,523	2	1,525	3	1,528	2	1,530	3	1,533	2	1,535	3	1,538	2	1,540	3	0,0009
0,04	2,024	3	2,027	3	2,030	4	2,034	3	2,037	3	2,040	4	2,044	3	2,047	3	2,050	4	2,054	3	0,0016
0,05	2,530	4	2,534	4	2,538	4	2,542	4	2,546	5	2,551	4	2,555	4	2,559	4	2,563	4	2,567	4	0,0025
0,06	3,036	5	3,041	5	3,046	5	3,051	5	3,056	5	3,061	5	3,066	5	3,071	4	3,075	5	3,080	5	0,0036
0,07	3,542	5	3,547	6	3,553	6	3,559	6	3,565	6	3,571	6	3,577	5	3,582	6	3,588	6	3,594	6	0,0049
0,08	4,048	6	4,054	7	4,061	6	4,067	7	4,074	7	4,081	6	4,087	7	4,094	7	4,101	6	4,107	7	0,0064
0,09	4,553	8	4,561	7	4,568	8	4,576	7	4,583	8	4,591	7	4,598	8	4,606	7	4,613	8	4,621	7	0,0081
0,10	5,059	9	5,068	8	5,076	8	5,084	9	5,093	8	5,101	8	5,109	8	5,117	9	5,126	8	5,134	8	0,0100
0,11	5,565	9	5,574	10	5,584	9	5,593	9	5,602	9	5,611	9	5,620	9	5,629	9	5,638	9	5,647	9	0,0121
0,12	6,071	10	6,081	10	6,091	10	6,101	10	6,111	10	6,121	10	6,131	10	6,141	10	6,151	10	6,161	9	0,0144
0,13	6,577	11	6,588	11	6,599	10	6,609	11	6,620	11	6,631	11	6,642	10	6,652	11	6,663	11	6,674	11	0,0169
0,14	7,083	11	7,094	12	7,106	12	7,118	11	7,129	12	7,141	12	7,153	11	7,164	12	7,176	11	7,187	12	0,0196
0,15	7,588	13	7,601	13	7,614	12	7,626	12	7,638	13	7,651	12	7,663	13	7,676	12	7,688	12	7,700	13	0,0225
0,16	8,094	14	8,108	13	8,121	13	8,134	14	8,148	13	8,161	13	8,174	13	8,187	14	8,201	13	8,214	13	0,0256
0,17	8,600	14	8,614	14	8,628	15	8,643	14	8,657	14	8,671	14	8,685	14	8,699	14	8,713	14	8,727	14	0,0289
0,18	9,106	15	9,121	15	9,136	15	9,151	15	9,166	15	9,181	15	9,196	14	9,210	15	9,225	15	9,240	15	0,0324
0,19	9,612	15	9,627	16	9,643	16	9,659	16	9,675	16	9,691	15	9,706	16	9,722	16	9,738	15	9,753	16	0,0361
0,20	10,117	17	10,134	17	10,151	16	10,167	17	10,184	16	10,200	17	10,217	16	10,233	17	10,250	16	10,266	17	0,0400
0,21	10,623	17	10,640	18	10,658	17	10,675	18	10,693	17	10,710	18	10,728	17	10,745	17	10,762	18	10,780	17	0,0441
0,22	11,129	18	11,147	18	11,165	19	11,184	18	11,202	18	11,220	18	11,238	18	11,256	19	11,275	18	11,293	18	0,0484
0,23	11,634	19	11,653	19	11,672	20	11,692	19	11,711	19	11,730	19	11,749	19	11,768	19	11,787	19	11,806	19	0,0529
0,24	12,140	20	12,160	20	12,180	20	12,200	20	12,220	20	12,240	19	12,259	20	12,279	20	12,299	20	12,319	20	0,0576
0,25	12,645	21	12,666	21	12,687	21	12,708	21	12,729	20	12,749	21	12,770	21	12,791	20	12,811	21	12,832	20	0,0625
0,26	13,151	21	13,172	22	13,194	22	13,216	21	13,237	22	13,259	21	13,280	22	13,302	21	13,323	22	13,345	21	0,0676
0,27	13,656	23	13,679	22	13,701	23	13,724	22	13,746	23	13,769	22	13,791	22	13,813	23	13,836	22	13,858	22	0,0729
0,28	14,162	23	14,185	23	14,208	24	14,232	23	14,255	23	14,278	23	14,301	23	14,324	24	14,348	23	14,371	23	0,0784
0,29	14,667	24	14,691	24	14,715	25	14,740	24	14,764	24	14,788	24	14,812	24	14,836	24	14,860	24	14,884	23	0,0841
0,30	15,172	25	15,197	25	15,222	25	15,247	25	15,272	25	15,297	25	15,322	25	15,347	25	15,372	24	15,396	25	0,0900
0,31	15,678	26	15,704	25	15,729	26	15,755	26	15,781	26	15,807	25	15,832	26	15,858	26	15,884	25	15,909	26	0,0961
0,32	16,183	27	16,210	26	16,236	27	16,263	27	16,290	26	16,316	27	16,343	26	16,369	27	16,396	26	16,422	26	0,1024
0,33	16,688	28	16,716	27	16,743	28	16,771	27	16,798	28	16,826	27	16,853	27	16,880	27	16,907	28	16,935	27	0,1089
0,34	17,193	29	17,222	28	17,250	28	17,278	29	17,307	28	17,335	28	17,363	28	17,391	28	17,419	28	17,447	28	0,1156
0,35	17,698	30	17,728	29	17,757	29	17,786	29	17,815	29	17,844	29	17,873	29	17,902	29	17,931	29	17,960	29	0,1225
0,36	18,204	30	18,234	30	18,264	30	18,294	30	18,324	29	18,353	30	18,383	30	18,413	30	18,443	30	18,473	29	0,1296
0,37	18,709	30	18,739	31	18,770	31	18,801	31	18,832	31	18,863	30	18,893	31	18,924	31	18,955	30	18,985	31	0,1369
0,38	19,214	31	19,245	32	19,277	32	19,309	31	19,340	32	19,372	31	19,403	32	19,435	31	19,466	32	19,498	31	0,1444
0,39	19,718	33	19,751	33	19,784	32	19,816	33	19,849	32	19,881	32	19,913	33	19,946	32	19,978	32	20,010	32	0,1521
0,40	20,223	34	20,257	33	20,290	34	20,324	33	20,357	33	20,390	33	20,423	33	20,456	33	20,489	33	20,522	33	0,1600
0,41	20,728	34	20,762	35	20,797	34	20,831	34	20,865	34	20,899	34	20,933	34	20,967	34	21,001	34	21,035	33	0,1681
0,42	21,233	35	21,268	35	21,303	35	21,338	35	21,373	35	21,408	35	21,443	35	21,478	34	21,512	35	21,547	35	0,1764
0,43	21,738	36	21,774	35	21,809	36	21,845	36	21,881	36	21,917	36	21,953	35	21,988	36	22,024	35	22,059	36	0,1849
0,44	22,242	37	22,279	37	22,316	36	22,352	37	22,389	37	22,426	36	22,462	37	22,499	36	22,535	36	22,571	37	0,1936
0,45	22,747	37	22,784	38	22,822	38	22,860	37	22,897	37	22,934	38	22,972	37	23,009	37	23,046	37	23,083	37	0,2025
0,50	25,269	42	25,311	41	25,352	42	25,394	42	25,436	41	25,477	42	25,519	41	25,560	42	25,602	41	25,643	41	0,2500
0,55	27,789	46	27,835	46	27,881	46	27,927	46	27,973	46	28,019	45	28,064	46	28,110	45	28,155	46	28,201	45	0,3025
0,60	30,307	50	30,357	51	30,408	50	30,458	50	30,508	50	30,558	50	30,608	49	30,657	50	30,707	50	30,757	49	0,3600
0,65	32,823	55	32,878	54	32,932	55	32,987	54	33,041	54	33,095	54	33,149	54	33,203	54	33,257	54	33,311	54	0,4225
0,70	35,337	59	35,396	59	35,455	58	35,513	59	35,572	58	35,630	58	35,688	59	35,747	58	35,805	58	35,863	58	0,4900
0,75	37,848	63	37,911	63	37,974	63	38,037	63	38,100	63	38,163	62	38,225	63	38,288	62	38,350	62	38,412	62	0,5625
0,80	40,357	67	40,424	68	40,492	67	40,559	67	40,626	67	40,693	66	40,759	67	40,826	67	40,893	66	40,959	66	0,6400
0,85	42,863	71	42,934	72	43,006	71	43,077	72	43,149	71	43,220	71	43,291	71	43,362	70	43,432	71	43,503	71	0,7225
0,90	45,365	76	45,441	76	45,517	76	45,593	75	45,668	76	45,744	75	45,819	75	45,894	75	45,969	75	46,044	75	0,8100
0,95	47,865	80	47,945	80	48,025	80	48,105	80	48,185	80	48,265	79	48,344	80	48,424	79	48,503	79	48,582	79	0,9025
1,00	50,361	84	50,445	85	50,530	84	50,614	84	50,698	84	50,782	84	50,866	84	50,950	83	51,033	84	51,117	83	1,0000
	4,5905		**4,6208**		**4,6513**		**4,6818**		**4,7125**		**4,7432**		**4,7741**		**4,8050**		**4,8361**		**4,8672**		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	504	505	506	507	508	509	510	511	512	513	514	
1	50	51	51	51	51	51	51	51	51	51	51	1
2	101	101	101	101	102	102	102	102	102	103	103	2
3	151	152	152	152	152	153	153	153	154	154	154	3
4	202	202	202	203	203	204	204	204	205	205	206	4
5	252	253	253	254	254	255	255	256	256	257	257	5
6	302	303	304	304	305	305	306	307	307	308	308	6
7	353	354	354	355	356	356	357	358	358	359	360	7
8	403	404	405	406	406	407	408	409	410	410	411	8
9	454	455	455	456	457	458	459	460	461	462	463	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	3,13 Days	dif.	3,14 Days	dif.	3,15 Days	dif.	3,16 Days	dif.	3,17 Days	dif.	3,18 Days	dif.	
	Sum of the Radii $r + r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,514	1	0,515	1	0,516	1	0,517	0	0,517	1	0,518	1	0,0001
0,02	1,028	2	1,030	2	1,032	1	1,033	2	1,035	2	1,037	1	0,0004
0,03	1,543	2	1,545	3	1,548	2	1,550	3	1,553	2	1,555	2	0,0009
0,04	2,057	3	2,060	3	2,063	4	2,067	3	2,070	3	2,073	4	0,0016
0,05	2,571	4	2,575	4	2,579	4	2,583	5	2,588	4	2,592	4	0,0025
0,06	3,085	5	3,090	5	3,095	5	3,100	5	3,105	5	3,110	5	0,0036
0,07	3,600	5	3,605	6	3,611	6	3,617	5	3,622	6	3,628	6	0,0049
0,08	4,114	6	4,120	7	4,127	6	4,133	7	4,140	6	4,146	7	0,0064
0,09	4,628	7	4,635	8	4,643	7	4,650	7	4,657	8	4,665	7	0,0081
0,10	5,142	8	5,150	9	5,159	8	5,167	8	5,175	8	5,183	8	0,0100
0,11	5,656	9	5,665	9	5,674	9	5,683	9	5,692	9	5,701	9	0,0121
0,12	6,170	10	6,180	10	6,190	10	6,200	10	6,210	10	6,220	9	0,0144
0,13	6,685	10	6,695	11	6,706	11	6,717	10	6,727	11	6,738	10	0,0169
0,14	7,199	11	7,210	12	7,222	11	7,233	12	7,245	11	7,256	11	0,0196
0,15	7,713	12	7,725	12	7,737	13	7,750	12	7,762	12	7,774	12	0,0225
0,16	8,227	13	8,240	13	8,253	13	8,266	13	8,279	13	8,292	13	0,0256
0,17	8,741	14	8,755	14	8,769	14	8,783	14	8,797	13	8,810	14	0,0289
0,18	9,255	15	9,270	14	9,284	15	9,299	15	9,314	15	9,329	14	0,0324
0,19	9,769	16	9,785	15	9,800	16	9,816	15	9,831	16	9,847	15	0,0361
0,20	10,283	16	10,299	17	10,316	16	10,332	16	10,348	17	10,365	16	0,0400
0,21	10,797	17	10,814	17	10,831	18	10,849	17	10,866	17	10,883	17	0,0441
0,22	11,311	18	11,329	18	11,347	18	11,365	18	11,383	18	11,401	18	0,0484
0,23	11,825	19	11,844	18	11,862	19	11,881	19	11,900	19	11,919	19	0,0529
0,24	12,339	19	12,358	20	12,378	20	12,398	19	12,417	20	12,437	19	0,0576
0,25	12,852	21	12,873	20	12,893	21	12,914	20	12,934	21	12,955	20	0,0625
0,26	13,366	22	13,388	21	13,409	21	13,430	21	13,451	22	13,473	21	0,0676
0,27	13,880	22	13,902	22	13,924	22	13,946	22	13,968	23	13,991	22	0,0729
0,28	14,394	23	14,417	23	14,440	23	14,463	23	14,486	22	14,508	23	0,0784
0,29	14,907	24	14,931	24	14,955	24	14,979	24	15,003	23	15,026	24	0,0841
0,30	15,421	25	15,446	24	15,470	25	15,495	24	15,519	25	15,544	24	0,0900
0,31	15,935	25	15,960	26	15,986	25	16,011	25	16,036	26	16,062	25	0,0961
0,32	16,448	27	16,475	26	16,501	26	16,527	26	16,553	26	16,579	26	0,1024
0,33	16,962	27	16,989	27	17,016	27	17,043	27	17,070	27	17,097	27	0,1089
0,34	17,475	28	17,503	28	17,531	28	17,559	28	17,587	28	17,615	27	0,1156
0,35	17,989	29	18,018	28	18,046	29	18,075	29	18,104	28	18,132	29	0,1225
0,36	18,502	30	18,532	29	18,561	30	18,591	29	18,620	30	18,650	29	0,1296
0,37	19,016	30	19,046	30	19,076	31	19,107	30	19,137	30	19,167	30	0,1369
0,38	19,529	31	19,560	31	19,591	31	19,622	32	19,654	31	19,685	31	0,1444
0,39	20,042	32	20,074	32	20,106	32	20,138	32	20,170	32	20,202	32	0,1521
0,40	20,555	33	20,588	33	20,621	33	20,654	33	20,687	32	20,719	33	0,1600
0,41	21,068	34	21,102	34	21,136	33	21,169	34	21,203	34	21,237	33	0,1681
0,42	21,582	34	21,616	35	21,651	34	21,685	34	21,719	35	21,754	34	0,1764
0,43	22,095	35	22,130	35	22,165	36	22,201	35	22,236	35	22,271	35	0,1849
0,44	22,608	36	22,644	36	22,680	36	22,716	36	22,752	36	22,788	36	0,1936
0,45	23,120	38	23,158	36	23,194	37	23,231	37	23,268	37	23,305	37	0,2025
0,50	25,684	41	25,725	41	25,766	42	25,808	40	25,848	41	25,889	41	0,2500
0,55	28,246	46	28,292	45	28,337	45	28,382	45	28,427	45	28,472	45	0,3025
0,60	30,806	50	30,856	49	30,905	50	30,955	49	31,004	49	31,053	49	0,3600
0,65	33,365	53	33,418	54	33,472	53	33,525	54	33,579	53	33,632	53	0,4225
0,70	35,921	57	35,978	58	36,036	58	36,094	57	36,151	58	36,209	57	0,4900
0,75	38,474	62	38,536	62	38,598	62	38,660	62	38,722	61	38,783	62	0,5625
0,80	41,025	66	41,091	67	41,158	66	41,224	65	41,289	66	41,355	66	0,6400
0,85	43,574	70	43,644	70	43,714	71	43,785	70	43,855	70	43,925	69	0,7225
0,90	46,119	75	46,194	74	46,268	75	46,343	74	46,417	74	46,491	74	0,8100
0,95	48,661	79	48,740	79	48,819	79	48,898	78	48,976	79	49,055	78	0,9025
1,00	51,200	83	57,283	83	51,366	83	51,449	83	51,532	83	51,615	82	1,0000
	4,8985		4,9298		4,9613		4,9928		5,0245		5,0562		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	512	513	514	515	516	517	518	519	
1	51	51	51	52	52	52	52	52	1
2	102	103	103	103	103	103	104	104	2
3	154	154	154	155	155	155	155	156	3
4	205	205	206	206	206	207	207	208	4
5	256	257	257	258	258	259	259	260	5
6	307	308	308	309	310	310	311	311	6
7	358	359	360	361	361	362	363	363	7
8	410	410	411	412	413	414	414	415	8
9	461	462	463	464	464	465	466	467	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	3,19 Days	dif.	3,20 Days	dif.	3,21 Days	dif.	3,22 Days	dif.	3,23 Days	dif.	3,24 Days	dif.	3,25 Days	dif.	3,26 Days	dif.	3,27 Days	dif.	3,28 Days	dif.	
	Sum of the Radii $r + r''$.																				
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,519	1	0,520	1	0,521	1	0,522	0	0,522	1	0,523	1	0,524	1	0,525	1	0,526	0	0,526	1	0,0001
0,02	1,038	2	1,040	2	1,042	1	1,043	2	1,045	1	1,046	2	1,048	2	1,050	1	1,051	2	1,053	1	0,0004
0,03	1,557	3	1,560	2	1,562	3	1,565	2	1,567	3	1,570	2	1,572	2	1,574	3	1,577	2	1,579	3	0,0009
0,04	2,077	3	2,080	3	2,083	3	2,086	4	2,090	3	2,093	3	2,096	3	2,099	3	2,102	4	2,106	3	0,0016
0,05	2,596	4	2,600	4	2,604	4	2,608	4	2,612	4	2,616	4	2,620	4	2,624	4	2,628	4	2,632	4	0,0025
0,06	3,115	5	3,120	5	3,125	4	3,129	5	3,134	5	3,139	5	3,144	5	3,149	5	3,154	4	3,158	5	0,0036
0,07	3,634	6	3,640	5	3,645	6	3,651	6	3,657	5	3,662	6	3,668	6	3,674	5	3,679	6	3,685	5	0,0049
0,08	4,153	7	4,160	6	4,166	6	4,172	7	4,179	6	4,185	7	4,192	6	4,198	7	4,205	6	4,211	7	0,0064
0,09	4,672	7	4,679	8	4,687	7	4,694	7	4,701	8	4,709	7	4,716	7	4,723	7	4,730	8	4,738	7	0,0081
0,10	5,191	8	5,199	8	5,207	9	5,216	8	5,224	8	5,232	8	5,240	8	5,248	8	5,256	8	5,264	8	0,0100
0,11	5,710	9	5,719	9	5,728	9	5,737	9	5,746	9	5,755	9	5,764	9	5,773	8	5,781	9	5,790	9	0,0121
0,12	6,229	10	6,239	10	6,249	10	6,259	9	6,268	10	6,278	10	6,288	9	6,297	10	6,307	10	6,317	9	0,0144
0,13	6,748	11	6,759	10	6,769	11	6,780	11	6,791	10	6,801	11	6,812	10	6,822	10	6,832	11	6,843	10	0,0169
0,14	7,267	12	7,279	11	7,290	11	7,301	12	7,313	11	7,324	11	7,335	12	7,347	11	7,358	11	7,369	11	0,0196
0,15	7,786	13	7,799	12	7,811	12	7,823	12	7,835	12	7,847	12	7,859	12	7,871	12	7,883	12	7,895	13	0,0225
0,16	8,305	13	8,318	13	8,331	13	8,344	13	8,357	13	8,370	13	8,383	13	8,396	13	8,409	13	8,422	13	0,0256
0,17	8,824	14	8,838	14	8,852	14	8,866	13	8,879	14	8,893	14	8,907	14	8,921	13	8,934	14	8,948	14	0,0289
0,18	9,343	15	9,358	15	9,373	14	9,387	15	9,402	14	9,416	15	9,431	14	9,445	15	9,460	14	9,474	15	0,0324
0,19	9,862	16	9,878	15	9,893	15	9,908	16	9,924	15	9,939	16	9,955	15	9,970	15	9,985	15	10,000	16	0,0361
0,20	10,381	16	10,397	17	10,414	16	10,430	16	10,446	16	10,462	16	10,478	16	10,494	17	10,511	16	10,527	16	0,0400
0,21	10,900	17	10,917	17	10,934	17	10,951	17	10,968	17	10,985	17	11,002	17	11,019	17	11,036	17	11,053	17	0,0441
0,22	11,419	18	11,437	18	11,455	17	11,472	18	11,490	18	11,508	18	11,526	17	11,543	18	11,561	18	11,579	17	0,0484
0,23	11,938	18	11,956	19	11,975	19	11,994	18	12,012	19	12,031	18	12,049	19	12,068	18	12,086	19	12,105	18	0,0529
0,24	12,456	20	12,476	19	12,495	20	12,515	19	12,534	20	12,554	19	12,573	19	12,592	20	12,612	19	12,631	19	0,0576
0,25	12,975	20	12,995	21	13,016	20	13,036	20	13,056	21	13,077	20	13,097	20	13,117	20	13,137	20	13,157	20	0,0625
0,26	13,494	21	13,515	21	13,536	21	13,557	21	13,578	21	13,599	21	13,620	21	13,641	21	13,662	21	13,683	21	0,0676
0,27	14,013	22	14,035	21	14,056	22	14,078	22	14,100	22	14,122	22	14,144	22	14,166	21	14,187	22	14,209	22	0,0729
0,28	14,531	23	14,554	23	14,577	22	14,599	23	14,622	23	14,645	22	14,667	23	14,690	23	14,713	22	14,735	23	0,0784
0,29	15,050	23	15,073	24	15,097	24	15,121	23	15,144	23	15,167	24	15,191	23	15,214	24	15,238	23	15,261	23	0,0841
0,30	15,568	25	15,593	24	15,617	25	15,642	24	15,666	24	15,690	24	15,714	25	15,739	24	15,763	24	15,787	24	0,0900
0,31	16,087	25	16,112	25	16,137	26	16,163	25	16,188	25	16,213	25	16,238	25	16,263	25	16,288	25	16,313	25	0,0961
0,32	16,605	27	16,632	26	16,658	25	16,683	26	16,709	26	16,735	26	16,761	26	16,787	26	16,813	25	16,838	26	0,1024
0,33	17,124	27	17,151	27	17,178	26	17,204	27	17,231	27	17,258	26	17,284	27	17,311	27	17,338	26	17,364	27	0,1089
0,34	17,642	28	17,670	28	17,698	27	17,725	28	17,753	27	17,780	28	17,808	27	17,835	28	17,863	27	17,890	27	0,1156
0,35	18,161	28	18,189	29	18,218	28	18,246	28	18,274	29	18,303	28	18,331	28	18,359	28	18,387	29	18,416	28	0,1225
0,36	18,679	29	18,708	30	18,738	29	18,767	29	18,796	29	18,825	29	18,854	29	18,883	29	18,912	29	18,941	29	0,1296
0,37	19,197	30	19,227	31	19,258	30	19,288	30	19,318	30	19,348	29	19,377	30	19,407	30	19,437	30	19,467	30	0,1369
0,38	19,716	31	19,747	30	19,777	31	19,808	31	19,839	31	19,870	31	19,901	30	19,931	31	19,962	30	19,992	31	0,1444
0,39	20,234	32	20,266	31	20,297	32	20,329	32	20,361	31	20,392	32	20,424	31	20,455	32	20,487	31	20,518	31	0,1521
0,40	20,752	32	20,784	33	20,817	33	20,850	32	20,882	32	20,914	33	20,947	32	20,979	32	21,011	32	21,043	33	0,1600
0,41	21,270	33	21,303	34	21,337	33	21,370	33	21,403	34	21,437	33	21,470	33	21,503	33	21,536	33	21,569	33	0,1681
0,42	21,788	34	21,822	34	21,856	35	21,891	34	21,925	34	21,959	34	21,993	33	22,026	34	22,060	34	22,094	34	0,1764
0,43	22,306	35	22,341	35	22,376	35	22,411	35	22,446	35	22,481	34	22,515	35	22,550	35	22,585	34	22,619	35	0,1849
0,44	22,824	36	22,860	36	22,896	35	22,931	36	22,967	36	23,003	35	23,038	36	23,074	35	23,109	36	23,145	35	0,1936
0,45	23,342	36	23,378	37	23,415	37	23,452	36	23,488	37	23,525	36	23,561	36	23,597	37	23,634	36	23,670	36	0,2025
0,50	25,930	41	25,971	41	26,012	40	26,052	41	26,093	40	26,133	41	26,174	40	26,214	41	26,255	40	26,295	40	0,2500
0,55	28,517	45	28,562	45	28,607	44	28,651	45	28,696	45	28,741	44	28,785	45	28,830	44	28,874	45	28,919	44	0,3025
0,60	31,102	49	31,151	49	31,200	49	31,249	49	31,298	48	31,346	49	31,395	48	31,443	49	31,492	48	31,540	49	0,3600
0,65	33,685	53	33,738	53	33,791	53	33,844	53	33,897	53	33,950	53	34,003	52	34,055	53	34,108	52	34,160	53	0,4225
0,70	36,266	57	36,323	58	36,381	57	36,438	57	36,495	57	36,552	56	36,608	57	36,665	57	36,722	56	36,778	57	0,4900
0,75	38,845	61	38,906	62	38,968	61	39,029	61	39,090	61	39,151	61	39,212	61	39,273	60	39,333	61	39,394	61	0,5625
0,80	41,421	66	41,487	65	41,552	65	41,617	66	41,683	65	41,748	65	41,813	65	41,878	65	41,943	64	42,007	65	0,6400
0,85	43,994	70	44,064	70	44,134	69	44,203	70	44,273	69	44,342	69	44,411	69	44,480	69	44,549	69	44,618	69	0,7225
0,90	46,565	74	46,639	74	46,713	74	46,787	73	46,860	74	46,934	73	47,007	73	47,080	73	47,153	73	47,226	73	0,8100
0,95	49,133	78	49,211	78	49,289	78	49,367	78	49,445	77	49,522	78	49,600	77	49,677	77	49,754	77	49,831	77	0,9025
1,00	51,697	83	51,780	82	51,862	82	51,944	82	52,026	82	52,108	81	52,189	82	52,271	81	52,352	82	52,434	81	1,0000
	5,0881		5,1200		5,1521		5,1842		5,2165		5,2488		5,2813		5,3138		5,3465		5,3792		c^2
	$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.																				

	518	519	520	521	522	523	524	525	526	527	
1	52	52	52	52	52	52	52	53	53	53	1
2	104	104	104	104	104	105	105	105	105	105	2
3	155	156	156	156	157	157	157	158	158	158	3
4	207	208	208	208	209	209	210	210	210	211	4
5	259	260	260	261	261	262	262	263	263	264	5
6	311	311	312	313	313	314	314	315	316	316	6
7	363	363	364	365	365	366	367	368	368	369	7
8	414	415	416	417	418	418	419	420	421	422	8
9	466	467	468	469	470	471	472	473	473	474	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord c.	3,29 Days	dif.	3,30 Days	dif.	3,31 Days	dif.	3,32 Days	dif.	3,33 Days	dif.	3,34 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,527	1	0,528	1	0,529	1	0,530	0	0,530	1	0,531	1	0,0001
0,02	1,054	2	1,056	2	1,058	1	1,059	2	1,061	1	1,062	2	0,0004
0,03	1,582	2	1,584	2	1,586	3	1,589	2	1,591	3	1,594	2	0,0009
0,04	2,109	3	2,112	3	2,115	3	2,118	4	2,122	3	2,125	3	0,0016
0,05	2,636	4	2,640	4	2,644	4	2,648	4	2,652	4	2,656	4	0,0025
0,06	3,163	5	3,168	5	3,173	5	3,178	4	3,182	5	3,187	5	0,0036
0,07	3,690	6	3,696	6	3,702	5	3,707	6	3,713	5	3,718	6	0,0049
0,08	4,218	6	4,224	6	4,230	7	4,237	6	4,243	7	4,250	6	0,0064
0,09	4,745	7	4,752	7	4,759	7	4,766	8	4,774	7	4,781	7	0,0081
0,10	5,272	8	5,280	8	5,288	8	5,296	8	5,304	8	5,312	8	0,0100
0,11	5,799	9	5,808	9	5,817	8	5,825	9	5,834	9	5,843	9	0,0121
0,12	6,326	10	6,336	9	6,345	10	6,355	10	6,365	9	6,374	10	0,0144
0,13	6,853	11	6,864	10	6,874	10	6,884	11	6,895	10	6,905	11	0,0169
0,14	7,380	12	7,392	11	7,403	11	7,414	11	7,425	11	7,436	11	0,0196
0,15	7,908	12	7,920	12	7,932	11	7,943	12	7,955	12	7,967	12	0,0225
0,16	8,435	12	8,447	13	8,460	13	8,473	13	8,486	12	8,498	13	0,0256
0,17	8,962	13	8,975	14	8,989	13	9,002	14	9,016	13	9,029	14	0,0289
0,18	9,489	14	9,503	14	9,517	15	9,532	14	9,546	15	9,561	14	0,0324
0,19	10,016	15	10,031	15	10,046	15	10,061	15	10,076	16	10,092	15	0,0361
0,20	10,543	16	10,559	16	10,575	16	10,591	16	10,607	16	10,623	15	0,0400
0,21	11,070	16	11,086	17	11,103	17	11,120	17	11,137	16	11,153	17	0,0441
0,22	11,596	18	11,614	18	11,632	17	11,649	18	11,667	17	11,684	18	0,0484
0,23	12,123	19	12,142	18	12,160	19	12,179	18	12,197	18	12,215	19	0,0529
0,24	12,650	20	12,670	19	12,689	19	12,708	19	12,727	19	12,746	19	0,0576
0,25	13,177	20	13,197	20	13,217	20	13,237	20	13,257	20	13,277	20	0,0625
0,26	13,704	21	13,725	21	13,746	20	13,766	21	13,787	21	13,808	20	0,0676
0,27	14,231	21	14,252	22	14,274	22	14,296	21	14,317	22	14,339	21	0,0729
0,28	14,758	22	14,780	22	14,802	23	14,825	22	14,847	22	14,869	23	0,0784
0,29	15,284	23	15,307	24	15,331	23	15,354	23	15,377	23	15,400	23	0,0841
0,30	15,811	24	15,835	24	15,859	24	15,883	24	15,907	24	15,931	24	0,0900
0,31	16,338	24	16,362	25	16,387	25	16,412	25	16,437	24	16,461	25	0,0961
0,32	16,864	26	16,890	25	16,915	26	16,941	26	16,967	25	16,992	25	0,1024
0,33	17,391	26	17,417	27	17,444	26	17,470	26	17,496	27	17,523	26	0,1089
0,34	17,917	28	17,945	27	17,972	27	17,999	27	18,026	27	18,053	27	0,1156
0,35	18,444	28	18,472	28	18,500	28	18,528	28	18,556	28	18,584	27	0,1225
0,36	18,970	29	18,999	29	19,028	29	19,057	28	19,085	29	19,114	29	0,1296
0,37	19,497	29	19,526	30	19,556	29	19,585	30	19,615	29	19,644	30	0,1369
0,38	20,023	30	20,053	31	20,084	30	20,114	31	20,145	30	20,175	30	0,1444
0,39	20,549	32	20,581	31	20,612	31	20,643	31	20,674	31	20,705	31	0,1521
0,40	21,076	32	21,108	32	21,140	32	21,172	32	21,204	31	21,235	32	0,1600
0,41	21,602	33	21,635	32	21,667	33	21,700	33	21,733	33	21,766	32	0,1681
0,42	22,128	34	22,162	33	21,195	34	22,229	33	22,262	34	22,296	33	0,1764
0,43	22,654	34	22,688	35	22,723	34	22,757	35	22,792	34	22,826	34	0,1849
0,44	23,180	35	23,215	36	23,251	35	23,286	35	23,321	35	23,356	35	0,1936
0,45	23,706	36	23,742	36	23,778	36	23,814	36	23,850	36	23,886	36	0,2025
0,50	26,335	40	26,375	40	26,415	40	26,455	40	26,495	40	26,535	40	0,2500
0,55	28,963	44	29,007	44	29,051	44	29,095	44	29,139	44	29,183	44	0,3025
0,60	31,589	48	31,637	48	31,685	48	31,733	48	31,781	48	31,829	48	0,3600
0,65	34,213	52	34,265	52	34,317	52	34,369	52	34,421	52	34,473	52	0,4225
0,70	36,835	56	36,891	56	36,947	57	37,004	56	37,060	56	37,116	56	0,4900
0,75	39,455	60	39,515	60	39,575	61	39,636	60	39,696	60	39,756	60	0,5625
0,80	42,072	64	42,136	65	42,201	64	42,265	64	42,329	65	42,394	64	0,6400
0,85	44,687	68	44,755	69	44,824	68	44,892	69	44,961	68	45,029	68	0,7225
0,90	47,299	73	47,372	73	47,445	72	47,517	72	47,589	73	47,662	72	0,8100
0,95	49,908	77	49,985	77	50,062	77	50,139	76	50,215	77	50,292	76	0,9025
1,00	52,515	81	52,596	81	52,677	80	52,757	81	52,838	81	52,919	80	1,0000
	5,4121		5,4450		5,4781		5,5112		5,5445		5,5778		c^2

Sum of the Radii $r+r''$. — $\frac{1}{2}\cdot(r+r'')^2$ or $r^2+r''^2$ nearly.

	526	527	528	529	530	531	532	
1	53	53	53	53	53	53	53	1
2	105	105	106	106	106	106	106	2
3	158	158	158	159	159	159	160	3
4	210	211	211	212	212	212	213	4
5	263	264	264	265	265	266	266	5
6	316	316	317	317	318	319	319	6
7	368	369	370	370	371	372	372	7
8	421	422	422	423	424	425	426	8
9	473	474	475	476	477	478	479	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	3,35 Days	dif.	3,36 Days	dif.	3,37 Days	dif.	3,38 Days	dif.	3,39 Days	dif.	3,40 Days	dif.	3,41 Days	dif.	3,42 Days	dif.	3,43 Days	dif.	3,44 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,532	1	0,533	1	0,534	0	0,534	1	0,535	1	0,536	1	0,537	1	0,538	0	0,538	1	0,539	1	0,0001
0,02	1,064	2	1,066	1	1,067	2	1,069	1	1,070	2	1,072	1	1,073	2	1,075	2	1,077	1	1,078	2	0,0004
0,03	1,596	2	1,598	3	1,601	2	1,603	2	1,605	3	1,608	2	1,610	3	1,613	2	1,615	2	1,617	3	0,0009
0,04	2,128	3	2,131	3	2,134	3	2,137	4	2,141	3	2,144	3	2,147	3	2,150	3	2,153	3	2,156	4	0,0016
0,05	2,660	4	2,664	4	2,668	4	2,672	4	2,676	4	2,680	4	2,684	4	2,688	4	2,692	3	2,695	4	0,0025
0,06	3,192	5	3,197	4	3,201	5	3,206	5	3,211	5	3,216	4	3,220	5	3,225	5	3,230	5	3,235	4	0,0036
0,07	3,724	5	3,729	6	3,735	6	3,741	5	3,746	6	3,752	5	3,757	6	3,763	5	3,768	6	3,774	5	0,0049
0,08	4,256	6	4,262	7	4,269	6	4,275	6	4,281	7	4,288	6	4,294	6	4,300	6	4,306	7	4,313	6	0,0064
0,09	4,788	7	4,795	7	4,802	7	4,809	7	4,816	7	4,823	8	4,831	7	4,838	7	4,845	7	4,852	7	0,0081
0,10	5,320	8	5,328	8	5,336	8	5,344	7	5,351	8	5,359	8	5,367	8	5,375	8	5,383	8	5,391	8	0,0100
0,11	5,852	8	5,860	9	5,869	9	5,878	9	5,887	8	5,895	9	5,904	9	5,913	8	5,921	9	5,930	8	0,0121
0,12	6,384	9	6,393	10	6,403	9	6,412	10	6,422	9	6,431	10	6,441	9	6,450	9	6,459	10	6,469	9	0,0144
0,13	6,916	10	6,926	10	6,936	10	6,946	11	6,957	10	6,967	10	6,977	10	6,987	11	6,998	10	7,008	10	0,0169
0,14	7,447	12	7,459	11	7,470	11	7,481	11	7,492	11	7,503	11	7,514	11	7,525	11	7,536	11	7,547	11	0,0196
0,15	7,979	12	7,991	12	8,003	12	8,015	12	8,027	12	8,039	11	8,050	12	8,062	12	8,074	12	8,086	12	0,0225
0,16	8,511	13	8,524	13	8,537	12	8,549	13	8,562	12	8,574	13	8,587	13	8,600	12	8,612	13	8,625	12	0,0256
0,17	9,043	13	9,056	14	9,070	13	9,083	14	9,097	13	9,110	14	9,124	13	9,137	13	9,150	14	9,164	13	0,0289
0,18	9,575	14	9,589	14	9,603	15	9,618	14	9,632	14	9,646	14	9,660	14	9,674	15	9,689	14	9,703	14	0,0324
0,19	10,107	15	10,122	15	10,137	15	10,152	15	10,167	15	10,182	15	10,197	15	10,212	15	10,227	15	10,242	14	0,0361
0,20	10,638	16	10,654	16	10,670	16	10,686	16	10,702	16	10,718	15	10,733	16	10,749	16	10,765	15	10,780	16	0,0400
0,21	11,170	17	11,187	16	11,203	17	11,220	17	11,237	16	11,253	17	11,270	16	11,286	17	11,303	16	11,319	17	0,0441
0,22	11,702	17	11,719	18	11,737	17	11,754	18	11,772	17	11,789	17	11,806	18	11,824	17	11,841	17	11,858	17	0,0484
0,23	12,234	18	12,252	18	12,270	18	12,288	18	12,306	19	12,325	18	12,343	18	12,361	18	12,379	19	12,397	18	0,0529
0,24	12,765	19	12,784	19	12,803	19	12,822	19	12,841	19	12,860	19	12,879	19	12,898	19	12,917	19	12,936	19	0,0576
0,25	13,297	20	13,317	20	13,337	19	13,356	20	13,376	20	13,396	20	13,416	19	13,435	20	13,455	19	13,474	20	0,0625
0,26	13,828	21	13,849	21	13,870	20	13,890	21	13,911	20	13,931	21	13,952	20	13,972	21	13,993	20	14,013	21	0,0676
0,27	14,360	21	14,381	22	14,403	21	14,424	22	14,446	21	14,467	21	14,488	21	14,509	22	14,531	21	14,552	21	0,0729
0,28	14,892	22	14,914	22	14,936	22	14,958	22	14,980	22	15,002	23	15,025	22	15,047	22	15,069	22	15,091	21	0,0784
0,29	15,423	23	15,446	23	15,469	23	15,492	23	15,515	23	15,538	23	15,561	23	15,584	22	15,606	23	15,629	23	0,0841
0,30	15,955	23	15,978	24	16,002	24	16,026	24	16,050	23	16,073	24	16,097	24	16,121	23	16,144	24	16,168	23	0,0900
0,31	16,486	25	16,511	24	16,535	25	16,560	24	16,584	25	16,609	24	16,633	25	16,658	24	16,682	24	16,706	25	0,0961
0,32	17,017	26	17,043	25	17,068	26	17,094	25	17,119	25	17,144	25	17,169	26	17,195	25	17,220	25	17,245	25	0,1024
0,33	17,549	26	17,575	26	17,601	26	17,627	26	17,653	27	17,680	26	17,706	26	17,732	25	17,757	26	17,783	26	0,1089
0,34	18,080	27	18,107	27	18,134	27	18,161	27	18,188	27	18,215	27	18,242	26	18,268	27	18,295	27	18,322	26	0,1156
0,35	18,611	28	18,639	28	18,667	28	18,695	27	18,722	28	18,750	28	18,778	27	18,805	28	18,833	27	18,860	28	0,1225
0,36	19,143	28	19,171	29	19,200	28	19,228	29	19,257	28	19,285	29	19,314	28	19,342	28	19,370	29	19,399	28	0,1296
0,37	19,674	29	19,703	30	19,733	29	19,762	29	19,791	29	19,820	30	19,850	29	19,879	29	19,908	29	19,937	29	0,1369
0,38	20,205	30	20,235	30	20,265	31	20,296	30	20,326	30	20,356	30	20,386	30	20,416	29	20,445	30	20,475	30	0,1444
0,39	20,736	31	20,767	31	20,798	31	20,829	31	20,860	31	20,891	31	20,922	30	20,952	31	20,983	31	21,014	30	0,1521
0,40	21,267	32	21,299	32	21,331	32	21,363	31	21,394	32	21,426	31	21,457	32	21,489	31	21,520	32	21,552	31	0,1600
0,41	21,798	33	21,831	32	21,863	33	21,896	32	21,928	33	21,961	32	21,993	32	22,025	33	22,058	32	22,090	32	0,1681
0,42	22,329	34	22,363	33	22,396	33	22,429	34	22,463	33	22,496	33	22,529	33	22,562	33	22,595	33	22,628	33	0,1764
0,43	22,860	34	22,894	35	22,929	34	22,963	34	22,997	34	23,031	34	23,065	33	23,098	34	23,132	34	23,166	34	0,1849
0,44	23,391	35	23,426	35	23,461	35	23,496	35	23,531	34	23,565	35	23,600	35	23,635	34	23,669	35	23,704	35	0,1936
0,45	23,922	36	23,958	35	23,993	36	24,029	36	24,065	35	24,100	36	24,136	35	24,171	36	24,207	35	24,242	35	0,2025
0,50	26,575	40	26,615	40	26,655	39	26,694	40	26,734	39	26,773	40	26,813	39	26,852	40	26,892	39	26,931	39	0,2500
0,55	29,227	44	29,271	43	29,314	44	29,358	44	29,402	43	29,445	44	29,489	43	29,532	43	29,575	44	29,619	43	0,3025
0,60	31,877	48	31,925	48	31,973	47	32,020	48	32,068	47	32,115	48	32,163	47	32,210	47	32,257	48	32,305	47	0,3600
0,65	34,525	52	34,577	52	34,629	51	34,680	52	34,732	52	34,784	51	34,835	51	34,886	52	34,938	51	34,989	51	0,4225
0,70	37,172	55	37,227	56	37,283	56	37,339	55	37,394	56	37,450	55	37,505	56	37,561	55	37,616	55	37,671	55	0,4900
0,75	39,816	60	39,876	59	39,935	60	39,995	60	40,055	59	40,114	60	40,174	59	40,233	59	40,292	59	40,351	60	0,5625
0,80	42,458	64	42,522	63	42,585	64	42,649	64	42,713	63	42,776	64	42,840	63	42,903	63	42,966	64	43,030	63	0,6400
0,85	45,097	68	45,165	68	45,233	68	45,301	67	45,368	68	45,436	67	45,503	68	45,571	67	45,638	67	45,705	67	0,7225
0,90	47,734	72	47,806	72	47,878	72	47,950	72	48,022	71	48,093	72	48,165	71	48,236	71	48,307	72	48,379	71	0,8100
0,95	50,368	76	50,444	76	50,520	76	50,596	76	50,672	76	50,748	75	50,823	76	50,899	75	50,974	75	51,049	75	0,9025
1,00	52,999	80	53,079	80	53,159	80	53,239	80	53,319	80	53,399	80	53,479	79	53,558	80	53,638	79	53,717	79	1,0000
	5,6113		5,6448		5,6785		5,7122		5,7461		5,7800		5,8141		5,8482		5,8825		5,9168		c^2

Sum of the Radii $r + r''$ (heading of the columns 3,35 … 3,44); the bottom row gives $\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	531	532	533	534	535	536	537	538	539	540	
1	53	53	53	53	54	54	54	54	54	54	1
2	106	106	107	107	107	107	107	108	108	108	2
3	159	160	160	160	161	161	161	161	162	162	3
4	212	213	213	214	214	214	215	215	216	216	4
5	266	266	267	267	268	268	269	269	270	270	5
6	319	319	320	320	321	322	322	323	323	324	6
7	372	372	373	374	375	375	376	377	377	378	7
8	425	426	426	427	428	429	430	430	431	432	8
9	478	479	480	481	482	482	483	484	485	486	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	3,45 Days	dif.	3,46 Days	dif.	3,47 Days	dif.	3,48 Days	dif.	3,49 Days	dif.	3,50 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,540	1	0,541	0	0,541	1	0,542	1	0,543	1	0,544	1	0,0001
0,02	1,080	1	1,081	2	1,083	1	1,084	2	1,086	2	1,088	1	0,0004
0,03	1,620	2	1,622	2	1,624	3	1,627	2	1,629	2	1,631	3	0,0009
0,04	2,160	3	2,163	3	2,166	3	2,169	3	2,172	3	2,175	3	0,0016
0,05	2,699	4	2,703	4	2,707	4	2,711	4	2,715	4	2,719	4	0,0025
0,06	3,239	5	3,244	5	3,249	4	3,253	5	3,258	5	3,263	4	0,0036
0,07	3,779	6	3,785	5	3,790	6	3,796	5	3,801	5	3,806	6	0,0049
0,08	4,319	6	4,325	7	4,332	6	4,338	6	4,344	6	4,350	6	0,0064
0,09	4,859	7	4,866	7	4,873	7	4,880	7	4,887	7	4,894	7	0,0081
0,10	5,399	7	5,406	8	5,414	8	5,422	8	5,430	8	5,438	7	0,0100
0,11	5,938	9	5,947	9	5,956	8	5,964	9	5,973	8	5,981	9	0,0121
0,12	6,478	10	6,488	9	6,497	9	6,506	10	6,516	9	6,525	9	0,0144
0,13	7,018	10	7,028	10	7,038	10	7,048	11	7,059	10	7,069	10	0,0169
0,14	7,558	11	7,569	11	7,580	11	7,591	10	7,601	11	7,612	11	0,0196
0,15	8,098	11	8,109	12	8,121	12	8,133	11	8,144	12	8,156	12	0,0225
0,16	8,637	13	8,650	12	8,662	13	8,675	12	8,687	13	8,700	12	0,0256
0,17	9,177	13	9,190	14	9,204	13	9,217	13	9,230	13	9,243	14	0,0289
0,18	9,717	14	9,731	14	9,745	14	9,759	14	9,773	14	9,787	14	0,0324
0,19	10,256	15	10,271	15	10,286	15	10,301	16	10,316	15	10,331	14	0,0361
0,20	10,796	16	10,812	15	10,827	16	10,843	16	10,859	15	10,874	16	0,0400
0,21	11,336	16	11,352	17	11,369	16	11,385	16	11,401	17	11,418	16	0,0441
0,22	11,875	18	11,893	17	11,910	17	11,927	17	11,944	17	11,961	17	0,0484
0,23	12,415	18	12,433	18	12,451	18	12,469	18	12,487	18	12,505	18	0,0529
0,24	12,955	18	12,973	19	12,992	19	13,011	18	13,029	19	13,048	19	0,0576
0,25	13,494	20	13,514	19	13,533	20	13,553	19	13,572	20	13,592	19	0,0625
0,26	14,034	20	14,054	20	14,074	21	14,095	20	14,115	20	14.135	20	0,0676
0,27	14,573	21	14,594	21	14,615	21	14,636	21	14,657	21	14,678	21	0,0729
0,28	15,112	22	15,134	22	15,156	22	15,178	22	15,200	22	15,222	21	0,0784
0,29	15,652	23	15,675	22	15,697	23	15,720	22	15,742	23	15,765	23	0,0841
0,30	16,191	24	16,215	23	16,238	24	16,262	23	16,285	23	16,308	24	0,0900
0,31	16,731	24	16,755	24	16,779	24	16,803	25	16,828	24	16,852	24	0,0961
0,32	17,270	25	17,295	25	17,320	25	17,345	25	17,370	25	17,395	25	0,1024
0,33	17,809	26	17,835	26	17,861	26	17,887	25	17,912	26	17,938	26	0,1089
0,34	18,348	27	18,375	27	18,402	26	18,428	27	18,455	26	18,481	27	0,1156
0,35	18,888	27	18,915	27	18,942	28	18,970	27	18,997	27	19,024	27	0,1225
0,36	19,427	28	19,455	28	19,483	28	19,511	28	19,539	28	19,567	28	0,1296
0,37	19,966	29	19,995	29	20,024	29	20,053	29	20,082	28	20,110	29	0,1369
0,38	20,505	30	20,535	30	20,565	29	20,594	30	20,624	29	20,653	30	0,1444
0,39	21,044	31	21,075	30	21,105	31	21,136	30	21,166	30	21,196	31	0,1521
0,40	21,583	31	21,614	32	21,646	31	21,677	31	21,708	31	21,739	31	0,1600
0,41	22.122	32	22,154	32	22,186	32	22,218	32	22,250	32	22,282	32	0,1681
0,42	22,661	33	22,694	33	22.727	32	22,759	33	22,792	33	22,825	33	0,1764
0,43	23,200	33	23,233	34	23,267	34	23,301	33	23,334	34	23,368	33	0,1849
0,44	23,739	34	23,773	34	23,807	35	23,842	34	23,876	34	23,910	35	0,1936
0,45	24,277	36	24,313	35	24,348	35	24,383	35	24,418	35	24,453	35	0,2025
0,50	26,970	40	27,010	39	27,049	39	27,088	39	27,127	39	27,166	39	0,2500
0,55	29,662	43	29,705	43	29,748	43	29,791	43	29,834	43	29,877	43	0,3025
0,60	32,352	47	32,399	47	32,446	47	32,493	47	32,540	47	32,587	46	0,3600
0,65	35,040	51	35,091	51	35,142	51	35,193	51	35,244	50	35,294	51	0,4225
0,70	37,726	55	37,781	55	37,836	55	37,891	55	37,946	55	38,001	54	0,4900
0,75	40,411	59	40,470	58	40,528	59	40,587	59	40,646	59	40,705	58	0,5625
0,80	43,093	63	43,156	63	43,219	62	43,281	63	43,344	63	43,407	62	0,6400
0,85	45,772	67	45,839	67	45,906	67	45,973	67	46,040	66	46,106	67	0,7225
0,90	48,450	71	48,521	71	48,592	70	48,662	71	48,733	71	48,804	70	0,8100
0,95	51,124	75	51,199	75	51,274	75	51,349	75	51,424	74	51,498	75	0,9025
1,00	53,796	79	53,875	79	53,954	79	54,033	78	54,111	79	54,190	78	1,0000
	5,9513		5,9858		6,0205		6,0552		6,0901		6,1250		c^2

$\frac{1}{2} \cdot (r + r')^2$ or $r^2 + r''^2$ nearly.

	538	539	540	541	542	543	544	
1	54	54	54	54	54	54	54	1
2	108	108	108	108	108	109	109	2
3	161	162	162	162	163	163	163	3
4	215	216	216	216	217	217	218	4
5	269	270	270	271	271	272	272	5
6	323	323	324	325	325	326	326	6
7	377	377	378	379	379	380	381	7
8	430	431	432	433	434	434	435	8
9	484	485	486	487	488	489	490	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	3,51 Days	dif.	3,52 Days	dif.	3,53 Days	dif.	3,54 Days	dif.	3,55 Days	dif.	3,56 Days	dif.	3,57 Days	dif.	3,58 Days	dif.	3,59 Days	dif.	3,60 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,545	0	0,545	1	0,546	1	0,547	1	0,548	0	0,548	1	0,549	1	0,550	1	0,551	1	0,552	0	0,0001
0,02	1,089	2	1,091	1	1,092	2	1,094	1	1,095	2	1,097	1	1,098	2	1,100	1	1,101	2	1,103	2	0,0004
0,03	1,634	2	1,636	2	1,638	3	1,641	2	1,643	2	1,645	3	1,648	2	1,650	2	1,652	2	1,654	3	0,0009
0,04	2,178	3	2,181	3	2,184	3	2,187	4	2,191	3	2,194	3	2,197	3	2,200	3	2,203	3	2,206	3	0,0016
0,05	2,723	4	2,727	3	2,730	4	2,734	4	2,738	4	2,742	4	2,746	4	2,750	4	2,754	3	2,757	4	0,0025
0,06	3,267	5	3,272	5	3,277	4	3,281	5	3,286	5	3,291	4	3,295	5	3,300	4	3,304	5	3,309	4	0,0036
0,07	3,812	5	3,817	6	3,823	5	3,828	6	3,834	5	3,839	5	3,844	6	3,850	5	3,855	5	3,860	6	0,0049
0,08	4,356	7	4,363	6	4,369	6	4,375	6	4,381	6	4,387	6	4,393	7	4,400	6	4,406	6	4,412	6	0,0064
0,09	4,901	7	4,908	7	4,915	7	4,922	7	4,929	7	4,936	7	4,943	6	4,949	7	4,956	7	4,963	7	0,0081
0,10	5,445	8	5,453	8	5,461	8	5,469	7	5,476	8	5,484	8	5,492	7	5,499	8	5,507	8	5,515	7	0,0100
0,11	5,990	8	5,998	9	6,007	8	6,015	9	6,024	8	6,032	9	6,041	8	6,049	9	6,058	8	6,066	9	0,0121
0,12	6,534	10	6,544	9	6,553	9	6,562	9	6,571	10	6,581	9	6,590	9	6,599	9	6,608	10	6,618	9	0,0144
0,13	7,079	10	7,089	10	7,099	10	7,109	10	7,119	10	7,129	10	7,139	10	7,149	10	7,159	10	7,169	10	0,0169
0,14	7,623	11	7,634	11	7,645	11	7,656	11	7,667	10	7,677	11	7,688	11	7,699	11	7,710	10	7,720	11	0,0196
0,15	8,168	11	8,179	12	8,191	12	8,203	11	8,214	12	8,226	11	8,237	12	8,249	11	8,260	12	8,272	11	0,0225
0,16	8,712	13	8,725	12	8,737	12	8,749	13	8,762	12	8,774	12	8,786	13	8,799	12	8,811	12	8,823	12	0,0256
0,17	9,257	13	9,270	13	9,283	13	9,296	13	9,309	13	9,322	13	9,335	13	9,348	13	9,361	14	9,375	13	0,0289
0,18	9,801	14	9,815	14	9,829	14	9,843	14	9,857	14	9,871	13	9,884	14	9,898	14	9,912	14	9,926	14	0,0324
0,19	10,345	15	10,360	15	10,375	14	10,389	15	10,404	15	10,419	14	10,433	15	10,448	15	10,463	14	10,477	15	0,0361
0,20	10,890	15	10,905	16	10,921	15	10,936	16	10,952	15	10,967	15	10,982	16	10,998	15	11,013	15	11,028	16	0,0400
0,21	11,434	16	11,450	16	11,466	17	11,483	16	11,499	16	11,515	16	11,531	16	11,547	17	11,564	16	11,580	16	0,0441
0,22	11,978	17	11,995	17	12,012	17	12,029	17	12,046	17	12,063	17	12,080	17	12,097	17	12,114	17	12,131	17	0,0484
0,23	12,523	17	12,540	18	12,558	18	12,576	18	12,594	17	12,611	18	12,629	18	12,647	18	12,665	17	12,682	18	0,0529
0,24	13,067	18	13,085	19	13,104	19	13,123	18	13,141	19	13,160	18	13,178	19	13,197	18	13,215	18	13,233	19	0,0576
0,25	13,611	19	13,630	20	13,650	19	13,669	19	13,688	20	13,708	19	13,727	19	13,746	19	13,765	20	13,785	19	0,0625
0,26	14,155	20	14,175	20	14,195	21	14,216	20	14,236	20	14,256	20	14,276	20	14,296	20	14,316	20	14,336	20	0,0676
0,27	14,699	21	14,720	21	14,741	21	14,762	21	14,783	21	14,804	21	14,825	20	14,845	21	14,866	21	14,887	20	0,0729
0,28	15,243	22	15,265	22	15,287	22	15,309	21	15,330	22	15,352	21	15,373	22	15,395	21	15,416	22	15,438	21	0,0784
0,29	15,788	22	15,810	23	15,833	22	15,855	22	15,877	23	15,900	22	15,922	22	15,944	23	15,967	22	15,989	22	0,0841
0,30	16,332	23	16,355	23	16,378	23	16,401	24	16,425	23	16,448	23	16,471	23	16,494	23	16,517	23	16,540	23	0,0900
0,31	16,876	24	16,900	24	16,924	24	16,948	24	16,972	24	16,996	24	17,020	23	17,043	24	17,067	24	17,091	24	0,0961
0,32	17,420	25	17,445	24	17,469	25	17,494	25	17,519	25	17,544	24	17,568	25	17,593	24	17,617	25	17,642	24	0,1024
0,33	17,964	25	17,989	26	18,015	25	18,040	26	18,066	25	18,091	26	18,117	25	18,142	26	18,168	25	18,193	25	0,1089
0,34	18,508	26	18,534	26	18,560	27	18,587	26	18,613	26	18,639	26	18,665	27	18,692	26	18,718	26	18,744	26	0,1156
0,35	19,051	28	19,079	27	19,106	27	19,133	27	19,160	27	19,187	27	19,214	27	19,241	27	19,268	27	19,295	26	0,1225
0,36	19,595	28	19,623	28	19,651	28	19,679	28	19,707	28	19,735	27	19,762	28	19,790	28	19,818	27	19,845	28	0,1296
0,37	20,139	29	20,168	29	20,197	28	20,225	29	20,254	28	20,282	29	20,311	28	20,339	29	20,368	28	20,396	29	0,1369
0,38	20,683	29	20,712	30	20,742	29	20,771	30	20,801	29	20,830	29	20,859	30	20,889	29	20,918	29	20,947	29	0,1444
0,39	21,227	30	21,257	30	21,287	30	21,317	31	21,348	30	21,378	30	21,408	30	21,438	30	21,468	30	21,498	30	0,1521
0,40	21,770	31	21,801	31	21,832	31	21,863	31	21,894	31	21,925	31	21,956	31	21,987	31	22,018	30	22,048	31	0,1600
0,41	22,314	32	22,346	32	22,378	31	22,409	32	22,441	32	22,473	31	22,504	32	22,536	31	22,567	32	22,599	31	0,1681
0,42	22,858	32	22,890	33	22,923	32	22,955	33	22,988	32	23,020	33	23,053	32	23,085	32	23,117	33	23,150	32	0,1764
0,43	23,401	34	23,435	33	23,468	33	23,501	33	23,534	34	23,568	33	23,601	33	23,634	33	23,667	33	23,700	33	0,1849
0,44	23,945	34	23,979	34	24,013	34	24,047	34	24,081	34	24,115	34	24,149	34	24,183	34	24,217	34	24,251	33	0,1936
0,45	24,488	35	24,523	35	24,558	35	24,593	35	24,628	34	24,662	35	24,697	35	24,732	34	24,766	35	24,801	34	0,2025
0,50	27,205	39	27,244	38	27,282	39	27,321	39	27,360	38	27,398	39	27,437	38	27,475	39	27,514	38	27,552	39	0,2500
0,55	29,920	42	29,962	43	30,005	43	30,048	42	30,090	43	30,133	42	30,175	43	30,218	42	30,260	42	30,302	43	0,3025
0,60	32,633	47	32,680	47	32,727	46	32,773	47	32,820	46	32,866	46	32,912	47	32,959	46	33,005	46	33,051	46	0,3600
0,65	35,345	51	35,396	50	35,446	51	35,497	50	35,547	50	35,597	51	35,648	50	35,698	50	35,748	50	35,798	50	0,4225
0,70	38,055	55	38,110	54	38,164	55	38,219	54	38,273	54	38,327	54	38,381	54	38,435	54	38,489	54	38,543	54	0,4900
0,75	40,763	59	40,822	58	40,880	58	40,938	59	40,997	58	41,055	58	41,113	58	41,171	58	41,229	57	41,286	58	0,5625
0,80	43,469	63	43,532	62	43,594	62	43,656	62	43,718	62	43,780	62	43,842	62	43,904	62	43,966	62	44,028	61	0,6400
0,85	46,173	66	46,239	66	46,305	67	46,372	66	46,438	66	46,504	66	46,570	65	46,635	66	46,701	66	46,767	65	0,7225
0,90	48,874	70	48,944	71	49,015	70	49,085	70	49,155	70	49,225	70	49,295	69	49,364	70	49,434	69	49,503	70	0,8100
0,95	51,573	74	51,647	74	51,721	74	51,795	74	51,869	74	51,943	74	52,017	74	52,091	73	52,164	74	52,238	73	0,9025
1,00	54,268	79	54,347	78	54,425	78	54,503	78	54,581	78	54,659	78	54,737	77	54,814	78	54,892	77	54,969	78	1,0000
	6,1601		**6,1952**		**6,2305**		**6,2658**		**6,3013**		**6,3368**		**6,3725**		**6,4082**		**6,4441**		**6,4800**		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	543	544	545	546	547	548	549	550	551	552	
1	54	54	55	55	55	55	55	55	55	55	1
2	109	109	109	109	109	110	110	110	110	110	2
3	163	163	164	164	164	164	165	165	165	166	3
4	217	218	218	218	219	219	220	220	220	221	4
5	272	272	273	273	274	274	275	275	276	276	5
6	326	326	327	328	328	329	329	330	331	331	6
7	380	381	382	382	383	384	384	385	386	386	7
8	434	435	436	437	438	438	439	440	441	442	8
9	489	490	491	491	492	493	494	495	496	497	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord c.	3,61 Days	dif.	3,62 Days	dif.	3,63 Days	dif	3,64 Days	dif.	3,65 Days	dif.	3,66 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,552	1	0,553	1	0,554	1	0,555	0	0,555	1	0,556	1	0,0001
0,02	1,105	1	1,106	2	1,108	1	1,109	2	1,111	1	1,112	2	0,0004
0,03	1,657	2	1,659	2	1,661	3	1,664	2	1,666	2	1,668	2	0,0009
0,04	2,209	3	2,212	3	2,215	3	2,218	3	2,221	3	2,224	3	0,0016
0,05	2,761	4	2,765	4	2,769	4	2,773	4	2,777	3	2,780	4	0,0025
0,06	3,313	5	3,318	5	3,323	4	3,327	5	3,332	4	3,336	5	0,0036
0,07	3,866	5	3,871	5	3,876	6	3,882	5	3,887	5	3,892	6	0,0049
0,08	4,418	6	4,424	6	4,430	6	4,436	6	4,442	6	4,448	7	0,0064
0,09	4,970	7	4,977	7	4,984	7	4,991	7	4,998	7	5,005	6	0,0081
0,10	5,522	8	5,530	8	5,538	7	5,545	8	5,553	8	5,561	7	0,0100
0,11	6,075	8	6,083	8	6,091	9	6,100	8	6,108	9	6,117	8	0,0121
0,12	6,627	9	6,636	9	6,645	9	6,654	9	6,663	10	6,673	9	0,0144
0,13	7,179	10	7,189	10	7,199	10	7,209	10	7,219	10	7,229	9	0,0169
0,14	7,731	11	7,742	11	7,753	10	7,763	11	7,774	11	7,785	10	0,0196
0,15	8,283	12	8,295	11	8,306	12	8,318	11	8,329	11	8,340	12	0,0225
0,16	8,835	13	8,848	12	8,860	12	8,872	12	8,884	12	8,896	13	0,0256
0,17	9,388	13	9,401	13	9,414	12	9,426	13	9,439	13	9,452	13	0,0289
0,18	9,940	13	9,953	14	9,967	14	9,981	14	9,995	13	10,008	14	0,0324
0,19	10,492	14	10,506	15	10,521	14	10,535	15	10,550	14	10,564	15	0,0361
0,20	11,044	15	11,059	15	11,074	16	11,090	15	11,105	15	11,120	15	0,0400
0,21	11,596	16	11,612	16	11,628	16	11,644	16	11,660	16	11,676	16	0,0441
0,22	12,148	17	12,165	16	12,181	17	12,198	17	12,215	17	12,232	16	0,0484
0,23	12,700	17	12,717	18	12,735	17	12,752	18	12,770	17	12,787	18	0,0529
0,24	13,252	18	13,270	18	13,288	19	13,307	18	13,325	18	13,343	19	0,0576
0,25	13,804	19	13,823	19	13,842	19	13,861	19	13,880	19	13,899	19	0,0625
0,26	14,356	19	14,375	20	14,395	20	14,415	20	14,435	20	14,455	20	0,0676
0,27	14,907	21	14,928	21	14,949	20	14,969	21	14,990	20	15,010	21	0,0729
0,28	15,459	22	15,481	21	15,502	22	15,524	21	15,545	21	15,566	21	0,0784
0,29	16,011	22	16,033	23	16,056	22	16,078	22	16,100	22	16,122	22	0,0841
0,30	16,563	23	16,586	23	16,609	23	16,632	23	16,655	22	16,677	23	0,0900
0,31	17,115	23	17,138	24	17,162	24	17,186	23	17,209	24	17,233	24	0,0961
0,32	17,666	25	17,691	24	17,715	25	17,740	24	17,764	25	17,789	24	0,1024
0,33	18,218	25	18,243	26	18,269	25	18,294	25	18,319	25	18,344	25	0,1089
0,34	18,770	26	18,796	26	18,822	26	18,848	26	18,874	26	18,900	25	0,1156
0,35	19,321	27	19,348	27	19,375	27	19,402	26	19,428	27	19,455	27	0,1225
0,36	19,873	28	19,901	27	19,928	28	19,956	27	19,983	27	20,010	28	0,1296
0,37	20,425	28	20,453	28	20,481	28	20,509	29	20,538	28	20,566	28	0,1369
0,38	20,976	29	21,005	29	21,034	29	21,063	29	21,092	29	21,121	29	0,1444
0,39	21,528	29	21,557	30	21,587	30	21,617	30	21,647	29	21,676	30	0,1521
0,40	22,079	31	22,110	30	22,140	31	22,171	30	22,201	31	22,232	30	0,1600
0,41	22,630	32	22,662	31	22,693	31	22,724	32	22,756	31	22,787	31	0,1681
0,42	23,182	32	23,214	32	23,246	32	23,278	32	23,310	32	23,342	32	0,1764
0,43	23,733	33	23,766	33	23,799	33	23,832	32	23,864	33	23,897	33	0,1849
0,44	24,284	34	24,318	34	24,352	33	24,385	34	24,419	33	24,452	34	0,1936
0,45	24,835	35	24,870	34	24,904	35	24,939	34	24,973	34	25,007	35	0,2025
0,50	27,591	38	27,629	38	27,667	38	27,705	39	27,744	38	27,782	38	0,2500
0,55	30,345	42	30,387	42	30,429	42	30,471	42	30,513	42	30,555	42	0,3025
0,60	33,097	46	33,143	46	33,189	46	33,235	46	33,281	46	33,327	45	0,3600
0,65	35,848	50	35,898	50	35,948	49	35,907	50	36,047	50	36,097	49	0,4225
0,70	38,597	54	38,651	53	38,704	54	38,758	54	38,812	53	38,865	53	0,4900
0,75	41,344	58	41,402	57	41,459	58	41,517	57	41,574	58	41,632	57	0,5625
0,80	44,089	62	44,151	61	44,212	62	44,274	61	44,335	61	44,396	61	0,6400
0,85	46,832	66	46,898	65	46,963	65	47,028	65	47,093	66	47,159	65	0,7225
0,90	49,573	69	49,642	70	49,712	69	49,781	69	49,850	69	49,919	68	0,8100
0,95	52,311	73	52,384	73	52,457	74	52,531	72	52,603	73	52,676	73	0,9025
1,00	55,047	77	55,124	77	55,201	77	55,278	77	55,355	76	55,431	77	1,0000
	6,5161		6,5522		6,5885		6,6248		6,6613		6,6978		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	551	552	553	554	555	556	557	
1	55	55	55	55	56	56	56	1
2	110	110	111	111	111	111	111	2
3	165	166	166	166	167	167	167	3
4	220	221	221	222	222	222	223	4
5	276	276	277	277	278	278	279	5
6	331	331	332	332	333	334	334	6
7	386	386	387	388	389	389	390	7
8	441	442	442	443	444	445	446	8
9	496	497	498	499	500	500	501	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	3,67 Days	dif.	3,68 Days	dif.	3,69 Days	dif.	3,70 Days	dif.	3,71 Days	dif.	3,72 Days	dif.	3,73 Days	dif.	3,74 Days	dif.	3,75 Days	dif.	3,76 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,557	1	0,558	0	0,558	1	0,559	1	0,560	1	0,561	0	0,561	1	0,562	1	0,563	1	0,564	0	0,0001
0,02	1,114	1	1,115	2	1,117	1	1,118	2	1,120	1	1,121	2	1,123	1	1,124	2	1,126	1	1,127	2	0,0004
0,03	1,670	3	1,673	2	1,675	2	1,677	3	1,680	2	1,682	2	1,684	2	1,686	3	1,689	2	1,691	2	0,0009
0,04	2,227	3	2,230	3	2,233	3	2,236	3	2,239	3	2,242	3	2,245	3	2,248	3	2,251	3	2,254	3	0,0016
0,05	2,784	4	2,788	4	2,792	3	2,795	4	2,799	4	2,803	4	2,807	4	2,811	3	2,814	4	2,818	4	0,0025
0,06	3,341	4	3,345	5	3,350	5	3,355	4	3,359	5	3,364	4	3,368	5	3,373	4	3,377	5	3,382	4	0,0036
0,07	3,898	5	3,903	5	3,908	6	3,914	5	3,919	5	3,924	5	3,929	6	3,935	5	3,940	5	3,945	5	0,0049
0,08	4,455	6	4,461	6	4,467	6	4,473	6	4,479	6	4,485	6	4,491	6	4,497	6	4,503	6	4,509	6	0,0064
0,09	5,011	7	5,018	7	5,025	7	5,032	7	5,039	6	5,045	7	5,052	7	5,059	7	5,066	6	5,072	7	0,0081
0,10	5,568	8	5,576	7	5,583	8	5,591	7	5,598	8	5,606	7	5,613	8	5,621	7	5,628	8	5,636	7	0,0100
0,11	6,125	8	6,133	9	6,142	8	6,150	8	6,158	8	6,166	9	6,175	8	6,183	8	6,191	9	6,200	8	0,0121
0,12	6,682	9	6,691	9	6,700	9	6,709	9	6,718	9	6,727	9	6,736	9	6,745	9	6,754	9	6,763	9	0,0144
0,13	7,238	10	7,248	10	7,258	10	7,268	10	7,278	10	7,288	9	7,297	10	7,307	10	7,317	10	7,327	9	0,0169
0,14	7,795	11	7,806	10	7,816	11	7,827	10	7,837	11	7,848	11	7,859	10	7,869	11	7,880	10	7,890	11	0,0196
0,15	8,352	11	8,363	12	8,375	11	8,386	11	8,397	12	8,409	11	8,420	11	8,431	11	8,442	12	8,454	11	0,0225
0,16	8,909	12	8,921	12	8,933	12	8,945	12	8,957	12	8,969	12	8,981	12	8,993	12	9,005	12	9,017	12	0,0256
0,17	9,465	13	9,478	13	9,491	13	9,504	13	9,517	13	9,530	12	9,542	13	9,555	13	9,568	13	9,581	12	0,0289
0,18	10,022	14	10,036	13	10,049	14	10,063	13	10,076	14	10,090	14	10,104	13	10,117	14	10,131	13	10,144	14	0,0324
0,19	10,579	14	10,593	14	10,607	15	10,622	14	10,636	14	10,650	15	10,665	14	10,679	14	10,693	15	10,708	14	0,0361
0,20	11,135	15	11,150	16	11,166	15	11,181	15	11,196	15	11,211	15	11,226	15	11,241	15	11,256	15	11,271	15	0,0400
0,21	11,692	16	11,708	16	11,724	16	11,740	15	11,755	16	11,771	16	11,787	16	11,803	16	11,819	15	11,834	16	0,0441
0,22	12,248	17	12,265	17	12,282	16	12,298	17	12,315	17	12,332	16	12,348	17	12,365	16	12,381	17	12,398	16	0,0484
0,23	12,805	17	12,822	18	12,840	17	12,857	18	12,875	17	12,892	17	12,909	18	12,927	17	12,944	17	12,961	17	0,0529
0,24	13,362	18	13,380	18	13,398	18	13,416	18	13,434	18	13,452	18	13,470	18	13,488	18	13,506	18	13,524	18	0,0576
0,25	13,918	19	13,937	19	13,956	19	13,975	19	13,994	19	14,013	18	14,031	19	14,050	19	14,069	19	14,088	19	0,0625
0,26	14,475	19	14,494	20	14,514	20	14,534	19	14,553	20	14,573	19	14,592	20	14,612	20	14,632	19	14,651	20	0,0676
0,27	15,031	20	15,051	21	15,072	20	15,092	21	15,113	20	15,133	20	15,153	21	15,174	20	15,194	20	15,214	21	0,0729
0,28	15,587	22	15,609	21	15,630	21	15,651	21	15,672	21	15,693	21	15,714	22	15,736	21	15,757	21	15,778	21	0,0784
0,29	16,144	22	16,166	22	16,188	22	16,210	22	16,232	22	16,254	21	16,275	22	16,297	22	16,319	22	16,341	21	0,0841
0,30	16,700	23	16,723	23	16,746	22	16,768	23	16,791	23	16,814	22	16,836	23	16,859	22	16,881	23	16,904	22	0,0900
0,31	17,257	23	17,280	24	17,304	23	17,327	23	17,350	24	17,374	23	17,397	24	17,421	23	17,444	23	17,467	23	0,0961
0,32	17,813	24	17,837	24	17,861	25	17,886	24	17,910	24	17,934	24	17,958	24	17,982	24	18,006	24	18,030	24	0,1024
0,33	18,369	25	18,394	25	18,419	25	18,444	25	18,469	25	18,494	25	18,519	25	18,544	25	18,569	24	18,593	25	0,1089
0,34	18,925	26	18,951	26	18,977	26	19,003	25	19,028	26	19,054	26	19,080	25	19,105	26	19,131	25	19,156	26	0,1156
0,35	19,482	26	19,508	27	19,535	26	19,561	27	19,588	26	19,614	26	19,640	27	19,667	26	19,693	26	19,719	27	0,1225
0,36	20,038	27	20,065	27	20,092	28	20,120	27	20,147	27	20,174	27	20,201	27	20,228	27	20,255	27	20,282	27	0,1296
0,37	20,594	28	20,622	28	20,650	28	20,678	28	20,706	28	20,734	28	20,762	28	20,790	28	20,818	27	20,845	28	0,1369
0,38	21,150	29	21,179	29	21,208	28	21,236	29	21,265	29	21,294	28	21,322	29	21,351	29	21,380	28	21,408	29	0,1444
0,39	21,706	30	21,736	29	21,765	30	21,795	29	21,824	30	21,854	29	21,883	29	21,912	30	21,942	29	21,971	29	0,1521
0,40	22,262	30	22,292	31	22,323	30	22,353	30	22,383	31	22,414	30	22,444	30	22,474	30	22,504	30	22,534	30	0,1600
0,41	22,818	31	22,849	31	22,880	31	22,911	31	22,942	31	22,973	31	23,004	31	23,035	31	23,066	31	23,097	30	0,1681
0,42	23,374	32	23,406	32	23,438	32	23,470	31	23,501	32	23,533	32	23,565	31	23,596	32	23,628	31	23,659	32	0,1764
0,43	23,930	33	23,963	32	23,995	33	24,028	32	24,060	33	24,093	32	24,125	33	24,158	32	24,190	32	24,222	32	0,1849
0,44	24,486	33	24,519	34	24,553	33	24,586	33	24,619	33	24,652	34	24,686	33	24,719	33	24,752	33	24,785	33	0,1936
0,45	25,042	34	25,076	34	25,110	34	25,144	34	25,178	34	25,212	34	25,246	34	25,280	34	25,314	34	25,347	34	0,2025
0,50	27,820	38	27,858	38	27,896	38	27,934	37	27,971	38	28,009	38	28,047	38	28,085	37	28,122	38	28,160	37	0,2500
0,55	30,597	42	30,639	41	30,680	42	30,722	42	30,764	41	30,805	42	30,847	41	30,888	42	30,930	41	30,971	41	0,3025
0,60	33,372	46	33,418	45	33,463	46	33,509	45	33,554	46	33,600	45	33,645	45	33,690	46	33,736	45	33,781	45	0,3600
0,65	36,146	50	36,196	49	36,245	49	36,294	50	36,344	49	36,393	49	36,442	49	36,491	49	36,540	49	36,589	49	0,4225
0,70	38,918	54	38,972	53	39,025	53	39,078	53	39,131	53	39,184	53	39,237	53	39,290	53	39,343	53	39,396	52	0,4900
0,75	41,689	57	41,746	57	41,803	57	41,860	57	41,917	57	41,974	57	42,031	56	42,087	57	42,144	56	42,200	57	0,5625
0,80	44,457	61	44,518	61	44,579	61	44,640	61	44,701	60	44,761	61	44,822	61	44,883	60	44,943	60	45,003	61	0,6400
0,85	47,224	64	47,288	65	47,353	65	47,418	64	47,482	65	47,547	64	47,611	65	47,676	64	47,740	64	47,804	64	0,7225
0,90	49,987	69	50,056	69	50,125	68	50,193	69	50,262	68	50,330	69	50,399	68	50,467	68	50,535	68	50,603	68	0,8100
0,95	52,749	73	52,822	72	52,894	73	52,967	72	53,039	72	53,111	72	53,183	72	53,255	72	53,327	72	53,399	72	0,9025
1,00	55,508	77	55,585	76	55,661	76	55,737	77	55,814	76	55,890	76	55,966	76	56,042	75	56,117	76	56,193	76	1,0000
	6,7345		**6,7712**		**6,8081**		**6,8450**		**6,8821**		**6,9192**		**6,9565**		**6,9938**		**7,0313**		**7,0688**		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	556	557	558	559	560	561	562	563	564	
1	56	56	56	56	56	56	56	56	56	1
2	111	111	112	112	112	112	112	113	113	2
3	167	167	167	168	168	168	169	169	169	3
4	222	223	223	224	224	224	225	225	226	4
5	278	279	279	280	280	281	281	282	282	5
6	334	334	335	335	336	337	337	338	338	6
7	389	390	391	391	392	393	393	394	395	7
8	445	446	446	447	448	449	450	450	451	8
9	500	501	502	503	504	505	506	507	508	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord	Sum of the Radii $r+r''$.												
	3,77		3,78		3,79		3,80		3,81		3,82		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,564	1	0,565	1	0,566	1	0,567	0	0,567	1	0,568	1	0,0001
0,02	1,129	1	1,130	2	1,132	1	1,133	2	1,135	1	1,136	2	0,0004
0,03	1,693	2	1,695	3	1,698	2	1,700	2	1,702	2	1,704	2	0,0009
0,04	2,257	3	2,260	3	2,263	3	2,266	3	2,269	3	2,272	3	0,0016
0,05	2,822	4	2,826	3	2,829	4	2,833	4	2,837	3	2,840	4	0,0025
0,06	3,386	5	3,391	4	3,395	5	3,400	4	3,404	5	3,409	4	0,0036
0,07	3,950	6	3,956	5	3,961	5	3,966	5	3,971	6	3,977	5	0,0049
0,08	4,515	6	4,521	6	4,527	6	4,533	6	4,539	6	4,545	6	0,0064
0,09	5,079	7	5,086	7	5,093	6	5,099	7	5,106	7	5,113	6	0,0081
0,10	5,643	8	5,651	7	5,658	8	5,666	7	5,673	8	5,681	7	0,0100
0,11	6,208	8	6,216	8	6,224	8	6,232	9	6,241	8	6,249	8	0,0121
0,12	6,772	9	6,781	9	6,790	9	6,799	9	6,808	9	6,817	9	0,0144
0,13	7,336	10	7,346	10	7,356	10	7,366	9	7,375	10	7,385	10	0,0169
0,14	7,901	10	7,911	11	7,922	10	7,932	10	7,942	11	7,953	10	0,0196
0,15	8,465	11	8,476	11	8,487	12	8,499	11	8,510	11	8,521	11	0,0225
0,16	9,029	12	9,041	12	9,053	12	9,065	12	9,077	12	9,089	12	0,0256
0,17	9,593	13	9,606	13	9,619	12	9,631	13	9,644	13	9,657	12	0,0289
0,18	10,158	13	10,171	13	10,184	14	10,198	13	10,211	14	10,225	13	0,0324
0,19	10,722	14	10,736	14	10,750	14	10,764	15	10,779	14	10,793	14	0,0361
0,20	11,286	15	11,301	15	11,316	15	11,331	15	11,346	15	11,361	14	0,0400
0,21	11,850	16	11,866	16	11,882	15	11,897	16	11,913	15	11,928	16	0,0441
0,22	12,414	17	12,431	16	12,447	17	12,464	16	12,480	16	12,496	17	0,0484
0,23	12,978	18	12,996	17	13,013	17	13,030	17	13,047	17	13,064	17	0,0529
0,24	13,542	18	13,560	18	13,578	18	13,596	18	13,614	18	13,632	18	0,0576
0,25	14,107	18	14,125	19	14,144	19	14,163	18	14,181	19	14,200	18	0,0625
0,26	14,671	19	14,690	19	14,709	20	14,729	19	14,748	20	14,768	19	0,0676
0,27	15,235	20	15,255	20	15,275	20	15,295	20	15,315	20	15,335	20	0,0729
0,28	15,799	20	15,819	21	15,840	21	15,861	21	15,882	21	15,903	21	0,0784
0,29	16,362	22	16,384	22	16,406	22	16,428	21	16,449	22	16,471	21	0,0841
0,30	16,926	23	16,949	22	16,971	23	16,994	22	17,016	22	17,038	23	0,0900
0,31	17,490	24	17,514	23	17,537	23	17,560	23	17,583	23	17,606	23	0,0961
0,32	18,054	24	18,078	24	18,102	24	18,126	24	18,150	24	18,174	23	0,1024
0,33	18,618	25	18,643	24	18,667	25	18,692	25	18,717	24	18,741	25	0,1089
0,34	19,182	25	19,207	26	19,233	25	19,258	25	19,283	26	19,309	25	0,1156
0,35	19,746	26	19,772	26	19,798	26	19,824	26	19,850	26	19,876	26	0,1225
0,36	20,309	27	20,336	27	20,363	27	20,390	27	20,417	27	20,444	27	0,1296
0,37	20,873	28	20,901	27	20,928	28	20,956	28	20,984	27	21,011	28	0,1369
0,38	21,437	28	21,465	29	21,494	28	21,522	28	21,550	29	21,579	28	0,1444
0,39	22,000	30	22,030	29	22,059	29	22,088	29	22,117	29	22,146	29	0,1521
0,40	22,564	30	22,594	30	22,624	30	22,654	30	22,684	29	22,713	30	0,1600
0,41	23,127	31	23,158	31	23,189	30	23,219	31	23,250	31	23,281	30	0,1681
0,42	23,691	31	23,722	32	23,754	31	23,785	32	23,817	31	23,848	31	0,1764
0,43	24,254	33	24,287	32	24,319	32	24,351	32	24,383	32	24,415	32	0,1849
0,44	24,818	33	24,851	33	24,884	33	24,917	32	24,949	33	24,982	33	0,1936
0,45	25,381	34	25,415	34	25,449	33	25,482	34	25,516	33	25,549	34	0,2025
0,50	28,197	38	28,235	37	28,272	38	28,310	37	28,347	37	28,384	38	0,2500
0,55	31,012	42	31,054	41	31,095	41	31,136	41	31,177	41	31,218	41	0,3025
0,60	33,826	45	33,871	45	33,916	45	33,961	45	34,006	44	34,050	45	0,3600
0,65	36,638	49	36,687	48	36,735	49	36,784	49	36,833	48	36,881	49	0,4225
0,70	39,448	53	39,501	52	39,553	53	39,606	52	39,658	53	39,711	52	0,4900
0,75	42,257	56	42,313	57	42,370	56	42,426	56	42,482	56	42,538	56	0,5625
0,80	45,064	60	45,124	60	45,184	60	45,244	60	45,304	60	45,364	60	0,6400
0,85	47,868	64	47,932	64	47,996	64	48,060	64	48,124	63	48,187	64	0,7225
0,90	50,671	68	50,739	67	50,806	68	50,874	67	50,941	68	51,009	67	0,8100
0,95	53,471	72	53,543	71	53,614	72	53,686	71	53,757	71	53,828	71	0,9025
1,00	56,269	75	56,344	75	56,419	76	56,495	75	56,570	75	56,645	75	1,0000
	7,1065		7,1442		7,1821		7,2200		7,2581		7,2962		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	563	564	565	566	567	568	569	
1	56	56	57	57	57	57	57	1
2	113	113	113	113	113	114	114	2
3	169	169	170	170	170	170	171	3
4	225	226	226	226	227	227	228	4
5	282	282	283	283	284	284	285	5
6	338	338	339	340	340	341	341	6
7	394	395	396	396	397	398	398	7
8	450	451	452	453	454	454	455	8
9	507	508	509	509	510	511	512	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord	3,83		3,84		3,85		3,86		3,87		3,88		3,89		3,90		3,91		3,92		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,569	1	0,570	0	0,570	1	0,571	1	0,572	1	0,573	0	0,573	1	0,574	1	0,575	0	0,575	1	0,0001
0,02	1,138	1	1,139	2	1,141	1	1,142	2	1,144	1	1,145	2	1,147	1	1,148	2	1,150	1	1,151	1	0,0004
0,03	1,706	3	1,709	2	1,711	2	1,713	2	1,715	3	1,718	2	1,720	2	1,722	2	1,724	2	1,726	3	0,0009
0,04	2,275	3	2,278	3	2,281	3	2,284	3	2,287	3	2,290	3	2,293	3	2,296	3	2,299	3	2,302	3	0,0016
0,05	2,844	4	2,848	4	2,852	3	2,855	4	2,859	4	2,863	3	2,866	4	2,870	4	2,874	3	2,877	4	0,0025
0,06	3,413	4	3,417	5	3,422	4	3,426	5	3,431	4	3,435	5	3,440	4	3,444	4	3,448	5	3,453	4	0,0036
0,07	3,982	5	3,987	5	3,992	5	3,997	6	4,003	5	4,008	5	4,013	5	4,018	5	4,023	5	4,028	5	0,0049
0,08	4,551	6	4,557	5	4,562	6	4,568	6	4,574	6	4,580	6	4,586	6	4,592	6	4,598	6	4,604	6	0,0064
0,09	5,119	7	5,126	7	5,133	6	5,139	7	5,146	7	5,153	6	5,159	7	5,166	7	5,173	6	5,179	7	0,0081
0,10	5,688	8	5,696	7	5,703	7	5,710	8	5,718	7	5,725	8	5,733	7	5,740	7	5,747	8	5,755	7	0,0100
0,11	6,257	8	6,265	8	6,273	8	6,281	9	6,290	8	6,298	8	6,306	8	6,314	8	6,322	8	6,330	8	0,0121
0,12	6,826	9	6,835	9	6,844	8	6,852	9	6,861	9	6,870	9	6,879	9	6,888	9	6,897	8	6,905	9	0,0144
0,13	7,395	9	7,404	10	7,414	9	7,423	10	7,433	10	7,443	9	7,452	10	7,462	9	7,471	10	7,481	9	0,0169
0,14	7,963	11	7,974	10	7,984	10	7,994	11	8,005	10	8,015	10	8,025	11	8,036	10	8,046	10	8,056	11	0,0196
0,15	8,532	11	8,543	11	8,554	11	8,565	11	8,576	12	8,588	11	8,599	11	8,610	11	8,621	11	8,632	11	0,0225
0,16	9,101	12	9,113	11	9,124	12	9,136	12	9,148	12	9,160	12	9,172	12	9,184	11	9,195	12	9,207	12	0,0256
0,17	9,669	13	9,682	13	9,695	12	9,707	13	9,720	12	9,732	13	9,745	12	9,757	13	9,770	12	9,782	13	0,0289
0,18	10,238	13	10,251	14	10,265	13	10,278	13	10,291	14	10,305	13	10,318	13	10,331	14	10,345	13	10,358	13	0,0324
0,19	10,807	14	10,821	14	10,835	14	10,849	14	10,863	14	10,877	14	10,891	14	10,905	14	10,919	14	10,933	14	0,0361
0,20	11,375	15	11,390	15	11,405	15	11,420	15	11,435	14	11,449	15	11,464	15	11,479	15	11,494	14	11,508	15	0,0400
0,21	11,944	16	11,960	15	11,975	16	11,991	15	12,006	16	12,022	15	12,037	16	12,053	15	12,068	16	12,084	15	0,0441
0,22	12,513	16	12,529	16	12,545	17	12,562	16	12,578	16	12,594	16	12,610	17	12,627	16	12,643	16	12,659	16	0,0484
0,23	13,081	17	13,098	17	13,115	17	13,132	17	13,149	17	13,166	17	13,183	17	13,200	17	13,217	17	13,234	17	0,0529
0,24	13,650	18	13,668	17	13,685	18	13,703	18	13,721	18	13,739	17	13,756	18	13,774	18	13,792	17	13,809	18	0,0576
0,25	14,218	19	14,237	18	14,255	19	14,274	18	14,292	19	14,311	18	14,329	19	14,348	18	14,366	19	14,385	18	0,0625
0,26	14,787	19	14,806	19	14,825	20	14,845	19	14,864	19	14,883	19	14,902	20	14,922	19	14,941	19	14,960	19	0,0676
0,27	15,355	20	15,375	20	15,395	20	15,415	20	15,435	20	14,455	20	14,475	20	15,495	20	15,515	20	15,535	20	0,0729
0,28	15,924	21	15,945	20	15,965	21	15,986	21	16,007	21	16,028	20	16,048	21	16,069	20	16,089	21	16,110	21	0,0784
0,29	16,492	22	16,514	21	16,535	22	16,557	21	16,578	22	16,600	21	16,621	21	16,642	22	16,664	21	16,685	21	0,0841
0,30	17,061	22	17,083	22	17,105	22	17,127	23	17,150	22	17,172	22	17,194	22	17,216	22	17,238	22	17,260	22	0,0900
0,31	17,629	23	17,652	23	17,675	23	17,698	23	17,721	23	17,744	23	17,767	23	17,790	22	17,812	23	17,835	23	0,0961
0,32	18,197	24	18,221	24	18,245	24	18,269	23	18,292	24	18,316	24	18,340	23	18,363	24	18,387	23	18,410	24	0,1024
0,33	18,766	24	18,790	25	18,815	24	18,839	25	18,864	24	18,888	24	18,912	25	18,937	24	18,961	24	18,985	25	0,1089
0,34	19,334	25	19,359	26	19,385	25	19,410	25	19,435	25	19,460	25	19,485	25	19,510	25	19,535	25	19,560	25	0,1156
0,35	19,902	26	19,928	26	19,954	26	19,980	26	20,006	26	20,032	26	20,058	26	20,084	25	20,109	26	20,135	26	0,1225
0,36	20,471	26	20,497	27	20,524	27	20,551	26	20,577	27	20,604	27	20,631	26	20,657	27	20,684	26	20,710	26	0,1296
0,37	21,039	27	21,066	28	21,094	27	21,121	28	21,149	27	21,176	27	21,203	27	21,230	28	21,258	27	21,285	27	0,1369
0,38	21,607	28	21,635	28	21,663	29	21,692	28	21,720	28	21,748	28	21,776	28	21,804	28	21,832	28	21,860	28	0,1444
0,39	22,175	29	22,204	29	22,233	29	22,262	29	22,291	29	22,320	28	22,348	29	22,377	29	22,406	29	22,435	28	0,1521
0,40	22,743	30	22,773	30	22,803	29	22,832	30	22,862	29	22,891	30	22,921	29	22,950	30	22,980	29	23,009	30	0,1600
0,41	23,311	31	23,342	30	23,372	30	23,402	31	23,433	30	23,463	30	23,493	31	23,524	30	23,554	30	23,584	30	0,1681
0,42	23,879	31	23,910	32	23,942	31	23,973	31	24,004	31	24,035	31	24,066	31	24,097	31	24,128	31	24,159	30	0,1764
0,43	24,447	32	24,479	32	24,511	32	24,543	32	24,575	31	24,606	32	24,638	32	24,670	32	24,702	31	24,733	32	0,1849
0,44	25,015	33	25,048	32	25,080	33	25,113	33	25,146	32	25,178	33	25,211	32	25,243	32	25,275	33	25,308	32	0,1936
0,45	25,583	33	25,616	34	25,650	33	25,683	33	25,716	34	25,750	33	25,783	33	25,816	33	25,849	33	25,882	33	0,2025
0,50	28,422	37	28,459	37	28,496	37	28,533	37	28,570	37	28,607	37	28,644	37	28,681	37	28,718	37	28,755	36	0,2500
0,55	31,259	41	31,300	41	31,341	41	31,382	40	31,422	41	31,463	41	31,504	40	31,544	41	31,585	40	31,625	41	0,3025
0,60	34,095	45	34,140	44	34,184	45	34,229	44	34,273	45	34,318	44	34,362	45	34,407	44	34,451	44	34,495	44	0,3600
0,65	36,930	48	36,978	48	37,026	49	37,075	48	37,123	48	37,171	48	37,219	48	37,267	48	37,315	48	37,363	48	0,4225
0,70	39,763	52	39,815	52	39,867	52	39,919	52	39,971	52	40,023	52	40,075	51	40,126	52	40,178	52	40,230	51	0,4900
0,75	42,594	56	42,650	56	42,706	56	42,762	55	42,817	56	42,873	56	42,929	55	42,984	55	43,039	56	43,095	55	0,5625
0,80	45,424	59	45,483	60	45,543	59	45,602	60	45,662	59	45,721	59	45,780	60	45,840	59	45,899	59	45,958	59	0,6400
0,85	48,251	63	48,314	64	48,378	63	48,441	63	48,504	63	48,567	63	48,630	64	48,694	62	48,756	63	48,819	63	0,7225
0,90	51,076	67	51,143	68	51,211	67	51,278	67	51,345	67	51,412	66	51,478	67	51,545	67	51,612	66	51,678	67	0,8100
0,95	53,899	71	53,970	71	54,041	71	54,112	71	54,183	71	54,254	70	54,324	71	54,395	70	54,465	70	54,535	71	0,9025
1,00	56,720	75	56,795	75	56,870	74	56,944	75	57,019	74	57,093	75	57,168	74	57,242	74	57,316	74	57,390	74	1,0000
	7,3345		7,3728		7,4113		7,4198		7,4885		7,5272		7,5661		7,6050		7,6441		7,6832		c^2

$\frac{1}{2}\cdot(r+r'')^2$ or $r^2+r''^2$ nearly.

	568	569	570	571	572	573	574	575	576	
1	57	57	57	57	57	57	57	58	58	1
2	114	114	114	114	114	115	115	115	115	2
3	170	171	171	171	172	172	172	173	173	3
4	227	228	228	228	229	229	230	230	230	4
5	284	285	285	286	286	287	287	288	288	5
6	341	341	342	343	343	344	344	345	346	6
7	398	398	399	400	400	401	402	403	403	7
8	454	455	456	457	458	458	459	460	461	8
9	511	512	513	514	515	516	517	518	518	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord c.	3,93 Days	dif.	3,94 Days	dif.	3,95 Days	dif.	3,96 Days	dif	3,97 Days	dif.	3,98 Days	dif.	
	Sum of the Radii $r+r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,576	1	0,577	1	0,578	0	0,578	1	0,579	1	0,580	1	0,0001
0,02	1,152	2	1,154	1	1,155	2	1,157	1	1,158	2	1,160	1	0,0004
0,03	1,729	2	1,731	2	1,733	2	1,735	2	1,737	3	1,740	2	0,0009
0,04	2,305	3	2,308	3	2,311	3	2,314	3	2,317	2	2,319	3	0,0016
0,05	2,881	4	2,885	3	2,888	4	2,892	4	2,896	3	2,899	4	0,0025
0,06	3,457	5	3,462	4	3,466	4	3,470	5	3,475	4	3,479	5	0,0036
0,07	4,033	6	4,039	5	4,044	5	4,049	5	4,054	5	4,059	5	0,0049
0,08	4,610	6	4,616	5	4,621	6	4,627	6	4,633	6	4,639	6	0,0064
0,09	5,186	6	5,192	7	5,199	7	5,206	6	5,212	7	5,219	6	0,0081
0,10	5,762	7	5,769	8	5,777	7	5,784	7	5,791	8	5,799	7	0,0100
0,11	6,338	8	6,346	8	6,354	8	6,362	8	6,370	8	6,378	8	0,0121
0,12	6,914	9	6,923	9	6,932	9	6,941	8	6,949	9	6,958	9	0,0144
0,13	7,490	10	7,500	9	7,509	10	7,519	9	7,528	10	7,538	9	0,0169
0,14	8,067	10	8,077	10	8,087	10	8,097	11	8,108	10	8,118	10	0,0196
0,15	8,643	11	8,654	11	8,665	11	8,676	11	8,687	11	8,698	10	0,0225
0,16	9,219	12	9,231	11	9,242	12	9,254	12	9,266	11	9,277	12	0,0256
0,17	9,795	12	9,807	13	9,820	12	9,832	13	9,845	12	9,857	12	0,0289
0,18	10,371	13	10,384	13	10,397	13	10,410	14	10,424	13	10,437	13	0,0324
0,19	10,947	14	10,961	14	10,975	14	10,989	14	11,003	13	11,016	14	0,0361
0,20	11,523	15	11,538	14	11,552	15	11,567	15	11,582	14	11,596	15	0,0400
0,21	12,099	15	12,114	16	12,130	15	12,145	16	12,161	15	12,176	15	0,0441
0,22	12,675	16	12,691	16	12,707	16	12,723	16	12,739	16	12,755	17	0,0484
0,23	13,251	17	13,268	17	13,285	17	13,302	16	13,318	17	13,335	17	0,0529
0,24	13,827	18	13,845	17	13,862	18	13,880	17	13,897	18	13,915	17	0,0576
0,25	14,403	18	14,421	19	14,440	18	14,458	18	14,476	18	14,494	19	0,0625
0,26	14,979	19	14,998	19	15,017	19	15,036	19	15,055	19	15,074	19	0,0676
0,27	15,555	20	15,575	19	15,594	20	15,614	20	15,634	19	15,653	20	0,0729
0,28	16,131	20	16,151	21	16,172	20	16,192	21	16,213	20	16,233	20	0,0784
0,29	16,706	22	16,728	21	16,749	21	16,770	21	16,791	21	16,812	22	0,0841
0,30	17,282	22	17,304	22	17,326	22	17,348	22	17,370	22	17,392	22	0,0900
0,31	17,858	23	17,881	22	17,903	23	17,926	23	17,949	22	17,971	23	0,0961
0,32	18,434	23	18,457	24	18,481	23	18,504	23	18,527	24	18,551	23	0,1024
0,33	19,010	24	19,034	24	19,058	24	19,082	24	19,106	24	19,130	24	0,1089
0,34	19,585	25	19,610	25	19,635	25	19,660	25	19,685	25	19,710	24	0,1156
0,35	20,161	26	20,187	25	20,212	26	20,238	25	20,263	26	20,289	25	0,1225
0,36	20,736	27	20,763	26	20,789	27	20,816	26	20,842	26	20,868	26	0,1296
0,37	21,312	27	21,339	27	21,366	27	21,393	27	21,420	27	21,447	27	0,1369
0,38	21,888	27	21,915	28	21,943	28	21,971	28	21,999	28	22,027	27	0,1444
0,39	22,463	29	22,492	28	22,520	29	22,549	28	22,577	29	22,606	28	0,1521
0,40	23,039	29	23,068	29	23,097	30	23,127	29	23,156	29	23,185	29	0,1600
0,41	23,614	30	23,644	30	23,674	30	23,704	30	23,734	30	23,764	30	0,1681
0,42	24,189	31	24,220	31	24,251	31	24,282	30	24,312	31	24,343	31	0,1764
0,43	24,765	31	24,796	32	24,828	31	24,859	32	24,891	31	24,922	32	0,1849
0,44	25,340	32	25,372	33	25,405	32	25,437	32	25,469	32	25,501	32	0,1936
0,45	25,915	33	25,948	33	25,981	33	26,014	33	26,047	33	26,080	33	0,2025
0,50	28,791	37	28,828	37	28,865	36	28,901	37	28,938	36	28,974	37	0,2500
0,55	31,666	40	31,706	41	31,747	40	31,787	40	31,827	40	31,867	40	0,3025
0,60	34,539	44	34,583	44	34,627	44	34,671	44	34,715	44	34,759	44	0,3600
0,65	37,411	48	37,459	48	37,507	47	37,554	48	37,602	47	37,649	48	0,4225
0,70	40,281	52	40,333	51	40,384	52	40,436	51	40,487	51	40,538	51	0,4900
0,75	43,150	55	43,205	55	43,260	55	43,315	55	43,370	55	43,425	55	0,5625
0,80	46,017	59	46,076	59	46,135	58	46,193	59	46,252	59	46,311	58	0,6400
0,85	48,882	63	48,945	62	49,007	63	49,070	62	49,132	62	49,194	63	0,7225
0,90	51,745	66	51,811	67	51,878	66	51,944	66	52,010	66	52,076	66	0,8100
0,95	54,606	70	54,676	70	54,746	70	54,816	70	54,886	69	54,955	70	0,9025
1,00	57,464	74	57,538	74	57,612	73	57,685	74	57,759	73	57,832	74	1,0000
	7,7225		7,7618		7,8013		7,8408		7,8805		7,9202		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r'^2$ nearly.

	575	576	577	578	579	580	
1	58	58	58	58	58	58	1
2	115	115	115	116	116	116	2
3	173	173	173	173	174	174	3
4	230	230	231	231	232	232	4
5	288	288	289	289	290	290	5
6	345	346	346	347	347	348	6
7	403	403	404	405	405	406	7
8	460	461	462	462	463	464	8
9	518	518	519	520	521	522	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	3,99		4,00		4,01		4,02		4,03		4,04		4,05		4,06		4,07		4,08		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,581	0	0,581	1	0,582	1	0,583	0	0,583	1	0,584	1	0,585	1	0,586	0	0,586	1	0,587	1	0,0001
0,02	1,161	2	1,163	1	1,164	2	1,166	1	1,167	1	1,168	2	1,170	1	1,171	2	1,173	1	1,174	2	0,0004
0,03	1,742	2	1,744	2	1,746	2	1,748	2	1,750	3	1,753	2	1,755	2	1,757	2	1,759	2	1,761	2	0,0009
0,04	2,322	3	2,325	3	2,328	3	2,331	3	2,334	3	2,337	3	2,340	3	2,343	3	2,346	2	2,348	3	0,0016
0,05	2,903	4	2,907	3	2,910	4	2,914	3	2,917	4	2,921	4	2,925	3	2,928	4	2,932	4	2,936	3	0,0025
0,06	3,484	4	3,488	4	3,492	5	3,497	4	3,501	4	3,505	5	3,510	4	3,514	4	3,518	5	3,523	4	0,0036
0,07	4,064	5	4,069	5	4,074	5	4,079	5	4,084	6	4,090	5	4,095	5	4,100	5	4,105	5	4,110	5	0,0049
0,08	4,645	6	4,651	5	4,656	6	4,662	6	4,668	6	4,674	6	4,680	5	4,685	6	4,691	6	4,697	6	0,0064
0,09	5,225	7	5,232	6	5,238	7	5,245	6	5,251	7	5,258	6	5,264	7	5,271	6	5,277	7	5,284	6	0,0081
0,10	5,806	7	5,813	7	5,820	8	5,828	7	5,835	7	5,842	7	5,849	8	5,857	7	5,864	7	5,871	7	0,0100
0,11	6,386	8	6,394	8	6,402	8	6,410	8	6,418	8	6,426	8	6,434	8	6,442	8	6,450	8	6,458	8	0,0121
0,12	6,967	9	6,976	8	6,984	9	6,993	9	7,002	8	7,010	9	7,019	9	7,028	8	7,036	9	7,045	9	0,0144
0,13	7,547	10	7,557	9	7,566	10	7,576	9	7,585	10	7,595	9	7,604	9	7,613	10	7,623	9	7,632	9	0,0169
0,14	8,128	10	8,138	10	8,148	10	8,158	11	8,169	10	8,179	10	8,189	10	8,199	10	8,209	10	8,219	10	0,0196
0,15	8,708	11	8,719	11	8,730	11	8,741	11	8,752	11	8,763	11	8,774	11	8,785	10	8,795	11	8,806	11	0,0225
0,16	9,289	12	9,301	11	9,312	12	9,324	11	9,335	12	9,347	12	9,359	11	9,370	12	9,382	11	9,393	12	0,0256
0,17	9,869	13	9,882	12	9,894	12	9,906	13	9,919	12	9,931	12	9,943	13	9,956	12	9,968	12	9,980	12	0,0289
0,18	10,450	13	10,463	13	10,476	13	10,489	13	10,502	13	10,515	13	10,528	13	10,541	13	10,554	13	10,567	13	0,0324
0,19	11,030	14	11,044	14	11,058	14	11,072	13	11,085	14	11,099	14	11,113	14	11,127	13	11,140	14	11,154	14	0,0361
0,20	11,611	14	11,625	15	11,640	14	11,654	15	11,669	14	11,683	15	11,698	14	11,712	15	11,727	14	11,741	14	0,0400
0,21	12,191	15	12,206	16	12,222	15	12,237	15	12,252	15	12,267	16	12,283	15	12,298	15	12,313	15	12,328	15	0,0441
0,22	12,772	16	12,788	16	12,804	15	12,819	16	12,835	16	12,851	16	12,867	16	12,883	16	12,899	16	12,915	16	0,0484
0,23	13,352	17	13,369	16	13,385	17	13,402	17	13,419	16	13,435	17	13,452	17	13,469	16	13,485	17	13,502	16	0,0529
0,24	13,932	18	13,950	17	13,967	18	13,985	17	14,002	17	14,019	18	14,037	17	14,054	17	14,071	18	14,089	17	0,0576
0,25	14,513	18	14,531	18	14,549	18	14,567	18	14,585	18	14,603	18	14,621	18	14,639	18	14,657	18	14,675	18	0,0625
0,26	15,093	19	15,112	19	15,131	19	15,150	18	15,168	19	15,187	19	15,206	19	15,225	18	15,243	19	15,262	19	0,0676
0,27	15,673	20	15,693	19	15,712	20	15,732	20	15,752	19	15,771	20	15,791	19	15,810	20	15,830	19	15,849	19	0,0729
0,28	16,253	21	16,274	20	16,294	20	16,314	21	16,335	20	16,355	20	16,375	20	16,395	21	16,416	20	16,436	20	0,0784
0,29	16,834	21	16,855	21	16,876	21	16,897	21	16,918	21	16,939	21	16,960	21	16,981	21	17,002	21	17,023	20	0,0841
0,30	17,414	22	17,436	21	17,457	22	17,479	22	17,501	22	17,523	21	17,544	22	17,566	22	17,588	21	17,609	22	0,0900
0,31	17,994	23	18,017	22	18,039	23	18,062	22	18,084	22	18,106	23	18,129	22	18,151	23	18,174	22	18,196	22	0,0961
0,32	18,574	23	18,597	24	18,621	23	18,644	23	18,667	23	18,690	23	18,713	24	18,737	23	18,760	23	18,783	23	0,1024
0,33	19,154	24	19,178	24	19,202	24	19,226	24	19,250	24	19,274	24	19,298	24	19,322	24	19,346	23	19,369	24	0,1089
0,34	19,734	25	19,759	25	19,784	24	19,808	25	19,833	25	19,858	24	19,882	25	19,907	24	19,931	25	19,956	24	0,1156
0,35	20,314	26	20,340	25	20,365	26	20,391	25	20,416	25	20,441	26	20,467	25	20,492	25	20,517	25	20,542	26	0,1225
0,36	20,894	27	20,921	26	20,947	26	20,973	26	20,999	26	21,025	26	21,051	26	21,077	26	21,103	26	21,129	26	0,1296
0,37	21,474	27	21,501	27	21,528	27	21,555	27	21,582	27	21,609	26	21,635	27	21,662	27	21,689	27	21,716	26	0,1369
0,38	22,054	28	22,082	28	22,110	27	22,137	28	22,165	27	22,192	28	22,220	27	22,247	28	22,275	27	22,302	27	0,1444
0,39	22,634	29	22,663	28	22,691	28	22,719	29	22,748	28	22,776	28	22,804	28	22,832	28	22,860	28	22,888	29	0,1521
0,40	23,214	29	23,243	29	23,272	29	23,301	29	23,330	29	23,359	29	23,388	29	23,417	29	23,446	29	23,475	29	0,1600
0,41	23,794	30	23,824	30	23,854	29	23,883	30	23,913	30	23,943	30	23,973	29	24,002	30	24,032	29	24,061	30	0,1681
0,42	24,374	30	24,404	31	24,435	30	24,465	31	24,496	30	24,526	31	24,557	30	24,587	30	24,617	31	24,648	30	0,1764
0,43	24,954	31	24,985	31	25,016	31	25,047	32	25,079	31	25,110	31	25,141	31	25,172	31	25,203	31	25,234	31	0,1849
0,44	25,533	32	25,565	32	25,597	32	25,629	32	25,661	32	25,693	32	25,725	32	25,757	31	25,788	32	25,820	32	0,1936
0,45	26,113	33	26,146	32	26,178	33	26,211	33	26,244	32	26,276	33	26,309	32	26,341	33	26,374	32	26,406	33	0,2025
0,50	29,011	36	29,047	37	29,084	36	29,120	36	29,156	36	29,192	37	29,229	36	29,265	36	29,301	36	29,337	36	0,2500
0,55	31,907	41	31,948	40	31,988	40	32,028	39	32,067	40	32,107	40	32,147	40	32,187	40	32,227	39	32,266	40	0,3025
0,60	34,803	44	34,847	43	34,890	44	34,934	44	34,978	43	35,021	44	35,065	43	35,108	43	35,151	44	35,195	43	0,3600
0,65	37,697	47	37,744	48	37,792	47	37,839	47	37,886	47	37,933	47	37,980	48	38,028	47	38,075	46	38,121	47	0,4225
0,70	40,589	51	40,640	51	40,691	51	40,742	51	40,793	51	40,844	51	40,895	51	40,946	50	40,996	51	41,047	50	0,4900
0,75	43,480	55	43,535	55	43,590	54	43,644	55	43,699	54	43,753	55	43,808	54	43,862	54	43,916	55	43,971	54	0,5625
0,80	46,369	59	46,428	58	46,486	58	46,544	59	46,603	58	46,661	58	46,719	58	46,777	58	46,835	58	46,893	58	0,6400
0,85	49,257	62	49,319	62	49,381	62	49,443	62	49,505	62	49,567	61	49,628	62	49,690	62	49,752	61	49,813	62	0,7225
0,90	52,142	66	52,208	65	52,273	66	52,339	66	52,405	65	52,470	66	52,536	65	52,601	65	52,666	66	52,732	65	0,8100
0,95	55,025	70	55,095	69	55,164	69	55,233	70	55,303	69	55,372	69	55,441	69	55,510	69	55,579	69	55,648	69	0,9025
1,00	57,906	73	57,979	73	58,052	73	58,125	73	58,198	73	58,271	73	58,344	73	58,417	73	58,490	72	58,562	73	1,0000
	7,9601		8,0000		8,0401		8,0802		8,1205		8,1608		8,2013		8,2418		8,2825		8,3232		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	579	580	581	582	583	584	585	586	587	588	
1	58	58	58	58	58	58	59	59	59	59	1
2	116	116	116	116	117	117	117	117	117	118	2
3	174	174	174	175	175	175	176	176	176	176	3
4	232	232	232	233	233	234	234	234	235	235	4
5	290	290	291	291	292	292	293	293	294	294	5
6	347	348	349	349	350	350	351	352	352	353	6
7	405	406	407	407	408	409	410	410	411	412	7
8	463	464	465	466	466	467	468	469	470	470	8
9	521	522	523	524	525	526	527	527	528	529	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord	Sum of the Radii $r + r''$.												
	4,09		4,10		4,11		4,12		4,13		4,14		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,588	1	0,589	0	0,589	1	0,590	1	0,591	0	0,591	1	0,0001
0,02	1,176	1	1,177	2	1,179	1	1,180	1	1,181	2	1,183	1	0,0004
0,03	1,763	3	1,766	2	1,768	2	1,770	2	1,772	2	1,774	2	0,0009
0,04	2,351	3	2.354	3	2,357	3	2,360	3	2,363	3	2,366	2	0,0016
0,05	2,939	4	2,943	3	2,946	4	2,950	3	2,953	4	2,957	4	0,0025
0,06	3,527	4	3,531	5	3,536	4	3,540	4	3,544	4	3,548	5	0,0036
0,07	4,115	5	4,120	5	4,125	5	4,130	5	4,135	5	4,140	5	0,0049
0,08	4,703	5	4,708	6	4,714	6	4,720	5	4,725	6	4,731	6	0,0064
0,09	5,290	7	5,297	6	5,303	7	5,310	6	5,316	7	5,323	6	0,0081
0,10	5,878	7	5,885	7	5,892	8	5,900	7	5,907	7	5,914	7	0,0100
0,11	6,466	8	6,474	8	6,482	8	6,490	7	6,497	8	6,505	8	0,0121
0,12	7,054	8	7,062	9	7,071	8	7,079	9	7,088	9	7,097	8	0,0144
0,13	7,641	10	7,651	9	7,660	9	7,669	10	7,679	9	7,688	9	0,0169
0,14	8,229	10	8,239	10	8,249	10	8,259	10	8,269	10	8,279	10	0,0196
0,15	8,817	11	8,828	10	8,838	11	8,849	11	8,860	11	8,871	10	0,0225
0,16	9,405	11	9,416	12	9,428	11	9,439	12	9,451	11	9,462	11	0,0256
0,17	9,992	13	10,005	12	10,017	12	10,029	12	10,041	12	10,053	12	0,0289
0,18	10,580	13	10,593	13	10,606	13	10,619	13	10,632	12	10,644	13	0,0324
0,19	11,168	13	11,181	14	11,195	14	11,209	13	11,222	14	11,236	13	0,0361
0,20	11,755	15	11,770	14	11,784	14	11,798	15	11,813	14	11,827	14	0,0400
0,21	12,343	15	12,358	15	12,373	15	12,388	15	12,403	15	12,418	15	0,0441
0,22	12,931	15	12,946	16	12,962	16	12,978	16	12,994	15	13,009	16	0,0484
0,23	13,518	17	13,535	16	13,551	17	13,568	16	13,584	17	13,601	16	0,0529
0,24	14,106	17	14,123	17	14,140	18	14,158	17	14,175	17	14,192	17	0,0576
0,25	14,693	18	14,711	18	14,729	18	14,747	18	14,765	18	14,783	18	0,0625
0,26	15,281	19	15,300	18	15,318	19	15,337	19	15,356	18	15,374	19	0,0676
0,27	15,868	20	15,888	19	15,907	20	15,927	19	15,946	19	15,965	20	0,0729
0,28	16,456	20	16,476	20	16,496	20	16,516	20	16,536	20	16,556	20	0,0784
0,29	17,043	21	17,064	21	17,085	21	17,106	21	17,127	20	17,147	21	0,0841
0,30	17,631	21	17,652	22	17,674	21	17,695	22	17,717	21	17,738	22	0,0900
0,31	18,218	23	18,241	22	18,263	22	18,285	22	18,307	22	18,329	23	0,0961
0,32	18,806	23	18,829	23	18,852	23	18,875	22	18,897	23	18,920	23	0,1024
0,33	19,393	24	19,417	23	19,440	24	19,464	24	19,488	23	19,511	24	0,1089
0,34	19,980	25	20,005	24	20,029	25	20,054	24	20,078	24	20,102	25	0,1156
0,35	20,568	25	20,593	25	20,618	25	20,643	25	20,668	25	20,693	25	0,1225
0,36	21,155	26	21,181	26	21,207	25	21,232	26	21,258	26	21,284	26	0,1296
0,37	21,742	27	21,769	26	21,795	27	21,822	26	21,848	27	21,875	26	0,1369
0,38	22,329	28	22,357	27	22,384	27	22,411	27	22,438	28	22,466	27	0,1444
0,39	22,917	28	22,945	28	22,973	28	23,001	28	23,029	27	23,056	28	0,1521
0,40	23,504	28	23,532	29	23,561	29	23,590	29	23,619	28	23,647	29	0,1600
0,41	24,091	29	24,120	30	24,150	29	24,179	30	24,209	29	24,238	29	0,1681
0,42	24,678	30	24,708	30	24,738	30	24,768	30	24,798	31	24,829	30	0,1764
0,43	25,265	31	25,296	31	25,327	31	25,358	30	25,388	31	25,419	31	0,1849
0,44	25,852	32	25,884	31	25,915	32	25,947	31	25,978	32	26,010	31	0,1936
0,45	26,439	32	26,471	33	26,504	32	26,536	32	26,568	32	26,600	32	0,2025
0,50	29,373	36	29,409	36	29,445	36	29,481	36	29,517	35	29,552	36	0,2500
0,55	32,306	40	32,346	39	32,385	40	32,425	39	32,464	40	32,504	39	0,3025
0,60	35,238	43	35,281	43	35,324	43	35,367	43	35,410	43	35,453	43	0,3600
0,65	38,168	47	38,215	47	38,262	47	38,309	46	38,355	47	38,402	46	0,4225
0,70	41,097	51	41,148	50	41,198	51	41,249	50	41,299	50	41,349	50	0,4900
0,75	44,025	54	44,079	54	44,133	54	44,187	54	44,241	54	44.295	53	0,5625
0,80	46,951	57	47,008	58	47,066	58	47,124	57	47,181	58	47,239	57	0,6400
0,85	49,875	61	49.936	61	49,997	62	50,059	61	50,120	61	50,181	61	0,7225
0,90	52,797	65	52,862	65	52,927	65	52,992	64	53,056	65	53,121	65	0,8100
0,95	55,717	68	55,785	69	55,854	69	55,923	68	55,991	68	56,059	69	0,9025
1,00	58,635	72	58,707	72	58,779	72	58,851	73	58,924	72	58,996	71	1,0000
	8,3641		8,4050		8,4461		8,4872		8,5285		8,5698		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	587	588	589	590	591	592	
1	59	59	59	59	59	59	1
2	117	118	118	118	118	118	2
3	176	176	177	177	177	178	3
4	235	235	236	236	236	237	4
5	294	294	295	295	296	296	5
6	352	353	353	354	355	355	6
7	411	412	412	413	414	414	7
8	470	470	471	472	473	474	8
9	528	529	530	531	532	533	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

	Sum of the Radii $r + r''$.																				
Chord	4,15		4,16		4,17		4,18		4,19		4,20		4,21		4,22		4,23		4,24		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,592	1	0,593	1	0,594	0	0,594	1	0,595	1	0,596	0	0,596	1	0,597	1	0,598	0	0,598	1	0,0001
0,02	1,184	2	1,186	1	1,187	2	1,189	1	1,190	1	1,191	2	1,193	1	1,194	2	1,196	1	1,197	1	0,0004
0,03	1,776	2	1,778	3	1,781	2	1,783	2	1,785	2	1,787	2	1,789	2	1,791	2	1,793	3	1,796	2	0,0009
0,04	2,368	3	2,371	3	2,374	3	2,377	3	2,380	3	2,383	3	2,386	2	2,388	3	2,391	3	2,394	3	0,0016
0,05	2,961	3	2,964	4	2,968	3	2,971	4	2,975	3	2,978	4	2,982	3	2,985	4	2,989	4	2,993	3	0,0025
0,06	3,553	4	3,557	4	3,561	5	3,566	4	3,570	4	3,574	4	3,578	5	3,583	4	3,587	4	3,591	4	0,0036
0,07	4,145	5	4,150	5	4,155	5	4,160	5	4,165	5	4,170	5	4,175	5	4,180	5	4,185	5	4,190	4	0,0049
0,08	4,737	6	4,743	5	4,748	6	4,754	6	4,760	5	4,765	6	4,771	6	4,777	5	4,782	6	4,788	6	0,0064
0,09	5,329	6	5,335	7	5,342	6	5,348	7	5,355	6	5,361	6	5,367	7	5,374	6	5,380	6	5,386	7	0,0081
0,10	5,921	7	5,928	7	5,935	8	5,943	7	5,950	7	5,957	7	5,964	7	5,971	7	5,978	7	5,985	7	0,0100
0,11	6,513	8	6,521	8	6,529	8	6,537	7	6,544	8	6,552	8	6,560	8	6,568	8	6,576	7	6,583	8	0,0121
0,12	7,105	9	7,114	8	7,122	9	7,131	8	7,139	9	7,148	8	7,156	9	7,165	8	7,173	9	7,182	8	0,0144
0,13	7,697	10	7,707	9	7,716	9	7,725	9	7,734	10	7,744	9	7,753	9	7,762	9	7,771	9	7,780	9	0,0169
0,14	8,289	10	8,299	10	8,309	10	8,319	10	8,329	10	8,339	10	8,349	10	8,359	10	8,369	10	8,379	10	0,0196
0,15	8,881	11	8,892	11	8,903	10	8,913	11	8,924	11	8,935	10	8,945	11	8,956	11	8,967	10	8,977	11	0,0225
0,16	9,473	12	9,485	11	9,496	12	9,508	11	9,519	11	9,530	12	9,542	11	9,553	11	9,564	12	9,576	11	0,0256
0,17	10,065	13	10,078	12	10,090	12	10,102	12	10,114	12	10,126	12	10,138	12	10,150	12	10,162	12	10,174	12	0,0289
0,18	10,657	13	10,670	13	10,683	13	10,696	13	10,709	12	10,721	13	10,734	13	10,747	13	10,760	12	10,772	13	0,0324
0,19	11,249	14	11,263	13	11,276	14	11,290	13	11,303	14	11,317	13	11,330	14	11,344	13	11,357	14	11,371	13	0,0361
0,20	11,841	15	11,856	14	11,870	14	11,884	14	11,898	14	11,912	15	11,927	14	11,941	14	11,955	14	11,969	14	0,0400
0,21	12,433	15	12,448	15	12,463	15	12,478	15	12,493	15	12,508	15	12,523	15	12,538	15	12,553	14	12,567	15	0,0441
0,22	13,025	16	13,041	16	13,057	15	13,072	16	13,088	15	13,103	16	13,119	16	13,135	15	13,150	16	13,166	15	0,0484
0,23	13,617	16	13,633	17	13,650	16	13,666	17	13,683	16	13,699	16	13,715	17	13,732	16	13,748	16	13,764	16	0,0529
0,24	14,209	17	14,226	17	14,243	17	14,260	17	14,277	17	14,294	17	14,311	17	14,328	17	14,345	17	14,362	17	0,0576
0,25	14,801	18	14,819	17	14,836	18	14,854	18	14,872	18	14,890	18	14,908	17	14,925	18	14,943	18	14,961	17	0,0625
0,26	15,393	18	15,411	19	15,430	18	15,448	19	15,467	18	15,485	19	15,504	18	15,522	18	15,540	19	15,559	18	0,0676
0,27	15,985	19	16,004	19	16,023	19	16,042	19	16,061	20	16,081	19	16,100	19	16,119	19	16,138	19	16,157	19	0,0729
0,28	16,576	20	16,596	20	16,616	20	16,636	20	16,656	20	16,676	20	16,696	20	16,716	19	16,735	20	16,755	20	0,0784
0,29	17,168	21	17,189	20	17,209	21	17,230	21	17,251	20	17,271	21	17,292	20	17,312	21	17,333	20	17,353	21	0,0841
0,30	17,760	21	17,781	22	17,803	21	17,824	21	17,845	22	17,867	21	17,888	21	17,909	21	17,930	22	17,952	21	0,0900
0,31	18,352	22	18,374	22	18,396	22	18,418	22	18,440	22	18,462	22	18,484	22	18,506	22	18,528	22	18,550	22	0,0961
0,32	18,943	23	18,966	23	18,989	23	19,012	22	19,034	23	19,057	23	19,080	23	19,103	22	19,125	23	19,148	22	0,1024
0,33	19,535	23	19,558	24	19,582	23	19,605	24	19,629	23	19,652	24	19,676	23	19,699	23	19,722	24	19,746	23	0,1089
0,34	20,127	24	20,151	24	20,175	24	20,199	24	20,223	25	20,248	24	20,272	24	20,296	24	20,320	24	20,344	24	0,1156
0,35	20,718	25	20,743	25	20,768	25	20,793	25	20,818	25	20,843	25	20,868	24	20,892	25	20,917	25	20,942	25	0,1225
0,36	21,310	25	21,335	26	21,361	26	21,387	25	21,412	26	21,438	25	21,463	26	21,489	25	21,514	26	21,540	25	0,1296
0,37	21,901	27	21,928	26	21,954	26	21,980	27	22,007	26	22,033	26	22,059	26	22,085	27	22,112	26	22,138	26	0,1369
0,38	22,493	27	22,520	27	22,547	27	22,574	27	22,601	27	22,628	27	22,655	27	22,682	27	22,709	27	22,736	27	0,1444
0,39	23,084	28	23,112	28	23,140	28	23,168	27	23,195	28	23,223	28	23,251	27	23,278	28	23,306	28	23,334	27	0,1521
0,40	23,676	28	23,704	29	23,733	28	23,761	29	23,790	28	23,818	29	23,847	28	23,875	28	23,903	29	23,932	28	0,1600
0,41	24,267	29	24,296	30	24,326	29	24,355	29	24,384	29	24,413	29	24,442	29	24,471	29	24,500	29	24,529	29	0,1681
0,42	24,859	30	24,889	29	24,918	30	24,948	30	24,978	30	25,008	30	25,038	30	25,068	29	25,097	30	25,127	30	0,1764
0,43	25,450	31	25,481	30	25,511	31	25,542	30	25,572	31	25,603	31	25,634	30	25,664	30	25,694	31	25,725	30	0,1849
0,44	26,041	32	26,073	31	26,104	31	26,135	32	26,167	31	26,198	31	26,229	31	26,260	31	26,291	32	26,323	31	0,1936
0,45	26,632	33	26,665	32	26,697	32	26,729	32	26,761	32	26,793	32	26,825	32	26,857	31	26,888	32	26,920	32	0,2025
0,50	29,588	36	29,624	36	29,660	35	29,695	36	29,731	35	29,766	36	29,802	35	29,837	36	29,873	35	29,908	35	0,2500
0,55	32,543	39	32,582	39	32,621	40	32,661	39	32,700	39	32,739	39	32,778	39	32,817	39	32,856	39	32,895	39	0,3025
0,60	35,496	43	35,539	43	35,582	43	35,625	43	35,668	42	35,710	43	35,753	42	35,795	43	35,838	42	35,880	43	0,3600
0,65	38,448	47	38,495	46	38,541	47	38,588	46	38,634	46	38,680	47	38,727	46	38,773	46	38,819	46	38,865	46	0,4225
0,70	41,399	50	41,449	50	41,499	50	41,549	50	41,599	50	41,649	50	41,699	50	41,749	49	41,798	50	41,848	49	0,4900
0,75	44,348	54	44,402	54	44,456	53	44,509	54	44,563	53	44,616	54	44,670	53	44,723	53	44,776	53	44,829	53	0,5625
0,80	47,296	57	47,353	58	47,411	57	47,468	57	47,525	57	47,582	57	47,639	57	47,696	56	47,752	57	47,809	57	0,6400
0,85	50,242	61	50,303	61	50,364	60	50,424	61	50,485	61	50,546	60	50,606	61	50,667	60	50,727	60	50,787	61	0,7225
0,90	53,186	64	53,250	65	53,315	64	53,379	64	53,443	65	53,508	64	53,572	64	53,636	64	53,700	64	53,764	64	0,8100
0,95	56,128	68	56,196	68	56,264	68	56,332	68	56,400	68	56,468	68	56,536	67	56,603	68	56,671	67	56,738	68	0,9025
1,00	59,067	72	59,139	72	59,211	72	59,283	71	59,354	72	59,426	71	59,497	71	59,568	72	59,640	71	59,711	71	1,0000
	8,6113		8,6528		8,6945		8,7362		8,7781		8,8200		8,8621		8,9042		8,9465		8,9888		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	591	592	593	594	595	596	597	598	599	
1	59	59	59	59	60	60	60	60	60	1
2	118	118	119	119	119	119	119	120	120	2
3	177	178	178	178	179	179	179	179	180	3
4	236	237	237	238	238	238	239	239	240	4
5	296	296	297	297	298	298	299	299	300	5
6	355	355	356	356	357	358	358	359	359	6
7	414	414	415	416	417	417	418	419	419	7
8	473	474	474	475	476	477	478	478	479	8
9	532	533	534	535	536	536	537	538	539	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord c.	4,25 Days	dif.	4,26 Days	dif.	4,27 Days	dif.	4,28 Days	dif.	4,29 Days	dif.	4,30 Days	dif.	
	Sum of the Radii $r+r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,599	1	0,600	1	0,601	0	0,601	1	0,602	1	0,603	0	0,0001
0,02	1,198	2	1,200	1	1,201	2	1,203	1	1,204	1	1,205	2	0,0004
0,03	1,798	2	1,800	2	1,802	2	1,804	2	1,806	2	1,808	2	0,0009
0,04	2,397	3	2,400	2	2,402	3	2,405	3	2,408	3	2,411	3	0,0016
0,05	2,996	4	3,000	3	3,003	4	3,007	3	3,010	4	3,014	3	0,0025
0,06	3,595	4	3,599	5	3,604	4	3,608	4	3,612	4	3,616	5	0,0036
0,07	4,194	5	4,199	5	4,204	5	4,209	5	4,214	5	4,219	5	0,0049
0,08	4,794	5	4,799	6	4,805	6	4,811	5	4,816	6	4,822	5	0,0064
0,09	5,393	6	5,399	7	5,406	6	5,412	6	5,418	6	5,424	7	0,0081
0,10	5,992	7	5,999	7	6,006	7	6,013	7	6,020	7	6,027	7	0,0100
0,11	6,591	8	6,599	8	6,607	7	6,614	8	6,622	8	6,630	8	0,0121
0,12	7,190	9	7,199	8	7,207	9	7,216	8	7,224	9	7,233	8	0,0144
0,13	7,789	10	7,799	9	7,808	9	7,817	9	7,826	9	7,835	9	0,0169
0,14	8,389	9	8,398	10	8,408	10	8,418	10	8,428	10	8,438	10	0,0196
0,15	8,988	10	8,998	11	9,009	10	9,019	11	9,030	10	9,040	11	0,0225
0,16	9,587	11	9,598	11	9,609	12	9,621	11	9,632	11	9,643	11	0,0256
0,17	10,186	12	10,198	12	10,210	12	10,222	12	10,234	12	10,246	12	0,0289
0,18	10,785	13	10,798	12	10,810	13	10,823	13	10,836	12	10,848	13	0,0324
0,19	11,384	14	11,398	13	11,411	13	11,424	14	11,438	13	11,451	13	0,0361
0,20	11,983	14	11,997	14	12,011	14	12,025	14	12,039	14	12,053	15	0,0400
0,21	12,582	15	12,597	15	12,612	15	12,627	14	12,641	15	12,656	15	0,0441
0,22	13,181	16	13,197	15	13,212	16	13,228	15	13,243	16	13,259	15	0,0484
0,23	13,780	16	13,796	17	13,813	16	13,829	16	13,845	16	13,861	16	0,0529
0,24	14,379	17	14,396	17	14,413	17	14,430	17	14,447	17	14,464	16	0,0576
0,25	14,978	18	14,996	17	15,013	18	15,031	18	15,049	17	15,066	18	0,0625
0,26	15,577	18	15,595	19	15,614	18	15,632	18	15,650	19	15,669	18	0,0676
0,27	16,176	19	16,195	19	16,214	19	16,233	19	16,252	19	16,271	19	0,0729
0,28	16,775	20	16,795	19	16,814	20	16,834	20	16,854	19	16,873	20	0,0784
0,29	17,374	20	17,394	21	17,415	20	17,435	20	17,455	21	17,476	20	0,0841
0,30	17,973	21	17,994	21	18,015	21	18,036	21	18,057	21	18,078	21	0,0900
0,31	18,572	21	18,593	22	18,615	22	18,637	22	18,659	22	18,681	21	0,0961
0,32	19,170	23	19,193	22	19,215	23	19,238	22	19,260	23	19,283	22	0,1024
0,33	19,769	23	19,792	24	19,816	23	19,839	23	19,862	23	19,885	23	0,1089
0,34	20,368	24	20,392	24	20,416	24	20,440	24	20,464	23	20,487	24	0,1156
0,35	20,967	24	20,991	25	21,016	25	21,041	24	21,065	25	21,090	24	0,1225
0,36	21,565	26	21,591	25	21,616	25	21,641	26	21,667	25	21,692	25	0,1296
0,37	22,164	26	22,190	26	22,216	26	22,242	26	22,268	26	22,294	26	0,1369
0,38	22,763	26	22,789	27	22,816	27	22,843	27	22,870	26	22,896	27	0,1444
0,39	23,361	28	23,389	27	23,416	28	23,444	27	23,471	27	23,498	28	0,1521
0,40	23,960	28	23,988	28	24,016	28	24,044	28	24,072	28	24,100	28	0,1600
0,41	24,558	29	24,587	29	24,616	29	24,645	29	24,674	29	24,703	28	0,1681
0,42	25,157	29	25,186	30	25,216	30	25,246	29	25,275	30	25,305	29	0,1764
0,43	25,755	31	25,786	30	25,816	30	25,846	30	25,876	31	25,907	30	0,1849
0,44	26,354	31	26,385	31	26,416	31	26,447	31	26,478	30	26,508	31	0,1936
0,45	26,952	32	26,984	31	27,015	32	27,047	32	27,079	31	27,110	32	0,2025
0,50	29,943	36	29,979	35	30,014	35	30,049	35	30,084	35	30,119	36	0,2500
0,55	32,934	39	32,973	38	33,011	39	33,050	39	33,089	38	33,127	39	0,3025
0,60	35,923	42	35,965	43	36,008	42	36,050	42	36,092	42	36,134	42	0,3600
0,65	38,911	46	38,957	46	39,003	45	39,048	46	39,094	46	39,140	46	0,4225
0,70	41,897	50	41,947	49	41,996	50	42,046	49	42,095	49	42,144	49	0,4900
0,75	44,882	54	44,936	52	44,988	53	45,041	53	45,094	53	45,147	53	0,5625
0,80	47,866	57	47,923	56	47,979	57	48,036	56	48,092	56	48,148	57	0,6400
0,85	50,848	60	50,908	60	50,968	60	51,028	60	51,088	60	51,148	60	0,7225
0,90	53,828	63	53,891	64	53,955	64	54,019	63	54,082	64	54,146	63	0,8100
0,95	56,806	67	56,873	67	56,940	68	57,008	67	57,075	67	57,142	67	0,9025
1,00	59,782	71	59,853	71	59,924	70	59,994	71	60,065	71	60,136	70	1,0000
	9,0313		9,0738		9,1165		9,1592		9,2021		9,2450		c^2
	$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.												

	598	599	600	601	602	603	
1	60	60	60	60	60	60	1
2	120	120	120	120	120	121	2
3	179	180	180	180	181	181	3
4	239	240	240	240	241	241	4
5	299	300	300	301	301	302	5
6	359	359	360	361	361	362	6
7	419	419	420	421	421	422	7
8	478	479	480	481	482	482	8
9	538	539	540	541	542	543	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord	4,31		4,32		4,33		4,34		4,35		4,36		4,37		4,38		4,39		4,40		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,603	1	0,604	1	0,605	1	0,606	0	0,606	1	0,607	1	0,608	0	0,608	1	0,609	1	0,610	0	0,0001
0,02	1,207	1	1,208	2	1,210	1	1,211	1	1,212	2	1,214	1	1,215	2	1,217	1	1,218	1	1,219	2	0,0004
0,03	1,810	2	1,812	2	1,814	3	1,817	2	1,819	2	1,821	2	1,823	2	1,825	2	1,827	2	1,829	2	0,0009
0,04	2,414	2	2,416	3	2,419	3	2,422	3	2,425	3	2,428	2	2,430	3	2,433	3	2,436	3	2,439	3	0,0016
0,05	3,017	4	3,021	3	3,024	4	3,028	3	3,031	4	3,035	3	3,038	4	3,042	3	3,045	3	3,048	4	0,0025
0,06	3,621	4	3,625	4	3,629	4	3,633	4	3,637	4	3,641	5	3,646	4	3,650	4	3,654	4	3,658	4	0,0036
0,07	4,224	5	4,229	5	4,234	5	4,239	5	4,244	4	4,248	5	4,253	5	4,258	5	4,263	5	4,268	5	0,0049
0,08	4,827	6	4,833	6	4,839	5	4,844	6	4,850	5	4,855	6	4,861	5	4,866	6	4,872	6	4,878	5	0,0064
0,09	5,431	6	5,437	6	5,443	7	5,450	6	5,456	6	5,462	6	5,468	7	5,475	6	5,481	6	5,487	6	0,0081
0,10	6,034	7	6,041	7	6,048	7	6,055	7	6,062	7	6,069	7	6,076	7	6,083	7	6,090	7	6,097	7	0,0100
0,11	6,638	7	6,645	8	6,653	8	6,661	7	6,668	8	6,676	8	6,684	7	6,691	8	6,699	7	6,706	8	0,0121
0,12	7,241	8	7,249	9	7,258	8	7,266	8	7,274	9	7,283	8	7,291	8	7,299	9	7,308	8	7,316	8	0,0144
0,13	7,844	9	7,853	9	7,862	10	7,872	9	7,881	9	7,890	9	7,899	9	7,908	9	7,917	9	7,926	9	0,0169
0,14	8,448	9	8,457	10	8,467	10	8,477	10	8,487	9	8,496	10	8,506	10	8,516	10	8,526	9	8,535	10	0,0196
0,15	9,051	10	9,061	11	9,072	10	9,082	11	9,093	10	9,103	11	9,114	10	9,124	11	9,135	10	9,145	10	0,0225
0,16	9,654	12	9,666	11	9,677	11	9,688	11	9,699	11	9,710	11	9,721	11	9,732	12	9,744	11	9,755	11	0,0256
0,17	10,258	12	10,270	11	10,281	12	10,293	12	10,305	12	10,317	12	10,329	12	10,341	11	10,352	12	10,364	12	0,0289
0,18	10,861	13	10,874	12	10,886	13	10,899	12	10,911	13	10,924	12	10,936	13	10,949	12	10,961	13	10,974	12	0,0324
0,19	11,464	14	11,478	13	11,491	13	11,504	13	11,517	14	11,531	13	11,544	13	11,557	13	11,570	13	11,583	13	0,0361
0,20	12,068	14	12,082	13	12,095	14	12,109	14	12,123	14	12,137	14	12,151	14	12,165	14	12,179	14	12,193	14	0,0400
0,21	12,671	14	12,685	15	12,700	15	12,715	14	12,729	15	12,744	15	12,759	14	12,773	15	12,788	14	12,802	15	0,0441
0,22	13,274	15	13,289	16	13,305	15	13,320	15	13,335	16	13,351	15	13,366	15	13,381	16	13,397	15	13,412	15	0,0484
0,23	13,877	16	13,893	16	13,909	16	13,925	17	13,942	16	13,958	16	13,974	16	13,990	16	14,006	15	14,021	16	0,0529
0,24	14,480	17	14,497	17	14,514	17	14,531	17	14,548	16	14,564	17	14,581	17	14,598	16	14,614	17	14,631	17	0,0576
0,25	15,084	17	15,101	18	15,119	17	15,136	17	15,153	18	15,171	17	15,188	18	15,206	17	15,223	17	15,240	18	0,0625
0,26	15,687	18	15,705	18	15,723	18	15,741	18	15,759	19	15,778	18	15,796	18	15,814	18	15,832	18	15,850	18	0,0676
0,27	16,290	19	16,309	19	16,328	19	16,347	18	16,365	19	16,384	19	16,403	19	16,422	19	16,441	18	16,459	19	0,0729
0,28	16,893	20	16,913	19	16,932	20	16,952	19	16,971	20	16,991	19	17,010	20	17,030	19	17,049	20	17,069	19	0,0784
0,29	17,496	20	17,516	21	17,537	20	17,557	20	17,577	20	17,597	21	17,618	20	17,638	20	17,658	20	17,678	20	0,0841
0,30	18,099	21	18,120	21	18,141	21	18,162	21	18,183	21	18,204	21	18,225	21	18,246	21	18,267	20	18,287	21	0,0900
0,31	18,702	22	18,724	22	18,746	21	18,767	22	18,789	22	18,811	21	18,832	22	18,854	21	18,875	22	18,897	21	0,0961
0,32	19,305	23	19,328	22	19,350	22	19,372	23	19,395	22	19,417	22	19,439	23	19,462	22	19,484	22	19,506	22	0,1024
0,33	19,908	23	19,931	23	19,954	24	19,978	23	20,001	23	20,024	23	20,047	22	20,069	23	20,092	23	20,115	23	0,1089
0,34	20,511	24	20,535	24	20,559	24	20,583	23	20,606	24	20,630	24	20,654	23	20,677	24	20,701	24	20,725	23	0,1156
0,35	21,114	25	21,139	24	21,163	25	21,188	24	21,212	24	21,236	25	21,261	24	21,285	24	21,309	25	21,334	24	0,1225
0,36	21,717	25	21,742	26	21,768	25	21,793	25	21,818	25	21,843	25	21,868	25	21,893	25	21,918	25	21,943	25	0,1296
0,37	22,320	26	22,346	26	22,372	26	22,398	25	22,423	26	22,449	26	22,475	26	22,501	25	22,526	26	22,552	26	0,1369
0,38	22,923	27	22,950	26	22,976	27	23,003	26	23,029	27	23,056	26	23,082	27	23,109	26	23,135	26	23,161	27	0,1444
0,39	23,526	27	23,553	27	23,580	28	23,608	27	23,635	27	23,662	27	23,689	27	23,716	27	23,743	27	23,770	27	0,1521
0,40	24,128	29	24,157	27	24,184	28	24,212	28	24,240	28	24,268	28	24,296	28	24,324	28	24,352	27	24,379	28	0,1600
0,41	24,731	29	24,760	29	24,789	28	24,817	29	24,846	29	24,875	28	24,903	29	24,932	28	24,960	29	24,989	28	0,1681
0,42	25,334	29	25,363	30	25,393	29	25,422	29	25,451	30	25,481	29	25,510	29	25,539	29	25,568	30	25,598	29	0,1764
0,43	25,937	30	25,967	30	25,997	30	26,027	30	26,057	30	26,087	30	26,117	30	26,147	30	26,177	30	26,207	29	0,1849
0,44	26,539	31	26,570	31	26,601	31	26,632	30	26,662	31	26,693	31	26,724	30	26,754	31	26,785	30	26,815	31	0,1936
0,45	27,142	31	27,173	32	27,205	31	27,236	32	27,268	31	27,299	32	27,331	31	27,362	31	27,393	31	27,424	32	0,2025
0,50	30,155	35	30,190	35	30,225	35	30,260	34	30,294	35	30,329	35	30,364	35	30,399	35	30,434	34	30,468	35	0,2500
0,55	33,166	39	33,205	38	33,243	39	33,282	38	33,320	38	33,358	39	33,397	38	33,435	38	33,473	38	33,511	39	0,3025
0,60	36,176	43	36,219	42	36,261	42	36,303	41	36,344	42	36,386	42	36,428	42	36,470	42	36,512	41	36,553	42	0,3600
0,65	39,186	45	39,231	46	39,277	45	39,322	46	39,368	45	39,413	45	39,458	46	39,504	45	39,549	45	39,594	45	0,4225
0,70	42,193	50	42,243	49	42,292	49	42,341	49	42,390	49	42,439	48	42,487	49	42,536	49	42,585	49	42,634	48	0,4900
0,75	45,200	52	45,252	53	45,305	53	45,358	52	45,410	53	45,463	52	45,515	52	45,567	52	45,619	53	45,672	52	0,5625
0,80	48,205	56	48,261	56	48,317	56	48,373	56	48,429	56	48,485	56	48,541	56	48,597	56	48,653	55	48,708	56	0,6400
0,85	51,208	60	51,268	59	51,327	60	51,387	59	51,446	60	51,506	59	51,565	60	51,625	59	51,684	59	51,743	59	0,7225
0,90	54,209	63	54,272	64	54,336	63	54,399	63	54,462	63	54,525	63	54,588	63	54,651	63	54,714	62	54,776	63	0,8100
0,95	57,209	67	57,276	66	57,342	67	57,409	67	57,476	66	57,542	67	57,609	66	57,675	67	57,742	66	57,808	66	0,9025
1,00	60,206	71	60,277	70	60,347	70	60,417	70	60,487	71	60,558	70	60,628	70	60,698	69	60,767	70	60,837	70	1,0000
	9,2881		9,3312		9,3745		9,4178		9,4613		9,5048		9,5185		9,5922		9,6361		9,6800		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	602	603	604	605	606	607	608	609	610	
1	60	60	60	61	61	61	61	61	61	1
2	120	121	121	121	121	121	122	122	122	2
3	181	181	181	182	182	182	182	183	183	3
4	241	241	242	242	242	243	243	244	244	4
5	301	302	302	303	303	304	304	305	305	5
6	361	362	362	363	364	364	365	365	366	6
7	421	422	423	424	424	425	426	426	427	7
8	482	482	483	484	485	486	486	487	488	8
9	542	543	544	545	545	546	547	548	549	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord c.	4,41 Days	dif.	4,42 Days	dif.	4,43 Days	dif.	4,44 Days	dif.	4,45 Days	dif.	4,46 Days	dif.	
	Sum of the Radii $r+r''$.												
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,610	1	0,611	1	0,612	0	0,612	1	0,613	1	0,614	1	0,0001
0,02	1,221	1	1,222	2	1,224	1	1,225	1	1,226	2	1,228	1	0,0004
0,03	1,831	2	1,833	2	1,835	2	1,837	2	1,839	3	1,842	2	0,0009
0,04	2,442	2	2,444	3	2,447	3	2,450	3	2,453	2	2,455	3	0,0016
0,05	3,052	3	3,055	4	3,059	3	3,062	4	3,066	3	3,069	4	0,0025
0,06	3,662	4	3,666	5	3,671	4	3,675	4	3,679	4	3,683	4	0,0036
0,07	4,273	5	4,278	4	4,282	5	4,287	5	4,292	5	4,297	5	0,0049
0,08	4,883	6	4,889	5	4,894	6	4,900	5	4,905	6	4,911	5	0,0064
0,09	5,493	7	5,500	6	5,506	6	5,512	6	5,518	6	5,524	7	0,0081
0,10	6,104	7	6,111	7	6,118	7	6,125	6	6,131	7	6,138	7	0,0100
0,11	6,714	8	6,722	7	6,729	8	6,737	7	6,744	8	6,752	8	0,0121
0,12	7,324	9	7,333	8	7,341	8	7,349	9	7,358	8	7,366	8	0,0144
0,13	7,935	9	7,944	9	7,953	9	7,962	9	7,971	9	7,980	9	0,0169
0,14	8,545	10	8,555	9	8,564	10	8,574	10	8,584	9	8,593	10	0,0196
0,15	9,155	11	9,166	10	9,176	11	9,187	10	9,197	10	9,207	11	0,0225
0,16	9,766	11	9,777	11	9,788	11	9,799	11	9,810	11	9,821	11	0,0256
0,17	10,376	12	10,388	11	10,399	12	10,411	12	10,423	12	10,435	11	0,0289
0,18	10,986	13	10,999	12	11,011	13	11,024	12	11,036	12	11,048	13	0,0324
0,19	11,596	14	11,610	13	11,623	13	11,636	13	11,649	13	11,662	13	0,0361
0,20	12,207	14	12,221	13	12,234	14	12,248	14	12,262	14	12,276	14	0,0400
0,21	12,817	15	12,832	14	12,846	15	12,861	14	12,875	14	12,889	15	0,0441
0,22	13,427	15	13,442	16	13,458	15	13,473	15	13,488	15	13,503	15	0,0484
0,23	14,037	16	14,053	16	14,069	16	14,085	16	14,101	16	14,117	16	0,0529
0,24	14,648	16	14,664	17	14,681	16	14,697	17	14,714	16	14,730	17	0,0576
0,25	15,258	17	15,275	17	15,292	18	15,310	17	15,327	17	15,344	17	0,0625
0,26	15,868	18	15,886	18	15,904	18	15,922	18	15,940	18	15,958	17	0,0676
0,27	16,478	19	16,497	18	16,515	19	16,534	19	16,553	18	16,571	19	0,0729
0,28	17,088	19	17,107	20	17,127	19	17,146	19	17,165	20	17,185	19	0,0784
0,29	17,698	20	17,718	20	17,738	20	17,758	20	17,778	20	17,798	20	0,0841
0,30	18,308	21	18,329	21	18,350	20	18,370	21	18,391	21	18,412	20	0,0900
0,31	18,918	22	18,940	21	18,961	21	18,982	22	19,004	21	19,025	22	0,0961
0,32	19,528	22	19,550	22	19,572	23	19,595	22	19,617	22	19,639	22	0,1024
0,33	20,138	23	20,161	23	20,184	23	20,207	22	20,229	23	20,252	23	0,1089
0,34	20,748	24	20,772	23	20,795	24	20,819	23	20,842	24	20,866	23	0,1156
0,35	21,358	24	21,382	24	21,406	25	21,431	24	21,455	24	21,479	24	0,1225
0,36	21,968	25	21,993	25	22,018	25	22,043	24	22,067	25	22,092	25	0,1296
0,37	22,578	25	22,603	26	22,629	26	22,655	25	22,680	26	22,706	25	0,1369
0,38	23,188	26	23,214	26	23,240	26	23,266	27	23,293	26	23,319	26	0,1444
0,39	23,797	27	23,824	27	23,851	27	23,878	27	23,905	27	23,932	27	0,1521
0,40	24,407	28	24,435	28	24,463	27	24,490	28	24,518	27	24,545	28	0,1600
0,41	25,017	28	25,045	29	25,074	28	25,102	28	25,130	29	25,159	28	0,1681
0,42	25,627	29	25,656	29	25,685	29	25,714	29	25,743	29	25,772	29	0,1764
0,43	26,236	30	26,266	30	26,296	30	26,326	29	26,355	30	26,385	30	0,1849
0,44	26,846	30	26,876	31	26,907	30	26,937	31	26,968	30	26,998	30	0,1936
0,45	27,456	31	27,487	31	27,518	31	27,549	31	27,580	31	27,611	31	0,2025
0,50	30,503	35	30,538	34	30,572	35	30,607	34	30,641	35	30,676	34	0,2500
0,55	33,550	38	33,588	38	33,626	38	33,664	38	33,702	38	33,740	38	0,3025
0,60	36,595	42	36,637	41	36,678	42	36,720	41	36,761	42	36,803	41	0,3600
0,65	39,639	45	39,684	45	39,729	45	39,774	45	39,819	45	39,864	45	0,4225
0,70	42,682	49	42,731	48	42,779	49	42,828	48	42,876	49	42,925	48	0,4900
0,75	45,724	52	45,776	52	45,828	52	45,880	52	45,932	51	45,983	52	0,5625
0,80	48,764	55	48,819	56	48,875	55	48,930	56	48,986	55	49,041	55	0,6400
0,85	51,802	59	51,861	59	51,920	59	51,979	59	52,038	59	52,097	59	0,7225
0,90	54,839	63	54,902	62	54,964	63	55,027	62	55,089	62	55,151	62	0,8100
0,95	57,874	66	57,940	66	58,006	66	58,072	66	58,138	66	58,204	65	0,9025
1,00	60,907	70	60,977	69	61,046	70	61,116	69	61,185	69	61,254	69	1,0000
	9,7241		9,7682		9,8125		9,8568		9,9013		9,9458		c^2

$\frac{1}{2} \cdot (r+r'')^2$ or $r^2+r''^2$ nearly.

	609	610	611	612	613	614	
1	61	61	61	61	61	61	1
2	122	122	122	122	123	123	2
3	183	183	183	184	184	184	3
4	244	244	244	245	245	246	4
5	305	305	306	306	307	307	5
6	365	366	367	367	368	368	6
7	426	427	428	428	429	430	7
8	487	488	489	490	490	491	8
9	548	549	550	551	552	553	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	4,47 Days	dif.	4,48 Days	dif.	4,49 Days	dif.	4,50 Days	dif.	4,51 Days	dif.	4,52 Days	dif.	4,53 Days	dif.	4,54 Days	dif.	4,55 Days	dif.	4,56 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,615	0	0,615	1	0,616	1	0,617	0	0,617	1	0,618	1	0,619	0	0,619	1	0,620	1	0,621	0	0,0001
0,02	1,229	1	1,230	2	1,232	1	1,233	2	1,235	1	1,236	1	1,237	2	1,239	1	1,240	1	1,241	2	0,0004
0,03	1,844	2	1,846	2	1,848	2	1,850	2	1,852	2	1,854	2	1,856	2	1,858	2	1,860	2	1,862	2	0,0009
0,04	2,458	3	2,461	3	2,464	2	2,466	3	2,469	3	2,472	3	2,475	2	2,477	3	2,480	3	2,483	2	0,0016
0,05	3,073	3	3,076	4	3,080	3	3,083	3	3,086	4	3,090	3	3,093	4	3,097	3	3,100	3	3,103	4	0,0025
0,06	3,687	4	3,691	4	3,695	4	3,699	5	3,704	4	3,708	4	3,712	4	3,716	4	3,720	4	3,724	4	0,0036
0,07	4,302	4	4,306	5	4,311	5	4,316	5	4,321	5	4,326	4	4,330	5	4,335	5	4,340	5	4,345	5	0,0049
0,08	4,916	6	4,922	5	4,927	6	4,933	5	4,938	6	4,944	5	4,949	6	4,955	5	4,960	5	4,965	6	0,0064
0,09	5,531	6	5,537	6	5,543	6	5,549	6	5,555	7	5,562	6	5,568	6	5,574	6	5,580	6	5,586	6	0,0081
0,10	6,145	7	6,152	7	6,159	7	6,166	7	6,173	6	6,179	7	6,186	7	6,193	7	6,200	7	6,207	7	0,0100
0,11	6,760	7	6,767	8	6,775	7	6,782	8	6,790	7	6,797	8	6,805	7	6,812	8	6,820	7	6,827	8	0,0121
0,12	7,374	8	7,382	9	7,391	8	7,399	8	7,407	8	7,415	8	7,423	9	7,432	8	7,440	8	7,448	8	0,0144
0,13	7,989	9	7,998	8	8,006	9	8,015	9	8,024	9	8,033	9	8,042	9	8,051	9	8,060	9	8,069	8	0,0169
0,14	8,603	10	8,613	9	8,622	10	8,632	9	8,641	10	8,651	10	8,661	9	8,670	10	8,680	9	8,689	10	0,0196
0,15	9,218	10	9,228	10	9,238	10	9,248	11	9,259	10	9,269	10	9,279	10	9,289	11	9,300	10	9,310	10	0,0225
0,16	9,832	11	9,843	11	9,854	11	9,865	11	9,876	11	9,887	11	9,898	11	9,909	11	9,920	10	9,930	11	0,0256
0,17	10,446	12	10,458	12	10,470	11	10,481	12	10,493	12	10,505	11	10,516	12	10,528	11	10,539	12	10,551	12	0,0289
0,18	11,061	12	11,073	12	11,085	13	11,098	12	11,110	12	11,122	13	11,135	12	11,147	12	11,159	13	11,172	12	0,0324
0,19	11,675	13	11,688	13	11,701	13	11,714	13	11,727	13	11,740	13	11,753	13	11,766	13	11,779	13	11,792	13	0,0361
0,20	12,290	13	12,303	14	12,317	14	12,331	13	12,344	14	12,358	14	12,372	13	12,385	14	12,399	14	12,413	13	0,0400
0,21	12,904	14	12,918	15	12,933	14	12,947	15	12,962	14	12,976	14	12,990	15	13,005	14	13,019	14	13,033	15	0,0441
0,22	13,518	15	13,533	15	13,548	16	13,564	15	13,579	15	13,594	15	13,609	15	13,624	15	13,639	15	13,654	15	0,0484
0,23	14,133	15	14,148	16	14,164	16	14,180	16	14,196	15	14,211	16	14,227	16	14,243	16	14,259	15	14,274	16	0,0529
0,24	14,747	16	14,763	17	14,780	16	14,796	17	14,813	16	14,829	17	14,846	16	14,862	16	14,878	17	14,895	16	0,0576
0,25	15,361	17	15,378	18	15,396	17	15,413	17	15,430	17	15,447	17	15,464	17	15,481	17	15,498	17	15,515	17	0,0625
0,26	15,975	18	15,993	18	16,011	18	16,029	18	16,047	18	16,065	17	16,082	18	16,100	18	16,118	18	16,136	17	0,0676
0,27	16,590	18	16,608	19	16,627	18	16,645	19	16,664	18	16,682	19	16,701	18	16,719	19	16,738	18	16,756	18	0,0729
0,28	17,204	19	17,223	19	17,242	20	17,262	19	17,281	19	17,300	19	17,319	19	17,338	19	17,357	19	17,376	19	0,0784
0,29	17,818	20	17,838	20	17,858	20	17,878	20	17,898	20	17,918	19	17,937	20	17,957	20	17,977	20	17,997	20	0,0841
0,30	18,432	21	18,453	21	18,474	20	18,494	21	18,515	20	18,535	21	18,556	20	18,576	21	18,597	20	18,617	21	0,0900
0,31	19,047	21	19,068	21	19,089	21	19,110	22	19,132	21	19,153	21	19,174	21	19,195	21	19,216	22	19,238	21	0,0961
0,32	19,661	22	19,683	22	19,705	22	19,727	22	19,749	21	19,770	22	19,792	22	19,814	22	19,836	22	19,858	22	0,1024
0,33	20,275	23	20,298	22	20,320	23	20,343	22	20,365	23	20,388	23	20,411	22	20,433	23	20,456	22	20,478	23	0,1089
0,34	20,889	23	21,912	24	20,936	23	20,959	23	20,982	24	21,006	23	21,029	23	21,052	23	21,075	23	21,098	23	0,1156
0,35	21,503	24	21,527	24	21,551	24	21,575	24	21,599	24	21,623	24	21,647	24	21,671	24	21,695	24	21,719	23	0,1225
0,36	22,117	25	22,142	24	22,166	25	22,191	25	22,216	25	22,241	24	22,265	25	22,290	24	22,314	25	22,339	24	0,1296
0,37	22,731	25	22,756	26	22,782	25	22,807	26	22,833	25	22,858	25	22,883	26	22,909	25	22,934	25	22,959	25	0,1369
0,38	23,345	26	23,371	26	23,397	26	23,423	26	23,440	26	23,475	26	23,501	26	23,527	26	23,553	26	23,579	26	0,1444
0,39	23,959	27	23,986	27	24,013	26	24,039	27	24,060	27	24,093	26	24,119	27	24,146	27	24,173	26	24,199	27	0,1521
0,40	24,573	27	24,600	28	24,628	27	24,655	28	24,683	27	24,710	27	24,737	28	24,765	27	24,792	27	24,819	28	0,1600
0,41	25,187	28	25,215	28	25,243	28	25,271	28	25,299	28	25,327	29	25,356	28	25,384	28	25,412	27	25,439	28	0,1681
0,42	25,801	29	25,830	28	25,858	29	25,887	29	25,916	29	25,945	29	25,974	28	26,002	29	26,031	29	26,060	28	0,1764
0,43	26,415	29	26,444	30	26,474	29	26,503	30	26,533	29	26,562	29	26,591	30	26,621	29	26,650	29	26,679	30	0,1849
0,44	27,028	31	27,059	30	27,089	30	27,119	30	27,149	30	27,179	30	27,209	30	27,239	30	27,269	30	27,299	30	0,1936
0,45	27,642	31	27,673	31	27,704	31	27,735	31	27,766	31	27,797	30	27,827	31	27,858	31	27,889	30	27,919	31	0,2025
0,50	30,710	35	30,745	34	30,779	34	30,813	35	30,848	34	30,882	34	30,916	34	30,950	35	30,985	34	31,019	34	0,2500
0,55	33,778	38	33,816	37	33,853	38	33,891	38	33,929	38	33,967	37	34,004	38	34,042	37	34,079	38	34,117	37	0,3025
0,60	36,844	41	36,885	41	36,926	42	36,968	41	37,009	41	37,050	41	37,091	41	37,132	41	37,173	41	37,214	41	0,3600
0,65	39,909	45	39,954	45	39,999	44	40,043	45	40,088	44	40,132	45	40,177	44	40,221	45	40,266	44	40,310	44	0,4225
0,70	42,973	48	43,021	48	43,069	48	43,117	48	43,165	48	43,213	48	43,261	48	43,309	48	43,357	48	43,405	48	0,4900
0,75	46,035	52	46,087	52	46,139	51	46,190	52	46,242	51	46,293	52	46,345	51	46,396	51	46,447	52	46,499	51	0,5625
0,80	49,096	55	49,151	56	49,207	55	49,262	55	49,317	55	49,372	54	49,426	55	49,481	55	49,536	55	49,591	54	0,6400
0,85	52,156	58	52,214	59	52,273	58	52,331	59	52,300	58	52,448	59	52,507	58	52,565	58	52,623	58	52,681	58	0,7225
0,90	55,213	63	55,276	62	55,338	62	55,400	62	55,462	61	55,523	62	55,585	62	55,647	62	55,709	61	55,770	62	0,8100
0,95	58,269	66	58,335	66	58,401	65	58,466	65	58,531	66	58,597	65	58,662	65	58,727	65	58,792	65	58,857	65	0,9025
1,00	61,323	70	61,393	69	61,462	69	61,531	69	61,600	68	61,668	69	61,737	69	61,806	68	61,874	69	61,943	68	1,0000
	9,9905		10,0352		10,0801		10,1250		10,1701		10,2152		10,2605		10,3058		10,3513		10,3968		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	613	614	615	616	617	618	619	620	621	
1	61	61	62	62	62	62	62	62	62	1
2	123	123	123	123	123	124	124	124	124	2
3	184	184	185	185	185	185	186	186	186	3
4	245	246	246	246	247	247	248	248	248	4
5	307	307	308	308	309	309	310	310	311	5
6	368	368	369	370	370	371	371	372	373	6
7	429	430	431	431	432	433	433	434	435	7
8	490	491	492	493	494	494	495	496	497	8
9	552	553	554	554	555	556	557	558	559	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Chord c.	4,57 Days	dif.	4,58 Days	dif.	4,59 Days	dif.	4,60 Days	dif	4,61 Days	dif.	4,62 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,621	1	0,622	1	0,623	0	0,623	1	0,624	1	0,625	0	0,0001
0,02	1,243	1	1,244	1	1,245	2	1,247	1	1,248	2	1,250	1	0,0004
0,03	1,864	2	1,866	2	1,868	2	1,870	2	1,872	2	1,874	2	0,0009
0,04	2,485	3	2,488	3	2,491	3	2,494	2	2,496	3	2,499	3	0,0016
0,05	3,107	3	3,110	4	3,114	3	3,117	3	3,120	4	3,124	3	0,0025
0,06	3,728	4	3,732	4	3,736	4	3,740	4	3,744	5	3,749	4	0,0036
0,07	4,350	4	4,354	5	4,359	5	4,364	4	4,368	5	4,373	5	0,0049
0,08	4,971	5	4,976	6	4,982	5	4,987	6	4,993	5	4,998	5	0,0064
0,09	5,592	6	5,598	6	5,604	7	5,611	6	5,617	6	5,623	6	0,0081
0,10	6,214	6	6,220	7	6,227	7	6,234	7	6,241	6	6,247	7	0,0100
0,11	6,835	7	6,842	8	6,850	7	6,857	8	6,865	7	6,872	8	0,0121
0,12	7,456	8	7,464	8	7,472	9	7,481	8	7,489	8	7,497	8	0,0144
0,13	8,077	9	8,086	9	8,095	9	8,104	9	8,113	9	8,122	8	0,0169
0,14	8,699	9	8,708	10	8,718	9	8,727	10	8,737	9	8,746	10	0,0196
0,15	9,320	10	9,330	10	9,340	11	9,351	10	9,361	10	9,371	10	0,0225
0,16	9,941	11	9,952	11	9,963	11	9,974	11	9,985	11	9,996	10	0,0256
0,17	10,563	11	10,574	12	10,586	11	10,597	12	10,609	11	10,620	12	0,0289
0,18	11,184	12	11,196	12	11,208	12	11,220	13	11,233	12	11,245	12	0,0324
0,19	11,805	13	11,818	13	11,831	13	11,844	13	11,857	13	11,870	12	0,0361
0,20	12,426	14	12,440	13	12,453	14	12,467	14	12,481	13	12,494	14	0,0400
0,21	13,048	14	13,062	14	13,076	14	13,090	14	13,104	15	13,119	14	0,0441
0,22	13,669	15	13,684	15	13,699	14	13,713	15	13,728	15	13,743	15	0,0484
0,23	14,290	15	14,305	16	14,321	16	14,337	15	14,352	16	14,368	15	0,0529
0,24	14,911	16	14,927	17	14,944	16	14,960	16	14,976	16	14,992	17	0,0576
0,25	15,532	17	15,549	17	15,566	17	15,583	17	15,600	17	15,617	17	0,0625
0,26	16,153	18	16,171	18	16,189	17	16,206	18	16,224	17	16,241	18	0,0676
0,27	16,774	19	16,793	18	16,811	18	16,829	19	16,848	18	16,866	18	0,0729
0,28	17,395	20	17,415	19	17,434	19	17,453	18	17,471	19	17,490	19	0,0784
0,29	18,017	19	18,036	20	18,056	20	18,076	19	18,095	20	18,115	20	0,0841
0,30	18,638	20	18,658	20	18,678	21	18,699	20	18,719	20	18,739	21	0,0900
0,31	19,259	21	19,280	21	19,301	21	19,322	21	19,343	21	19,364	21	0,0961
0,32	19,880	21	19,901	22	19,923	22	19,945	21	19,966	22	19,988	22	0,1024
0,33	20,501	22	20,523	22	20,545	23	20,568	22	20,590	22	20,612	23	0,1089
0,34	21,121	24	21,145	23	21,168	23	21,191	23	21,214	23	21,237	23	0,1156
0,35	21,742	24	21,766	24	21,790	24	21,814	23	21,837	24	21,861	24	0,1225
0,36	22,363	25	22,388	24	22,412	25	22,437	24	22,461	24	22,485	25	0,1296
0,37	22,984	25	23,009	25	23,034	26	23,060	25	23,085	25	23,110	25	0,1369
0,38	23,605	26	23,631	26	23,657	25	23,682	26	23,708	26	23,734	26	0,1444
0,39	24,226	26	24,252	27	24,279	26	24,305	27	24,332	26	24,358	27	0,1521
0,40	24,847	27	24,874	27	24,901	27	24,928	27	24,955	27	24,982	27	0,1600
0,41	25,467	28	25,495	28	25,523	28	25,551	28	25,579	27	25,606	28	0,1681
0,42	26,088	29	26,117	28	26,145	29	26,174	28	26,202	29	26,231	28	0,1764
0,43	26,709	29	26,738	29	26,767	29	26,796	30	26,826	29	26,855	29	0,1849
0,44	27,329	30	27,359	30	27,389	30	27,419	30	27,449	30	27,479	30	0,1936
0,45	27,950	31	27,981	30	28,011	31	28,042	30	28,072	31	28,103	30	0,2025
0,50	31,053	34	31,087	34	31,121	34	31,155	34	31,189	33	31,222	34	0,2500
0,55	34,154	38	34,192	37	34,229	38	34,267	37	34,304	37	34,341	37	0,3025
0,60	37,255	41	37,296	41	37,337	40	37,377	41	37,418	41	37,459	40	0,3600
0,65	40,354	45	40,399	44	40,443	44	40,487	44	40,531	44	40,575	44	0,4225
0,70	43,453	47	43,500	48	43,548	48	43,596	47	43,643	48	43,691	47	0,4900
0,75	46,550	51	46,601	51	46,652	51	46,703	51	46,754	51	46,805	51	0,5625
0,80	49,645	55	49,700	54	49,754	55	49,809	54	49,863	55	49,918	54	0,6400
0,85	52,739	58	52,797	58	52,855	58	52,913	58	52,971	58	53,029	57	0,7225
0,90	55,832	61	55,893	62	55,955	61	56,016	61	56,077	61	56,138	61	0,8100
0,95	58,922	65	58,987	65	59,052	65	59,117	65	59,182	64	59,246	65	0,9025
1,00	62,011	69	62,080	68	62,148	68	62,216	68	62,284	68	62,352	68	1,0000
	10,4125		10,4882		10,5341		10,5800		10,6261		10.6722		c^2

$\frac{1}{2}\cdot(r+r'')^2$ or $r^2+r''^2$ nearly.

	620	621	622	623	624	625	
1	62	62	62	62	62	63	1
2	124	124	124	125	125	125	2
3	186	186	187	187	187	188	3
4	248	248	249	249	250	250	4
5	310	311	311	312	312	313	5
6	372	373	373	374	374	375	6
7	434	435	435	436	437	438	7
8	496	497	498	498	499	500	8
9	558	559	560	561	562	563	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r+r''$. 4,63 Days	dif.	4,64 Days	dif.	4,65 Days	dif.	4,66 Days	dif.	4,67 Days	dif.	4,68 Days	dif.	4,69 Days	dif.	4,70 Days	dif.	4,71 Days	dif.	4,72 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,625	1	0,626	1	0,627	0	0,627	1	0,628	1	0,629	0	0,629	1	0,630	1	0,631	0	0,631	1	0,0001
0,02	1,251	1	1,252	2	1,254	1	1,255	1	1,256	2	1,258	1	1,259	1	1,260	2	1,262	1	1,263	1	0,0004
0,03	1,876	2	1,878	2	1,880	2	1,882	2	1,884	2	1,886	2	1,888	2	1,890	2	1,892	2	1,894	2	0,0009
0,04	2,502	2	2,504	3	2,507	3	2,510	2	2,512	3	2,515	3	2,518	3	2,521	2	2,523	3	2,526	3	0,0016
0,05	3,127	3	3,130	4	3,134	3	3,137	4	3,141	3	3,144	3	3,147	4	3,151	3	3,154	3	3,157	4	0,0025
0,06	3,753	4	3,757	4	3,761	4	3,765	4	3,769	4	3,773	4	3,777	4	3,781	4	3,785	4	3,789	4	0,0036
0,07	4,378	5	4,383	4	4,387	5	4,392	5	4,397	5	4,402	4	4,406	5	4,411	5	4,416	4	4,420	5	0,0049
0,08	5,003	6	5,009	5	5,014	6	5,020	5	5,025	5	5,030	6	5,036	5	5,041	5	5,046	6	5,052	5	0,0064
0,09	5,629	6	5,635	6	5,641	6	5,647	6	5,653	6	5,659	6	5,665	6	5,671	6	5,677	6	5,683	6	0,0081
0,10	6,254	7	6,261	7	6,268	6	6,274	7	6,281	7	6,288	7	6,295	6	6,301	7	6,308	7	6,315	6	0,0100
0,11	6,880	7	6,887	7	6,894	8	6,902	7	6,909	8	6,917	7	6,924	7	6,931	8	6,939	7	6,946	7	0,0121
0,12	7,505	8	7,513	8	7,521	8	7,529	8	7,537	8	7,545	8	7,553	8	7,561	9	7,570	8	7,578	8	0,0144
0,13	8,130	9	8,139	9	8,148	9	8,157	8	8,165	9	8,174	9	8,183	9	8,192	8	8,200	9	8,209	9	0,0169
0,14	8,756	9	8,765	10	8,775	9	8,784	9	8,793	10	8,803	9	8,812	10	8,822	9	8,831	9	8,840	10	0,0196
0,15	9,381	10	9,391	10	9,401	10	9,411	10	9,421	11	9,432	10	9,442	10	9,452	10	9,462	10	9,472	10	0,0225
0,16	10,006	11	10,017	11	10,028	11	10,039	11	10,050	10	10,060	11	10,071	11	10,082	10	10,092	11	10,103	11	0,0256
0,17	10,632	11	10,643	12	10,655	11	10,666	12	10,678	11	10,689	11	10,700	12	10,712	11	10,723	12	10,735	11	0,0289
0,18	11,257	12	11,269	12	11,281	12	11,293	13	11,306	12	11,318	12	11,330	12	11,342	12	11,354	12	11,366	12	0,0324
0,19	11,882	13	11,895	13	11,908	13	11,921	13	11,934	12	11,946	13	11,959	13	11,972	13	11,985	12	11,997	13	0,0361
0,20	12,508	13	12,521	14	12,535	13	12,548	14	12,562	13	12,575	13	12,588	14	12,602	13	12,615	14	12,629	13	0,0400
0,21	13,133	14	13,147	14	13,161	14	13,175	15	13,190	14	13,204	14	13,218	14	13,232	14	13,246	14	13,260	14	0,0441
0,22	13,758	15	13,773	15	13,788	15	13,803	14	13,817	15	13,832	15	13,847	15	13,862	15	13,877	14	13,891	15	0,0484
0,23	14,383	16	14,399	15	14,414	16	14,430	15	14,445	16	14,461	15	14,476	16	14,492	15	14,507	16	14,523	15	0,0529
0,24	15,009	16	15,025	16	15,041	16	15,057	16	15,073	16	15,089	17	15,106	16	15,122	16	15,138	16	15,154	16	0,0576
0,25	15,634	17	15,651	17	15,668	16	15,684	17	15,701	17	15,718	17	15,735	17	15,752	16	15,768	17	15,785	17	0,0625
0,26	16,259	18	16,277	17	16,294	18	16,312	17	16,329	18	16,347	17	16,364	18	16,382	17	16,399	17	16,416	18	0,0676
0,27	16,884	18	16,902	19	16,921	18	16,939	18	16,957	18	16,975	18	16,993	18	17,011	19	17,030	18	17,048	18	0,0729
0,28	17,509	19	17,528	19	17,547	19	17,566	19	17,585	19	17,604	19	17,623	18	17,641	19	17,660	19	17,679	19	0,0784
0,29	18,135	19	18,154	20	18,174	19	18,193	20	18,213	19	18,232	20	18,252	19	18,271	20	18,291	19	18,310	19	0,0841
0,30	18,760	20	18,780	20	18,800	20	18,820	21	18,841	20	18,861	20	18,881	20	18,901	20	18,921	20	18,941	20	0,0900
0,31	19,385	21	19,406	21	19,427	20	19,447	21	19,468	21	19,489	21	19,510	21	19,531	21	19,552	20	19,572	21	0,0961
0,32	20,010	21	20,031	22	20,053	22	20,075	21	20,096	22	20,118	21	20,139	22	20,161	21	20,182	21	20,203	22	0,1024
0,33	20,635	22	20,657	22	20,679	23	20,702	22	20,724	22	20,746	22	20,768	22	20,790	22	20,812	23	20,835	22	0,1089
0,34	21,260	23	21,283	23	21,306	23	21,329	23	21,352	22	21,374	23	21,397	23	21,420	23	21,443	23	21,466	22	0,1156
0,35	21,885	23	21,908	24	21,932	24	21,956	23	21,979	24	22,003	23	22,026	24	22,050	23	22,073	24	22,097	23	0,1225
0,36	22,510	24	22,534	24	22,558	25	22,583	24	22,607	24	22,631	24	22,655	24	22,679	25	22,704	24	22,728	24	0,1296
0,37	23,135	25	23,160	25	23,185	25	23,210	25	23,235	24	23,259	25	23,284	25	23,309	25	23,334	25	23,359	24	0,1369
0,38	23,760	25	23,785	26	23,811	26	23,837	25	23,862	26	23,888	25	23,913	26	23,939	25	23,964	26	23,990	25	0,1444
0,39	24,385	26	24,411	26	24,437	27	24,464	26	24,490	26	24,516	26	24,542	26	24,568	27	24,595	26	24,621	26	0,1521
0,40	25,009	27	25,036	27	25,063	27	25,090	27	25,117	27	25,144	27	25,171	27	25,198	27	25,225	27	25,252	26	0,1600
0,41	25,634	28	25,662	28	25,690	27	25,717	28	25,745	27	25,772	28	25,800	28	25,828	27	25,855	27	25,882	28	0,1681
0,42	26,259	28	26,287	29	26,316	28	26,344	28	26,372	29	26,401	28	26,429	28	26,457	28	26,485	28	26,513	28	0,1764
0,43	26,884	29	26,913	29	26,942	29	26,971	29	27,000	29	27,029	29	27,058	29	27,087	28	27,115	29	27,144	29	0,1849
0,44	27,509	29	27,538	30	27,568	30	27,598	29	27,627	30	27,657	29	27,686	30	27,716	30	27,746	29	27,775	29	0,1936
0,45	28,133	31	28,164	30	28,194	30	28,224	31	28,255	30	28,285	30	28,315	30	28,345	31	28,376	30	28,406	30	0,2025
0,50	31,256	34	31,290	34	31,324	34	31,358	33	31,391	34	31,425	34	31,459	33	31,492	34	31,526	33	31,559	34	0,2500
0,55	34,378	38	34,416	37	34,453	37	34,490	37	34,527	37	34,564	37	34,601	37	34,638	37	34,675	37	34,712	36	0,3025
0,60	37,499	41	37,540	41	37,581	40	37,621	41	37,662	40	37,702	40	37,742	41	37,783	40	37,823	40	37,863	40	0,3600
0,65	40,619	44	40,663	44	40,707	44	40,751	44	40,795	44	40,839	44	40,883	43	40,926	44	40,970	44	41,014	43	0,4225
0,70	43,738	48	43,786	47	43,833	47	43,880	47	43,927	48	43,975	47	44,022	47	44,069	47	44,116	47	44,163	47	0,4900
0,75	46,856	50	46,906	51	46,957	51	47,008	51	47,059	50	47,109	51	47,160	50	47,210	50	47,260	51	47,311	50	0,5625
0,80	49,972	54	50,026	54	50,080	54	50,134	54	50,188	54	50,242	54	50,296	54	50,350	54	50,404	54	50,458	53	0,6400
0,85	53,086	58	53,144	58	53,202	57	53,259	57	53,316	58	53,374	57	53,431	57	53,488	58	53,546	57	53,603	57	0,7225
0,90	56,199	61	56,260	61	56,321	61	56,382	61	56,443	61	56,504	61	56,565	60	56,625	61	56,686	60	56,746	61	0,8100
0,95	59,311	64	59,375	65	59,440	64	59,504	64	59,568	64	59,632	65	59,697	64	59,761	64	59,825	63	59,888	64	0,9025
1,00	62,420	68	62,488	68	62,556	68	62,624	67	62,691	68	62,759	68	62,827	67	62,894	67	62,961	68	63,029	67	1,0000
	10,7185		10,7648		10,8113		10,8578		10,9045		10,9512		10,9981		11,0450		11,0921		11,1392		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	624	625	626	627	628	629	630	631	632	
1	62	63	63	63	63	63	63	63	63	1
2	125	125	125	125	126	126	126	126	126	2
3	187	188	188	188	188	189	189	189	190	3
4	250	250	250	251	251	252	252	252	253	4
5	312	313	313	314	314	315	315	316	316	5
6	374	375	376	376	377	377	378	379	379	6
7	437	438	438	439	440	440	441	442	442	7
8	499	500	501	502	502	503	504	505	506	8
9	562	563	563	564	565	566	567	568	569	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord c.	4,73 Days	dif.	4,74 Days	dif.	4,75 Days	dif.	4,76 Days	dif.	4,77 Days	dif.	4,78 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,632	1	0,633	1	0,634	0	0,634	1	0,635	0	0,635	1	0,0001
0,02	1,264	2	1,266	1	1,267	1	1,268	2	1,270	1	1,271	1	0,0004
0,03	1,896	2	1,898	2	1,900	2	1,902	2	1,904	2	1,906	2	0,0009
0,04	2,529	2	2,531	3	2,534	3	2,537	2	2,539	3	2,542	3	0,0016
0,05	3,161	3	3,164	3	3,167	4	3,171	3	3,174	3	3,177	4	0,0025
0,06	3,793	4	3,797	4	3,801	4	3,805	4	3,809	4	3,813	4	0,0036
0,07	4,425	5	4,430	4	4,434	5	4,439	5	4,444	4	4,448	5	0,0049
0,08	5,057	5	5,062	6	5,068	5	5,073	5	5,078	6	5,084	5	0,0064
0,09	5,689	6	5,695	6	5,701	6	5,707	6	5,713	6	5,719	6	0,0081
0,10	6,321	7	6,328	7	6,335	6	6,341	7	6,348	7	6,355	6	0,0100
0,11	6,953	8	6,961	7	6,968	7	6,975	8	6,983	7	6,990	7	0,0121
0,12	7,586	8	7,594	8	7,602	8	7,610	8	7,618	8	7,626	8	0,0144
0,13	8,218	8	8,226	9	8,235	9	8,244	8	8,252	9	8,261	9	0,0169
0,14	8,850	9	8,859	9	8,868	10	8,878	9	8,887	9	8,896	10	0,0196
0,15	9,482	10	9,492	10	9,502	10	9,512	10	9,522	10	9,532	10	0,0225
0,16	10,114	11	10,125	10	10,135	11	10,146	11	10,157	10	10,167	11	0,0256
0,17	10,746	11	10,757	12	10,769	11	10,780	11	10,791	12	10,803	11	0,0289
0,18	11,378	12	11,390	12	11,402	12	11,414	12	11,426	12	11,438	12	0,0324
0,19	12,010	13	12,023	12	12,035	13	12,048	13	12,061	12	12,073	13	0,0361
0,20	12,642	13	12,655	14	12,669	13	12,682	13	12,695	14	12,709	13	0,0400
0,21	13,274	14	13,288	14	13,302	14	13,316	14	13,330	14	13,344	14	0,0441
0,22	13,906	15	13,921	14	13,935	15	13,950	15	13,965	14	13,979	15	0,0484
0,23	14,538	15	14,553	16	14,569	15	14,584	15	14,599	16	14,615	15	0,0529
0,24	15,170	16	15,186	16	15,202	16	15,218	16	15,234	16	15,250	16	0,0576
0,25	15,802	17	15,819	16	15,835	17	15,852	17	15,869	16	15,885	17	0,0625
0,26	16,434	17	16,451	18	16,469	17	16,486	17	16,503	17	16,520	18	0,0676
0,27	17,066	18	17,084	18	17,102	18	17,120	18	17,138	18	17,156	18	0,0729
0,28	17,698	18	17,716	19	17,735	19	17,754	18	17,772	19	17,791	19	0,0784
0,29	18,329	20	18,349	19	18,368	20	18,388	19	18,407	19	18,426	19	0,0841
0,30	18,961	20	18,981	20	19,001	20	19,021	20	19,041	20	19,061	20	0,0900
0,31	19,593	21	19,614	20	19,634	21	19,655	21	19,676	20	19,696	21	0,0961
0,32	20,225	21	20,246	22	20,268	21	20,289	21	20,310	22	20,332	21	0,1024
0,33	20,857	22	20,879	22	20,901	22	20,923	22	20,945	22	20,967	22	0,1089
0,34	21,488	23	21,511	23	21,534	22	21,556	23	21,579	23	21,602	22	0,1156
0,35	22,120	24	22,144	23	22,167	23	22,190	24	22,214	23	22,237	23	0,1225
0,36	22,752	24	22,776	24	22,800	24	22,824	24	22,848	24	22,872	24	0,1296
0,37	23,383	25	23,408	25	23,433	25	23,458	24	23,482	25	23,507	24	0,1369
0,38	24,015	26	24,041	25	24,066	25	24,091	26	24,117	25	24,142	25	0,1444
0,39	24,647	26	24,673	26	24,699	26	24,725	26	24,751	26	24,777	26	0,1521
0,40	25,278	27	25,305	27	25,332	27	25,359	26	25,385	27	25,412	26	0,1600
0,41	25,910	27	25,937	28	25,965	27	25,992	27	26,019	28	26,047	27	0,1681
0,42	26,541	29	26,570	28	26,598	28	26,626	28	26,654	28	26,682	27	0,1764
0,43	27,173	29	27,202	28	27,230	29	27,259	29	27,288	28	27,316	29	0,1849
0,44	27,804	30	27,834	29	27,863	30	27,893	29	27,922	29	27,951	30	0,1936
0,45	28,436	30	28,466	30	28,496	30	28,526	30	28,556	30	28,586	30	0,2025
0,50	31,593	33	31,626	33	31,659	34	31,693	33	31,726	33	31,759	34	0,2500
0,55	34,748	37	34,785	37	34,822	37	34,859	36	34,895	37	34,932	37	0,3025
0,60	37,903	41	37,944	40	37,984	40	38,024	40	38,064	40	38,104	40	0,3600
0,65	41,057	44	41,101	43	41,144	44	41,188	43	41,231	43	41,274	44	0,4225
0,70	44,210	47	44,257	46	44,303	47	44,350	47	44,397	47	44,444	46	0,4900
0,75	47,361	50	47,411	51	47,462	50	47,512	50	47,562	50	47,612	50	0,5625
0,80	50,511	54	50,565	53	50,618	54	50,672	53	50,725	54	50,779	53	0,6400
0,85	53,660	57	53,717	57	53,774	57	53,831	56	53,887	57	53,944	57	0,7225
0,90	56,807	60	56,867	61	56,928	60	56,988	60	57,048	60	57,108	60	0,8100
0,95	59,952	64	60,016	64	60,080	63	60,143	64	60,207	63	60,270	64	0,9025
1,00	63,096	67	63,163	67	63,230	67	63,297	67	63,364	67	63,431	67	1,0000
	11,1865		11,2338		11,2813		11,3288		11,3765		11,4242		c^2

$\frac{1}{2}\cdot(r+r'')^2$ or $r^2+r''^2$ nearly.

	631	632	633	634	635	636	
1	63	63	63	63	64	64	1
2	126	126	127	127	127	127	2
3	189	190	190	190	191	191	3
4	252	253	253	254	254	254	4
5	316	316	317	317	318	318	5
6	379	379	380	380	381	382	6
7	442	442	443	444	445	445	7
8	505	506	506	507	508	509	8
9	568	569	570	571	572	572	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

	Sum of the Radii $r + r''$.																				
Chord c.	4,79		4,80		4,81		4,82		4,83		4,84		4,85		4,86		4,87		4,88		
	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,636	1	0,637	0	0,637	1	0,638	1	0,639	0	0,639	1	0,640	1	0,641	0	0,641	1	0,642	1	0,0001
0,02	1,272	2	1,274	1	1,275	1	1,276	2	1,278	1	1,279	1	1,280	2	1,282	1	1,283	1	1,284	2	0,0004
0,03	1,908	2	1,910	2	1,912	2	1,914	2	1,916	2	1,918	2	1,920	2	1,922	2	1,924	2	1,926	2	0,0009
0,04	2,545	2	2,547	3	2,550	3	2,553	2	2,555	3	2,558	2	2,560	3	2,563	3	2,566	2	2,568	3	0,0016
0,05	3,181	3	3,184	3	3,187	4	3,191	3	3,194	3	3,197	4	3,201	3	3,204	3	3,207	3	3,210	4	0,0025
0,06	3,817	4	3,821	4	3,825	4	3,829	4	3,833	4	3,837	4	3,841	4	3,845	4	3,849	4	3,853	3	0,0036
0,07	4,453	5	4,458	4	4,462	5	4,467	5	4,472	4	4,476	5	4,481	4	4,485	5	4,490	5	4,495	4	0,0049
0,08	5,089	5	5,094	6	5,100	5	5,105	5	5,110	6	5,116	5	5,121	5	5,126	5	5,131	6	5,137	5	0,0064
0,09	5,725	6	5,731	6	5,737	6	5,743	6	5,749	6	5,755	6	5,761	6	5,767	6	5,773	6	5,779	6	0,0081
0,10	6,361	7	6,368	7	6,375	6	6,381	7	6,388	6	6,394	7	6,401	7	6,408	6	6,414	7	6,421	6	0,0100
0,11	6,997	8	7,005	7	7,012	7	7,019	8	7,027	7	7,034	7	7,041	7	7,048	8	7,056	7	7,063	7	0,0121
0,12	7,634	7	7,641	8	7,649	8	7,657	8	7,665	8	7,673	8	7,681	8	7,689	8	7,697	8	7,705	8	0,0144
0,13	8,270	8	8,278	9	8,287	9	8,296	8	8,304	9	8,313	8	8,321	9	8,330	8	8,338	9	8,347	9	0,0169
0,14	8,906	9	8,915	9	8,924	10	8,934	9	8,943	9	8,952	9	8,961	10	8,971	9	8,980	9	8,989	9	0,0196
0,15	9,542	10	9,552	10	9,562	10	9,572	10	9,582	9	9,591	10	9,601	10	9,611	10	9,621	10	9,631	10	0,0225
0,16	10,178	10	10,188	11	10,199	11	10,210	10	10,220	11	10,231	10	10,241	11	10,252	11	10,263	10	10,273	11	0,0256
0,17	10,814	11	10,825	11	10,836	12	10,848	11	10,859	11	10,870	11	10,881	12	10,893	11	10,904	11	10,915	11	0,0289
0,18	11,450	12	11,462	12	11,474	12	11,486	12	11,498	12	11,510	11	11,521	12	11,533	12	11,545	12	11,557	12	0,0324
0,19	12,086	13	12,099	12	12,111	13	12,124	12	12,136	13	12,149	12	12,161	13	12,174	12	12,186	13	12,199	12	0,0361
0,20	12,722	13	12,735	14	12,749	13	12,762	13	12,775	13	12,788	13	12,801	14	12,815	13	12,828	13	12,841	13	0,0400
0,21	13,358	14	13,372	14	13,386	14	13,400	14	13,414	14	13,428	13	13,441	14	13,455	14	13,469	14	13,483	14	0,0441
0,22	13,994	15	14,009	14	14,023	15	14,038	14	14,052	15	14,067	14	14,081	15	14,096	14	14,110	15	14,125	14	0,0484
0,23	14,630	15	14,645	15	14,660	16	14,676	15	14,691	15	14,706	15	14,721	15	14,736	16	14,752	15	14,767	15	0,0529
0,24	15,266	16	15,282	16	15,298	16	15,314	16	15,330	15	15,345	16	15,361	16	15,377	16	15,393	16	15,409	15	0,0576
0,25	15,902	16	15,918	17	15,935	17	15,952	16	15,968	17	15,985	16	16,001	17	16,018	16	16,034	17	16,051	16	0,0625
0,26	16,538	17	16,555	17	16,572	17	16,589	18	16,607	17	16,624	17	16,641	17	16,658	17	16,675	17	16,692	18	0,0676
0,27	17,174	18	17,192	17	17,209	18	17,227	18	17,245	18	17,263	18	17,281	18	17,299	18	17,317	17	17,334	18	0,0729
0,28	17,810	18	17,828	19	17,847	18	17,865	19	17,884	18	17,902	19	17,921	18	17,939	19	17,958	18	17,976	19	0,0784
0,29	18,445	20	18,465	19	18,484	19	18,503	19	18,522	19	18,541	20	18,561	19	18,580	19	18,599	19	18,618	19	0,0841
0,30	19,081	20	19,101	20	19,121	20	19,141	20	19,161	20	19,181	19	19,200	20	19,220	20	19,240	20	19,260	19	0,0900
0,31	19,717	21	19,738	20	19,758	21	19,779	20	19,799	21	19,820	20	19,840	21	19,861	20	19,881	21	19,902	20	0,0961
0,32	20,353	21	20,374	21	20,395	22	20,417	21	20,438	21	20,459	21	20,480	21	20,501	21	20,522	21	20,543	21	0,1024
0,33	20,989	22	21,011	21	21,032	22	21,054	22	21,076	22	21,098	22	21,120	22	21,142	21	21,163	22	21,185	22	0,1089
0,34	21,624	23	21,647	23	21,670	22	21,692	23	21,715	22	21,737	23	21,760	22	21,782	22	21,804	23	21,827	22	0,1156
0,35	22,260	23	22,283	24	22,307	23	22,330	23	22,353	23	22,376	23	22,399	23	22,422	23	22,445	23	22,468	23	0,1225
0,36	22,896	24	22,920	24	22,944	23	22,967	24	22,991	24	23,015	24	23,039	24	23,063	23	23,086	24	23,110	24	0,1296
0,37	23,531	25	23,556	25	23,581	24	23,605	25	23,630	24	23,654	25	23,679	24	23,703	24	23,727	25	23,752	24	0,1369
0,38	24,167	25	24,192	26	24,218	25	24,243	25	24,268	25	24,293	25	24,318	25	24,343	25	24,368	25	24,393	25	0,1444
0,39	24,803	26	24,829	26	24,855	25	24,880	26	24,906	26	24,932	26	24,958	26	24,984	25	25,009	26	25,035	26	0,1521
0,40	25,438	27	25,465	26	25,491	27	25,518	27	25,545	26	25,571	26	25,597	27	25,624	26	25,650	27	25,677	26	0,1600
0,41	26,074	27	26,101	27	26,128	28	26,156	27	26,183	27	26,210	27	26,237	27	26,264	27	26,291	27	26,318	27	0,1681
0,42	26,709	28	26,737	28	26,765	28	26,793	28	26,821	28	26,849	27	26,876	28	26,904	28	26,932	28	26,960	27	0,1764
0,43	27,345	29	27,374	28	27,402	29	27,431	28	27,459	29	27,488	28	27,516	28	27,544	29	27,573	28	27,601	28	0,1849
0,44	27,981	29	28,010	29	28,039	29	28,068	29	28,097	29	28,126	29	28,155	29	28,184	30	28,214	29	28,243	28	0,1936
0,45	28,616	30	28,646	30	28,676	30	28,706	29	28,735	30	28,765	30	28,795	30	28,825	29	28,854	30	28,884	30	0,2025
0,50	31,793	33	31,826	33	31,859	33	31,892	33	31,925	34	31,959	33	31,992	33	32,025	33	32,058	33	32,091	33	0,2500
0,55	34,969	36	35,005	37	35,042	36	35,078	37	35,115	36	35,151	36	35,187	37	35,224	36	35,260	36	35,296	37	0,3025
0,60	38,144	40	38,184	39	38,223	40	38,263	40	38,303	40	38,343	39	38,382	40	38,422	40	38,462	39	38,501	40	0,3600
0,65	41,318	43	41,361	43	41,404	43	41,447	43	41,490	43	41,533	43	41,576	43	41,619	43	41,662	43	41,705	43	0,4225
0,70	44,490	47	44,537	46	44,583	47	44,630	46	44,676	47	44,723	46	44,769	46	44,815	47	44,862	46	44,908	46	0,4900
0,75	47,662	50	47,712	50	47,762	50	47,812	49	47,861	50	47,911	50	47,961	49	48,010	50	48,060	49	48,109	50	0,5625
0,80	50,832	53	50,885	54	50,939	53	50,992	53	51,045	53	51,098	53	51,151	53	51,204	53	51,257	53	51,310	52	0,6400
0,85	54,001	57	54,058	56	54,114	57	54,171	56	54,227	57	54,284	56	54,340	56	54,396	56	54,452	57	54,509	56	0,7225
0,90	57,168	60	57,228	60	57,288	60	57,348	60	57,408	60	57,468	59	57,527	60	57,587	59	57,646	60	57,706	59	0,8100
0,95	60,334	63	60,397	64	60,461	63	60,524	63	60,587	63	60,650	63	60,713	63	60,776	63	60,839	63	60,902	63	0,9025
1,00	63,498	67	63,565	66	63,631	67	63,698	66	63,764	67	63,831	66	63,897	67	63,964	66	64,030	66	64,096	66	1,0000
	11,4721		11,5200		11,5681		11,6162		11,6645		11,7128		11,7613		11,8098		11,8585		11,9072		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	635	636	637	638	639	640	641	642	643	
1	64	64	64	64	64	64	64	64	64	1
2	127	127	127	128	128	128	128	128	129	2
3	191	191	191	191	192	192	192	193	193	3
4	254	254	255	255	256	256	256	257	257	4
5	318	318	319	319	320	320	321	321	322	5
6	381	382	382	383	383	384	385	385	386	6
7	445	445	446	447	447	448	449	449	450	7
8	508	509	510	510	511	512	513	514	514	8
9	572	572	573	574	575	576	577	578	579	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	Sum of the Radii $r+r''$. 4,89 Days	dif.	4,90 Days	dif.	4,91 Days	dif.	4,92 Days	dif.	4,93 Days	dif.	4,94 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,643	0	0,643	1	0,644	1	0,645	0	0,645	1	0,646	1	0,0001
0,02	1,286	1	1,287	1	1,288	1	1,289	2	1,291	1	1,292	1	0,0004
0,03	1,928	2	1,930	2	1,932	2	1,934	2	1,936	2	1,938	2	0,0009
0,04	2,571	3	2,574	2	2,576	3	2,579	2	2,581	3	2,584	3	0,0016
0,05	3,214	3	3,217	3	3,220	4	3,224	3	3,227	3	3,230	3	0,0025
0,06	3,856	4	3,860	4	3,864	4	3,868	4	3,872	4	3,876	4	0,0036
0,07	4,499	5	4,504	4	4,508	5	4,513	5	4,518	4	4,522	5	0,0049
0,08	5,142	5	5,147	5	5,152	6	5,158	5	5,163	5	5,168	5	0,0064
0,09	5,785	6	5,791	5	5,796	6	5,802	6	5,808	6	5,814	6	0,0081
0,10	6,427	7	6,434	7	6,441	6	6,447	7	6,454	6	6,460	7	0,0100
0,11	7,070	7	7,077	8	7,085	7	7,092	7	7,099	7	7,106	7	0,0121
0,12	7,713	8	7,721	8	7,729	7	7,736	8	7,744	8	7,752	8	0,0144
0,13	8,356	8	8,364	9	8,373	8	8,381	9	8,390	8	8,398	9	0,0169
0,14	8,998	9	9,007	10	9,017	9	9,026	9	9,035	9	9,044	9	0,0196
0,15	9,641	10	9,651	10	9,661	9	9,670	10	9,680	10	9,690	10	0,0225
0,16	10,284	10	10,294	11	10,305	10	10,315	11	10,326	10	10,336	10	0,0256
0,17	10,926	11	10,937	12	10,949	11	10,960	11	10,971	11	10,982	11	0,0289
0,18	11,569	12	11,581	12	11,593	11	11,604	12	11,616	12	11,628	12	0,0324
0,19	12,211	13	12,224	12	12,236	13	12,249	12	12,261	13	12,274	12	0,0361
0,20	12,854	13	12,867	13	12,880	14	12,894	13	12,907	13	12,920	13	0,0400
0,21	13,497	14	13,511	13	13,524	14	13,538	14	13,552	14	13,566	13	0,0441
0,22	14,139	15	14,154	14	14,168	15	14,183	14	14,197	14	14,211	15	0,0484
0,23	14,782	15	14,797	15	14,812	15	14,827	15	14,842	15	14,857	15	0,0529
0,24	15,424	16	15,440	16	15,456	16	15,472	16	15,488	15	15,503	16	0,0576
0,25	16,067	16	16,083	17	16,100	16	16,116	17	16,133	16	16,149	16	0,0625
0,26	16,710	17	16,727	17	16,744	17	16,761	17	16,778	17	16,795	17	0,0676
0,27	17,352	18	17,370	18	17,388	17	17,405	18	17,423	18	17,441	17	0,0729
0,28	17,995	18	18,013	18	18,031	19	18,050	18	18,068	18	18,086	19	0,0784
0,29	18,637	19	18,656	19	18,675	19	18,694	19	18,713	19	18,732	19	0,0841
0,30	19,279	20	19,299	20	19,319	20	19,339	19	19,358	20	19,378	19	0,0900
0,31	19,922	20	19,942	21	19,963	20	19,983	20	20,003	21	20,024	20	0,0961
0,32	20,564	21	20,585	21	20,606	21	20,627	21	20,648	21	20,669	21	0,1024
0,33	21,207	21	21,228	22	21,250	22	21,272	21	21,293	22	21,315	22	0,1089
0,34	21,849	22	21,871	23	21,894	22	21,916	22	21,938	23	21,961	22	0,1156
0,35	22,491	23	22,514	23	22,537	23	22,560	23	22,583	23	22,606	23	0,1225
0,36	23,134	23	23,157	24	23,181	24	23,205	23	23,228	24	23,252	23	0,1296
0,37	23,776	24	23,800	25	23,825	24	23,849	24	23,873	24	23,897	25	0,1369
0,38	24,418	25	24,443	25	24,468	25	24,493	25	24,518	25	24,543	25	0,1444
0,39	25,061	25	25,086	26	25,112	25	25,137	26	25,163	26	25,189	25	0,1521
0,40	25,703	26	25,729	26	25,755	27	25,782	26	25,808	26	25,834	26	0,1600
0,41	26,345	27	26,372	27	26,399	27	26,426	27	26,453	27	26,480	26	0,1681
0,42	26,987	28	27,015	27	27,042	28	27,070	28	27,098	27	27,125	27	0,1764
0,43	27,629	29	27,658	28	27,686	28	27,714	28	27,742	28	27,770	29	0,1849
0,44	28,271	29	28,300	29	28,329	29	28,358	29	28,387	29	28,416	29	0,1936
0,45	28,914	29	28,943	30	28,973	29	29,002	30	29,032	29	29,061	30	0,2025
0,50	32,124	32	32,156	33	32,189	33	32,222	33	32,255	33	32,288	32	0,2500
0,55	35,333	36	35,369	36	35,405	36	35,441	36	35,477	36	35,513	36	0,3025
0,60	38,541	39	38,580	40	38,620	39	38,659	39	38,698	40	38,738	39	0,3600
0,65	41,748	43	41,791	42	41,833	43	41,876	43	41,919	42	41,961	43	0,4225
0,70	44,954	46	45,000	46	45,046	46	45,092	46	45,138	46	45,184	46	0,4900
0,75	48,159	49	48,208	50	48,258	49	48,307	49	48,356	49	48,405	50	0,5625
0,80	51,362	53	51,415	53	51,468	52	51,520	53	51,573	53	51,626	52	0,6400
0,85	54,565	56	54,621	56	54,677	56	54,733	55	54,788	56	54,844	56	0,7225
0,90	57,765	60	57,825	59	57,884	59	57,943	59	58,002	60	58,062	59	0,8100
0,95	60,965	62	61,027	63	61,090	62	61,152	63	61,215	62	61,277	63	0,9025
1,00	64,162	66	64,228	66	64,294	66	64,360	66	64,426	66	64,492	65	1,0000
	11,9561		12,0050		12,0541		12,1032		12,1525		12,2018		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	642	643	644	645	646	
1	64	64	64	65	65	1
2	128	129	129	129	129	2
3	193	193	193	194	194	3
4	257	257	258	258	258	4
5	321	322	322	323	323	5
6	385	386	386	387	388	6
7	449	450	451	452	452	7
8	514	514	515	516	517	8
9	578	579	580	581	581	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord	Sum of the Radii $r+r''$.																				
	4,95		4,96		4,97		4,98		4,99		5,00		5,01		5,02		5,03		5,04		
c.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,647	0	0,647	1	0,648	1	0,649	0	0,649	1	0,650	1	0,651	0	0,651	1	0,652	1	0,653	0	0,0001
0,02	1,293	2	1,295	1	1,296	1	1,297	2	1,299	1	1,300	1	1,301	1	1,302	2	1,304	1	1,305	1	0,0004
0,03	1,940	2	1,942	2	1,944	2	1,946	2	1,948	2	1,950	2	1,952	2	1,954	2	1,956	2	1,958	2	0,0009
0,04	2,587	2	2,589	3	2,592	3	2,595	2	2,597	3	2,600	2	2,602	3	2,605	3	2,608	2	2,610	3	0,0016
0,05	3,233	4	3,237	3	3,240	3	3,243	3	3,246	4	3,250	3	3,253	3	3,256	3	3,259	4	3,263	3	0,0025
0,06	3,880	4	3,884	4	3,888	4	3,892	4	3,896	4	3,900	4	3,904	3	3,907	4	3,911	4	3,915	4	0,0036
0,07	4,527	4	4,531	5	4,536	4	4,540	5	4,545	5	4,550	4	4,554	5	4,559	4	4,563	5	4,568	4	0,0049
0,08	5,173	6	5,179	5	5,184	5	5,189	5	5,194	5	5,199	6	5,205	5	5,210	5	5,215	5	5,220	5	0,0064
0,09	5,820	6	5,826	6	5,832	6	5,838	6	5,844	5	5,849	6	5,855	6	5,861	6	5,867	6	5,873	6	0,0081
0,10	6,467	6	6,473	7	6,480	6	6,486	7	6,493	6	6,499	7	6,506	6	6,512	7	6,519	6	6,525	7	0,0100
0,11	7,113	8	7,121	7	7,128	7	7,135	7	7,142	7	7,149	7	7,156	7	7,163	8	7,171	7	7,178	7	0,0121
0,12	7,760	8	7,768	8	7,776	7	7,783	8	7,791	8	7,799	8	7,807	8	7,815	7	7,822	8	7,830	8	0,0144
0,13	8,407	8	8,415	9	8,424	8	8,432	9	8,441	8	8,449	8	8,457	9	8,466	8	8,474	9	8,483	8	0,0169
0,14	9,053	9	9,062	10	9,072	9	9,081	9	9,090	9	9,099	9	9,108	9	9,117	9	9,126	9	9,135	9	0,0196
0,15	9,700	10	9,710	9	9,719	10	9,729	10	9,739	10	9,749	9	9,758	10	9,768	10	9,778	10	9,788	9	0,0225
0,16	10,346	11	10,357	10	10,367	11	10,378	10	10,388	11	10,399	10	10,409	10	10,419	11	10,430	10	10,440	10	0,0256
0,17	10,993	11	11,004	11	11,015	11	11,026	11	11,037	11	11,048	11	11,059	12	11,071	11	11,082	11	11,093	11	0,0289
0,18	11,640	11	11,651	12	11,663	12	11,675	12	11,687	11	11,698	12	11,710	12	11,722	11	11,733	12	11,745	12	0,0324
0,19	12,286	13	12,299	12	12,311	12	12,323	13	12,336	12	12,348	12	12,360	13	12,373	12	12,385	12	12,397	13	0,0361
0,20	12,933	13	12,946	13	12,959	13	12,972	13	12,985	13	12,998	13	13,011	13	13,024	13	13,037	13	13,050	13	0,0400
0,21	13,579	14	13,593	14	13,607	13	13,620	14	13,634	14	13,648	13	13,661	14	13,675	14	13,689	13	13,702	14	0,0441
0,22	14,226	14	14,240	15	14,255	14	14,269	14	14,283	15	14,298	14	14,312	14	14,326	14	14,340	15	14,355	14	0,0484
0,23	14,872	15	14,887	15	14,902	15	14,917	15	14,932	15	14,947	15	14,962	15	14,977	15	14,992	15	15,007	15	0,0529
0,24	15,519	16	15,535	15	15,550	16	15,566	15	15,581	16	15,597	16	15,613	15	15,628	16	15,644	15	15,659	16	0,0576
0,25	16,165	17	16,182	16	16,198	16	16,214	17	16,231	16	16,247	16	16,263	16	16,279	16	16,295	17	16,312	16	0,0625
0,26	16,812	17	16,829	17	16,846	17	16,863	17	16,880	17	16,897	16	16,913	17	16,930	17	16,947	17	16,964	17	0,0676
0,27	17,458	18	17,476	17	17,493	18	17,511	18	17,529	17	17,546	18	17,564	17	17,581	18	17,599	17	17,616	18	0,0729
0,28	18,105	18	18,123	18	18,141	18	18,159	19	18,178	18	18,196	18	18,214	18	18,232	18	18,250	19	18,269	18	0,0784
0,29	18,751	19	18,770	19	18,789	19	18,808	19	18,827	19	18,846	18	18,864	19	18,883	19	18,902	19	18,921	19	0,0841
0,30	19,397	20	19,417	20	19,437	19	19,456	20	19,476	19	19,495	20	19,515	19	19,534	20	19,554	19	19,573	20	0,0900
0,31	20,044	20	20,064	20	20,084	21	20,105	20	20,125	20	20,145	20	20,165	20	20,185	20	20,205	20	20,225	20	0,0961
0,32	20,690	21	20,711	21	20,732	21	20,753	21	20,774	21	20,795	20	20,815	21	20,836	21	20,857	21	20,878	20	0,1024
0,33	21,337	21	21,358	22	21,380	21	21,401	22	21,423	21	21,444	22	21,466	21	21,487	21	21,508	22	21,530	21	0,1089
0,34	21,983	22	22,005	22	22,027	22	22,049	23	22,072	22	22,094	22	22,116	22	22,138	22	22,160	22	22,182	22	0,1156
0,35	22,629	23	22,652	23	22,675	23	22,698	22	22,720	23	22,743	23	22,766	23	22,789	22	22,811	23	22,834	23	0,1225
0,36	23,275	24	23,299	23	23,322	24	23,346	23	23,369	24	23,393	23	23,416	24	23,440	23	23,463	23	23,486	24	0,1296
0,37	23,922	24	23,946	24	23,970	24	23,994	24	24,018	24	24,042	24	24,066	24	24,090	24	24,114	24	24,138	24	0,1369
0,38	24,568	25	24,593	25	24,618	24	24,642	25	24,667	25	24,692	24	24,716	25	24,741	25	24,766	24	24,790	25	0,1444
0,39	25,214	26	25,240	25	25,265	25	25,290	26	25,316	25	25,341	26	25,367	25	25,392	25	25,417	25	25,442	26	0,1521
0,40	25,860	26	25,886	26	25,912	27	25,939	26	25,965	26	25,991	26	26,017	26	26,043	26	26,069	26	26,095	25	0,1600
0,41	26,506	27	26,533	27	26,560	27	26,587	26	26,613	27	26,640	27	26,667	26	26,693	27	26,720	27	26,747	26	0,1681
0,42	27,152	28	27,180	27	27,207	28	27,235	27	27,262	27	27,289	28	27,317	27	27,344	27	27,371	27	27,398	28	0,1764
0,43	27,799	28	27,827	28	27,855	28	27,883	28	27,911	28	27,939	28	27,967	28	27,995	28	28,023	27	28,050	28	0,1849
0,44	28,445	28	28,473	29	28,502	29	28,531	28	28,559	29	28,588	29	28,617	28	28,645	29	28,674	28	28,702	29	0,1936
0,45	29,091	29	29,120	29	29,149	30	29,179	29	29,208	29	29,237	30	29,267	29	29,296	29	29,325	29	29,354	29	0,2025
0,50	32,320	33	32,353	33	32,386	32	32,418	33	32,451	32	32,483	33	32,516	32	32,548	33	32,581	32	32,613	33	0,2500
0,55	35,549	36	35,585	36	35,621	36	35,657	36	35,693	36	35,729	35	35,764	36	35,800	36	35,836	36	35,872	35	0,3025
0,60	38,777	39	38,816	40	38,856	39	38,895	39	38,934	39	38,973	39	39,012	39	39,051	39	39,090	39	39,129	39	0,3600
0,65	42,004	43	42,047	42	42,089	42	42,131	43	42,174	42	42,216	43	42,259	42	42,301	42	42,343	42	42,385	42	0,4225
0,70	45,230	46	45,276	45	45,321	46	45,367	46	45,413	45	45,458	46	45,504	46	45,550	45	45,595	46	45,641	45	0,4900
0,75	48,455	49	48,504	49	48,553	49	48,602	49	48,651	49	48,700	48	48,748	49	48,797	49	48,846	49	48,895	48	0,5625
0,80	51,678	52	51,730	53	51,783	52	51,835	52	51,887	52	51,939	53	51,992	52	52,044	52	52,096	52	52,148	52	0,6400
0,85	54,900	56	54,956	55	55,011	56	55,067	56	55,123	55	55,178	55	55,233	56	55,289	55	55,344	55	55,399	56	0,7225
0,90	58,121	59	58,180	59	58,239	59	58,298	58	58,356	59	58,415	59	58,474	58	58,532	59	58,591	59	58,650	58	0,8100
0,95	61,340	62	61,402	62	61,464	63	61,527	62	61,589	62	61,651	62	61,713	62	61,775	62	61,837	61	61,898	62	0,9025
1,00	64,557	66	64,623	66	64,689	65	64,754	65	64,819	66	64,885	65	64,950	65	65,015	65	65,080	66	65,146	65	1,0000
	12,2513		12,3008		12,3505		12,4002		12,4501		12,5000		12,5501		12,6002		12,6505		12,7008		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	646	647	648	649	650	651	652	653	
1	65	65	65	65	65	65	65	65	1
2	129	129	130	130	130	130	130	131	2
3	194	194	194	195	195	195	196	196	3
4	258	259	259	260	260	260	261	261	4
5	323	324	324	325	325	326	326	327	5
6	388	388	389	389	390	391	391	392	6
7	452	453	454	454	455	456	456	457	7
8	517	518	518	519	520	521	522	522	8
9	581	582	583	584	585	586	587	588	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord c.	5,05 Days	dif.	5,06 Days	dif.	5,07 Days	dif.	5,08 Days	dif.	5,09 Days	dif.	5,10 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,653	1	0,654	0	0,654	1	0,655	1	0,656	0	0,656	1	0,0001
0,02	1,306	2	1,308	1	1,309	1	1,310	2	1,312	1	1,313	1	0,0004
0,03	1,960	1	1,961	2	1,963	2	1,965	2	1,967	2	1,969	2	0,0009
0,04	2,613	2	2,615	3	2,618	2	2,620	3	2,623	3	2,626	2	0,0016
0,05	3,266	3	3,269	3	3,272	4	3,276	3	3,279	3	3,282	3	0,0025
0,06	3,919	4	3,923	4	3,927	4	3,931	4	3,935	3	3,938	4	0,0036
0,07	4,572	5	4,577	4	4,581	5	4,586	4	4,590	5	4,595	4	0,0049
0,08	5,225	6	5,231	5	5,236	5	5,241	5	5,246	5	5,251	5	0,0064
0,09	5,879	5	5,884	6	5,890	6	5,896	6	5,902	6	5,908	5	0,0081
0,10	6,532	6	6,538	7	6,545	6	6,551	7	6,558	6	6,564	6	0,0100
0,11	7,185	7	7,192	7	7,199	7	7,206	7	7,213	7	7,220	7	0,0121
0,12	7,838	8	7,846	8	7,854	7	7,861	8	7,869	8	7,877	7	0,0144
0,13	8,491	9	8,500	8	8,508	8	8,516	9	8,525	8	8,533	8	0,0169
0,14	9,144	9	9,153	9	9,162	9	9,171	9	9,180	9	9,189	9	0,0196
0,15	9,797	10	9,807	10	9,817	9	9,826	10	9,836	10	9,846	9	0,0225
0,16	10,450	11	10,461	10	10,471	10	10,481	11	10,492	10	10,502	10	0,0256
0,17	11,104	11	11,115	11	11,126	10	11,136	11	11,147	11	11,158	11	0,0289
0,18	11,757	11	11,768	12	11,780	12	11,792	11	11,803	12	11,815	11	0,0324
0,19	12,410	12	12,422	12	12,434	13	12,447	12	12,459	12	12,471	12	0,0361
0,20	13,063	13	13,076	13	13,089	13	13,102	12	13,114	13	13,127	13	0,0400
0,21	13,716	13	13,729	14	13,743	14	13,757	13	13,770	14	13,784	13	0,0441
0,22	14,369	14	14,383	14	14,397	14	14,411	15	14,426	14	14,440	14	0,0484
0,23	15,022	15	15,037	15	15,052	14	15,066	15	15,081	15	15,096	15	0,0529
0,24	15,675	15	15,690	16	15,706	15	15,721	16	15,737	15	15,752	16	0,0576
0,25	16,328	16	16,344	16	16,360	16	16,376	16	16,392	17	16,409	16	0,0625
0,26	16,981	17	16,998	16	17,014	17	17,031	17	17,048	17	17,065	16	0,0676
0,27	17,634	17	17,651	18	17,669	17	17,686	18	17,704	17	17,721	17	0,0729
0,28	18,287	18	18,305	18	18,323	18	18,341	18	18,359	18	18,377	18	0,0784
0,29	18,940	18	18,958	19	18,977	19	18,996	19	19,015	18	19,033	19	0,0841
0,30	19,593	19	19,612	19	19,631	20	19,651	19	19,670	19	19,689	20	0,0900
0,31	20,245	20	20,265	21	20,286	20	20,306	20	20,326	19	20,345	20	0,0961
0,32	20,898	21	20,919	21	20,940	20	20,960	21	20,981	21	21,002	20	0,1024
0,33	21,551	22	21,573	21	21,594	21	21,615	21	21,636	22	21,658	21	0,1089
0,34	22,204	22	22,226	22	22,248	22	22,270	22	22,292	22	22,314	22	0,1156
0,35	22,857	22	22,879	23	22,902	23	22,925	22	22,947	23	22,970	22	0,1225
0,36	23,510	23	23,533	23	23,556	23	23,579	24	23,603	23	23,626	23	0,1296
0,37	24,162	24	24,186	24	24,210	24	24,234	24	24,258	24	24,282	24	0,1369
0,38	24,815	25	24,840	24	24,864	25	24,889	24	24,913	25	24,938	24	0,1444
0,39	25,468	25	25,493	25	25,518	25	25,543	25	25,568	26	25,594	25	0,1521
0,40	26,120	26	26,146	26	26,172	26	26,198	26	26,224	26	26,250	25	0,1600
0,41	26,773	27	26,800	26	26,826	27	26,853	26	26,879	26	26,905	27	0,1681
0,42	27,426	27	27,453	27	27,480	27	27,507	27	27,534	27	27,561	27	0,1764
0,43	28,078	28	28,106	28	28,134	28	28,162	27	28,189	28	28,217	28	0,1849
0,44	28,731	28	28,759	29	28,788	28	28,816	29	28,845	28	28,873	28	0,1936
0,45	29,383	30	29,413	29	29,442	29	29,471	29	29,500	29	29,529	29	0,2025
0,50	32,646	32	32,678	32	32,710	33	32,743	32	32,775	32	32,807	32	0,2500
0,55	35,907	36	35,943	35	35,978	36	36,014	35	36,049	36	36,085	35	0,3025
0,60	39,168	39	39,207	38	39,245	39	39,284	39	39,323	39	39,362	38	0,3600
0,65	42,427	42	42,469	43	42,512	42	42,554	42	42,596	41	42,637	42	0,4225
0,70	45,686	45	45,731	46	45,777	45	45,822	45	45,867	45	45,912	45	0,4900
0,75	48,943	49	48,992	49	49,041	48	49,089	49	49,138	48	49,186	48	0,5625
0,80	52,200	51	52,251	52	52,303	52	52,355	52	52,407	51	52,458	52	0,6400
0,85	55,455	55	55,510	55	55,565	55	55,620	55	55,675	55	55,730	55	0,7225
0,90	58,708	58	58,766	59	58,825	58	58,883	58	58,941	59	59,000	58	0,8100
0,95	61,960	62	62,022	61	62,083	62	62,145	61	62,206	62	62,268	61	0,9025
1,00	65,211	65	65,276	64	65,340	65	65,405	65	65,470	65	65,535	64	1,0000
	12,7513		12,8018		12,8525		12,9032		12,9541		13,0050		c^2

$\frac{1}{2} \cdot (r+r'')^2$ or $r^2+r''^2$ nearly.

	652	653	654	655	656	657	
1	65	65	65	66	66	66	1
2	130	131	131	131	131	131	2
3	196	196	196	197	197	197	3
4	261	261	262	262	262	263	4
5	326	327	327	328	328	329	5
6	391	392	392	393	394	394	6
7	456	457	458	459	459	460	7
8	522	522	523	524	525	526	8
9	587	588	589	590	590	591	9

Prop. parts for the sum of the Radii.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	1	1	1	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	1	2	2	2	3	3	4
5	1	1	2	2	3	3	4	4	5
6	1	1	2	2	3	4	4	5	5
7	1	1	2	3	4	4	5	6	6
8	1	2	2	3	4	5	6	6	7
9	1	2	3	4	5	5	6	7	8
10	1	2	3	4	5	6	7	8	9
11	1	2	3	4	6	7	8	9	10
12	1	2	4	5	6	7	8	10	11
13	1	3	4	5	7	8	9	10	12
14	1	3	4	6	7	8	10	11	13
15	2	3	5	6	8	9	11	12	14
16	2	3	5	6	8	10	11	13	14
17	2	3	5	7	9	10	12	14	15
18	2	4	5	7	9	11	13	14	16
19	2	4	6	8	10	11	13	15	17
20	2	4	6	8	10	12	14	16	18
21	2	4	6	8	11	13	15	17	19
22	2	4	7	9	11	13	15	18	20
23	2	5	7	9	12	14	16	18	21
24	2	5	7	10	12	14	17	19	22
25	3	5	8	10	13	15	18	20	23
26	3	5	8	10	13	16	18	21	23
27	3	5	8	11	14	16	19	22	24
28	3	6	8	11	14	17	20	22	25
29	3	6	9	12	15	17	20	23	26
30	3	6	9	12	15	18	21	24	27
31	3	6	9	12	16	19	22	25	28
32	3	6	10	13	16	19	22	26	29
33	3	7	10	13	17	20	23	26	30
34	3	7	10	14	17	20	24	27	31
35	4	7	11	14	18	21	25	28	32
36	4	7	11	14	18	22	25	29	32
37	4	7	11	15	19	22	26	30	33
38	4	8	11	15	19	23	27	30	34
39	4	8	12	16	20	23	27	31	35
40	4	8	12	16	20	24	28	32	36
41	4	8	12	16	21	25	29	33	37
42	4	8	13	17	21	25	29	34	38
43	4	9	13	17	22	26	30	34	39
44	4	9	13	18	22	26	31	35	40
45	5	9	14	18	23	27	32	36	41
46	5	9	14	18	23	28	32	37	41
47	5	9	14	19	24	28	33	38	42
48	5	10	14	19	24	29	34	38	43
49	5	10	15	20	25	29	34	39	44
50	5	10	15	20	25	30	35	40	45
51	5	10	15	20	26	31	36	41	46
52	5	10	16	21	26	31	36	42	47
53	5	11	16	21	27	32	37	42	48
54	5	11	16	22	27	32	38	43	49
55	6	11	17	22	28	33	39	44	50
56	6	11	17	22	28	34	39	45	50
57	6	11	17	23	29	34	40	46	51
58	6	12	17	23	29	35	41	46	52
59	6	12	18	24	30	35	41	47	53
60	6	12	18	24	30	36	42	48	54
61	6	12	18	24	31	37	43	49	55
62	6	12	19	25	31	37	43	50	56
63	6	13	19	25	32	38	44	50	57
64	6	13	19	26	32	38	45	51	58
65	7	13	20	26	33	39	46	52	59
66	7	13	20	26	33	40	46	53	59
67	7	13	20	27	34	40	47	54	60
68	7	14	20	27	34	41	48	54	61
69	7	14	21	28	35	41	48	55	62
70	7	14	21	28	35	42	49	56	63
80	8	16	24	32	40	48	56	64	72
90	9	18	27	36	45	54	63	72	81
100	10	20	30	40	50	60	70	80	90

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Chord c.	5,11 Days	dif.	5,12 Days	dif.	5,13 Days	dif.	5,14 Days	dif.	5,15 Days	dif.	5,16 Days	dif.	5,17 Days	dif.	5,18 Days	dif.	5,19 Days	dif.	5,20 Days.	
	Sum of the Radii $r + r''$.																			
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000	0,0000
0,01	0,657	1	0,658	0	0,658	1	0,659	1	0,660	0	0,660	1	0,661	1	0,662	0	0,662	1	0,663	0,0001
0,02	1,314	1	1,315	2	1,317	1	1,318	1	1,319	2	1,321	1	1,322	1	1,323	1	1,324	2	1,326	0,0004
0,03	1,971	2	1,973	2	1,975	2	1,977	2	1,979	2	1,981	2	1,983	2	1,985	2	1,987	1	1,988	0,0009
0,04	2,628	3	2,631	2	2,633	3	2,636	2	2,638	3	2,641	3	2,644	2	2,646	3	2,649	2	2,651	0,0016
0,05	3,285	3	3,288	4	3,292	3	3,295	3	3,298	3	3,301	3	3,304	4	3,308	3	3,311	3	3,314	0,0025
0,06	3,942	4	3,946	4	3,950	4	3,954	4	3,958	4	3,962	3	3,965	4	3,969	4	3,973	4	3,977	0,0036
0,07	4,599	5	4,604	4	4,608	5	4,613	4	4,617	5	4,622	4	4,626	5	4,631	4	4,635	5	4,640	0,0049
0,08	5,256	5	5,261	6	5,267	5	5,272	5	5,277	5	5,282	5	5,287	5	5,292	5	5,297	5	5,302	0,0064
0,09	5,913	6	5,919	6	5,925	6	5,931	5	5,936	6	5,942	6	5,948	6	5,954	5	5,959	6	5,965	0,0081
0,10	6,570	7	6,577	6	6,583	7	6,590	6	6,596	6	6,602	7	6,609	6	6,615	7	6,622	6	6,628	0,0100
0,11	7,227	7	7,234	8	7,242	7	7,249	7	7,256	7	7,263	7	7,270	7	7,277	7	7,284	7	7,291	0,0121
0,12	7,884	8	7,892	8	7,900	8	7,908	7	7,915	8	7,923	8	7,931	7	7,938	8	7,946	8	7,954	0,0144
0,13	8,541	9	8,550	8	8,558	8	8,566	9	8,575	8	8,583	8	8,591	9	8,600	8	8,608	8	8,616	0,0169
0,14	9,198	9	9,207	9	9,216	9	9,225	9	9,234	9	9,243	9	9,252	9	9,261	9	9,270	9	9,279	0,0196
0,15	9,855	10	9,865	10	9,875	9	9,884	10	9,894	10	9,904	9	9,913	10	9,923	9	9,932	10	9,942	0,0225
0,16	10,512	11	10,523	10	10,533	10	10,543	10	10,553	11	10,564	10	10,574	10	10,584	10	10,594	11	10,605	0,0256
0,17	11,169	11	11,180	11	11,191	11	11,202	11	11,213	11	11,224	11	11,235	11	11,246	10	11,256	11	11,267	0,0289
0,18	11,826	12	11,838	11	11,849	12	11,861	11	11,872	12	11,884	12	11,896	11	11,907	12	11,919	11	11,930	0,0324
0,19	12,483	12	12,495	13	12,508	12	12,520	12	12,532	12	12,544	12	12,556	12	12,568	13	12,581	12	12,593	0,0361
0,20	13,140	13	13,153	13	13,166	13	13,179	13	13,192	12	13,204	13	13,217	13	13,230	13	13,243	12	13,255	0,0400
0,21	13,797	14	13,811	13	13,824	14	13,838	13	13,851	13	13,864	14	13,878	13	13,891	14	13,905	13	13,918	0,0441
0,22	14,454	14	14,468	14	14,482	14	14,496	14	14,510	15	14,525	14	14,539	14	14,553	14	14,567	14	14,581	0,0484
0,23	15,111	15	15,126	14	15,140	15	15,155	15	15,170	15	15,185	14	15,199	15	15,214	15	15,229	14	15,243	0,0529
0,24	15,768	15	15,783	16	15,799	15	15,814	15	15,829	16	15,845	15	15,860	15	15,875	16	15,891	15	15,906	0,0576
0,25	16,425	16	16,441	16	16,457	16	16,473	16	16,489	16	16,505	16	16,521	16	16,537	16	16,553	16	16,569	0,0625
0,26	17,081	17	17,098	17	17,115	17	17,132	16	17,148	17	17,165	16	17,181	17	17,198	17	17,215	16	17,231	0,0676
0,27	17,738	18	17,756	17	17,773	17	17,790	18	17,808	17	17,825	17	17,842	17	17,859	18	17,877	17	17,894	0,0729
0,28	18,395	18	18,413	18	18,431	18	18,449	18	18,467	18	18,485	18	18,503	18	18,521	18	18,539	17	18,556	0,0784
0,29	19,052	19	19,071	18	19,089	19	19,108	18	19,126	19	19,145	18	19,163	19	19,182	19	19,201	18	19,219	0,0841
0,30	19,709	19	19,728	19	19,747	19	19,766	20	19,786	19	19,805	19	19,824	19	19,843	19	19,862	20	19,882	0,0900
0,31	20,365	20	20,385	20	20,405	20	20,425	20	20,445	20	20,465	20	20,485	20	20,505	19	20,524	20	20,544	0,0961
0,32	21,022	21	21,043	20	21,063	21	21,084	20	21,104	21	21,125	20	21,145	21	21,166	20	21,186	21	21,207	0,1024
0,33	21,679	21	21,700	21	21,721	21	21,742	22	21,764	21	21,785	21	21,806	21	21,827	21	21,848	21	21,869	0,1089
0,34	22,336	21	22,357	22	22,379	22	22,401	22	22,423	22	22,445	21	22,466	22	22,488	22	22,510	22	22,532	0,1156
0,35	22,992	23	23,015	22	23,037	23	23,060	22	23,082	23	23,105	22	23,127	22	23,149	23	23,172	22	23,194	0,1225
0,36	23,649	23	23,672	23	23,695	23	23,718	23	23,741	23	23,764	23	23,787	23	23,810	23	23,833	23	23,856	0,1296
0,37	24,306	23	24,329	24	24,353	24	24,377	24	24,401	23	24,424	24	24,448	24	24,472	23	24,495	24	24,519	0,1369
0,38	24,962	25	24,987	24	25,011	24	25,035	25	25,060	24	25,084	24	25,108	25	25,133	24	25,157	24	25,181	0,1444
0,39	25,619	25	25,644	25	25,669	25	25,694	25	25,719	25	25,744	25	25,769	25	25,794	25	25,819	25	25,844	0,1521
0,40	26,275	26	26,301	26	26,327	25	26,352	26	26,378	26	26,404	25	26,429	26	26,455	25	26,480	26	26,506	0,1600
0,41	26,932	26	26,958	27	26,985	26	27,011	26	27,037	26	27,063	27	27,090	26	27,116	26	27,142	26	27,168	0,1681
0,42	27,588	27	27,615	27	27,642	27	27,669	27	27,696	27	27,723	27	27,750	27	27,777	27	27,804	26	27,830	0,1764
0,43	28,245	27	28,272	28	28,300	28	28,328	27	28,355	28	28,383	27	28,410	28	28,438	27	28,465	28	28,493	0,1849
0,44	28,901	29	28,930	28	28,958	28	28,986	28	29,014	28	29,042	29	29,071	28	29,099	28	29,127	28	29,155	0,1936
0,45	29,558	29	29,587	29	29,616	28	29,644	29	29,673	29	29,702	29	29,731	29	29,760	28	29,788	29	29,817	0,2025
0,50	32,839	33	32,872	32	32,904	32	32,936	32	32,968	32	33,000	32	33,032	32	33,064	32	33,096	32	33,128	0,2500
0,55	36,120	36	36,156	35	36,191	35	36,226	36	36,262	35	36,297	35	36,332	35	36,367	35	36,402	36	36,438	0,3025
0,60	39,400	39	39,439	39	39,478	38	39,516	39	39,555	38	39,593	38	39,631	39	39,670	38	39,708	39	39,747	0,3600
0,65	42,679	42	42,721	42	42,763	42	42,805	42	42,847	41	42,888	42	42,930	41	42,971	42	43,013	42	43,055	0,4225
0,70	45,957	45	46,002	46	46,048	45	46,093	45	46,138	44	46,182	45	46,227	45	46,272	45	46,317	45	46,362	0,4900
0,75	49,234	49	49,283	48	49,331	48	49,379	48	49,427	48	49,475	49	49,524	48	49,572	48	49,620	48	49,668	0,5625
0,80	52,510	52	52,562	51	52,613	52	52,665	51	52,716	51	52,767	52	52,819	51	52,870	51	52,921	51	52,972	0,6400
0,85	55,785	54	55,839	55	55,894	55	55,949	54	56,003	55	56,058	55	56,113	54	56,167	55	56,222	54	56,276	0,7225
0,90	59,058	58	59,116	58	59,174	58	59,232	58	59,290	57	59,347	58	59,405	58	59,463	58	59,521	57	59,578	0,8100
0,95	62,329	62	62,391	61	62,452	61	62,513	61	62,574	61	62,635	61	62,696	61	62,757	61	62,818	61	62,879	0,9025
1,00	65,599	65	65,664	64	65,728	65	65,793	64	65,857	65	65,922	64	65,986	64	66,050	64	66,114	64	66,178	1,0000
	13,0561		13,1072		13,1585		13,2098		13,2613		13,3128		13,3645		13,4162		13,4681		13,5200	c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	656	657	658	659	660	661	662	663	
1	66	66	66	66	66	66	66	66	1
2	131	131	132	132	132	132	132	133	2
3	197	197	197	198	198	198	199	199	3
4	262	263	263	264	264	264	265	265	4
5	328	329	329	330	330	331	331	332	5
6	394	394	395	395	396	397	397	398	6
7	459	460	461	461	462	463	463	464	7
8	525	526	526	527	528	529	530	530	8
9	590	591	592	593	594	595	596	597	9

TABLE II.—To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord c.	5,20		5,30		5,40		5,50		5,60		5,70		5,80		5,90		6,00		6,10		
	Days.	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,663	6	0,669	6	0,675	7	0,682	6	0,688	6	0,694	6	0,700	6	0,706	6	0,712	6	0,718	6	0,0001
0,02	1,326	12	1,338	13	1,351	12	1,363	13	1,376	12	1,388	12	1,400	12	1,412	12	1,424	12	1,436	11	0,0004
0,03	1,988	19	2,007	19	2,026	19	2,045	18	2,063	19	2,082	18	2,100	18	2,118	18	2,136	18	2,154	17	0,0009
0,04	2,651	26	2,677	25	2,702	25	2,727	24	2,751	25	2,776	24	2,800	24	2,824	24	2,848	24	2,872	23	0,0016
0,05	3,314	32	3,346	31	3,377	31	3,408	31	3,439	31	3,470	30	3,500	30	3,530	30	3,560	29	3,589	30	0,0025
0,06	3,977	38	4,015	38	4,053	37	4,090	37	4,127	37	4,164	36	4,200	36	4,236	36	4,272	35	4,307	35	0,0036
0,07	4,640	44	4,684	44	4,728	44	4,772	43	4,815	43	4,858	42	4,900	42	4,942	42	4,984	41	5,025	41	0,0049
0,08	5,302	51	5,353	50	5,403	50	5,453	50	5,503	49	5,552	48	5,600	48	5,648	48	5,696	47	5,743	47	0,0064
0,09	5,965	57	6,022	57	6,079	56	6,135	55	6,190	55	6,245	55	6,300	54	6,354	54	6,408	53	6,461	53	0,0081
0,10	6,628	63	6,691	63	6,754	63	6,817	61	6,878	61	6,939	61	7,000	60	7,060	60	7,120	59	7,179	58	0,0100
0,11	7,291	70	7,361	69	7,430	68	7,498	68	7,566	67	7,633	67	7,700	66	7,766	66	7,832	65	7,897	64	0,0121
0,12	7,954	76	8,030	75	8,105	75	8,180	74	8,254	73	8,327	73	8,400	72	8,472	72	8,544	70	8,614	71	0,0144
0,13	8,616	83	8,699	81	8,780	81	8,861	81	8,942	79	9,021	79	9,100	78	9,178	77	9,255	77	9,332	76	0,0169
0,14	9,279	89	9,368	88	9,456	87	9,543	86	9,629	86	9,715	85	9,800	84	9,884	83	9,967	83	10,050	82	0,0196
0,15	9,942	95	10,037	94	10,131	94	10,225	92	10,317	92	10,409	91	10,500	90	10,590	89	10,679	89	10,768	88	0,0225
0,16	10,605	101	10,706	101	10,807	99	10,906	99	11,005	98	11,103	97	11,200	96	11,296	95	11,391	95	11,486	94	0,0256
0,17	11,267	108	11,375	107	11,482	106	11,588	105	11,693	104	11,797	103	11,900	102	12,002	101	12,103	101	12,204	99	0,0289
0,18	11,930	114	12,044	113	12,157	112	12,269	111	12,380	111	12,491	109	12,600	108	12,708	107	12,815	106	12,921	106	0,0324
0,19	12,593	120	12,713	120	12,833	118	12,951	117	13,068	116	13,184	116	13,300	114	13,414	113	13,527	112	13,639	112	0,0361
0,20	13,255	127	13,382	126	13,508	125	13,633	123	13,756	122	13,878	121	13,999	121	14,120	119	14,239	118	14,357	117	0,0400
0,21	13,918	133	14,051	132	14,183	131	14,314	130	14,444	128	14,572	127	14,699	127	14,826	125	14,951	124	15,075	123	0,0441
0,22	14,581	139	14,720	139	14,859	137	14,996	135	15,131	135	15,266	133	15,399	132	15,531	132	15,663	130	15,793	129	0,0484
0,23	15,243	146	15,389	145	15,534	143	15,677	142	15,819	141	15,960	139	16,099	138	16,237	137	16,374	136	16,510	135	0,0529
0,24	15,906	152	16,058	151	16,209	150	16,359	148	16,507	146	16,653	146	16,799	144	16,943	143	17,086	142	17,228	141	0,0576
0,25	16,569	158	16,727	157	16,884	156	17,040	154	17,194	153	17,347	152	17,499	150	17,649	149	17,798	148	17,946	146	0,0625
0,26	17,231	165	17,396	164	17,560	162	17,722	160	17,882	159	18,041	158	18,199	156	18,355	155	18,510	154	18,664	152	0,0676
0,27	17,894	171	18,065	170	18,235	168	18,403	167	18,570	165	18,735	163	18,898	163	19,061	161	19,222	159	19,381	158	0,0729
0,28	18,556	178	18,734	176	18,910	174	19,084	173	19,257	172	19,429	169	19,598	169	19,767	166	19,933	166	20,099	164	0,0784
0,29	19,219	184	19,403	182	19,585	181	19,766	179	19,945	177	20,122	176	20,298	174	20,472	173	20,645	172	20,817	170	0,0841
0,30	19,882	190	20,072	189	20,261	186	20,447	185	20,632	184	20,816	182	20,998	180	21,178	179	21,357	177	21,534	176	0,0900
0,31	20,544	197	20,741	195	20,936	193	21,129	191	21,320	190	21,510	188	21,698	186	21,884	185	22,069	183	22,252	182	0,0961
0,32	21,207	203	21,410	201	21,611	199	21,810	198	22,008	195	22,203	194	22,397	193	22,590	190	22,780	190	22,970	187	0,1024
0,33	21,869	210	22,079	207	22,286	205	22,491	204	22,695	202	22,897	200	23,097	198	23,295	197	23,492	195	23,687	194	0,1089
0,34	22,532	215	22,747	214	22,961	212	23,173	210	23,383	208	23,591	206	23,797	204	24,001	203	24,204	201	24,405	199	0,1156
0,35	23,194	222	23,416	220	23,636	218	23,854	216	24,070	214	24,284	213	24,497	210	24,707	209	24,916	206	25,122	206	0,1225
0,36	23,856	229	24,085	226	24,311	224	24,535	223	24,758	220	24,978	218	25,196	217	25,413	214	25,627	213	25,840	211	0,1296
0,37	24,519	235	24,754	232	24,986	231	25,217	228	25,445	226	25,671	225	25,896	222	26,118	221	26,339	219	26,558	216	0,1369
0,38	25,181	241	25,422	239	25,661	237	25,898	235	26,133	232	26,365	230	26,595	229	26,824	226	27,050	225	27,275	223	0,1444
0,39	25,844	247	26,091	245	26,336	243	26,579	241	26,820	239	27,059	236	27,295	235	27,530	232	27,762	231	27,993	228	0,1521
0,40	26,506	254	26,760	251	27,011	249	27,260	247	27,507	245	27,752	243	27,995	240	28,235	239	28,474	236	28,710	235	0,1600
0,41	27,168	260	27,428	258	27,686	256	27,942	253	28,195	251	28,446	248	28,694	247	28,941	244	29,185	243	29,428	240	0,1681
0,42	27,830	267	28,097	264	28,361	262	28,623	259	28,882	257	29,139	255	29,394	252	29,646	251	29,897	248	30,145	246	0,1764
0,43	28,493	273	28,766	270	29,036	268	29,304	265	29,569	264	29,833	260	30,093	259	30,352	256	30,608	255	30,863	252	0,1849
0,44	29,155	279	29,434	277	29,711	274	29,985	272	30,257	269	30,526	267	30,793	264	31,057	263	31,320	260	31,580	258	0,1936
0,45	29,817	286	30,103	283	30,386	280	30,666	278	30,944	275	31,219	273	31,492	271	31,763	268	32,031	266	32,297	264	0,2025
0,50	33,128	317	33,445	315	33,760	311	34,071	309	34,380	306	34,686	303	34,989	301	35,290	298	35,588	296	35,884	293	0,2500
0,55	36,438	349	36,787	346	37,133	343	37,476	339	37,815	337	38,152	334	38,486	331	38,817	328	39,145	325	39,470	323	0,3025
0,60	39,747	381	40,128	377	40,505	374	40,879	371	41,250	367	41,617	365	41,982	361	42,343	358	42,701	355	43,056	352	0,3600
0,65	43,055	413	43,468	409	43,877	405	44,282	402	44,684	398	45,082	395	45,477	391	45,868	388	46,256	384	46,640	382	0,4225
0,70	46,362	445	46,807	440	47,247	437	47,684	433	48,117	429	48,546	425	48,971	421	49,392	418	49,810	414	50,224	411	0,4900
0,75	49,668	476	50,144	473	50,617	468	51,085	464	51,549	459	52,008	456	52,464	451	52,915	448	53,363	444	53,807	440	0,5625
0,80	52,972	509	53,481	504	53,985	500	54,485	495	54,980	490	55,470	486	55,956	482	56,438	478	56,916	473	57,389	470	0,6400
0,85	56,276	541	56,817	536	57,353	530	57,883	526	58,409	521	58,930	517	59,447	512	59,959	508	60,467	503	60,970	500	0,7225
0,90	59,578	573	60,151	568	60,719	562	61,281	557	61,838	552	62,390	547	62,937	542	63,479	538	64,017	534	64,551	528	0,8100
0,95	62,879	605	63,484	599	64,083	594	64,677	588	65,265	583	65,848	578	66,426	573	66,999	567	67,566	564	68,130	558	0,9025
1,00	66,178	637	66,815	632	67,447	625	68,072	619	68,691	614	69,305	608	69,913	604	70,517	597	71,114	593	71,707	588	1,0000
	13,5200		14,0450		14,5800		15,1250		15,6800		16,2450		16,8200		17,4050		18,0000		18,6050		c^2

$\frac{1}{2}\cdot(r+r'')^2$ or $r^2+r''^2$ nearly.

	662	665	668	671	674	677	680	683	686	689	692	695	698	701	704	707	710	713	716	719	
1	66	67	67	67	67	68	68	68	69	69	69	70	70	70	70	71	71	71	72	72	1
2	132	133	134	134	135	135	136	137	137	138	138	139	140	140	141	141	142	143	143	144	2
3	199	200	200	201	202	203	204	205	206	207	208	209	209	210	211	212	213	214	215	216	3
4	265	266	267	268	270	271	272	273	274	276	277	278	279	280	282	283	284	285	286	288	4
5	331	333	334	336	337	339	340	342	343	345	346	348	349	351	352	354	355	357	358	360	5
6	397	399	401	403	404	406	408	410	412	413	415	417	419	421	422	424	426	428	430	431	6
7	463	466	468	470	472	474	476	478	480	482	484	487	489	491	493	495	497	499	501	503	7
8	530	532	534	537	539	542	544	546	549	551	554	556	558	561	563	566	568	570	573	575	8
9	596	599	601	604	607	609	612	615	617	620	623	626	628	631	634	636	639	642	644	647	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	6,20 Days	dif.	6,30 Days	dif.	6,40 Days	dif.	6,50 Days	dif.	6,60 Days	dif.	6,70 Days	dif.	6,80 Days	dif.	6,90 Days	dif.	7,00 Days	dif.	7,10 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,724	6	0,730	5	0,735	6	0,741	6	0,747	5	0,752	6	0,758	6	0,764	5	0,769	5	0,774	6	0,0001
0,02	1,447	12	1,459	12	1,471	11	1,482	11	1,493	12	1,505	11	1,516	11	1,527	11	1,538	11	1,549	11	0,0004
0,03	2,171	18	2,189	17	2,206	17	2,223	17	2,240	17	2,257	17	2,274	17	2,291	16	2,307	16	2,323	17	0,0009
0,04	2,895	23	2,918	23	2,941	23	2,964	23	2,987	22	3,009	23	3,032	22	3,054	22	3,076	22	3,098	22	0,0016
0,05	3,619	29	3,648	29	3,677	28	3,705	29	3,734	28	3,762	28	3,790	28	3,818	27	3,845	27	3,872	28	0,0025
0,06	4,342	35	4,377	35	4,412	34	4,446	34	4,480	34	4,514	34	4,548	33	4,581	33	4,614	33	4,647	33	0,0036
0,07	5,066	41	5,107	40	5,147	40	5,187	40	5,227	40	5,267	39	5,306	39	5,345	38	5,383	38	5,421	38	0,0049
0,08	5,790	46	5,836	47	5,883	45	5,928	46	5,974	45	6,019	45	6,064	44	6,108	44	6,152	44	6,196	43	0,0064
0,09	6,514	52	6,566	52	6,618	51	6,669	51	6,720	51	6,771	51	6,822	50	6,872	49	6,921	49	6,970	49	0,0081
0,10	7,237	58	7,295	58	7,353	57	7,410	57	7,467	57	7,524	55	7,579	56	7,635	55	7,690	55	7,745	54	0,0100
0,11	7,961	64	8,025	63	8,088	63	8,151	63	8,214	62	8,276	61	8,337	61	8,398	61	8,459	60	8,519	60	0,0121
0,12	8,685	70	8,755	69	8,824	68	8,892	69	8,961	67	9,028	67	9,095	67	9,162	66	9,228	66	9,294	65	0,0144
0,13	9,408	76	9,484	75	9,559	74	9,633	74	9,707	74	9,781	72	9,853	72	9,925	72	9,997	71	10,068	71	0,0169
0,14	10,132	82	10,214	80	10,294	80	10,374	80	10,454	79	10,533	78	10,611	78	10,689	77	10,766	77	10,843	76	0,0196
0,15	10,856	87	10,943	87	11,030	85	11,115	86	11,201	84	11,285	84	11,369	83	11,452	83	11,535	82	11,617	82	0,0225
0,16	11,580	93	11,673	92	11,765	91	11,856	91	11,947	90	12,037	90	12,127	89	12,216	88	12,304	88	12,392	87	0,0256
0,17	12,303	99	12,402	98	12,500	97	12,597	97	12,694	96	12,790	95	12,885	94	12,979	94	13,073	93	13,166	92	0,0289
0,18	13,027	105	13,132	103	13,235	103	13,338	103	13,441	101	13,542	101	13,643	100	13,743	99	13,842	98	13,940	98	0,0324
0,19	13,751	110	13,861	110	13,971	108	14,079	108	14,187	107	14,294	107	14,401	105	14,506	105	14,611	104	14,715	103	0,0361
0,20	14,474	116	14,590	116	14,706	114	14,820	114	14,934	113	15,047	112	15,159	111	15,270	110	15,380	109	15,489	109	0,0400
0,21	15,198	122	15,320	121	15,441	120	15,561	120	15,681	118	15,799	117	15,916	117	16,033	116	16,149	115	16,264	114	0,0441
0,22	15,922	127	16,049	127	16,176	126	16,302	125	16,427	124	16,551	123	16,674	122	16,796	122	16,918	120	17,038	120	0,0484
0,23	16,645	134	16,779	133	16,912	131	17,043	131	17,174	129	17,303	129	17,432	128	17,560	127	17,687	126	17,813	125	0,0529
0,24	17,369	139	17,508	139	17,647	137	17,784	136	17,920	136	18,056	134	18,190	133	18,323	133	18,456	131	18,587	130	0,0576
0,25	18,092	146	18,238	144	18,382	143	18,525	142	18,667	141	18,808	140	18,948	139	19,087	137	19,224	137	19,361	136	0,0625
0,26	18,816	151	18,967	150	19,117	149	19,266	148	19,414	146	19,560	146	19,706	144	19,850	143	19,993	143	20,136	141	0,0676
0,27	19,539	157	19,696	156	19,852	155	20,007	153	20,160	152	20,312	151	20,463	150	20,613	149	20,762	148	20,910	147	0,0729
0,28	20,263	163	20,426	161	20,587	161	20,748	159	20,907	158	21,065	156	21,221	156	21,377	154	21,531	153	21,684	153	0,0784
0,29	20,987	168	21,155	168	21,323	166	21,489	164	21,653	164	21,817	162	21,979	161	22,140	160	22,300	159	22,459	157	0,0841
0,30	21,710	175	21,885	173	22,058	171	22,229	171	22,400	169	22,569	168	22,737	166	22,903	166	23,069	164	23,233	163	0,0900
0,31	22,434	180	22,614	179	22,793	177	22,970	176	23,146	175	23,321	174	23,495	172	23,667	171	23,838	169	24,007	169	0,0961
0,32	23,157	186	23,343	185	23,528	183	23,711	182	23,893	180	24,073	179	24,252	178	24,430	176	24,606	176	24,782	174	0,1024
0,33	23,881	192	24,073	190	24,263	189	24,452	187	24,639	186	24,825	185	25,010	183	25,193	182	25,375	181	25,556	179	0,1089
0,34	24,604	198	24,802	196	24,998	195	25,193	193	25,386	191	25,577	191	25,768	189	25,957	187	26,144	186	26,330	185	0,1156
0,35	25,328	203	25,531	202	25,733	200	25,933	199	26,132	198	26,330	195	26,525	195	26,720	193	26,913	191	27,104	191	0,1225
0,36	26,051	209	26,260	208	26,468	206	26,674	205	26,879	203	27,082	201	27,283	200	27,483	199	27,682	197	27,879	195	0,1296
0,37	26,774	216	26,990	213	27,203	212	27,415	210	27,625	209	27,834	207	28,041	205	28,246	204	28,450	203	28,653	201	0,1369
0,38	27,498	221	27,719	219	27,938	218	28,156	216	28,372	214	28,586	212	28,798	212	29,010	209	29,219	208	29,427	207	0,1444
0,39	28,221	227	28,448	225	28,673	223	28,896	222	29,118	220	29,338	218	29,556	217	29,773	215	29,988	213	30,201	212	0,1521
0,40	28,945	232	29,177	231	29,408	229	29,637	227	29,864	226	30,090	224	30,314	222	30,536	221	30,757	219	30,976	217	0,1600
0,41	29,668	238	29,906	237	30,143	235	30,378	233	30,611	231	30,842	229	31,071	228	31,299	226	31,525	225	31,750	223	0,1681
0,42	30,391	245	30,636	242	30,878	240	31,118	239	31,357	237	31,594	235	31,829	233	32,062	232	32,294	230	32,524	228	0,1764
0,43	31,115	250	31,365	248	31,613	246	31,859	244	32,103	243	32,346	240	32,586	239	32,825	238	33,063	235	33,298	234	0,1849
0,44	31,838	256	32,094	254	32,348	252	32,600	250	32,850	248	33,098	246	33,344	245	33,589	242	33,831	241	34,072	239	0,1936
0,45	32,561	262	32,823	260	33,083	257	33,340	256	33,596	254	33,850	252	34,102	250	34,352	248	34,600	246	34,846	245	0,2025
0,50	36,177	291	36,468	289	36,757	286	37,043	284	37,327	282	37,609	280	37,889	278	38,167	276	38,443	274	38,717	272	0,2500
0,55	39,793	320	40,113	317	40,430	315	40,745	313	41,058	310	41,368	308	41,676	306	41,982	303	42,285	301	42,586	300	0,3025
0,60	43,408	349	43,757	346	44,103	344	44,447	341	44,788	338	45,126	336	45,462	334	45,796	331	46,127	329	46,456	326	0,3600
0,65	47,022	378	47,400	375	47,775	373	48,148	369	48,517	367	48,884	364	49,248	362	49,610	358	49,968	356	50,324	354	0,4225
0,70	50,635	408	51,043	404	51,447	401	51,848	398	52,246	395	52,641	392	53,033	389	53,422	387	53,809	383	54,192	381	0,4900
0,75	54,247	437	54,684	434	55,118	429	55,547	427	55,974	423	56,397	421	56,818	417	57,235	414	57,649	411	58,060	408	0,5625
0,80	57,859	466	58,325	462	58,787	459	59,246	455	59,701	452	60,153	448	60,601	445	61,046	442	61,488	439	61,927	43	0,6400
0,85	61,470	495	61,965	491	62,456	488	62,944	484	63,428	480	63,908	476	64,384	473	64,857	469	65,326	466	65,792	46	0,7225
0,90	65,079	525	65,604	520	66,124	517	66,641	512	67,153	508	67,661	505	68,166	501	68,667	497	69,164	494	69,658	490	0,8100
0,95	68,688	554	69,242	549	69,791	545	70,336	541	70,877	537	71,414	533	71,947	529	72,476	525	73,001	521	73,522	517	0,9025
1,00	72,295	584	72,879	578	73,457	574	74,031	570	74,601	565	75,166	561	75,727	557	76,284	552	76,836	549	77,385	545	1,0000
	19,2200		19,8450		20,4800		21,1250		21,7800		22,4450		23,1200		23,8050		24,5000		25,2050		r^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	723	726	729	732	735	738	741	744	747	750	753	756	759	762	765	768	771	774	777	
1	72	73	73	73	74	74	74	74	75	75	75	76	76	76	77	77	77	77	78	1
2	145	145	146	146	147	148	148	149	149	150	151	151	152	152	153	154	154	155	155	2
3	217	218	219	220	221	221	222	223	224	225	226	227	228	229	230	230	231	232	233	3
4	289	290	292	293	294	295	296	298	299	300	301	302	304	305	306	307	308	310	311	4
5	362	363	365	366	368	369	371	372	374	375	377	378	380	381	383	384	386	387	389	5
6	434	436	437	439	441	443	445	446	448	450	452	454	455	457	459	461	463	464	466	6
7	506	508	510	512	515	517	519	521	523	525	527	529	531	533	536	538	540	542	544	7
8	578	581	583	586	588	590	593	595	598	600	602	605	607	610	612	614	617	619	622	8
9	651	653	656	659	662	664	667	670	672	675	678	680	683	686	689	691	694	697	699	9

TABLE II. — To find the time T; the sum of the radii $r+r''$, and the chord c being given.

Sum of the Radii $r+r''$.

Chord c.	7,20 Days	dif.	7,30 Days	dif.	7,40 Days	dif.	7,50 Days	dif	7,60 Days	dif.	7,70 Days	dif.	7,80 Days	dif.	7,90 Days	dif.	8,00 Days	dif.	8,10 Days	dif.	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,0000
0,01	0,780	5	0,785	6	0,791	5	0,796	5	0,801	6	0,807	5	0,812	5	0,817	5	0,822	5	0,827	5	0,0001
0,02	1,560	11	1,571	10	1,581	11	1,592	11	1,603	10	1,613	11	1,624	10	1,634	10	1,644	10	1,654	11	0,0004
0,03	2,340	16	2,356	16	2,372	16	2,388	16	2,404	16	2,420	15	2,435	16	2,451	15	2,466	16	2,482	15	0,0009
0,04	3,120	21	3,141	22	3,163	21	3,184	21	3,205	21	3,226	21	3,247	21	3,268	20	3,288	21	3,309	20	0,0016
0,05	3,900	27	3,927	26	3,953	27	3,980	27	4,007	26	4,033	26	4,059	26	4,085	26	4,111	25	4,136	26	0,0025
0,06	4,680	32	4,712	32	4,744	32	4,776	32	4,808	31	4,839	32	4,871	31	4,902	31	4,933	30	4,963	31	0,0036
0,07	5,459	38	5,497	38	5,535	37	5,572	37	5,609	37	5,646	36	5,682	37	5,719	36	5,755	36	5,791	35	0,0049
0,08	6,239	44	6,283	42	6,325	43	6,368	42	6,410	42	6,452	42	6,494	42	6,530	41	6,577	41	6,618	41	0,0064
0,09	7,019	49	7,068	48	7,116	48	7,164	48	7,212	47	7,259	47	7,306	47	7,353	46	7,399	46	7,445	46	0,0081
0,10	7,799	54	7,853	54	7,907	53	7,960	53	8,013	52	8,065	53	8,118	52	8,170	51	8,221	51	8,272	51	0,0100
0,11	8,579	59	8,638	59	8,697	59	8,756	58	8,814	58	8,872	57	8,929	58	8,987	56	9,043	57	9,100	56	0,0121
0,12	9,359	65	9,424	64	9,488	64	9,552	63	9,615	64	9,679	62	9,741	62	9,803	62	9,865	62	9,927	61	0,0144
0,13	10,139	70	10,209	70	10,279	69	10,348	69	10,417	68	10,485	68	10,553	67	10,620	67	10,687	67	10,754	66	0,0169
0,14	10,919	75	10,994	75	11,069	75	11,144	74	11,218	74	11,292	73	11,365	72	11,437	72	11,509	72	11,581	71	0,0196
0,15	11,699	81	11,780	80	11,860	80	11,940	79	12,019	79	12,098	78	12,176	78	12,254	78	12,332	76	12,408	77	0,0225
0,16	12,479	86	12,565	86	12,651	85	12,736	85	12,821	84	12,905	83	12,988	83	13,071	83	13,154	82	13,236	81	0,0256
0,17	13,258	92	13,350	91	13,441	91	13,532	90	13,622	89	13,711	89	13,800	88	13,888	88	13,976	87	14,063	86	0,0289
0,18	14,038	98	14,136	96	14,232	96	14,328	95	14,423	95	14,518	94	14,612	93	14,705	93	14,798	92	14,890	92	0,0324
0,19	14,818	103	14,921	102	15,023	101	15,124	100	15,224	100	15,324	99	15,423	99	15,522	98	15,620	97	15,717	97	0,0361
0,20	15,598	108	15,706	107	15,813	107	15,920	106	16,026	105	16,131	104	16,235	104	16,339	103	16,442	102	16,544	102	0,0400
0,21	16,378	113	16,491	113	16,604	112	16,716	111	16,827	110	16,937	110	17,047	109	17,156	108	17,264	108	17,372	106	0,0441
0,22	17,158	118	17,276	118	17,394	118	17,512	116	17,628	116	17,744	114	17,858	115	17,973	113	18,086	113	18,199	112	0,0484
0,23	17,938	124	18,062	123	18,185	123	18,308	121	18,429	121	18,550	120	18,670	119	18,789	119	18,908	118	19,026	117	0,0529
0,24	18,717	130	18,847	129	18,976	127	19,103	127	19,230	127	19,357	125	19,482	124	19,606	124	19,730	123	19,853	122	0,0576
0,25	19,497	135	19,632	134	19,766	133	19,899	133	20,032	131	20,163	130	20,293	130	20,423	129	20,552	128	20,680	127	0,0625
0,26	20,277	140	20,417	140	20,557	138	20,695	138	20,833	136	20,969	136	21,105	135	21,240	134	21,374	133	21,507	133	0,0676
0,27	21,057	146	21,203	144	21,347	144	21,491	143	21,634	142	21,776	141	21,917	140	22,057	139	22,196	138	22,334	138	0,0729
0,28	21,837	151	21,988	150	22,138	140	22,287	148	22,435	147	22,582	146	22,728	146	22,874	144	23,018	144	23,162	142	0,0784
0,29	22,616	157	22,773	155	22,928	155	23,083	153	23,236	153	23,389	151	23,540	151	23,691	149	23,840	149	23,989	147	0,0841
0,30	23,396	162	23,558	161	23,719	160	23,879	158	24,037	158	24,195	157	24,352	155	24,507	155	24,662	154	24,816	152	0,0900
0,31	24,176	167	24,343	166	24,509	166	24,675	164	24,839	162	25,001	162	25,163	161	25,324	160	25,484	159	25,643	158	0,0961
0,32	24,956	172	25,128	172	25,300	170	25,470	170	25,640	168	25,808	167	25,975	166	26,141	165	26,306	164	26,470	163	0,1024
0,33	25,735	179	25,914	176	26,090	176	26,266	175	26,441	173	26,614	173	26,787	171	26,958	170	27,128	169	27,297	168	0,1089
0,34	26,515	184	26,699	182	26,881	181	27,062	180	27,242	179	27,421	177	27,598	177	27,775	175	27,950	174	28,124	173	0,1156
0,35	27,295	189	27,484	187	27,671	187	27,858	185	28,043	184	28,227	183	28,410	181	28,591	181	28,772	179	28,951	178	0,1225
0,36	28,074	195	28,269	193	28,462	192	28,654	190	28,844	189	29,033	188	29,221	187	29,408	186	29,594	184	29,778	183	0,1296
0,37	28,854	200	29,054	198	29,252	197	29,449	196	29,645	195	29,840	193	30,033	192	30,225	191	30,416	189	30,605	189	0,1369
0,38	29,634	205	29,839	204	30,043	202	30,245	201	30,446	200	30,646	198	30,844	198	31,042	195	31,237	195	31,432	194	0,1444
0,39	30,413	211	30,624	209	30,833	208	31,041	206	31,247	205	31,452	204	31,656	202	31,858	201	32,059	200	32,259	199	0,1521
0,40	31,193	216	31,409	215	31,624	213	31,837	211	32,048	211	32,259	208	32,467	208	32,675	206	32,881	205	33,086	204	0,1600
0,41	31,973	221	32,194	220	32,414	218	32,632	217	32,849	216	33,065	214	33,279	213	33,492	211	33,703	210	33,913	209	0,1681
0,42	32,752	227	32,979	225	33,204	224	33,428	222	33,650	221	33,871	219	34,090	218	34,308	217	34,525	215	34,740	214	0,1764
0,43	33,532	232	33,764	231	33,995	229	34,224	227	34,451	226	34,677	225	34,902	223	35,125	222	35,347	220	35,567	219	0,1849
0,44	34,311	238	34,549	236	34,785	234	35,019	233	35,252	232	35,484	229	35,713	229	35,942	227	36,169	225	36,394	224	0,1936
0,45	35,091	243	35,334	241	35,575	240	35,815	238	36,053	237	36,290	235	36,525	233	36,758	232	36,990	231	37,221	229	0,2025
0,50	38,989	270	39,259	268	39,527	266	39,793	265	40,058	263	40,321	261	40,582	259	40,841	258	41,099	256	41,355	255	0,2500
0,55	42,886	297	43,183	295	43,478	293	43,771	291	44,062	289	44,351	287	44,638	286	44,924	283	45,207	[illegible]	45,489	280	0,3025
0,60	46,782	324	47,106	322	47,428	320	47,748	317	48,065	316	48,381	313	48,694	312	49,006	309	49,315	308	49,623	306	0,3600
0,65	50,678	351	51,029	349	51,378	346	51,724	345	52,069	341	52,410	340	52,750	337	53,087	336	53,423	333	53,756	331	0,4225
0,70	54,573	379	54,952	375	55,327	373	55,700	371	56,071	368	56,439	366	56,805	364	57,169	361	57,530	359	57,889	356	0,4900
0,75	58,468	405	58,873	403	59,276	400	59,676	397	60,073	395	60,468	392	60,860	389	61,249	387	61,636	385	62,021	382	0,5625
0,80	62,362	432	62,794	430	63,224	427	63,651	423	64,074	421	64,495	418	64 913	416	65,329	413	65,742	410	66,152	408	0,6400
0,85	66,255	460	66,715	456	67,171	454	67,625	450	68,075	447	68,522	445	68,967	441	69,408	439	69,847	436	70,283	433	0,7225
0,90	70,148	486	70,634	484	71,118	480	71,598	477	72,075	473	72,548	471	73,019	468	73,487	464	73,951	462	74,413	459	0,8100
0,95	74,039	514	74,553	510	75,063	507	75,570	504	76,074	500	76,574	497	77,071	493	77,564	491	78,055	487	78,542	485	0,9025
1,00	77,930	541	78,471	537	79,008	534	79,542	530	80,072	527	80,599	523	81,122	519	81,641	517	82,158	513	82,671	510	1,0000
	25,9200		26,6450		27,3800		28,1250		28,8800		29,6450		30,4200		31,2050		32,0000		32,8050		c^2

$\frac{1}{2} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	779	782	785	788	791	794	797	800	803	806	809	812	815	818	821	824	827	830	
1	78	78	79	79	79	79	80	80	80	81	81	81	82	82	82	82	83	83	1
2	156	156	157	158	158	159	159	160	161	161	162	162	163	164	164	165	165	166	2
3	234	235	236	236	237	238	239	240	241	242	243	244	245	245	246	247	248	249	3
4	312	313	314	315	316	318	319	320	321	322	324	325	326	327	328	330	331	332	4
5	390	391	393	394	396	397	399	400	402	403	405	406	408	409	411	412	414	415	5
6	467	469	471	473	475	476	478	480	482	484	485	487	489	491	493	494	496	498	6
7	545	547	550	552	554	556	558	560	562	564	566	568	571	573	575	577	579	581	7
8	623	626	628	630	633	635	638	640	642	645	647	650	652	654	657	659	662	664	8
9	701	704	707	709	712	715	717	720	723	725	728	731	734	736	739	742	744	747	9

TABLE II. — To find the time T; the sum of the radii $r + r''$, and the chord c being given.

Sum of the Radii $r + r''$.

Chord c.	8,20 Days	dif.	8,30 Days	dif.	8,40 Days	dif.	8,50 Days	dif.	8,60 Days	dif.	8,70 Days	dif.	8,80 Days	dif.	8,90 Days	dif.	9,00 Days	dif.	9,10 Days	
0,00	0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000		0,000	0,0000
0,01	0,832	5	0,837	5	0,842	5	0,847	5	0,852	5	0,857	5	0,862	5	0,867	5	0,872	5	0,877	0,0001
0,02	1,665	10	1,675	10	1,685	10	1,695	10	1,705	10	1,715	9	1,724	10	1,734	10	1,744	10	1,754	0,0004
0,03	2,497	15	2,512	15	2,527	15	2,542	15	2,557	15	2,572	15	2,587	14	2,601	15	2,616	14	2,630	0,0009
0,04	3,329	21	3,350	20	3,370	20	3,390	20	3,410	19	3,429	20	3,449	20	3,469	19	3,488	19	3,507	0,0016
0,05	4,162	25	4,187	25	4,212	25	4,237	25	4,262	25	4,287	24	4,311	25	4,336	24	4,360	24	4,384	0,0025
0,06	4,994	30	5,024	30	5,054	30	5,084	30	5,114	30	5,144	29	5,173	30	5,203	29	5,232	29	5,261	0,0036
0,07	5,826	36	5,862	35	5,897	35	5,932	35	5,967	34	6,001	35	6,036	34	6,070	34	6,104	34	6,138	0,0049
0,08	6,659	40	6,699	40	6,739	40	6,779	40	6,819	40	6,859	39	6,898	39	6,937	39	6,976	39	7,015	0,0064
0,09	7,491	45	7,536	46	7,582	45	7,627	44	7,671	45	7,716	44	7,760	44	7,804	44	7,848	43	7,891	0,0081
0,10	8,323	51	8,374	50	8,424	50	8,474	50	8,524	49	8,573	49	8,622	49	8,671	49	8,720	48	8,768	0,0100
0,11	9,156	55	9,211	56	9,267	55	9,322	54	9,376	55	9,431	54	9,485	53	9,538	54	9,592	53	9,645	0,0121
0,12	9,988	61	10,049	60	10,109	60	10,169	60	10,229	59	10,288	59	10,347	58	10,405	59	10,464	58	10,522	0,0144
0,13	10,820	66	10,886	65	10,951	65	11,016	65	11,081	64	11,145	64	11,209	64	11,273	63	11,336	63	11,399	0,0169
0,14	11,652	71	11,723	71	11,794	70	11,864	69	11,933	69	12,002	69	12,071	69	12,140	68	12,208	67	12,275	0,0196
0,15	12,485	76	12,561	75	12,636	75	12,711	75	12,786	74	12,860	73	12,933	74	13,007	73	13,080	72	13,152	0,0225
0,16	13,317	81	13,398	80	13,478	80	13,558	80	13,638	79	13,717	79	13,796	78	13,874	78	13,952	77	14,029	0,0256
0,17	14,149	86	14,235	86	14,321	85	14,406	84	14,490	84	14,574	84	14,658	83	14,741	83	14,824	82	14,906	0,0289
0,18	14,982	91	15,073	90	15,163	90	15,253	90	15,343	89	15,432	88	15,520	88	15,608	87	15,695	87	15,782	0,0324
0,19	15,814	96	15,910	96	16,006	95	16,101	94	16,195	94	16,289	93	16,382	93	16,475	92	16,567	92	16,659	0,0361
0,20	16,646	101	16,747	101	16,848	100	16,948	99	17,047	99	17,146	98	17,244	98	17,342	97	17,439	97	17,536	0,0400
0,21	17,478	107	17,585	105	17,690	105	17,795	105	17,900	103	18,003	104	18,107	102	18,209	102	18,311	102	18,413	0,0441
0,22	18,311	111	18,422	111	18,533	110	18,643	109	18,752	109	18,861	108	18,969	107	19,076	107	19,183	106	19,289	0,0484
0,23	19,143	116	19,259	116	19,375	115	19,490	114	19,604	114	19,718	113	19,831	112	19,943	112	20,055	111	20,166	0,0529
0,24	19,975	122	20,097	120	20,217	120	20,337	120	20,457	118	20,575	118	20,693	117	20,810	117	20,927	116	21,043	0,0576
0,25	20,807	127	20,934	126	21,060	125	21,185	124	21,309	124	21,433	122	21,555	122	21,677	122	21,799	121	21,920	0,0625
0,26	21,640	131	21,771	131	21,902	130	22,032	129	22,161	129	22,290	127	22,417	128	22,545	126	22,671	125	22,796	0,0676
0,27	22,472	136	22,608	136	22,744	135	22,879	135	23,014	133	23,147	133	23,280	132	23,412	131	23,543	130	23,673	0,0729
0,28	23,304	142	23,446	141	23,587	140	23,727	139	23,866	138	24,004	138	24,142	137	24,279	136	24,415	135	24,550	0,0784
0,29	24,136	147	24,283	146	24,429	145	24,574	144	24,718	143	24,861	143	25,004	142	25,146	140	25,286	141	25,427	0,0841
0,30	24,968	152	25,120	151	25,271	150	25,421	149	25,570	149	25,719	147	25,866	147	26,013	145	26,158	145	26,303	0,0900
0,31	25,801	157	25,958	155	26,113	155	26,268	155	26,423	153	26,576	152	26,728	152	26,880	150	27,030	150	27,180	0,0961
0,32	26,633	162	26,795	161	26,956	160	27,116	159	27,275	158	27,433	157	27,590	157	27,747	155	27,902	155	28,057	0,1024
0,33	27,465	167	27,632	166	27,798	165	27,963	164	28,127	163	28,290	162	28,452	162	28,614	160	28,774	159	28,933	0,1089
0,34	28,297	172	28,469	171	28,640	170	28,810	169	28,979	168	29,147	167	29,314	167	29,481	165	29,646	164	29,810	0,1156
0,35	29,129	177	29,306	176	29,482	176	29,658	174	29,832	172	30,004	172	30,176	172	30,348	170	30,518	169	30,687	0,1225
0,36	29,961	183	30,144	181	30,325	180	30,505	179	30,684	178	30,862	177	31,039	175	31,214	175	31,389	174	31,563	0,1296
0,37	30,794	187	30,981	186	31,167	185	31,352	184	31,536	183	31,719	182	31,901	180	32,081	180	32,261	179	32,440	0,1369
0,38	31,626	192	31,818	191	32,009	190	32,199	189	32,388	188	32,576	187	32,763	185	32,948	185	33,133	184	33,317	0,1444
0,39	32,458	197	32,655	196	32,851	195	33,046	194	33,240	193	33,433	192	33,625	190	33,815	190	34,005	188	34,193	0,1521
0,40	33,290	202	33,492	202	33,694	200	33,894	198	34,092	198	34,290	197	34,487	195	34,682	195	34,877	193	35,070	0,1600
0,41	34,122	207	34,329	207	34,536	205	34,741	204	34,945	202	35,147	202	35,349	200	35,549	199	35,748	198	35,946	0,1681
0,42	34,954	213	35,167	211	35,378	210	35,588	209	35,797	207	36,004	207	36,211	205	36,416	204	36,620	203	36,823	0,1764
0,43	35,786	218	36,004	216	36,220	215	36,435	214	36,649	212	36,861	212	37,073	210	37,283	209	37,492	208	37,700	0,1849
0,44	36,618	223	36,841	221	37,062	220	37,282	219	37,501	217	37,718	217	37,935	215	38,150	214	38,364	212	38,576	0,1936
0,45	37,450	228	37,678	226	37,904	225	38,129	224	38,353	223	38,576	221	38,797	220	39,017	218	39,235	218	39,453	0,2025
0,50	41,610	253	41,863	252	42,115	250	42,365	248	42,613	248	42,861	245	43,106	245	43,351	243	43,594	241	43,835	0,2500
0,55	45,769	279	46,048	277	46,325	275	46,600	273	46,873	272	47,145	271	47,416	268	47,684	268	47,952	266	48,218	0,3025
0,60	49,929	303	50,232	302	50,534	300	50,834	299	51,133	297	51,430	295	51,725	293	52,018	291	52,309	290	52,599	0,3600
0,65	54,087	329	54,416	327	54,743	326	55,069	323	55,392	321	55,713	320	56,033	318	56,351	316	56,667	314	56,981	0,4225
0,70	58,245	355	58,600	352	58,952	350	59,302	349	59,651	346	59,997	344	60,341	342	60,683	341	61,024	338	61,362	0,4900
0,75	62,403	380	62,783	377	63,160	376	63,536	373	63,909	371	64,280	369	64,649	366	65,015	365	65,380	363	65,743	0,5625
0,80	66,560	405	66,965	403	67,368	400	67,768	398	68,166	396	68,562	394	68,956	391	69,347	389	69,736	387	70,123	0,6400
0,85	70,716	431	71,147	428	71,575	425	72,000	423	72,423	421	72,844	418	73,262	416	73,678	413	74,091	411	74,502	0,7225
0,90	74,872	456	75,328	453	75,781	451	76,232	448	76,680	445	77,125	443	77,568	440	78,008	438	78,446	435	78,881	0,8100
0,95	79,027	481	79,508	479	79,987	476	80,463	473	80,936	470	81,406	467	81,873	465	82,338	462	82,800	460	83,260	0,9025
1,00	83,181	507	83,688	504	84,192	501	84,693	498	85,191	495	85,686	492	86,178	489	86,667	487	87,154	483	87,637	1,0000
	33,6200		34,4450		35,2800		36,1250		36,9800		37,8450		38,7200		39,6050		40,5000		41,4050	c^2

$\frac{1}{4} \cdot (r + r'')^2$ or $r^2 + r''^2$ nearly.

	832	835	838	841	844	847	850	853	856	859	862	865	868	871	874	877	
1	83	84	84	84	84	85	85	85	85	85	86	87	87	87	87	88	1
2	166	167	168	168	169	169	170	171	171	172	172	173	174	174	175	175	2
3	250	251	251	252	253	254	255	256	257	258	259	260	260	261	262	263	3
4	333	334	335	336	338	339	340	341	342	344	345	346	347	348	350	351	4
5	416	418	419	421	422	424	425	427	428	430	431	433	434	436	437	439	5
6	499	501	503	505	506	508	510	512	514	515	517	519	521	523	524	526	6
7	582	585	587	589	591	593	595	597	599	601	603	606	608	610	612	614	7
8	666	668	670	673	675	678	680	682	685	687	690	692	694	697	699	702	8
9	749	752	754	757	760	762	765	768	770	773	776	779	781	784	787	789	9

TABLE III.

This table gives the true anomaly U of a comet, moving in a parabolic orbit, whose perihelion distance is equal to the mean distance of the sun from the earth *or unity;* the time from the perihelion being t' days. It was computed by Burckhardt, by means of the formula in book ii. § 23, Mécanique Céleste, [693 &c.], namely,

$$t'=27^{\text{days}},4038\ \ldots \times\{3\,\text{tang.}\,\tfrac{1}{2}U+\text{tang.}^3\,\tfrac{1}{2}U\}.$$

If the perihelion distance be D, and the time from the perihelion t days, we must put $t=D^{\frac{3}{2}}\,t'$ [693*a*]. If U be given, we must find, in this table, the corresponding value of log. t', and then the value of t from the formula,

$$\text{log.}\ t=\text{log.}\ t'+\tfrac{3}{2}\,\text{log.}\ D.$$

But if t be given, we must first find

$$\text{log.}\ t'=\text{log.}\ t-\tfrac{3}{2}\,\text{log.}\ D\,;$$

and then from this table the value of U, corresponding to this value of log. t'.

When t' is less than 5 days, the differences of log. t' vary so rapidly, that it is found convenient to vary the form of this part of the table. This is done by two different methods; the one proposed by Burckhardt, the other by Carlini; by means of the first six columns of the first page of Table III. Burckhardt's method consists in finding t', from log. t'; and then, with the argument t', we obtain the corresponding value of U, as in the first three columns of the table. In the next three columns, which contain the table of Carlini, the argument is log. t', as in all the rest of the table, and the corresponding number is log. $\frac{U}{t'}$, or log. $U-$ log. t'; U being expressed in sexagesimal minutes, and t', in days. This method of Carlini is very convenient, in the case which most frequently occurs; namely, where t is given to find U; for we have,

$$\text{log.}\ t'=\text{log.}\ t-\tfrac{3}{2}\,\text{log.}\ D\,;$$

$$\text{log.}\ U \text{ in minutes}=\text{log.}\ t'+\text{tab. number corresponding to log.}\ t'.$$

In the determination of the constant factor $27^{\text{days}},4038$, in the above value of t', we have neglected, as in [692′], the mass of the earth in comparison with that of the sun; as is usually done in computing tables of this kind. This omission may be rectified, by adding 0,0000006 to the argument log. t' in the table; or by subtracting 0,0000006 from the logarithm of t, in finding the log. t'.

Tables of this kind have been given by several authors, as Halley, La Caille, Zach, Pingré, &c.; but above all, by Delambre, who improved and extended this table very much, giving the values of U, corresponding to the argument t', taken at convenient intervals from $t'=0$ to $t'=200{,}000$ days. Burckhardt made an important improvement in Table III.; by taking for the argument log. t', *which is given by a previous calculation,* and by this means he saves the labor of finding t' from log. t'.

Barker published a general table of the parabolic motion of a comet, in which the argument is the true anomaly U, taken at intervals of 5^m; the corresponding numbers are what he calls the logarithms of the mean motion represented by

$$\text{log. mean motion}=\text{log.}\ t'-0{,}0398716,$$

and the numbers in Barker's table may be deduced from those of Burckhardt's in Table III., by putting

$$\text{Barker's log.}=\text{Burckhardt's log. (Table III.)}-0{,}0398716\,;$$

so that Table III., may be considered as an improvement on Barker's table, and may be used for the same purposes; the arguments, however, are in an inverted order. The argument in Barker's table being the true anomaly U; and in Burckhardt's table, the argument is the logarithm of the time t'.

EXAMPLES OF THE USE OF TABLE III.

EXAMPLE I.

Given the log. of perihelion distance, or log. $D.=9{,}7656500$	
Time from perihelion	$t=49^{\text{days}},25281$
To find the true anomaly U.	
$\frac{3}{2}\times$ log. D,	9,6484750
log. t,	1,6924310
log. $t'=$ log. $t-\frac{3}{2}$ log. D,	2,0439560
In Table III. $90^d\,16^m\,29^s,3$ corresponds to	2,043
$5\ \ 01\ ,9=315^s,8\times 0{,}9561$	
$U=90\ \ 21\ \ 31\ ,2$	

EXAMPLE II.

Given the log. of perihelion distance, or log. $D=9{,}7656500$	
True anomaly,	$U=90^d\,21^m\,31^s,2$
To find t.	
Log. 2,043 Table III., corresponds to	90 16 29 ,3
Difference, $301^s,9=$	5 01 ,9
Tabular difference, $315^s,8 : 301^s,9 :: 0{,}001$	: 0,0009560
Hence log. t',	$=2{,}0439560$
Add $\frac{3}{2}$ log. D,	$=9{,}6484750$
Sum is log. $t=$ log. $49^{\text{days}},2528$	$=1{,}6924310$

EXAMPLE III.

Given the log. of perihelion distance, or log. $D=0{,}1250000$	
Time from perihelion, $t=2$ days.	
To find the true anomaly U, by Burckhardt's method.	
$\frac{3}{2}\times$ log. D,	0,1875000
log. t,	0,3010300
log. $t'=$ log. $t-\frac{3}{2}$ log. D,	$=0{,}1135300$
$t'=1^{\text{day}},2987633$	
In Table III. $1^{\text{day}},2$ corresponds to	$1^d\,40^m\,20^s,6$
Tab. diff. $501^s,6\times 0{,}987633=495^s,4$	= 8 15 ,4
Sum is $U=$	1 48 36 ,0

EXAMPLE IV.

Given D, t' as in example iii., to find U, by Carlini's method.

. . . .	0,113530
Table III., Carlini, log. $U-$ log. t',	1,922298
Sum is log. $U=108^m,600=1^d\,48^m\,36^s,0$	2,035828

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Days t'.	True Anom. U.	Diff.	Log. of t' days.	Log. U in minutes, *minus* log. t' in days.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
days.	d m s	s					d m s	s		d m s	s		d m s	s
0,0	0,00,00,0	501,8	9,00	1,922370	0	0,700	6,58,07,1	57,5	0,760	7,59,41,3	65,9	0,820	9,10,11,4	75,4
0,1	0,08,21,8	501,8	9,10	1,922370	0	0,701	6,59,04,6	57,7	0,761	8,00,47,2	66,1	0,821	9,11,26,8	75,6
0,2	0,16,43,6	501,8	9,20	1,922370	1	0,702	7,00,02,3	57,8	0,762	8,01,53,3	66,2	0,822	9,12,42,4	75,8
0,3	0,25,05,4	501,8	9,30	1,922369	1	0,703	7,01,00,1	57,9	0,763	8,02,59,5	66,4	0,823	9,13,58,2	76,0
0,4	0,33,27,2		9,40	1,922368		0,704	7,01,58,0		0,764	8,04,05,9		0,824	9,15,14,2	
		501,7			2			58,1			66,5			76,1
0,5	0,41,48,9	501,8	9,50	1,922366	2	0,705	7,02,56,1	58,2	0,765	8,05,12,4	66,7	0,825	9,16,30,3	76,3
0,6	0,50,10,7	501,7	9,60	1,922364	4	0,706	7,03,54,3	58,4	0,766	8,06,19,1	66,8	0,826	9,17,46,6	76,5
0,7	0,58,32,4	501,7	9,70	1,922360	6	0,707	7,04,52,7	58,4	0,767	8,07,25,9	66,9	0,827	9,19,03,1	76,6
0,8	1,06,54,1	501,7	9,80	1,922354	10	0,708	7,05,51,1	58,6	0,768	8,08,32,8	67,1	0,828	9,20,19,7	76,9
0,9	1,15,15,8		9,90	1,922344		0,709	7,06,49,7		0,769	8,09,39,9		0,829	9,21,36,6	
		501,6			16			58,8			67,3			76,9
1,0	1,23,37,4	501,6	0,00	1,922328	25	0,710	7,07,48,5	58,8	0,770	8,10,47,2	67,4	0,830	9,22,53,5	77,2
1,1	1,31,59,0	501,6	0,10	1,922303	39	0,711	7,08,47,3	59,0	0,771	8,11,54,6	67,6	0,831	9,24,10,7	77,3
1,2	1,40,20,6	501,6	0,20	1,922264	63	0,712	7,09,46,3	59,2	0,772	8,13,02,2	67,8	0,832	9,25,28,0	77,5
1,3	1,48,42,2	501,5	0,30	1,922201	99	0,713	7,10,45,5	59,3	0,773	8,14,10,0	67,8	0,833	9,26,45,5	77,7
1,4	1,57,03,7		0,40	1,922102		0,714	7,11,44,8		0,774	8,15,17,8		0,834	9,28,03,2	
		501,5						59,4			68,1			77,9
1,5	2,05,25,2	501,4				0,715	7,12,44,2	59,5	0,775	8,16,25,9	68,1	0,835	9,29,21,1	78,0
1,6	2,13,46,6	501,4				0,716	7,13,43,7	59,7	0,776	8,17,34,0	68,4	0,836	9,30,39,1	78,2
1,7	2,22,08,0	501,4				0,717	7,14,43,4	59,8	0,777	8,18,42,4	68,5	0,837	9,31,57,3	78,4
1,8	2,30,29,4	501,3				0,718	7,15,43,2	60,0	0,778	8,19,50,9	68,6	0,838	9,33,15,7	78,6
1,9	2,38,50,7					0,719	7,16,43,2		0,779	8,20,59,5		0,839	9,34,34,3	
		501,2						60,0			68,8			78,7
2,0	2,47,11,9	501,2	0,40	1,922102	13	0,720	7,17,43,2	60,2	0,780	8,22,08,3	69,0	0,840	9,35,53,0	78,9
2,1	2,55,33,1	501,1	0,41	1,922089	13	0,721	7,18,43,4	60,3	0,781	8,23,17,3	69,1	0,841	9,37,11,9	79,1
2,2	3,03,54,2	501,0	0,42	1,922076	14	0,722	7,19,43,7	60,5	0,782	8,24,26,4	69,3	0,842	9,38,31,0	79,3
2,3	3,12,15,2	501,0	0,43	1,922062	14	0,723	7,20,44,2	60,6	0,783	8,25,35,7	69,4	0,843	9,39,50,3	79,4
2,4	3,20,36,2		0,44	1,922048		0,724	7,21,44,8		0,784	8,26,45,1		0,844	9,41,09,7	
		500,9			16			60,8			69,6			79,6
2,5	3,28,57,1	500,8	0,45	1,922032	16	0,725	7,22,45,6	60,9	0,785	8,27,54,7	69,7	0,845	9,42,29,3	79,8
2,6	3,37,17,9	500,7	0,46	1,922016	17	0,726	7,23,46,5	61,0	0,786	8,29,04,4	69,9	0,846	9,43,49,1	80,0
2,7	3,45,38,6	500,7	0,47	1,921999	18	0,727	7,24,47,5	61,2	0,787	8,30,14,3	70,1	0,847	9,45,09,1	80,2
2,8	3,53,39,3	500,6	0,48	1,921981	18	0,728	7,25,48,7	61,3	0,788	8,31,24,4	70,2	0,848	9,46,29,3	80,3
2,9	4,02,19,9		0,49	1,921963		0,729	7,26,50,0		0,789	8,32,34,6		0,849	9,47,49,6	
		500,5			19			61,5			70,3			80,5
3,0	4,10,40,4	500,4	0,50	1,921944	20	0,730	7,27,51,5	61,6	0,790	8,33,44,9	70,6	0,850	9,49,10,1	80,7
3,1	4,19,00,8	500,3	0,51	1,921924	21	0,731	7,28,53,1	61,7	0,791	8,34,55,5	70,7	0,851	9,50,30,8	80,9
3,2	4,27,21,1	500,2	0,52	1,921903	22	0,732	7,29,54,8	61,9	0,792	8,36,06,2	70,8	0,852	9,51,51,7	81,0
3,3	4,35,41,3	500,1	0,53	1,921881	23	0,733	7,30,56,7	62,0	0,793	8,37,17,0	71,0	0,853	9,53,12,7	81,3
3,4	4,44,01,4		0,54	1,921858		0,734	7,31,58,7		0,794	8,38,28,0		0,854	9,54,34,0	
		500,1			24			62,2			71,2			81,4
3,5	4,52,21,5	499,9	0,55	1,921834	26	0,735	7,33,00,0	62,3	0,795	8,39,39,2	71,3	0,855	9,55,55,4	81,6
3,6	5,00,41,4	499,8	0,56	1,921808	26	0,736	7,34,03,2	62,4	0,796	8,40,50,5	71,5	0,856	9,57,17,0	81,7
3,7	5,09,01,2	499,7	0,57	1,921782	28	0,737	7,35,05,6	62,6	0,797	8,42,02,0	71,6	0,857	9,58,38,7	82,0
3,8	5,17,20,9	499,7	0,58	1,921754	29	0,738	7,36,08,2	62,7	0,798	8,43,13,6	71,9	0,858	10,00,00,7	82,2
3,9	5,25,40,6		0,59	1,921725		0,739	7,37,10,9		0,799	8,44,25,5		0,859	10,01,22,9	
		499,5			30			62,9			71,9			82,3
4,0	5,34,00,1	499,4	0,60	1,921695	32	0,740	7,38,13,8	63,0	0,800	8,45,37,4	72,2	0,860	10,02,45,2	82,5
4,1	5,42,19,5	499,2	0,61	1,921663	33	0,741	7,39,16,8	63,1	0,801	8,46,49,6	72,3	0,861	10,04,07,7	82,7
4,2	5,50,38,7	499,1	0,62	1,921630	35	0,742	7,40,19,9	63,3	0,802	8,48,01,9	72,4	0,862	10,05,30,4	82,9
4,3	5,58,57,8	499,0	0,63	1,921595	37	0,743	7,41,23,2	63,4	0,803	8,49,14,3	72,6	0,863	10,06,53,3	83,1
4,4	6,07,16,8		0,64	1,921558		0,744	7,42,26,6		0,804	8,50,26,9		0,864	10,08,16,4	
		498,9			38			63,6			72,8			83,2
4,5	6,15,35,7	498,7	0,65	1,921520	40	0,745	7,43,30,2	63,7	0,805	8,51,39,7	73,0	0,865	10,09,39,6	83,5
4,6	6,23,54,4	498,6	0,66	1,921480	43	0,746	7,44,33,9	63,9	0,806	8,52,52,7	73,1	0,866	10,11,03,1	83,6
4,7	6,32,13,0	498,5	0,67	1,921437	44	0,747	7,45,37,8	64,0	0,807	8,54,05,8	73,3	0,867	10,12,26,7	83,8
4,8	6,40,31,5	498,3	0,68	1,921393	46	0,748	7,46,41,8	64,1	0,808	8,55,19,1	73,4	0,868	10,13,50,5	84,0
4,9	6,48,49,8		0,69	1,921347		0,749	7,47,45,9		0,809	8,56,32,5		0,869	10,15,14,5	
		498,2			47			64,3			73,6			84,2
5,0	6,57,08,0	498,0	0,70	1,921300	49	0,750	7,48,50,2	64,5	0,810	8,57,46,1	73,8	0,870	10,16,38,7	84,4
5,1	7,05,26,0	497,9	0,71	1,921251		0,751	7,49,54,7	64,6	0,811	8,58,59,9	73,9	0,871	10,18,03,1	84,5
5,2	7,13,43,9	497,8				0,752	7,50,59,3	64,7	0,812	9,00,13,8	74,1	0,872	10,19,27,6	84,8
5,3	7,22,01,7	497,6				0,753	7,52,04,0	64,9	0,813	9,01,27,9	74,3	0,873	10,20,52,4	84,9
5,4	7,30,19,3					0,754	7,53,08,9		0,814	9,02,42,2		0,874	10,22,17,3	
		497,4						65,0			74,5			85,2
5,5	7,38,36,7	497,2				0,755	7,54,13,9	65,2	0,815	9,03,56,7	74,6	0,875	10,23,42,5	85,3
5,6	7,46,53,9	497,1				0,756	7,55,19,1	65,3	0,816	9,05,11,3	74,7	0,876	10,25,07,8	85,5
5,7	7,55,11,0	496,9				0,757	7,56,24,4	65,5	0,817	9,06,26,0	75,0	0,877	10,26,33,3	85,7
5,8	8,03,27,9	496,8				0,758	7,57,29,9	65,6	0,818	9,07,41,0	75,1	0,878	10,27,59,0	85,9
5,9	8,11,44,7	496,6				0,759	7,58,35,5	65,8	0,819	9,08,56,1	75,3	0,879	10,29,24,9	86,1
6,0	8,20,01,3					0,760	7,59,41,3	65,9	0,820	9,10,11,4	75,4	0,880	10,30,51,0	86,3

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
	d m s	s		d m s	s		d m s	s		d m s	s		d m s	s
0,880	10,30,51,0	86,3	0,940	12,03,01,8	98,5	1,000	13,48,13,4	112,3	1,060	15,48,02,6	127,8	1,120	18,04,11,5	145,0
0,881	10,32,17,3	86,4	0,941	12,04,40,3	98,7	1,001	13,50,05,7	112,6	1,061	15,50,10,4	128,1	1,121	18,06,36,5	145,4
0,882	10,33,43,7	86,7	0,942	12,06,19,0	99,0	1,002	13,51,58,3	112,9	1,062	15,52,18,5	128,4	1,122	18,09,01,9	145,6
0,883	10,35,10,4	86,8	0,943	12,07,58,0	99,2	1,003	13,53,51,2	113,1	1,063	15,54,26,9	128,6	1,123	18,11,27,5	146,0
0,884	10,36,37,2	87,1	0,944	12,09,37,2	99,4	1,004	13,55,44,3	113,3	1,064	15,56,35,5	128,9	1,124	18,13,53,5	146,2
0,885	10,38,04,3	87,2	0,945	12,11,16,6	99,7	1,005	13,57,37,6	113,6	1,065	15,58,44,4	129,2	1,125	18,16,19,7	146,6
0,886	10,39,31,5	87,5	0,946	12,12,56,3	99,8	1,006	13,59,31,2	113,8	1,066	16,00,53,6	129,4	1,126	18,18,46,3	146,8
0,887	10,40,59,0	87,6	0,947	12,14,36,1	100,1	1,007	14,01,25,0	114,1	1,067	16,03,03,0	129,8	1,127	18,21,13,1	147,2
0,888	10,42,26,6	87,8	0,948	12,16,16,2	100,2	1,008	14,03,19,1	114,3	1,068	16,05,12,8	130,0	1,128	18,23,40,3	147,5
0,889	10,43,54,4	88,0	0,949	12,17,56,4	100,6	1,009	14,05,13,4	114,5	1,069	16,07,22,8	130,3	1,129	18,26,07,8	147,7
0,890	10,45,22,4	88,3	0,950	12,19,37,0	100,7	1,010	14,07,07,9	114,8	1,070	16,09,33,1	130,6	1,130	18,28,35,5	148,1
0,891	10,46,50,7	88,4	0,951	12,21,17,7	100,9	1,011	14,09,02,7	115,1	1,071	16,11,43,7	130,9	1,131	18,31,03,6	148,4
0,892	10,48,19,1	88,6	0,952	12,22,58,6	101,2	1,012	14,10,57,8	115,3	1,072	16,13,54,6	131,1	1,132	18,33,32,0	148,7
0,893	10,49,47,7	88,8	0,953	12,24,39,8	101,4	1,013	14,12,53,1	115,5	1,073	16,16,05,7	131,4	1,133	18,36,00,7	149,0
0,894	10,51,16,5	89,0	0,954	12,26,21,2	101,6	1,014	14,14,48,6	115,8	1,074	16,18,17,1	131,7	1,134	18,38,29,7	149,3
0,895	10,52,45,5	89,2	0,955	12,28,02,8	101,9	1,015	14,16,44,4	116,1	1,075	16,20,28,8	132,0	1,135	18,40,59,0	149,6
0,896	10,54,14,7	89,4	0,956	12,29,44,7	102,0	1,016	14,18,40,5	116,3	1,076	16,22,40,8	132,3	1,136	18,43,28,6	149,9
0,897	10,55,44,1	89,6	0,957	12,31,26,7	102,3	1,017	14,20,36,8	116,6	1,077	16,24,53,1	132,5	1,137	18,45,58,5	150,3
0,898	10,57,13,7	89,8	0,958	12,33,09,0	102,5	1,018	14,22,33,4	116,8	1,078	16,27,05,6	132,9	1,138	18,48,28,8	150,5
0,899	10,58,43,5	90,0	0,959	12,34,51,5	102,8	1,019	14,24,30,2	117,1	1,079	16,29,18,5	133,1	1,139	18,50,59,3	150,9
0,900	11,00,13,5	90,2	0,960	12,36,34,3	102,9	1,020	14,26,27,3	117,3	1,080	16,31,31,6	133,3	1,140	18,53,30,2	151,2
0,901	11,01,43,7	90,4	0,961	12,38,17,2	103,2	1,021	14,28,24,6	117,6	1,081	16,33,44,9	133,7	1,141	18,56,01,4	151,4
0,902	11,03,14,1	90,6	0,962	12,40,00,4	103,4	1,022	14,30,22,2	117,8	1,082	16,35,58,6	133,9	1,142	18,58,32,8	151,8
0,903	11,04,44,7	90,8	0,963	12,41,43,8	103,7	1,023	14,32,20,0	118,1	1,083	16,38,12,5	134,2	1,143	19,01,04,6	152,1
0,904	11,06,15,5	91,0	0,964	12,43,27,5	103,8	1,024	14,34,18,1	118,4	1,084	16,40,26,7	134,5	1,144	19,03,36,7	152,4
0,905	11,07,46,5	91,2	0,965	12,45,11,3	104,1	1,025	14,36,16,5	118,6	1,085	16,42,41,2	134,7	1,145	19,06,09,1	152,7
0,906	11,09,17,7	91,4	0,966	12,46,55,4	104,4	1,026	14,38,15,1	118,8	1,086	16,44,55,9	135,2	1,146	19,08,41,8	153,0
0,907	11,10,49,1	91,6	0,967	12,48,39,8	104,5	1,027	14,40,13,9	119,1	1,087	16,47,11,1	135,4	1,147	19,11,14,8	153,4
0,908	11,12,20,7	91,8	0,968	12,50,24,3	104,8	1,028	14,42,13,0	119,4	1,088	16,49,26,5	135,6	1,148	19,13,48,2	153,7
0,909	11,13,52,5	92,1	0,969	12,52,09,1	105,0	1,029	14,44,12,4	119,6	1,089	16,51,42,1	135,9	1,149	19,16,21,9	154,0
0,910	11,15,24,6	92,2	0,970	12,53,54,1	105,2	1,030	14,46,12,0	119,9	1,090	16,53,58,0	136,3	1,150	19,18,55,9	154,3
0,911	11,16,56,8	92,4	0,971	12,55,39,3	105,5	1,031	14,48,11,9	120,1	1,091	16,56,14,3	136,5	1,151	19,21,30,2	154,6
0,912	11,18,29,2	92,7	0,972	12,57,24,8	105,7	1,032	14,50,12,0	120,4	1,092	16,58,30,8	136,8	1,152	19,24,04,8	155,0
0,913	11,20,01,9	92,8	0,973	12,59,10,5	105,9	1,033	14,52,12,4	120,7	1,093	17,00,47,6	137,0	1,153	19,26,39,8	155,2
0,914	11,21,34,7	93,0	0,974	13,00,56,4	106,2	1,034	14,54,13,1	120,9	1,094	17,03,04,6	137,4	1,154	19,29,15,0	155,6
0,915	11,23,07,7	93,3	0,975	13,02,42,6	106,4	1,035	14,56,14,0	121,2	1,095	17,05,22,0	137,7	1,155	19,31,50,6	155,9
0,916	11,24,41,0	93,4	0,976	13,04,29,0	106,6	1,036	14,58,15,2	121,5	1,096	17,07,39,7	138,0	1,156	19,34,26,5	156,2
0,917	11,26,14,4	93,7	0,977	13,06,15,6	106,9	1,037	15,00,16,7	121,7	1,097	17,09,57,7	138,2	1,157	19,37,02,7	156,5
0,918	11,27,48,1	93,9	0,978	13,08,02,5	107,1	1,038	15,02,18,4	122,0	1,098	17,12,15,9	138,5	1,158	19,39,39,2	156,8
0,919	11,29,22,0	94,0	0,979	13,09,49,6	107,3	1,039	15,04,20,4	122,2	1,099	17,14,34,4	138,8	1,159	19,42,16,0	157,2
0,920	11,30,56,0	94,3	0,980	13,11,36,9	107,6	1,040	15,06,22,6	122,5	1,100	17,16,53,2	139,1	1,160	19,44,53,2	157,6
0,921	11,32,30,3	94,5	0,981	13,13,24,5	107,8	1,041	15,08,25,1	122,7	1,101	17,19,12,3	139,4	1,161	19,47,30,8	157,9
0,922	11,34,04,8	94,7	0,982	13,15,12,3	108,0	1,042	15,10,27,8	122,9	1,102	17,21,31,7	139,7	1,162	19,50,08,7	158,2
0,923	11,35,39,5	94,9	0,983	13,17,00,3	108,3	1,043	15,12,30,7	123,3	1,103	17,23,51,4	140,0	1,163	19,52,46,9	158,4
0,924	11,37,14,4	95,1	0,984	13,18,48,6	108,5	1,044	15,14,34,0	123,5	1,104	17,26,11,4	140,3	1,164	19,55,25,3	158,8
0,925	11,38,49,5	95,3	0,985	13,20,37,1	108,8	1,045	15,16,37,5	123,8	1,105	17,28,31,7	140,6	1,165	19,58,04,1	159,1
0,926	11,40,24,8	95,6	0,986	13,22,25,9	108,9	1,046	15,18,41,3	124,1	1,106	17,30,52,3	140,8	1,166	20,00,43,2	159,4
0,927	11,42,00,4	95,7	0,987	13,24,14,8	109,3	1,047	15,20,45,4	124,3	1,107	17,33,13,1	141,2	1,167	20,03,22,6	159,7
0,928	11,43,36,1	96,0	0,988	13,26,04,1	109,4	1,048	15,22,49,7	124,6	1,108	17,35,34,3	141,5	1,168	20,06,02,3	160,1
0,929	11,45,12,1	96,2	0,989	13,27,53,5	109,7	1,049	15,24,54,3	124,8	1,109	17,37,55,8	141,8	1,169	20,08,42,4	160,4
0,930	11,46,48,3	96,3	0,990	13,29,43,2	110,0	1,050	15,26,59,1	125,1	1,110	17,40,17,6	142,0	1,170	20,11,22,8	160,7
0,931	11,48,24,6	96,6	0,991	13,31,33,2	110,1	1,051	15,29,04,2	125,4	1,111	17,42,39,6	142,3	1,171	20,14,03,5	161,0
0,932	11,50,01,2	96,9	0,992	13,33,23,3	110,4	1,052	15,31,09,6	125,7	1,112	17,45,01,9	142,7	1,172	20,16,44,5	161,4
0,933	11,51,38,1	97,0	0,993	13,35,13,7	110,7	1,053	15,33,15,3	125,9	1,113	17,47,24,6	142,9	1,173	20,19,25,9	161,7
0,934	11,53,15,1	97,2	0,994	13,37,04,4	110,9	1,054	15,35,21,2	126,2	1,114	17,49,47,5	143,3	1,174	20,22,07,6	162,0
0,935	11,54,52,3	97,5	0,995	13,38,55,3	111,1	1,055	15,37,27,4	126,5	1,115	17,52,10,8	143,5	1,175	20,24,49,6	162,4
0,936	11,56,29,8	97,7	0,996	13,40,46,4	111,4	1,056	15,39,33,9	126,8	1,116	17,54,34,3	143,8	1,176	20,27,32,0	162,7
0,937	11,58,07,5	97,8	0,997	13,42,37,8	111,6	1,057	15,41,40,7	127,0	1,117	17,56,58,1	144,2	1,177	20,30,14,7	163,0
0,938	11,59,45,3	98,2	0,998	13,44,29,4	111,9	1,058	15,43,47,7	127,3	1,118	17,59,22,3	144,4	1,178	20,32,57,7	163,3
0,939	12,01,23,5	98,3	0,999	13,46,21,3	112,1	1,059	15,45,55,0	127,6	1,119	18,01,46,7	144,8	1,179	20,35,41,0	163,7
0,940	12,03,01,8	98,5	1,000	13,48,13,4	112,3	1,060	15,48,02,6	127,8	1,120	18,04,11,5	145,0	1,180	20,38,24,7	164,0

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Log. of t days.	True Anom. U.	Diff.	Log. of t days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
	d m s	s		d m s	s		d m s	s		d m s	s		d m s	s
1,180	20,38,24,7	164,0	1,240	23,32,24,9	184,6	1,300	20.47,44,5	206,6	1,360	30,25,38,7	229,7	1,420	34,26,51,8	253,1
1,181	20,41,08,7	164,3	1,241	23,35,29,5	185,0	1,301	26,51,11,1	207,0	1,361	30,29,28,4	230,1	1,421	34,31,04,9	253,6
1,182	20,43,53,0	164,6	1,242	23,38,34,5	185,3	1,302	26,54,38,1	207,4	1,362	30,33,18,5	230,4	1,422	34,35,18,5	253,9
1,183	20,46,37,6	165,0	1,243	23,41,39,8	185,6	1,303	26,58,05,5	207,8	1,363	30,37,08,9	230,9	1,423	34,39,32,4	254,3
1,184	20,49,22,6	165,3	1,244	23,44,45,4	186,0	1,304	27,01,33,3	208,1	1,364	30,40,59,8	231,2	1,424	34,43,46,7	254,7
1,185	20,52,07,9	165,7	1,245	23,47,51,4	186,4	1,305	27,05,01,4	208,5	1,365	30,44,51,0	231,7	1,425	34,48,01,4	255,1
1,186	20,54,53,6	166,0	1,246	23,50,57,8	186,7	1,306	27,08,29,9	208,9	1,366	30,48,42,7	232,0	1,426	34,52,16,5	255,5
1,187	20,57,39,6	166,3	1,247	23,54,04,5	187,1	1,307	27,11,58,8	209,3	1,367	30,52,34,7	232,4	1,427	34,56,32,0	255,8
1,188	21,00,25,9	166,7	1,248	23,57,11,6	187,4	1,308	27,15,28,1	209,7	1,368	30,56,27,1	232,8	1,428	35,00,47,8	256,3
1,189	21,03,12,6	167,0	1,249	24,00,19,0	187,8	1,309	27,18,57,8	210,0	1,369	31,00,19,9	233,2	1,429	35,05,04,1	256,6
1,190	21,05,59,6	167,3	1,250	24,03,26,8	188,2	1,310	27,22,27,8	210,4	1,370	31,04,13,1	233,6	1,430	35,09,20,7	257,0
1,191	21,08,46,9	167,7	1,251	24,06,35,0	188,5	1,311	27,25,58,2	210,8	1,371	31,08,06,7	234,0	1,431	35,13,37,7	257,5
1,192	21,11,34,6	168,0	1,252	24,09,43,5	188,9	1,312	27,29,29,0	211,2	1,372	31,12,00,7	234,4	1,432	35,17,55,2	257,8
1,193	21,14,22,6	168,3	1,253	24,12,52,4	189,2	1,313	27,33,00,2	211,7	1,373	31,15,55,1	234,7	1,433	35,22,13,0	258,2
1,194	21,17,10,9	168,7	1,254	24,16,01,6	189,6	1,314	27,36,31,9	211,9	1,374	31,19,49,8	235,2	1,434	35,26,31,2	258,5
1,195	21,19,59,6	169,0	1,255	24,19,11,2	190,0	1,315	27,40,03,8	212,3	1,375	31,23,45,0	235,5	1,435	35,30,49,7	259,0
1,196	21,22,48,6	169,3	1,256	24,22,21,2	190,3	1,316	27,43,36,1	212,7	1,376	31,27,40,5	236,0	1,436	35,35,08,7	259,4
1,197	21,25,37,9	169,7	1,257	24,25,31,5	190,7	1,317	27,47,08,8	213,1	1,377	31,31,36,5	236,3	1,437	35,39,28,1	259,7
1,198	21,28,27,6	170,0	1,258	24,28,42,2	191,1	1,318	27,50,41,9	213,5	1,378	31,35,32,8	236,7	1,438	35,43,47,8	260,2
1,199	21,31,17,6	170,4	1,259	24,31,53,3	191,4	1,319	27,54,15,4	213,9	1,379	31,39,29,5	237,1	1,439	35,48,08,0	260,5
1,200	21,34,08,0	170,7	1,260	24,35,04,7	191,8	1,320	27,57,49,3	214,2	1,380	31,43,26,6	237,5	1,440	35,52,28,5	260,9
1,201	21,36,58,7	171,0	1,261	24,38,16,5	192,1	1,321	28,01,23,5	214,6	1,381	31,47,24,1	237,9	1,441	35,56,49,4	261,3
1,202	21,39,49,7	171,4	1,262	24,41,28,6	192,5	1,322	28,04,58,1	215,0	1,382	31,51,22,0	238,3	1,442	36,01,10,7	261,7
1,203	21,42,41,1	171,7	1,263	24,44,41,1	192,9	1,323	28,08,33,1	215,4	1,383	31,55,20,3	238,7	1,443	36,05,32,4	262,0
1,204	21,45,32,8	172,0	1,264	24,47,54,0	193,3	1,324	28,12,08,5	215,7	1,384	31,59,19,0	239,0	1,444	36,09,54,4	262,5
1,205	21,48,24,8	172,4	1,265	24,51,07,3	193,6	1,325	28,15,44,2	216,1	1,385	32,03,18,0	239,5	1,445	36,14,16,9	262,9
1,206	21,51,17,2	172,8	1,266	24,54,20,9	194,0	1,326	28,19,20,3	216,6	1,386	32,07,17,5	239,8	1,446	36,18,39,8	263,2
1,207	21,54,10,0	173,1	1,267	24,57,34,9	194,3	1,327	28,22,56,9	216,9	1,387	32,11,17,3	240,2	1,447	36,23,03,0	263,6
1,208	21,57,03,1	173,5	1,268	25,00,49,2	194,7	1,328	28,26,33,8	217,3	1,388	32,15,17,5	240,7	1,448	36,27,26,6	264,0
1,209	21,59,56,6	173,8	1,269	25,04,03,9	195,1	1,329	28,30,11,1	217,6	1,389	32,19,18,2	241,0	1,449	36,31,50,6	264,4
1,210	22,02,50,4	174,1	1,270	25,07,19,0	195,4	1,330	28,33,48,7	218,1	1,390	32,23,19,2	241,4	1,450	36,36,15,0	264,8
1,211	22,05,44,5	174,5	1,271	25,10,34,4	195,8	1,331	28,37,26,8	218,5	1,391	32,27,20,6	241,8	1,451	36,40,39,8	265,1
1,212	22,08,39,0	174,8	1,272	25,13,50,2	196,2	1,332	28,41,05,3	218,8	1,392	32,31,22,4	242,2	1,452	36,45,04,9	265,6
1,213	22,11,33,8	175,2	1,273	25,17,06,4	196,6	1,333	28,44,44,1	219,2	1,393	32,35,24,6	242,6	1,453	36,40,30,5	265,9
1,214	22,14,29,0	175,5	1,274	25,20,23,0	196,9	1,334	28,48,23,3	219,6	1,394	32,39,27,2	242,9	1,454	36,53,56,4	266,3
1,215	22,17,24,5	175,8	1,275	25,23,39,9	197,3	1,335	28,52,02,9	220,0	1,395	32,43,30,1	243,4	1,455	36,58,22,7	266,7
1,216	22,20,20,3	176,2	1,276	25,26,57,2	197,7	1,336	28,55,42,9	220,4	1,396	32,47,33,5	243,8	1,456	37,02,49,4	267,1
1,217	22,23,16,5	176,5	1,277	25,30,14,9	198,0	1,337	28,59,23,3	220,7	1,397	32,51,37,3	244,1	1,457	37,07,16,5	267,4
1,218	22,26,13,0	176,9	1,278	25,33,32,9	198,4	1,338	29,03,04,0	221,2	1,398	32,55,41,4	244,6	1,458	37,11,43,9	267,9
1,219	22,29,09,9	177,2	1,279	25,36,51,3	198,8	1,339	29,06,45,2	221,5	1,399	32,59,46,0	244,9	1,459	37,16,11,8	268,2
1,220	22,32,07,1	177,6	1,280	25,40,10,1	199,2	1,340	29,10,26,7	221,9	1,400	33,03,50,9	245,3	1,460	37,20,40,0	268,6
1,221	22,35,04,7	177,9	1,281	25,43,29,3	199,5	1,341	29,14,08,6	222,3	1,401	33,07,56,2	245,8	1,461	37,25,08,6	269,0
1,222	22,38,02,6	178,3	1,282	25,46,48,8	199,9	1,342	29,17,50,9	222,7	1,402	33,12,02,0	246,1	1,462	37,29,37,6	269,4
1,223	22,41,00,9	178,6	1,283	25,50,08,7	200,3	1,343	29,21,33,6	223,1	1,403	33,16,08,1	246,5	1,463	37,34,07,0	269,7
1,224	22,43,59,5	179,0	1,284	25,53,29,0	200,6	1,344	29,25,16,7	223,5	1,404	33,20,14,6	246,9	1,464	37,38,36,7	270,2
1,225	22,46,58,5	179,3	1,285	25,56,49,6	201,0	1,345	29,29,00,2	223,8	1,405	33,24,21,5	247,2	1,465	37,43,06,9	270,5
1,226	22,49,57,8	179,6	1,286	26,00,10,6	201,4	1,346	29,32,44,0	224,3	1,406	33,28,28,7	247,7	1,466	37,47,37,4	270,9
1,227	22,52,57,4	180,0	1,287	26,03,32,0	201,8	1,347	29,36,28,3	224,6	1,407	33,32,36,4	248,1	1,467	37,52,08,3	271,3
1,228	22,55,57,4	180,4	1,288	26,06,53,8	202,1	1,348	29,40,12,9	225,0	1,408	33,36,44,5	248,4	1,468	37,56,39,6	271,7
1,229	22,58,57,8	180,7	1,289	26,10,15,9	202,6	1,349	29,43,57,9	225,4	1,409	33,40,52,9	248,9	1,469	38,01,11,3	272,0
1,230	23,01,58,5	181,1	1,290	26,13,38,5	202,9	1,350	29,47,43,3	225,8	1,410	33,45,01,8	249,2	1,470	38,05,43,3	272,4
1,231	23,04,59,6	181,4	1,291	26,17,01,4	203,3	1,351	29,51,29,1	226,2	1,411	33,49,11,0	249,7	1,471	38,10,15,7	272,8
1,232	23,08,01,0	181,7	1,292	26,20,24,7	203,7	1,352	29,55,15,3	226,5	1,412	33,53,20,7	250,0	1,472	38,14,48,5	273,2
1,233	23,11,02,7	182,1	1,293	26,23,48,4	204,0	1,353	29,59,01,8	227,0	1,413	33,57,30,7	250,4	1,473	38,19,21,7	273,6
1,234	23,14,04,8	182,5	1,294	26,27,12,4	204,4	1,354	30,02,48,8	227,3	1,414	34,01,41,1	250,8	1,474	38,23,55,3	273,9
1,235	23,17,07,3	182,8	1,295	26,30,36,8	204,8	1,355	30,06,36,1	227,8	1,415	34,05,51,9	251,2	1,475	38,28,29,2	274,3
1,236	23,20,10,1	183,2	1,296	26,34,01,6	205,1	1,356	30,10,23,9	228,1	1,416	34,10,03,1	251,6	1,476	38,33,03,5	274,7
1,237	23,23,13,3	183,5	1,297	26,37,26,7	205,6	1,357	30,14,12,0	228,5	1,417	34,14,14,7	151,9	1,477	38,37,38,2	275,0
1,238	23,26,16,8	183,9	1,298	26,40,52,3	205,9	1,358	30,18,00,5	228,9	1,418	34,18,26,6	252,4	1,478	38,42,13,2	275,5
1,239	23,29,20,7	184,2	1,299	26,44,18,2	206,3	1,359	30,21,49,4	229,3	1,419	34,22,39,0	252,8	1,479	38,46,48,7	275,8
1,240	23,32,24,9	184,6	1,300	26,47,44,5	206,6	1,360	30,25,38,7	229,7	1,420	34,26,51,8	253,1	1,480	38,51,24,5	276,1

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
	d m s	s		d m s	s		d m s	s		d m s	s		d m s	s
1,480	38,51,24,5	276,1	1,540	43,38,21,7	297,8	1,600	48,45,41,9	316,7	1,660	54,10,15,6	331,9	1,720	59,47,50,4	342,5
1,481	38,56,00,6	276,6	1,541	43,43,19,5	298,0	1,601	48,50,58,6	317,0	1,661	54,15,47,5	332,2	1,721	59,53,32,9	342,7
1,482	39,00,37,2	276,9	1,542	43,48,17,5	298,3	1,602	48,56,15,6	317,3	1,662	54,21,19,7	332,4	1,722	59,59,15,6	342,8
1,483	39,05,14,1	277,3	1,543	43,53,15,8	298,7	1,603	49,01,32,9	317,5	1,663	54,26,52,1	332,6	1,723	60,04,58,4	342,9
1,484	39,09,51,4	277,7	1,544	43,58,14,5	299,1	1,604	49,06,50,4	317,9	1,664	54,32,24,7	332,8	1,724	60,10,41,3	343,1
1,485	39,14,29,1	278,0	1,545	44,03,13,6	299,4	1,605	49,12,08,3	318,2	1,665	54,37,57,5	333,0	1,725	60,16,24,4	343,2
1,486	39,19,07,1	278,4	1,546	44,08,13,0	299,7	1,606	49,17,26,5	318,4	1,666	54,43,30,5	333,2	1,726	60,22,07,6	343,3
1,487	39,23,45,5	278,8	1,547	44,13,12,7	300,1	1,607	49,22,44,9	318,7	1,667	54,49,03,7	333,4	1,727	60,27,50,9	343,4
1,488	39,28,24,3	279,2	1,548	44,18,12,8	300,4	1,608	49,28,03,6	319,0	1,668	54,54,37,1	333,7	1,728	60,33,34,3	343,5
1,489	39,33,03,5	279,5	1,549	44,23,13,2	300,9	1,609	49,33,22,6	319,2	1,669	55,00,10,8	333,9	1,729	60,39,17,8	343,7
1,490	39,37,43,0	279,9	1,550	44,28,14,1	301,1	1,610	49,38,41,8	319,6	1,670	55,05,44,7	334,1	1,730	60,45,01,5	343,8
1,491	39,42,22,9	280,3	1,551	44,33,15,2	301,4	1,611	49,44,01,4	319,8	1,671	55,11,18,8	334,2	1,731	60,50,45,3	343,9
1,492	39,47,03,2	280,6	1,552	44,38,16,6	301,8	1,612	49,49,21,2	320,1	1,672	55,16,53,0	334,5	1,732	60,56,29,2	344,0
1,493	39,51,43,8	281,0	1,553	44,43,18,4	302,1	1,613	49,54,41,3	320,4	1,673	55,22,27,5	334,7	1,733	61,02,13,2	344,2
1,494	39,56,24,8	281,4	1,554	44,48,20,5	302,4	1,614	50,00,01,7	320,6	1,674	55,28,02,2	334,9	1,734	61,07,57,4	344,3
1,495	40,01,06,2	281,8	1,555	44,53,22,9	302,8	1,615	50,05,22,3	320,9	1,675	55,33,37,1	335,1	1,735	61,13,41,7	344,4
1,496	40,05,48,0	282,1	1,556	44,58,25,7	303,1	1,616	50,10,43,2	321,2	1,676	55,39,12,2	335,2	1,736	61,19,26,1	344,5
1,497	40,10,30,1	282,5	1,557	45,03,28,8	303,4	1,617	50,16,04,4	321,5	1,677	55,44,47,4	335,5	1,737	61,25,10,6	344,6
1,498	40,15,12,6	282,8	1,558	45,08,32,2	303,8	1,618	50,21,25,9	321,7	1,678	55,50,22,9	335,7	1,738	61,30,55,2	344,7
1,499	40,19,55,4	283,2	1,559	45,13,36,0	304,2	1,619	50,26,47,6	322,0	1,679	55,55,58,6	335,8	1,739	61,36,39,9	344,9
1,500	40,24,38,6	283,6	1,560	45,18,40,2	304,4	1,620	50,32,09,6	322,2	1,680	56,01,34,4	336,0	1,740	61,42,24,8	345,0
1,501	40,29,22,2	284,0	1,561	45,23,44,6	304,8	1,621	50,37,31,8	322,5	1,681	56,07,10,4	336,3	1,741	61,48,09,8	345,1
1,502	40,34,06,2	284,3	1,562	45,28,49,4	305,1	1,622	50,42,54,3	322,8	1,682	56,12,46,7	336,4	1,742	61,53,54,9	345,2
1,503	40,38,50,5	284,7	1,563	45,33,54,5	305,4	1,623	50,48,17,1	323,1	1,683	56,18,23,1	336,6	1,743	61,59,40,1	345,2
1,504	40,43,35,2	285,0	1,564	45,38,59,9	305,7	1,624	50,53,40,2	323,3	1,684	56,23,59,7	336,8	1,744	62,05,25,3	345,3
1,505	40,48,20,2	285,4	1,565	45,44,05,6	306,1	1,625	50,59,03,5	323,6	1,685	56,29,36,5	337,0	1,745	62,11,10,6	345,4
1,506	40,53,05,6	285,8	1,566	45,49,11,7	306,3	1,626	51,04,27,1	323,8	1,686	56,35,13,5	337,1	1,746	62,16,56,0	345,5
1,507	40,57,51,4	286,2	1,567	45,54,18,0	306,7	1,627	51,09,50,9	324,1	1,687	56,40,50,6	337,3	1,747	62,22,41,5	345,6
1,508	41,02,37,6	286,5	1,568	45,59,24,7	307,0	1,628	51,15,15,0	324,4	1,688	56,46,27,9	337,5	1,748	62,28,27,1	345,7
1,509	41,07,24,1	286,8	1,569	46,04,31,7	307,3	1,629	51,20,39,4	324,6	1,689	56,52,05,4	337,8	1,749	62,34,12,8	345,8
1,510	41,12,10,9	287,2	1,570	46,09,39,0	307,6	1,630	51,26,04,0	324,9	1,690	56,57,43,2	337,9	1,750	62,39,58,6	345,9
1,511	41,16,58,1	287,6	1,571	46,14,46,6	308,0	1,631	51,31,28,9	325,1	1,691	57,03,21,1	338,0	1,751	62,45,44,5	346,0
1,512	41,21,45,7	287,9	1,572	46,19,54,6	308,3	1,632	51,36,54,0	325,4	1,692	57,08,59,1	338,2	1,752	62,51,30,5	346,1
1,513	41,26,33,6	288,3	1,573	46,25,02,9	308,6	1,633	51,42,19,4	325,6	1,693	57,14,37,3	338,4	1,753	62,57,16,6	346,2
1,514	41,31,21,9	288,7	1,574	46,30,11,5	308,8	1,634	51,47,45,0	325,9	1,694	57,20,15,7	338,6	1,754	63,03,02,8	346,2
1,515	41,36,10,6	289,0	1,575	46,35,20,3	309,2	1,635	51,53,10,9	326,1	1,695	57,25,54,3	338,7	1,755	63,08,49,0	346,3
1,516	41,40,59,6	289,4	1,576	46,40,29,5	309,5	1,636	51,58,37,0	326,4	1,696	57,31,33,0	338,9	1,756	63,14,35,3	346,4
1,517	41,45,49,0	289,7	1,577	46,45,39,0	309,8	1,637	52,04,03,4	326,6	1,697	57,37,11,9	339,0	1,757	63,20,21,7	346,5
1,518	41,50,38,7	290,1	1,578	46,50,48,8	310,2	1,638	52,09,30,0	326,9	1,698	57,42,50,9	339,2	1,758	63,26,08,2	346,6
1,519	41,55,28,8	290,4	1,579	46,55,59,0	310,4	1,639	52,14,56,9	327,1	1,699	57,48,30,1	339,4	1,759	63,31,54,8	346,6
1,520	42,00,19,2	290,8	1,580	47,01,09,4	310,7	1,640	52,20,24,0	327,3	1,700	57,54,09,5	339,7	1,760	63,37,41,4	346,7
1,521	42,05,10,0	291,1	1,581	47,06,20,1	311,1	1,641	52,25,51,3	327,6	1,701	57,59,49,2	339,8	1,761	63,43,28,1	346,8
1,522	42,10,01,1	291,5	1,582	47,11,31,2	311,4	1,642	52,31,18,9	327,8	1,702	58,05,29,0	339,9	1,762	63,49,14,9	346,9
1,523	42,14,52,6	291,8	1,583	47,16,42,6	311,7	1,643	52,36,46,7	328,1	1,703	58,11,08,9	340,1	1,763	63,55,01,8	346,9
1,524	42,19,44,4	292,2	1,584	47,21,54,3	312,0	1,644	52,42,14,8	328,3	1,704	58,16,49,0	340,2	1,764	64,00,48,7	347,0
1,525	42,24,36,6	292,6	1,585	47,27,06,3	312,3	1,645	52,47,43,1	328,5	1,705	58,22,29,2	340,4	1,765	64,06,35,7	347,1
1,526	42,29,29,2	292,9	1,586	47,32,18,6	312,6	1,646	52,53,11,6	328,8	1,706	58,28,09,6	340,5	1,766	64,12,22,8	347,1
1,527	42,34,22,1	293,2	1,587	47,37,31,2	312,9	1,647	52,58,40,4	329,0	1,707	58,33,50,1	340,7	1,767	64,18,09,9	347,2
1,528	42,39,15,3	293,6	1,588	47,42,44,1	313,1	1,648	53,04,09,4	329,3	1,708	58,39,30,8	340,8	1,768	64,23,57,1	347,3
1,529	42,44,08,9	294,0	1,589	47,47,57,2	313,5	1,649	53,09,38,7	329,5	1,709	58,45,11,6	340,9	1,769	64,29,44,4	347,4
1,530	42,49,02,9	294,3	1,590	47,53,10,7	313,8	1,650	53,15,08,2	329,7	1,710	58,50,52,5	341,1	1,770	64,35,31,8	347,4
1,531	42,53,57,2	294,7	1,591	47,58,24,5	314,1	1,651	53,20,37,9	330,0	1,711	58,56,33,6	341,3	1,771	64,41,19,2	347,5
1,532	42,58,51,9	295,0	1,592	48,03,38,6	314,3	1,652	53,26,07,9	330,2	1,712	59,02,14,9	341,4	1,772	64,47,06,7	347,5
1,533	43,03,46,9	295,4	1,593	48,08,52,9	314,7	1,653	53,31,38,1	330,4	1,713	59,07,56,3	341,6	1,773	64,52,54,2	347,5
1,534	43,08,42,3	295,7	1,594	48,14,07,6	315,0	1,654	53,37,08,5	330,6	1,714	59,13,37,9	341,8	1,774	64,58,41,7	347,6
1,535	43,13,38,0	296,0	1,595	48,19,22,6	315,3	1,655	53,42,39,1	330,8	1,715	59,19,19,7	341,9	1,775	65,04,29,3	347,6
1,536	43,18,34,0	296,4	1,596	48,24,37,9	315,5	1,656	53,48,09,9	331,1	1,716	59,25,01,6	342,0	1,776	65,10,16,9	347,7
1,537	43,23,30,4	296,7	1,597	48,29,53,4	315,9	1,657	53,53,41,0	331,3	1,717	59,30,43,6	342,1	1,777	65,16,04,6	347,8
1,538	43,28,27,1	297,1	1,598	48,35,09,3	316,1	1,658	53,59,12,3	331,6	1,718	59,36,25,7	342,3	1,778	65,21,52,4	347,8
1,539	43,33,24,2	297,5	1,599	48,40,25,4	316,5	1,659	54,04,43,9	331,7	1,719	59,42,08,0	342,4	1,779	65,27,40,2	347,9
1,540	43,38,21,7	297,8	1,600	48,45,41,9	316,7	1,660	54,10,15,6	331,9	1,720	59,47,50,4	342,5	1,780	65,33,28,1	347,9

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
	d m s	s		d m s	s		d m s	s		d m s	s		d m s	s
1,780	65,33,28,1	347,9	1,840	71,21,49,8	348,0	1,900	77,07,43,8	343,0	1,960	82,46,30,5	333,8	2,020	88,14,20,9	321,2
1,781	65,39,16,0	347,9	1,841	71,27,37,8	347,9	1,901	77,13,26,8	342,9	1,961	82,52,04,3	333,6	2,021	88,19,42,1	321,0
1,782	65,45,03,9	348,0	1,842	71,33,25,7	347,9	1,902	77,19,09,7	342,8	1,962	82,57,37,9	333,4	2,022	88,25,03,1	320,8
1,783	65,50,51,9	348,0	1,843	71,39,13,6	347,8	1,903	77,24,52,5	342,7	1,963	83,03,11,3	333,2	2,023	88,30,23,9	320,5
1,784	65,56,39,9	348,1	1,844	71,45,01,4	347,7	1,904	77,30,35,2	342,6	1,964	83,08,44,5	333,0	2,024	88,35,44,4	320,3
1,785	66,02,28,0	348,1	1,845	71,50,49,1	347,7	1,905	77,36,17,8	342,4	1,965	83,14,17,5	332,8	2,025	88,41,04,7	320,1
1,786	66,08,16,1	348,2	1,846	71,56,36,8	347,7	1,906	77,42,00,2	342,3	1,966	83,19,50,3	332,7	2,026	88,46,24,8	319,8
1,787	66,14,04,3	348,2	1,847	72,02,24,5	347,6	1,907	77,47,42,5	342,1	1,967	83,25,23,0	332,5	2,027	88,51,44,6	319,6
1,788	66,19,52,5	348,2	1,848	72,08,12,1	347,6	1,908	77,53,24,6	342,0	1,968	83,30,55,5	332,3	2,028	88,57,04,2	319,4
1,789	66,25,40,7	348,3	1,849	72,13,59,7	347,5	1,909	77,59,06,6	341,9	1,969	83,36,27,8	332,1	2,029	89,02,23,6	319,1
1,790	66,31,29,0	348,3	1,850	72,19,47,2	347,5	1,910	78,04,48,5	341,7	1,970	83,41,59,9	331,8	2,030	89,07,42,7	318,9
1,791	66,37,17,3	348,3	1,851	72,25,34,7	347,4	1,911	78,10,30,2	341,5	1,971	83,47,31,7	331,7	2,031	89,13,01,6	318,6
1,792	66,43,05,6	348,3	1,852	72,31,22,1	347,3	1,912	78,16,11,7	341,4	1,972	83,53,03,4	331,5	2,032	89,18,20,2	318,4
1,793	66,48,53,9	348,4	1,853	72,37,09,4	347,3	1,913	78,21,53,1	341,3	1,973	83,58,34,9	331,3	2,033	89,23,38,6	318,1
1,794	66,54,42,3	348,4	1,854	72,42,56,7	347,2	1,914	78,27,34,4	341,1	1,974	84,04,06,2	331,1	2,034	89,28,56,7	317,9
1,795	67,00,30,7	348,5	1,855	72,48,43,9	347,1	1,915	78,33,15,5	341,0	1,975	84,09,37,3	330,9	2,035	89,34,14,6	317,7
1,796	67,06,19,2	348,5	1,856	72,54,31,0	347,0	1,916	78,38,56,5	340,9	1,976	84,15,08,2	330,7	2,036	89,39,32,3	317,5
1,797	67,12,07,7	348,4	1,857	73,00,18,0	347,0	1,917	78,44,37,4	340,8	1,977	84,20,38,9	330,5	2,037	89,44,49,8	317,2
1,798	67,17,56,1	348,4	1,858	73,06,05,0	347,0	1,918	78,50,18,2	340,7	1,978	84,26,09,4	330,4	2,038	89,50,07,0	316,9
1,799	67,23,44,5	348,4	1,859	73,11,52,0	346,9	1,919	78,55,58,9	340,5	1,979	84,31,39,8	330,1	2,039	89,55,23,9	316,7
1,800	67,29,32,9	348,5	1,860	73,17,38,9	346,8	1,920	79,01,39,4	340,4	1,980	84,37,09,9	329,9	2,040	90,00,40,6	316,5
1,801	67,35,21,4	348,5	1,861	73,23,25,7	346,8	1,921	79,07,19,8	340,2	1,981	84,42,39,8	329,7	2,041	90,05,57,1	316,2
1,802	67,41,09,9	348,5	1,862	73,29,12,5	346,7	1,922	79,13,00,0	340,0	1,982	84,48,09,5	329,5	2,042	90,11,13,3	316,0
1,803	67,46,58,4	348,6	1,863	73,34,59,2	346,6	1,923	79,18,40,0	339,9	1,983	84,53,39,0	329,3	2,043	90,16,29,3	315,8
1,804	67,52,47,0	348,6	1,864	73,40,45,8	346,5	1,924	79,24,19,9	339,7	1,984	84,59,08,3	329,1	2,044	90,21,45,1	315,5
1,805	67,58,35,6	348,5	1,865	73,46,32,3	346,4	1,925	79,29,59,6	339,6	1,985	85,04,37,4	328,9	2,045	90,27,00,6	315,2
1,806	68,04,24,1	348,6	1,866	73,52,18,7	346,4	1,926	79,35,39,2	339,5	1,986	85,10,06,3	328,7	2,046	90,32,15,8	315,0
1,807	68,10,12,7	348,6	1,867	73,58,05,1	346,3	1,927	79,41,18,7	339,3	1,987	85,15,35,0	328,5	2,047	90,37,30,8	314,8
1,808	68,16,01,3	348,6	1,868	74,03,51,4	346,2	1,928	79,46,58,0	339,2	1,988	85,21,03,5	328,3	2,048	99,42,45,6	314,5
1,809	68,21,49,9	348,6	1,869	74,09,37,6	346,1	1,929	79,52,37,2	339,0	1,989	85,26,31,8	328,1	2,049	90,48,00,1	314,3
1,810	68,27,38,5	348,6	1,870	74,15,23,7	346,0	1,930	79,58,16,2	338,9	1,990	85,31,59,9	327,8	2,050	90,53,14,4	314,0
1,811	68,33,27,1	348,6	1,871	74,21,09,7	345,9	1,931	80,03,55,1	338,7	1,991	85,37,27,7	327,7	2,051	90,58,28,4	313,7
1,812	68,39,15,7	348,5	1,872	74,26,55,6	345,9	1,932	80,09,33,8	338,5	1,992	85,42,55,4	327,4	2,052	91,03,42,1	313,5
1,813	68,45,04,2	348,6	1,873	74,32,41,5	345,8	1,933	80,15,12,3	338,4	1,993	85,48,22,8	327,2	2,053	91,08,55,6	313,3
1,814	68,50,52,8	348,6	1,874	74,38,27,3	345,7	1,934	80,20,50,7	338,2	1,994	85,53,50,0	327,0	2,054	91,14,08,9	313,1
1,815	68,56,41,4	348,5	1,875	74,44,13,0	345,6	1,935	80,26,28,9	338,0	1,995	85,59,17,0	326,8	2,055	91,19,22,0	312,8
1,816	69,02,29,9	348,5	1,876	74,49,58,6	345,5	1,936	80,32,06,9	337,9	1,996	86,04,43,8	326,6	2,056	91,24,34,8	312,5
1,817	69,08,18,4	348,6	1,877	74,55,44,1	345,5	1,937	80,37,44,8	337,7	1,997	86,10,10,4	326,4	2,057	91,29,47,3	312,3
1,818	69,14,07,0	348,5	1,878	75,01,29,6	345,4	1,938	80,43,22,5	337,6	1,998	86,15,36,8	326,1	2,058	91,34,59,6	312,0
1,819	69,19,55,5	348,5	1,879	75,07,15,0	345,3	1,939	80,49,00,1	337,5	1,999	86,21,02,9	325,9	2,059	91,40,11,6	311,8
1,820	69,25,44,0	348,5	1,880	75,13,00,3	345,2	1,940	80,54,37,6	337,3	2,000	86,26,28,8	325,7	2,060	91,45,23,4	311,5
1,821	69,31,32,5	348,5	1,881	75,18,45,5	345,1	1,941	81,00,14,9	337,1	2,001	86,31,54,5	325,5	2,061	91,50,34,9	311,2
1,822	69,37,21,0	348,5	1,882	75,24,30,6	344,9	1,942	81,05,52,0	336,9	2,002	86,37,20,0	325,3	2,062	91,55,46,1	311,0
1,823	69,43,09,5	348,5	1,883	75,30,15,5	344,9	1,943	81,11,28,9	336,8	2,003	86,42,45,3	325,1	2,063	92,00,57,1	310,7
1,824	69,48,58,0	348,5	1,884	75,36,00,4	344,8	1,944	81,17,05,7	336,6	2,004	86,48,10,4	324,8	2,064	92,06,07,8	310,5
1,825	69,54,46,5	348,4	1,885	75,41,45,2	344,7	1,945	81,22,42,3	336,5	2,005	86,53,35,2	324,6	2,065	92,11,18,3	310,2
1,826	70,00,34,9	348,4	1,886	75,47,29,9	344,5	1,946	81,28,18,8	336,3	2,006	86,58,59,8	324,4	2,066	92,16,28,5	310,0
1,827	70,06,23,3	348,4	1,887	75,53,14,4	344,5	1,947	81,33,55,1	336,1	2,007	87,04,24,2	324,2	2,067	92,21,38,5	309,7
1,828	70,12,11,7	348,3	1,888	75,58,58,9	344,4	1,948	81,39,31,2	335,9	2,008	87,09,48,4	323,9	2,068	92,26,48,2	309,5
1,829	70,18,00,0	348,3	1,889	76,04,43,3	344,2	1,949	81,45,07,1	335,7	2,009	87,15,12,3	323,7	2,069	92,31,57,7	309,2
1,830	70,23,48,3	348,3	1,890	76,10,27,5	344,1	1,950	81,50,42,8	335,6	2,010	87,20,36,0	323,5	2,070	92,37,06,9	308,9
1,831	70,29,36,6	348,3	1,891	76,16,11,6	344,0	1,951	81,56,18,4	335,4	2,011	87,25,59,5	323,3	2,071	92,42,15,8	308,7
1,832	70,35,24,9	348,2	1,892	76,21,55,6	343,9	1,952	82,01,53,8	335,2	2,012	87,31,22,8	323,1	2,072	92,47,24,5	308,4
1,833	70,41,13,1	348,2	1,893	76,27,39,5	343,8	1,953	82,07,29,0	335,1	2,013	87,36,45,9	322,8	2,073	92,52,32,9	308,2
1,834	70,47,01,3	348,2	1,894	76,33,23,3	343,7	1,954	82,13,04,1	334,9	2,014	87,42,08,7	322,6	2,074	92,57,41,1	307,9
1,835	70,52,49,5	348,1	1,895	76,39,07,0	343,6	1,955	82,18,39,0	334,7	2,015	87,47,31,3	322,4	2,075	93,02,49,0	307,7
1,836	70,58,37,6	348,1	1,896	76,44,50,6	343,5	1,956	82,24,13,7	334,5	2,016	87,52,53,7	322,1	2,076	93,07,56,7	307,4
1,837	71,04,25,7	348,1	1,897	76,50,34,1	343,4	1,957	82,29,48,2	334,3	2,017	87,58,15,8	321,9	2,077	93,13,04,1	307,1
1,838	71,10,13,8	348,0	1,898	76,56,17,5	343,2	1,958	82,35,22,5	334,1	2,018	88,03,37,7	321,7	2,078	93,18,11,2	306,8
1,839	71,16,01,8	348,0	1,899	77,02,00,7	343,1	1,959	82,40,56,6	333,9	2,019	88,08,59,4	321,5	2,079	93,23,18,0	306,5
1,840	71,21,49,8	348,0	1,900	77,07,43,8	343,0	1,960	82,46,30,5	333,8	2,020	88,14,20,9	321,2	2,080	93,28,24,5	306,3

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
	d m s	s		d m s	s		d m s	s		d m s	s		d m s	s
2,080	93,28,24,5	306,3	2,140	98,26,49,4	290,1	2,200	103,08,34,9	273,1	2,260	107,33,22,7	256,2	2,320	111,41,25,5	239,7
2,081	93,33,30,8	306,1	2,141	98,31,39,5	289,8	2,201	103,13,08,0	272,9	2,261	107,37,38,9	255,9	2,321	111,45,25,2	239,4
2,082	93,38,36,9	305,8	2,142	98,36,29,3	289,5	2,202	103,17,40,9	272,6	2,262	107,41,54,8	255,7	2,322	111,49,24,6	239,2
2,083	93,43,42,7	305,6	2,143	98,41,18,8	289,2	2,203	103,22,13,5	272,3	2,263	107,46,10,5	255,4	2,323	111,53,23,8	238,9
2,084	93,48,48,3	305,3	2,144	98,46,08,0	288,9	2,204	103,26,45,8	272,0	2,264	107,50,25,9	255,1	2,324	111,57,22,7	238,6
2,085	93,53,53,6	305,0	2,145	98,50,56,9	288,6	2,205	103,31,17,8	271,7	2,265	107,54,41,0	254,8	2,325	112,01,21,3	238,4
2,086	93,58,58,6	304,8	2,146	98,55,45,5	288,4	2,206	103,35,49,5	271,4	2,266	107,58,55,8	254,5	2,326	112,05,19,7	238,1
2,087	94,04,03,4	304,5	2,147	99,00,33,9	288,1	2,207	103,40,20,9	271,2	2,267	108,03,10,3	254,3	2,327	112,09,17,8	237,8
2,088	94,09,07,9	304,2	2,148	99,05,22,0	287,8	2,208	103,44,52,1	270,9	2,268	108,07,24,6	253,9	2,328	112,13,15,6	237,5
2,089	94,14,12,1	304,0	2,149	99,10,09,8	287,5	2,209	103,49,23,0	270,6	2,269	108,11,38,5	253,7	2,329	112,17,13,1	237,3
2,090	94,19,16,1	303,7	2,150	99,14,57,3	287,3	2,210	103,53,53,6	270,3	2,270	108,15,52,2	253,4	2,330	112,21,10,4	237,0
2,091	94,24,19,8	303,4	2,151	99,19,44,6	286,9	2,211	103,58,23,9	270,0	2,271	108,20,05,6	253,1	2,331	112,25,07,4	236,8
2,092	94,29,23,2	303,2	2,152	99,24,31,5	286,7	2,212	104,02,53,9	269,7	2,272	108,24,18,7	252,9	2,332	112,29,04,2	236,5
2,093	94,34,26,4	302,9	2,153	99,29,18,2	286,5	2,213	104,07,23,6	269,5	2,273	108,28,31,6	252,6	2,333	112,33,00,7	236,2
2,094	94,39,29,3	302,7	2,154	99,34,04,7	286,1	2,214	104,11,53,1	269,2	2,274	108,32,44,2	252,3	2,334	112,36,56,9	235,9
2,095	94,44,32,0	302,4	2,155	99,38,50,8	285,8	2,215	104,16,22,3	268,9	2,275	108,36,56,5	252,0	2,335	112,40,52,8	235,7
2,096	94,49,34,4	302,1	2,156	99,43,36,6	285,6	2,216	104,20,51,2	268,6	2,276	108,41,08,5	251,8	2,336	112,44,48,5	235,4
2,097	94,54,36,5	301,8	2,157	99,48,22,2	285,3	2,217	104,25,19,8	268,3	2,277	108,45,20,3	251,4	2,337	112,48,43,9	235,2
2,098	94,59,38,3	301,5	2,158	99,53,07,5	285,0	2,218	104,29,48,1	268,0	2,278	108,49,31,7	251,2	2,338	112,52,39,1	234,9
2,099	95,04,39,8	301,3	2,159	99,57,52,5	284,7	2,219	104,34,16,1	267,8	2,279	108,53,42,9	250,9	2,339	112,56,34,0	234,6
2,100	95,09,41,1	301,0	2,160	100,02,37,2	284,5	2,220	104,38,43,9	267,4	2,280	108,57,53,8	250,6	2,340	113,00,28,6	234,4
2,101	95,14,42,1	300,8	2,161	100,07,21,7	284,2	2,221	104,43,11,3	267,2	2,281	109,02,04,4	250,4	2,341	113,04,23,0	234,0
2,102	95,19,42,9	300,5	2,162	100,12,05,9	283,9	2,222	104,47,38,5	266,9	2,282	109,06,14,8	250,1	2,342	113,08,17,0	233,8
2,103	95,24,43,4	300,2	2,163	100,16,49,8	283,6	2,223	104,52,05,4	266,7	2,283	109,10,24,9	249,8	2,343	113,12,10,8	233,6
2,104	95,29,43,6	299,9	2,164	100,21,33,4	283,3	2,224	104,56,32,1	266,3	2,284	109,14,34,7	249,5	2,344	113,16,04,4	233,3
2,105	95,34,43,5	299,7	2,165	100,26,16,7	283,0	2,225	105,00,58,4	266,1	2,285	109,18,44,2	249,3	2,345	113,19,57,7	233,0
2,106	95,39,43,2	299,4	2,166	100,30,59,7	282,8	2,226	105,05,24,5	265,7	2,286	109,22,53,5	248,9	2,346	113,23,50,7	232,7
2,107	95,44,42,6	299,1	2,167	100,35,42,5	282,5	2,227	105,09,50,2	265,5	2,287	109,27,02,4	248,7	2,347	113,27,43,4	232,5
2,108	95,49,41,7	298,9	2,168	100,40,25,0	282,2	2,228	105,14,15,7	265,2	2,288	109,31,11,1	248,5	2,348	113,31,35,9	232,2
2,109	95,54,40,6	298,6	2,169	100,45,07,2	281,9	2,229	105,18,40,9	265,0	2,289	109,35,19,6	248,1	2,349	113,35,28,1	232,0
2,110	95,59,39,2	298,3	2,170	100,49,49,1	281,6	2,230	105,23,05,9	264,6	2,290	109,39,27,7	247,9	2,350	113,39,20,1	231,7
2,111	96,04,37,5	298,1	2,171	100,54,30,7	281,3	2,231	105,27,30,5	264,4	2,291	109,43,35,6	247,6	2,351	113,43,11,8	231,5
2,112	96,09,35,6	297,8	2,172	100,59,12,0	281,1	2,232	105,31,54,9	264,1	2,292	109,47,43,2	247,4	2,352	113,47,03,3	231,2
2,113	96,14,33,4	297,5	2,173	101,03,53,1	280,8	2,233	105,36,19,0	263,8	2,293	109,51,50,6	247,0	2,353	113,50,54,5	230,9
2,114	96,19,30,9	297,2	2,174	101,08,33,9	280,5	2,234	105,40,42,8	263,5	2,294	109,55,57,6	246,8	2,354	113,54,45,4	230,7
2,115	96,24,28,1	297,0	2,175	101,13,14,4	280,2	2,235	105,45,06,3	263,2	2,295	110,00,04,4	246,6	2,355	113,58,36,1	230,4
2,116	96,29,25,1	296,7	2,176	101,17,54,6	280,0	2,236	105,49,29,5	263,0	2,296	110,04,11,0	246,2	2,356	114,02,26,5	230,1
2,117	96,34,21,8	296,4	2,177	101,22,34,6	279,6	2,237	105,53,52,5	262,6	2,297	110,08,17,2	246,0	2,357	114,06,16,6	229,9
2,118	96,39,18,2	296,1	2,178	101,27,14,2	279,4	2,238	105,58,15,1	262,4	2,298	110,12,23,2	245,7	2,358	114,10,06,5	229,6
2,119	96,44,14,3	295,9	2,179	101,31,53,6	279,1	2,239	106,02,37,5	262,1	2,299	110,16,28,9	245,5	2,359	114,13,56,1	229,4
2,120	96,49,10,2	295,6	2,180	101,36,32,7	278,8	2,240	106,06,59,6	261,9	2,300	110,20,34,4	245,1	2,360	114,17,45,5	229,1
2,121	96,54,05,8	295,3	2,181	101,41,11,5	278,5	2,241	106,11,21,5	261,5	2,301	110,24,39,5	244,9	2,361	114,21,34,6	228,8
2,122	96,59,01,1	295,0	2,182	101,45,50,0	278,2	2,242	106,15,43,0	261,3	2,302	110,28,44,4	244,6	2,362	114,25,23,4	228,5
2,123	97,03,56,1	294,8	2,183	101,50,28,2	278,0	2,243	106,20,04,3	261,0	2,303	110,32,49,0	244,3	2,363	114,29,11,9	228,3
2,124	97,08,50,9	294,5	2,184	101,55,06,2	277,7	2,244	106,24,25,3	260,7	2,304	110,36,53,3	244,1	2,364	114,33,00,2	228,1
2,125	97,13,45,4	294,2	2,185	101,59,43,9	277,4	2,245	106,28,46,0	260,4	2,305	110,40,57,4	243,7	2,365	114,36,48,3	227,8
2,126	97,18,39,6	293,9	2,186	102,04,21,3	277,1	2,246	106,33,06,4	260,1	2,306	110,45,01,1	243,5	2,366	114,40,36,1	227,5
2,127	97,23,33,5	293,7	2,187	102,08,58,4	276,8	2,247	106,37,26,5	259,8	2,307	110,49,04,6	243,3	2,367	114,44,23,6	227,3
2,128	97,28,27,2	293,4	2,188	102,13,35,2	276,5	2,248	106,41,46,3	259,6	2,308	110,53,07,9	242,9	2,368	114,48,10,9	227,0
2,129	97,33,20,6	293,1	2,189	102,18,11,7	276,3	2,249	106,46,05,9	259,3	2,309	110,57,10,8	242,7	2,369	114,51,57,9	226,7
2,130	97,38,13,7	292,8	2,190	102,22,48,0	276,0	2,250	106,50,25,2	259,0	2,310	111,01,13,5	242,4	2,370	114,55,44,6	226,5
2,131	97,43,06,5	292,6	2,191	102,27,24,0	275,6	2,251	106,54,44,2	258,7	2,311	111,05,15,9	242,2	2,371	114,59,31,1	226,2
2,132	97,47,59,1	292,2	2,192	102,31,59,6	275,4	2,252	106,59,02,9	258,5	2,312	111,09,18,1	241,8	2,372	115,03,17,3	226,0
2,133	97,52,51,3	292,0	2,193	102,36,35,0	275,1	2,253	107,03,21,4	258,2	2,313	111,13,19,9	241,6	2,373	115,07,03,3	225,7
2,134	97,57,43,3	291,7	2,194	102,41,10,1	274,9	2,254	107,07,39,6	257,8	2,314	111,17,21,5	241,4	2,374	115,10,49,0	225,4
2,135	98,02,35,0	291,5	2,195	102,45,45,0	274,6	2,255	107,11,57,4	257,6	2,315	111,21,22,9	241,0	2,375	115,14,34,4	225,2
2,136	98,07,26,5	291,1	2,196	102,50,19,6	274,2	2,256	107,16,15,0	257,4	2,316	111,25,23,9	240,8	2,376	115,18,19,6	225,0
2,137	98,12,17,6	290,9	2,197	102,54,53,8	274,0	2,257	107,20,32,4	257,0	2,317	111,20,24,7	240,5	2,377	115,22,04,6	224,7
2,138	98,17,08,5	290,6	2,198	102,59,27,8	273,7	2,258	107,24,49,4	256,8	2,318	111,33,25,2	240,3	2,378	115,25,49,3	224,4
2,139	98,21,59,1	290,3	2,199	103,04,01,5	273,4	2,259	107,29,06,2	256,5	2,319	111,37,25,5	240,0	2,379	115,29,33,7	224,2
2,140	98,26,49,4	290,1	2,200	103,08,34,9	273,1	2,260	107,33,22,7	256,2	2,320	111,41,25,5	239,7	2,380	115,33,17,9	223,9

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
	d m s	s		d m s	s		d m s	s		d m s	s		d m s	s
2,380	115,33,17,9	223,9	2,440	119,09,48,1	209,0	2,500	122,31,52,0	195,1	2,560	125,40,28,5	182,1	2,620	128,36,36,7	170,2
2,381	115,37,01,8	223,7	2,441	119,13,17,1	208,8	2,501	122,35,07,1	194,9	2,561	125,43,30,6	181,9	2,621	128,39,26,9	169,9
2,382	115,40,45,5	223,4	2,442	119,16,45,9	208,5	2,502	122,38,22,0	194,6	2,562	125,46,32,5	181,7	2,622	128,42,16,8	169,7
2,383	115,44,28,9	223,1	2,443	119,20,14,4	208,3	2,503	122,41,36,6	194,4	2,563	125,49,34,2	181,5	2,623	128,45,06,5	169,5
2,384	115,48,12,0	222,9	2,444	119,23,42,7	208,0	2,504	122,44,51,0	194,2	2,564	125,52,35,7	181,3	2,624	128,47,56,0	169,4
2,385	115,51,54,9	222,7	2,445	119,27,10,7	207,8	2,505	122,48,05,2	194,0	2,565	125,55,37,0	181,1	2,625	128,50,45,4	169,1
2,386	115,55,37,6	222,4	2,446	119,30,38,5	207,6	2,506	122,51,19,2	193,8	2,566	125,58,38,1	180,8	2,626	128,53,34,5	169,0
2,387	115,59,20,0	222,1	2,447	119,34,06,1	207,3	2,507	122,54,33,0	193,6	2,567	126,01,38,9	180,7	2,627	128,56,23,5	168,8
2,388	116,03,02,1	221,8	2,448	119,37,33,4	207,1	2,508	122,57,46,6	193,3	2,568	126,04,39,6	180,5	2,628	128,59,12,3	168,6
2,389	116,06,43,9	221,6	2,449	119,41,00,5	206,8	2,509	123,00,59,9	193,1	2,569	126,07,40,1	180,2	2,629	129,02,00,9	168,4
2,390	116,10,25,5	221,4	2,450	119,44,27,3	206,6	2,510	123,04,13,0	192,8	2,570	126,10,40,3	180,1	2,630	129,04,49,3	168,2
2,391	116,14,06,9	221,1	2,451	119,47,53,9	206,4	2,511	123,07,25,8	192,6	2,571	126,13,40,4	179,8	2,631	129,07,37,5	168,0
2,392	116,17,48,0	220,8	2,452	119,51,20,3	206,1	2,512	123,10,38,4	192,3	2,572	126,16,40,2	179,6	2,632	129,10,25,5	167,8
2,393	116,21,28,8	220,6	2,453	119,54,46,4	205,9	2,513	123,13,50,7	192,2	2,573	126,19,39,8	179,5	2,633	129,13,13,3	167,7
2,394	116,25,09,4	220,3	2,454	119,58,12,3	205,7	2,514	123,17,02,9	191,9	2,574	126,22,39,3	179,2	2,634	129,16,01,0	167,4
2,395	116,28,49,7	220,1	2,455	120,01,38,0	205,4	2,515	123,20,14,8	191,7	2,575	126,25,38,5	179,0	2,635	129,18,48,4	167,3
2,396	116,32,29,8	219,8	2,456	120,05,03,4	205,2	2,516	123,23,26,5	191,5	2,576	126,28,37,5	178,8	2,636	129,21,35,7	167,1
2,397	116,36,09,6	219,6	2,457	120,08,28,6	204,9	2,517	123,26,38,0	191,2	2,577	126,31,36,3	178,7	2,637	129,24,22,8	166,9
2,398	116,39,49,2	219,3	2,458	120,11,53,5	204,7	2,518	123,29,49,2	191,1	2,578	126,34,35,0	178,4	2,638	129,27,09,7	166,7
2,399	116,43,28,5	219,2	2,459	120,15,18,2	204,5	2,519	123,33,00,3	190,8	2,579	126,37,33,4	178,2	2,639	129,29,56,4	166,6
2,400	116,47,07,7	218,8	2,460	120,18,42,7	204,3	2,520	123,36,11,1	190,6	2,580	126,40,31,6	178,0	2,640	129,32,43,0	166,3
2,401	116,50,46,5	218,7	2,461	120,22,07,0	204,0	2,521	123,39,21,7	190,4	2,581	126,43,29,6	177,8	2,641	129,35,29,3	166,2
2,402	116,54,25,2	218,3	2,462	120,25,31,0	203,8	2,522	123,42,32,1	190,2	2,582	126,46,27,4	177,6	2,642	129,38,15,5	165,9
2,403	116,58,03,5	218,1	2,463	120,28,54,8	203,5	2,523	123,45,42,3	190,0	2,583	126,49,25,0	177,4	2,643	129,41,01,4	165,8
2,404	117,01,41,6	217,9	2,464	120,32,18,3	203,3	2,524	123,48,52,3	189,7	2,584	126,52,22,4	177,2	2,644	129,43,47,2	165,6
2,405	117,05,19,5	217,7	2,465	120,35,41,6	203,1	2,525	123,52,02,0	189,5	2,585	126,55,19,6	177,0	2,645	129,46,32,8	165,5
2,406	117,08,57,2	217,3	2,466	120,39,04,7	202,8	2,526	123,55,11,5	189,3	2,586	126,58,16,6	176,8	2,646	129,49,18,3	165,2
2,407	117,12,34,5	217,1	2,467	120,42,27,5	202,6	2,527	123,58,20,8	189,1	2,587	127,01,13,4	176,6	2,647	129,52,03,5	165,1
2,408	117,16,11,6	216,9	2,468	120,45,50,1	202,4	2,528	124,01,29,9	188,9	2,588	127,04,10,0	176,4	2,648	129,54,48,6	164,8
2,409	117,19,48,5	216,7	2,469	120,49,12,5	202,2	2,529	124,04,38,8	188,7	2,589	127,07,06,4	176,2	2,649	129,57,33,4	164,7
2,410	117,23,25,2	216,3	2,470	120,52,34,7	201,9	2,530	124,07,47,5	188,4	2,590	127,10,02,6	176,0	2,650	130,00,18,1	164,5
2,411	117,27,01,5	216,2	2,471	120,55,56,6	201,7	2,531	124,10,55,9	188,3	2,591	127,12,58,6	175,8	2,651	130,03,02,6	164,2
2,412	117,30,37,7	215,9	2,472	120,59,18,3	201,4	2,532	124,14,04,2	188,0	2,592	127,15,54,4	175,6	2,652	130,05,46,8	164,1
2,413	117,34,13,6	215,5	2,473	121,02,39,7	201,2	2,533	124,17,12,2	187,8	2,593	127,18,50,0	175,4	2,653	130,08,30,9	163,9
2,414	117,37,49,1	215,3	2,474	121,06,00,9	201,0	2,534	124,20,20,0	187,6	2,594	127,21,45,4	175,1	2,654	130,11,14,8	163,8
2,415	117,41,24,4	215,1	2,475	121,09,21,9	200,8	2,535	124,23,27,6	187,4	2,595	127,24,40,5	175,0	2,655	130,13,58,6	163,5
2,416	117,44,59,5	214,8	2,476	121,12,42,7	200,5	2,536	124,26,35,0	187,2	2,596	127,27,35,5	174,8	2,656	130,16,42,1	163,4
2,417	117,48,34,3	214,6	2,477	121,16,03,2	200,3	2,537	124,29,42,2	186,9	2,597	127,30,30,3	174,6	2,657	130,19,25,5	163,2
2,418	117,52,08,9	214,4	2,478	121,19,23,5	200,1	2,538	124,32,49,1	186,8	2,598	127,33,24,9	174,4	2,658	130,22,08,7	163,0
2,419	117,55,43,3	214,0	2,479	121,22,43,6	199,8	2,539	124,35,55,9	186,5	2,599	127,36,19,3	174,2	2,659	130,24,51,7	162,8
2,420	117,59,17,3	213,9	2,480	121,26,03,4	199,6	2,540	124,39,02,4	186,3	2,600	127,39,13,5	174,0	2,660	130,27,34,5	162,7
2,421	118,02,51,2	213,6	2,481	121,29,23,0	199,4	2,541	124,42,08,7	186,1	2,601	127,42,07,5	173,8	2,661	130,30,17,2	162,4
2,422	118,06,24,8	213,3	2,482	121,32,42,4	199,1	2,542	124,45,14,8	185,9	2,602	127,45,01,3	173,6	2,662	130,32,59,6	162,3
2,423	118,09,58,1	213,1	2,483	121,36,01,5	198,9	2,543	124,48,20,7	185,7	2,603	127,47,54,9	173,4	2,663	130,35,41,9	162,1
2,424	118,13,31,2	212,8	2,484	121,39,20,4	198,7	2,544	124,51,26,4	185,4	2,604	127,50,48,3	173,3	2,664	130,38,24,0	162,0
2,425	118,17,04,0	212,7	2,485	121,42,39,1	198,5	2,545	124,54,31,8	185,3	2,605	127,53,41,6	173,0	2,665	130,41,06,0	161,7
2,426	118,20,36,7	212,4	2,486	121,45,57,6	198,2	2,546	124,57,37,1	185,0	2,606	127,56,34,6	172,8	2,666	130,43,47,7	161,6
2,427	118,24,09,1	212,1	2,487	121,49,15,8	198,0	2,547	125,00,42,1	184,9	2,607	127,59,27,4	172,7	2,667	130,46,29,3	161,4
2,428	118,27,41,2	211,9	2,488	121,52,33,8	197,8	2,548	125,03,47,0	184,6	2,608	128,02,20,1	172,4	2,668	130,49,10,7	161,3
2,429	118,31,13,1	211,7	2,489	121,55,51,6	197,5	2,549	125,06,51,6	184,4	2,609	128,05,12,5	172,2	2,669	130,51,52,0	161,0
2,430	118,34,44,8	211,4	2,490	121,59,09,1	197,3	2,550	125,09,56,0	184,2	2,610	128,08,04,7	172,1	2,670	130,54,33,0	160,8
2,431	118,38,16,2	211,2	2,491	122,02,26,4	197,1	2,551	125,13,00,2	184,0	2,611	128,10,56,8	171,9	2,671	130,57,13,8	160,7
2,432	118,41,47,4	210,9	2,492	122,05,43,5	196,9	2,552	125,16,04,2	183,8	2,612	128,13,48,7	171,6	2,672	130,59,54,5	160,5
2,433	118,45,18,3	210,7	2,493	122,09,00,4	196,6	2,553	125,19,08,0	183,5	2,613	128,16,40,3	171,5	2,673	131,02,35,0	160,3
2,434	118,48,49,0	210,5	2,494	122,12,17,0	196,4	2,554	125,22,11,5	183,4	2,614	128,19,31,8	171,3	2,674	131,05,15,3	160,2
2,435	118,52,19,5	210,2	2,495	122,15,33,4	196,2	2,555	125,25,14,9	183,1	2,615	128,22,23,1	171,1	2,675	131,07,55,5	160,0
2,436	118,55,49,7	210,0	2,496	122,18,49,6	195,9	2,556	125,28,18,0	183,0	2,616	128,25,14,2	170,9	2,676	131,10,35,5	159,8
2,437	118,59,19,7	209,7	2,497	122,22,05,5	195,8	2,557	125,31,21,0	182,7	2,617	128,28,05,1	170,7	2,677	131,13,15,3	159,6
2,438	119,02,49,4	209,5	2,498	122,25,21,3	195,5	2,558	125,34,23,7	182,5	2,618	128,30,55,8	170,5	2,678	131,15,54,9	159,4
2,439	119,06,18,9	209,2	2,499	122,28,36,8	195,2	2,559	125,37,26,2	182,3	2,619	128,33,46,3	170,4	2,679	131,18,34,3	159,3
2,440	119,09,48,1	209,0	2,500	122,31,52,0	195,1	2,560	125,40,28,5	182,1	2,620	128,36,36,7	170,2	2,680	131,21,13,6	159,1

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
	d m s	s		d m s	s		d m s	s		d m s	s		d m s	s
2,680	131,21,13,6	159,0	2,740	133,55,12,9	148,9	2,800	136,19,25,8	139,5	2,860	138,34,38,2	130,9	2,920	140,41,32,6	122,9
2,681	131,23,52,6	158,9	2,741	133,57,41,8	148,8	2,801	136,21,45,3	139,4	2,861	138,36,49,1	130,7	2,921	140,43,35,5	122,8
2,682	131,26,31,5	158,7	2,742	134,00,10,6	148,6	2,802	136,24,04,7	139,2	2,862	138,38,59,8	130,6	2,922	140,45,38,3	122,7
2,683	131,29,10,2	158,6	2,743	134,02,39,2	148,4	2,803	136,26,23,9	139,0	2,863	138,41,10,4	130,5	2,923	140,47,41,0	122,5
2,684	131,31,48,8	158,3	2,744	134,05,07,6	148,2	2,804	136,28,42,9	138,9	2,864	138,43,20,9	130,3	2,924	140,49,43,5	122,4
2,685	131,34,27,1	158,2	2,745	134,07,35,8	148,1	2,805	136,31,01,8	138,8	2,865	138,45,31,2	130,2	2,925	140,51,45,9	122,3
2,686	131,37,05,3	158,1	2,746	134,10,03,9	147,9	2,806	136,33,20,6	138,6	2,866	138,47,41,4	130,0	2,926	140,53,48,2	122,1
2,687	131,39,43,4	157,8	2,747	134,12,31,8	147,8	2,807	136,35,39,2	138,5	2,867	138,49,51,4	129,9	2,927	140,55,50,3	122,1
2,688	131,42,21,2	157,6	2,748	134,14,59,6	147,6	2,808	136,37,57,7	138,3	2,868	138,52,01,3	129,8	2,928	140,57,52,4	121,9
2,689	131,44,58,8	157,5	2,749	134,17,27,2	147,5	2,809	136,40,16,0	138,2	2,869	138,54,11,1	129,7	2,929	140,59,54,3	121,7
2,690	131,47,36,3	157,3	2,750	134,19,54,7	147,3	2,810	136,42,34,2	138,0	2,870	138,56,20,8	129,5	2,930	141,01,56,0	121,7
2,691	131,50,13,6	157,2	2,751	134,22,22,0	147,1	2,811	136,44,52,2	137,9	2,871	138,58,30,3	129,3	2,931	141,03,57,7	121,5
2,692	131,52,50,8	156,9	2,752	134,24,49,1	146,9	2,812	136,47,10,1	137,7	2,872	139,00,39,6	129,3	2,932	141,05,59,2	121,4
2,693	131,55,27,7	156,8	2,753	134,27,16,0	146,8	2,813	136,49,27,8	137,6	2,873	139,02,48,9	129,1	2,933	141,08,00,6	121,3
2,694	131,58,04,5	156,6	2,754	134,29,42,8	146,6	2,814	136,51,45,4	137,4	2,874	139,04,58,0	128,9	2,934	141,10,01,9	121,1
2,695	132,00,41,1	156,5	2,755	134,32,09,4	146,5	2,815	136,54,02,8	137,3	2,875	139,07,06,9	128,8	2,935	141,12,03,0	121,1
2,696	132,03,17,6	156,2	2,756	134,34,35,9	146,3	2,816	136,56,20,1	137,1	2,876	139,09,15,7	128,7	2,936	141,14,04,1	120,9
2,697	132,05,53,8	156,1	2,757	134,37,02,2	146,2	2,817	136,58,37,2	137,0	2,877	139,11,24,4	128,6	2,937	141,16,05,0	120,7
2,698	132,08,29,9	155,9	2,758	134,39,28,4	146,0	2,818	137,00,54,2	136,9	2,878	139,13,33,0	128,4	2,938	141,18,05,7	120,7
2,699	132,11,05,8	155,8	2,759	134,41,54,4	145,8	2,819	137,03,11,1	136,7	2,879	139,15,41,4	128,3	2,939	141,20,06,4	120,5
2,700	132,13,41,6	155,6	2,760	134,44,20,2	145,7	2,820	137,05,27,8	136,6	2,880	139,17,49,7	128,1	2,940	141,22,06,9	120,4
2,701	132,16,17,2	155,4	2,761	134,46,45,9	145,5	2,821	137,07,44,4	136,4	2,881	139,19,57,8	128,0	2,941	141,24,07,3	120,3
2,702	132,18,52,6	155,2	2,762	134,49,11,4	145,4	2,822	137,10,00,8	136,2	2,882	139,22,05,8	127,9	2,942	141,26,07,6	120,2
2,703	132,21,27,8	155,1	2,763	134,51,36,8	145,2	2,823	137,12,17,0	136,1	2,883	139,24,13,7	127,8	2,943	141,28,07,8	120,0
2,704	132,24,02,9	154,9	2,764	134,54,02,0	145,0	2,824	137,14,33,1	136,0	2,884	139,26,21,5	127,6	2,944	141,30,07,8	119,9
2,705	132,26,37,8	154,7	2,765	134,56,27,0	144,9	2,825	137,16,49,1	135,8	2,885	139,28,29,1	127,5	2,945	141,32,07,7	119,8
2,706	132,29,12,5	154,6	2,766	134,58,51,9	144,7	2,826	137,19,04,9	135,7	2,886	139,30,36,6	127,3	2,946	141,34,07,5	119,6
2,707	132,31,47,1	154,4	2,767	135,01,16,6	144,6	2,827	137,21,20,6	135,6	2,887	139,32,43,9	127,2	2,947	141,36,07,1	119,6
2,708	132,34,21,5	154,2	2,768	135,03,41,2	144,4	2,828	137,23,36,2	135,4	2,888	139,34,51,1	127,1	2,948	141,38,06,7	119,4
2,709	132,36,55,7	154,0	2,769	135,06,05,6	144,2	2,829	137,25,51,6	135,2	2,889	139,36,58,2	126,9	2,949	141,40,06,1	119,3
2,710	132,39,29,7	153,9	2,770	135,08,29,8	144,1	2,830	137,28,06,8	135,1	2,890	139,39,05,1	126,8	2,950	141,42,05,4	119,2
2,711	132,42,03,6	153,7	2,771	135,10,53,9	144,0	2,831	137,30,21,9	135,0	2,891	139,41,11,9	126,7	2,951	141,44,04,6	119,0
2,712	132,44,37,3	153,5	2,772	135,13,17,9	143,8	2,832	137,32,36,9	134,8	2,892	139,43,18,6	126,6	2,952	141,46,03,6	118,9
2,713	132,47,10,8	153,4	2,773	135,15,41,7	143,6	2,833	137,34,51,7	134,7	2,893	139,45,25,2	126,4	2,953	141,48,02,5	118,8
2,714	132,49,44,2	153,2	2,774	135,18,05,3	143,5	2,834	137,37,06,4	134,5	2,894	139,47,31,6	126,3	2,954	141,50,01,3	118,7
2,715	132,52,17,4	153,1	2,775	135,20,28,8	143,3	2,835	137,39,20,9	134,4	2,895	139,49,37,9	126,1	2,955	141,52,00,0	118,6
2,716	132,54,50,5	152,8	2,776	135,22,52,1	143,1	2,836	137,41,35,3	134,3	2,896	139,51,44,0	126,0	2,956	141,53,58,6	118,4
2,717	132,57,23,3	152,7	2,777	135,25,15,2	143,0	2,837	137,43,49,6	134,1	2,897	139,53,50,0	125,9	2,957	141,55,57,0	118,3
2,718	132,59,56,0	152,5	2,778	135,27,38,2	142,9	2,838	137,46,03,7	133,9	2,898	139,55,55,9	125,8	2,958	141,57,55,3	118,2
2,719	133,02,28,5	152,4	2,779	135,30,01,1	142,7	2,839	137,48,17,6	133,8	2,899	139,58,01,7	125,6	2,959	141,59,53,5	118,1
2,720	133,05,00,9	152,2	2,780	135,32,23,8	142,5	2,840	137,50,31,4	133,7	2,900	140,00,07,3	125,5	2,960	142,01,51,6	118,0
2,721	133,07,33,1	152,0	2,781	135,34,46,3	142,4	2,841	137,52,45,1	133,6	2,901	140,02,12,8	125,4	2,961	142,03,49,6	117,8
2,722	133,10,05,1	151,8	2,782	135,37,08,7	142,2	2,842	137,54,58,7	133,4	2,902	140,04,18,2	125,2	2,962	142,05,47,4	117,7
2,723	133,12,36,9	151,7	2,783	135,39,30,9	142,1	2,843	137,57,12,1	133,2	2,903	140,06,23,4	125,1	2,963	142,07,45,1	117,6
2,724	133,15,08,6	151,5	2,784	135,41,53,0	141,9	2,844	137,59,25,3	133,1	2,904	140,08,28,5	125,0	2,964	142,09,42,7	117,5
2,725	133,17,40,1	151,3	2,785	135,44,14,9	141,8	2,845	138,01,38,4	133,0	2,905	140,10,33,5	124,8	2,965	142,11,40,2	117,3
2,726	133,20,11,4	151,2	2,786	135,46,36,7	141,6	2,846	138,03,51,4	132,8	2,906	140,12,38,3	124,7	2,966	142,13,37,5	117,3
2,727	133,22,42,6	151,0	2,787	135,48,58,3	141,5	2,847	138,06,04,2	132,7	2,907	140,14,43,0	124,6	2,967	142,15,34,8	117,1
2,728	133,25,13,6	150,8	2,788	135,51,19,8	141,4	2,848	138,08,16,9	132,5	2,908	140,16,47,6	124,4	2,968	142,17,31,9	117,0
2,729	133,27,44,4	150,7	2,789	135,53,41,2	141,1	2,849	138,10,29,4	132,4	2,909	140,18,52,0	124,4	2,969	142,19,28,9	116,8
2,730	133,30,15,1	150,5	2,790	135,56,02,3	141,0	2,850	138,12,41,8	132,3	2,910	140,20,56,4	124,2	2,970	142,21,25,7	116,8
2,731	133,32,45,6	150,4	2,791	135,58,23,3	140,9	2,851	138,14,54,1	132,1	2,911	140,23,00,6	124,0	2,971	142,23,22,5	116,6
2,732	133,35,16,0	150,2	2,792	136,00,44,2	140,7	2,852	138,17,06,2	132,0	2,912	140,25,04,6	124,0	2,972	142,25,19,1	116,5
2,733	133,37,46,2	150,0	2,793	136,03,04,9	140,6	2,853	138,19,18,2	131,8	2,913	140,27,08,6	123,8	2,973	142,27,15,6	116,4
2,734	133,40,16,2	149,9	2,794	136,05,25,5	140,4	2,854	138,21,30,0	131,7	2,914	140,29,12,4	123,7	2,974	142,29,12,0	116,3
2,735	133,42,46,1	149,7	2,795	136,07,45,9	140,3	2,855	138,23,41,7	131,6	2,915	140,31,16,1	123,5	2,975	142,31,08,3	116,1
2,736	133,45,15,8	149,5	2,796	136,10,06,2	140,1	2,856	138,25,53,3	131,4	2,916	140,33,19,6	123,5	2,976	142,33,04,4	116,0
2,737	133,47,45,3	149,4	2,797	136,12,26,3	140,0	2,857	138,28,04,7	131,3	2,917	140,35,23,1	123,3	2,977	142,35,00,4	116,0
2,738	133,50,14,7	149,2	2,798	136,14,46,3	139,8	2,858	138,30,16,0	131,2	2,918	140,37,26,4	123,2	2,978	142,36,56,4	115,8
2,739	133,52,43,9	149,0	2,799	136,17,06,1	139,7	2,859	138,32,27,2	131,0	2,919	140,39,29,6	123,0	2,979	142,38,52,2	115,6
2,740	133,55,12,9	148,9	2,800	136,19,25,8	139,5	2,860	138,34,38,2	130,9	2,920	140,41,32,6	122,9	2,980	142,40,47,8	115,6

TABLE III.

To find the true anomaly U, corresponding to the time t' from the perihelion in days, in a parabolic orbit, whose perihelion distance is the same as the mean distance of the sun from the earth.

Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.	Log. of t' days.	True Anom. U.	Diff.
	d m s	s		d m s	s		d m s	s		d m s	s		d m s	s
2,980	142,40,47,8	115,6	3,00	143,18,57,3	1127,4	3,25	150,17,18,2	883,7	3,50	155,47,35,6	703,1	3,75	160,11,49,1	565,6
2,981	142,42,43,4	115,4	3,01	143,37,44,7	1116,1	3,26	150,32,01,9	875,4	3,51	155,59,18,7	696,8	3,76	160,21,14,7	560,9
2,982	142,44,38,8	115,4	3,02	143,56,20,8	1104,8	3,27	150,46,37,1	867,2	3,52	156,10,55,5	690,7	3,77	160,30,35,6	556,1
2,983	142,46,34,2	115,2	3,03	144,14,45,6	1093,8	3,28	151,01,04,5	859,0	3,53	156,22,26,2	684,6	3,78	160,39,51,7	551,3
2,984	142,48,29,4	115,1	3,04	144,32,59,4	1083,1	3,29	151,15,23,5	851,2	3,54	156,33,50,8	678,6	3,79	160,49,03,0	546,7
2,985	142,50,24,5	114,9	3,05	144,51,02,5	1072,3	3,30	151,29,34,7	843,3	3,55	156,45,09,4	672,7	3,80	160,58,09,7	542,2
2,986	142,52,19,4	114,9	3,06	145,08,54,8	1061,7	3,31	151,43,38,0	835,5	3,56	156,56,22,1	666,7	3,81	161,07,11,9	537,6
2,987	142,54,14,3	114,8	3,07	145,26,36,5	1051,2	3,32	151,57,33,5	827,6	3,57	157,07,28,8	660,9	3,82	161,16,09,5	533,0
2,988	142,56,09,1	114,6	3,08	145,44,07,7	1040,9	3,33	152,11,21,1	820,2	3,58	157,18,29,7	655,1	3,83	161,25,02,5	528,6
2,989	142,58,03,7	114,5	3,09	146,01,28,6	1030,7	3,34	152,25,01,3	812,7	3,59	157,29,24,8	649,5	3,84	161,33,51,1	524,1
2,990	142,59,58,2	114,4	3,10	146,18,39,3	1020,6	3,35	152,38,34,0	805,1	3,60	157,40,14,3	643,7	3,85	161,42,35,2	519,8
2,991	143,01,52,6	114,3	3,11	146,35,39,9	1010,6	3,36	152,51,59,1	797,8	3,61	157,50,58,0	638,2	3,86	161,51,15,0	515,4
2,992	143,03,46,9	114,2	3,12	146,52,30,5	1000,8	3,37	153,05,16,9	790,6	3,62	158,01,36,2	632,6	3,87	161,59,50,4	511,1
2,993	143,05,41,1	114,1	3,13	147,09,11,3	991,2	3,38	153,18,27,5	783,3	3,63	158,12,08,8	627,2	3,88	162,08,21,5	506,8
2,994	143,07,35,2	113,9	3,14	147,25,42,5	981,6	3,39	153,31,30,8	776,3	3,64	158,22,36,0	621,7	3,89	162,16,48,3	502,7
2,995	143,09,29,1	113,9	3,15	147,42,04,1	972,1	3,40	153,44,27,1	769,2	3,65	158,32,57,7	616,4	3,90	162,25,11,0	498,4
2,996	143,11,23,0	113,7	3,16	147,58,16,2	962,8	3,41	153,57,16,3	762,2	3,66	158,43,14,1	611,1	3,91	162,33,29,4	494,2
2,997	143,13,16,7	113,6	3,17	148,14,19,0	953,5	3,42	154,09,58,5	755,4	3,67	158,53,25,2	605,8	3,92	162,41,43,6	490,2
2,998	143,15,10,3	113,5	3,18	148,30,12,5	944,4	3,43	154,22,33,9	748,7	3,68	159,03,31,0	600,6	3,93	162,49,53,8	486,2
2,999	143,17,03,8	113,5	3,19	148,45,56,9	935,4	3,44	154,35,02,6	741,8	3,69	159,13,31,6	595,5	3,94	162,58,00,0	482,0
3,000	143,18,57,3		3,20	149,01,32,3	926,6	3,45	154,47,24,4	735,2	3,70	159,23,27,1	590,4	3,95	163,06,02,0	478,1
			3,21	149,16,58,9	917,7	3,46	154,59,39,6	728,8	3,71	159,33,17,5	585,3	3,96	163,14,00,1	474,1
			3,22	149,32,16,6	909,1	3,47	155,11,48,4	722,1	3,72	159,43,02,8	580,4	3,97	163,21,54,2	470,3
			3,23	149,47,25,7	900,5	3,48	155,23,50,5	715,7	3,73	159,52,43,2	575,4	3,98	163,29,44,5	466,3
			3,24	150,02,26,2	892,0	3,49	155,35,46,2	709,4	3,74	160,02,18,6	570,5	3,99	163,37,30,8	462,6
			3,25	150,17,18,2	883,7	3,50	155,47,35,6	703,1	3,75	160,11,49,1	565,6	4,00	163,45,13,4	
												5,00	172,32,09,2	

TABLE IV.

This table is given for the purpose of computing the true anomaly v, from the time t from the perihelion; in a very excentrical orbit, whether it be an ellipsis or hyperbola. The excentricity is represented by e, and the perihelion distance by D. In using this table we must first compute, by means of Table III., the anomaly U, corresponding to the time t from the perihelion, in a parabola, whose perihelion distance is D. To this value of U, we must apply a correction, of the *first order*, $S.\,(1-e)$; first proposed by Simpson, and which corresponds to the function [697]. When $1-e$ is somewhat large, and great accuracy is required. we must apply a correction of the *second order* $B.\,(1-e)^2$; first computed by Bessel. The logarithms of the values of S, B, in sexagesimal seconds, are given in Table IV., for every degree of the anomaly U, with their differences; and *when any one of these values is negative, the letter n is annexed to its logarithm.* For intermediate values of U, we must use the common rules of interpolation. The logarithms of S are given to *seven* places of decimals, and those of B to *five* places; but in most cases it will be sufficiently accurate, if we reject the two last of these figures. The logarithm of S, added to the log. $(1-e)$, gives the logarithm of Simpson's correction; and the logarithm of B, added to 2 log. $(1-e)$, gives the logarithm of Bessel's correction. In symbols we have,

$$v = U + S.(1-e) + B.(1-e)^2; \qquad \text{[In an ellipsis].}$$

$$v = U - S.(e-1) + B.(e-1)^2; \qquad \text{[In a hyperbola].}$$

EXAMPLE.

We shall suppose that with the time t from passing the perihelion, and the perihelion distance D, the anomaly in a parabola is found, by means of Table III., to be $U=50^d$. Then it is required to find the true anomaly v; in an ellipsis, whose excentricity is $e=0{,}99$; and in a hyperbola, whose excentricity is $e=1,01$.

In an ellipsis.

Given $e=0{,}99$ and $U=50^d$ to find v.

S	log.	$4{,}38317_n$	B	log.	$3{,}82417_n$
$1-e$	log.	$8{,}00000$	$(1-e)$	log.	$8{,}00000$
$-241^s{,}6$	log.	$2{,}38317_n$		same	$8{,}00000$
			$-0^s{,}7$	log.	$9{,}82417_n$

	$U=$	$50^d\ 00^m\ 00^s{,}0$
Simpson's correction,	$-$	$4\ 01\ {,}6$
Bessel's correction,	$-$	$0\ {,}7$
True anomaly $v=$		$49\ 55\ 57\ {,}7$

In a hyperbola.

Given $e=1,01$ and $U=50^d$ to find v.

The calculation of the corrections of Simpson and Bessel, is the same as in the ellipsis; the only difference is in the sign of Simpson's correction.

	$U=$	$50^d\ 00^m\ 00^s{,}0$
Simpson's correction,	$+$	$4\ 01\ {,}6$
Bessel's correction,	$-$	$0\ {,}7$
True anomaly $v=$		$50\ 04\ 00\ {,}9$

TABLE IV.

To find the true anomaly v, in a very eccentric ellipsis or hyperbola, from the corresponding anomaly U in a parabola; according to Simpson's method, improved by Bessel.

U.	Log. of S. sex. seconds.	First Diff.	Second Diff.	Log. of B.	Diff.
d					
0	Infin. neg.	Infinite.		Infin. neg.	Infin.
1	2,95425,35$_n$	30066,59	Infin. neg.	2,05194$_n$	30139
2	3,25491,94$_n$	17565,04	— 12501,55	2,35333$_n$	17676
3	3,43056,98$_n$	12432,07	— 5132,97	2,53009$_n$	12601
4	3,55489,05$_n$	9611,39	— 2820,68	2,65610$_n$	9840
5	3,65100,44$_n$	7840,72	— 1770,67	2,75450$_n$	8111
6	3,72941,16$_n$	6559,40	— 1281,32	2,83561$_n$	6912
7	3,79500,56$_n$	5665,92	— 893,48	2,90473$_n$	6040
8	3,85166,48$_n$	4963,91	— 702,01	2,96513$_n$	5390
9	3,90130,39$_n$	4406,23	— 557,68	3,01903$_n$	4874
10	3,94536,62$_n$	3951,45	— 454,78	3,06777$_n$	4463
11	3,98488,07$_n$	3572,47	— 378,98	3,11240$_n$	4124
12	4,02060,54$_n$	3251,27	— 321,20	3,15364$_n$	3841
13	4,05311,81$_n$	2974,14	— 277,13	3,19205$_n$	3600
14	4,08285,95$_n$	2733,98	— 240,16	3,22805$_n$	3393
15	4,11019,93$_n$	2520,74	— 213,24	3,26198$_n$	3211
16	4,13540,67$_n$	2331,31	— 189,43	3,29409$_n$	3049
17	4,15871,98$_n$	2160,97	— 170,34	3,32458$_n$	2907
18	4,18032,95$_n$	2006,73	— 154,24	3,35365$_n$	2777
19	4,20039,68$_n$	1865,93	— 140,80	3,38142$_n$	2658
20	4,21905,61$_n$	1736,51	— 129,42	3,40800$_n$	2548
21	4,23642,12$_n$	1616,94	— 119,57	3,43348$_n$	2446
22	4,25259,06$_n$	1505,72	— 111,22	3,45794$_n$	2352
23	4,26764,78$_n$	1401,68	— 104,04	3,48146$_n$	2261
24	4,28166,46$_n$	1304,08	— 97,60	3,50407$_n$	2174
25	4,29470,54$_n$	1211,79	— 92,29	3,52581$_n$	2091
26	4,30682,33$_n$	1124,28	— 87,51	3,54672$_n$	2012
27	4,31806,61$_n$	1040,89	— 83,39	3,56684$_n$	1935
28	4,32847,50$_n$	961,17	— 79,72	3,58619$_n$	1860
29	4,33808,67$_n$	884,55	— 76,62	3,60479$_n$	1785
30	4,34693,22$_n$	810,59	— 73,96	3,62264$_n$	1712
31	4,35503,81$_n$	739,02	— 71,57	3,63976$_n$	1641
32	4,36242,83$_n$	669,49	— 69,53	3,65617$_n$	1570
33	4,36912,32$_n$	601,61	— 67,88	3,67187$_n$	1498
34	4,37513,93$_n$	535,21	— 66,40	3,68685$_n$	1428
35	4,38049,14$_n$	469,89	— 65,32	3,70113$_n$	1359
36	4,38519,03$_n$	405,49	— 64,40	3,71472$_n$	1286
37	4,38924,52$_n$	341,80	— 63,69	3,72758$_n$	1213
38	4,39266,32$_n$	278,41	— 63,39	3,73971$_n$	1139
39	4,39544,73$_n$	215,33	— 63,08	3,75110$_n$	1064
40	4,39760,06$_n$	152,20	— 63,13	3,76174$_n$	991
41	4,39912,26$_n$	88,92	— 63,28	3,77165$_n$	915
42	4,40001,18$_n$	24,83	— 64,09	3,78080$_n$	839
43	4,40026,01$_n$	— 39,69	— 64,52	3,78919$_n$	760
44	4,39986,32$_n$	— 105,29	— 65,60	3,79679$_n$	677
45	4,39881,03$_n$	— 172,10	— 66,81	3,80356$_n$	593
46	4,39708,93$_n$	— 240,53	— 68,43	3,80949$_n$	505
47	4,39468,40$_n$	— 310,50	— 69,97	3,81454$_n$	416
48	4,39157,90$_n$	— 382,96	— 72,46	3,81870$_n$	324
49	4,38774,94$_n$	— 457,88	— 74,92	3,82194$_n$	223
50	4,38317,06$_n$	— 535,74	— 77,86	3,82417$_n$	121
51	4,37781,32$_n$	— 616,92	— 81,18	3,82538$_n$	10
52	4,37164,40$_n$	— 702,21	— 85,29	3,82548$_n$	— 103
53	4,36462,19$_n$	— 791,95	— 89,74	3,82445$_n$	— 225
54	4,35670,24$_n$	— 887,07	— 95,12	3,82220$_n$	— 357
55	4,34783,17$_n$	— 988,14	— 101,07	3,81863$_n$	— 498
56	4,33795,03$_n$	— 1096,29	— 108,15	3,81365$_n$	— 652
57	4,32698,74$_n$	— 1212,66	— 116,37	3,80713$_n$	— 816
58	4,31486,08$_n$	— 1338,63	— 125,97	3,79897$_n$	— 1000
59	4,30147,45$_n$	— 1475,69	— 137,06	3,78897$_n$	— 1202
60	4,28671,76$_n$	— 1626,06	— 150,37	3,77695$_n$	— 1428

U.	Log. of S. sex. seconds	First Diff.	Second Diff.	Log. of B. sex. seconds.	Diff.
d					
60	4,28671,76$_n$	— 1626,06	— 150,37	3,77695$_n$	— 1428
61	4,27045,70$_n$	— 1792,30	— 166,24	3,76267$_n$	— 1686
62	4,25253,40$_n$	— 1977,47	— 185,17	3,74581$_n$	— 1979
63	4,23275,93$_n$	— 2185,58	— 208,11	3,72602$_n$	— 2320
64	4,21090,35$_n$	— 2421,97	— 236,39	3,70282$_n$	— 2721
65	4,18668,38$_n$	— 2693,44	— 271,47	3,67561$_n$	— 3207
66	4,15974,94$_n$	— 3009,36	— 315,92	3,64354$_n$	— 3799
67	4,12965,58$_n$	— 3382,62	— 373,26	3,60555$_n$	— 4548
68	4,09582,96$_n$	— 3831,30	— 448,68	3,56007$_n$	— 5536
69	4,05751,66$_n$	— 4382,78	— 551,48	3,50471$_n$	— 6892
70	4,01368,88$_n$	— 5077,77	— 694,99	3,43579$_n$	— 8874
71	3,96291,11$_n$	— 5983,59	— 905,82	3,34705$_n$	— 12093
72	3,90307,52$_n$	— 7216,09	— 1232,50	3,22612$_n$	— 18261
73	3,83091,43$_n$	— 8996,44	— 1780,35	3,04351$_n$	— 35397
74	3,74094,99$_n$	— 11802,33	— 2805,89	2,68954$_n$	
75	3,62292,66$_n$	— 16910,21		2,21860	+ 71557
76	3,45382,45$_n$	— 28624,88		2,93417	26840
77	3,16757,57$_n$	— 2,29739,84		3,20257	17235
78	0,87017,73$_n$			3,37492	12888
79	3,16902,43	30814,31		3,50380	10390
80	3,47716,74	18262,12		3,60770	8763
81	3,65978,86	13119,81	— 5142,31	3,69533	7621
82	3,79098,67	10301,88	— 2817,93	3,77154	6774
83	3,89400,55	8517,07	— 1784,81	3,83928	6114
84	3,97917,62	7284,17	— 1232,90	3,90042	5588
85	4,05201,79	6380,22	— 903,95	3,95630	5152
86	4,11582,01	5688,57	— 691,65	4,00782	4798
87	4,17270,58	5141,71	— 546,86	4,05580	4498
88	4,22412,29	4699,51	— 442,20	4,10078	4236
89	4,27111,80	4330,71	— 368,80	4,14314	4011
90	4,31442,51	4023,08	— 307,63	4,18325	3804
91	4,35465,59	3759,67	— 263,41	4,22129	3636
92	4,39225,26	3532,27	— 227,40	4,25765	3479
93	4,42757,53	3333,42	— 198,85	4,29244	3341
94	4,46090,95	3159,56	— 173,86	4,32585	3214
95	4,49250,51	3004,37	— 155,19	4,35799	3093
96	4,52254,88	2866,04	— 138,33	4,38892	2995
97	4,55120,92	2741,90	— 124,14	4,41887	2899
98	4,57862,82	2629,89	— 112,01	4,44786	2813
99	4,60492,71	2528,07	— 101,82	4,47599	2734
100	4,63020,78	2435,74	— 92,33	4,50333	2659
101	4,65456,52	2351,03	— 84,71	4,52992	2593
102	4,67807,55	2273,42	— 77,61	4,55585	2531
103	4,70080,97	2202,22	— 71,20	4,58116	2475
104	4,72283,19	2136,55	— 65,67	4,60591	2421
105	4,74419,74	2075,91	— 60,64	4,63012	2371
106	4,76495,65	2019,80	— 56,11	4,65383	2328
107	4,78515,45	1967,83	— 51,97	4,67711	2288
108	4,80483,28	1919,59	— 48,24	4,69999	2249
109	4,82402,87	1874,82	— 44,77	4,72248	2215
110	4,84277,69	1833,19	— 41,63	4,74463	2184
111	4,86110,88	1794,50	— 38,69	4,76647	2155
112	4,87905,38	1758,60	— 35,90	4,78802	2128
113	4,89663,98	1725,15	— 33,45	4,80930	2106
114	4,91389,13	1694,13	— 31,02	4,83036	2085
115	4,93083,26	1665,32	— 28,81	4,85121	2068
116	4,94748,58	1638,62	— 26,70	4,87189	2053
117	4,96387,20	1613,91	— 24,71	4,89242	2039
118	4,98001,11	1591,12	— 22,79	4,91281	2027
119	4,99592,23	1570,11	— 21,01	4,93308	2021
120	5,01162,34	1550,91	— 19,20	4,95329	2015

TABLE IV.

To find the true anomaly v, in a very eccentric ellipsis or hyperbola, from the corresponding anomaly U in a parabola; according to Simpson's method, improved by Bessel.

U.	Log. of S. sex. seconds.	First Diff.	Second Diff.	Log. of B. sex. seconds.	Diff.
d					
120	5,01162,34	1550,91	− 19,20	4,95329	2015
121	5,02713,25	1533,19	− 17,72	4,97344	2011
122	5,04246,44	1517,43	− 15,76	4,99355	2012
123	5,05763,87	1502,94	− 14,49	5,01367	2012
124	5,07266,81	1490,25	− 12,69	5,03379	2016
125	5,08757,06	1478,81	− 11,44	5,05395	2024
126	5,10235,87	1469,14	− 9,67	5,07419	2032
127	5,11705,01	1460,34	− 8,80	5,09451	2045
128	5,13165,35	1453,56	− 6,78	5,11496	2060
129	5,14618,91	1448,03	− 5,53	5,13556	2078
130	5,16066,94	1443,82	− 4,21	5,15634	2100
131	5,17510,76	1441,39	− 2,43	5,17734	2121
132	5,18952,15	1440,17	− 1,22	5,19855	2149
133	5,20392,32	1440,18	+ 0,01	5,22004	2178
134	5,21832,50	1442,16	1,98	5,24182	2213
135	5,23274,66	1445,23	3,07	5,26395	2254
136	5,24719,89	1450,16	4,93	5,28649	2292
137	5,26170,05	1456,59	6,43	5,30941	2337
138	5,27626,64	1464,66	8,07	5,33278	2386
139	5,29091,30	1474,48	9,82	5,35664	2442
140	5,30565,78	1486,09	11,61	5,38106	2505
141	5,32051,87	1499,56	13,47	5,40611	2569
142	5,33551,43	1514,99	15,43	5,43180	2641
143	5,35066,42	1532,45	17,46	5,45821	2718
144	5,36598,87	1552,30	19,85	5,48539	2802
145	5,38151,17	1574,26	21,96	5,51341	2895
146	5,39725,43	1598,81	24,55	5,54236	2994
147	5,41324,24	1626,03	27,22	5,57230	3101
148	5,42950,27	1656,14	30,11	5,60331	3216
149	5,44606,41	1689,34	33,20	5,63547	3346
150	5,46295,75	1725,91	36,57	5,66893	3482
151	5,48021,66	1766,31	40,40	5,70375	3635
152	5,49787,97	1810,61	44,30	5,74010	3796
153	5,51598,58	1859,45	48,84	5,77806	3975
154	5,53458,03	1913,22	53,77	5,81781	4167
155	5,55371,25	1972,49	59,27	5,85948	4384
156	5,57343,74	2037,95	65,46	5,90332	4614
157	5,59381,69	2110,33	72,38	5,94946	4870
158	5,61492,02	2190,64	80,31	5,99816	5149
159	5,63682,66	2279,86	89,22	6,04965	5461
160	5,65962,52	2379,47	99,61	6,10426	5806
161	5,68341,99	2491,01	111,54	6,16232	6182
162	5,70833,00	2616,49	125,48	6,22414	6602
163	5,73449,49	2758,43	141,94	6,29016	7095
164	5,76207,92	2920,00	161,57	6,36111	7626
165	5,79127,92	3105,20	185,20	6,43737	8239
166	5,82233,12	3319,21	214,01	6,51976	8938
167	5,85552,33	3568,97	249,76	6,60914	9752
168	5,89121,30	3863,72	294,75	6,70666	10698
169	5,92985,02	4216,40	352,68	6,81364	11826
170	5,97201,42	4645,21	428,81	6,93190	13181
171	6,01846,63	5177,17	531,96	7,06371	14849
172	6,07023,80	5853,62	676,45	7,21220	16953
173	6,12877,42	6741,70	888,08	7,38173	19694
174	6,19619,12	7957,82	1216,12	7,57867	23421
175	6,27576,94	9723,30	1765,48	7,81288	28797
176	6,37300,24	12519,06	2795,76	8,10085	37265
177	6,49819,30	17727,12	5208,66	8,47350	52672
178	6,67546,42	30013,76	12286,64	9,00022	
179	6,97560,18	Infinite.	Infinite.		
180	Infinite.				

In the extreme and middle parts of the table, the first differences vary rapidly, in which case we may use the values of S, B, instead of their logarithms, as in the following auxiliary table.

AUXILIARY TABLE IV.

U.	S. sex. seconds.	Diff.	B. sex. seconds.	Diff.
d	s		s	
0	− 0,0	− 900,0	− 0,0	− 112,7
1	− 900,0	− 898,5	− 112,7	− 112,9
2	− 1798,5	− 896,5	− 225,6	− 113,3
3	− 2695,0	− 893,3	− 338,9	− 114,1
4	− 3588,3	− 888,9	− 453,0	− 115,2
5	− 4477,2		− 568,2	
70			− 2727,7	504,1
71			− 2223,6	540,5
72			− 1683,1	577,7
73			− 1105,4	616,1
74	− 5507,4	1310,5	− 489,3	654,7
75	− 4196,9	1353,6	165,4	693,9
76	− 2843,3	1372,4	859,3	735,0
77	− 1470,9	1463,5	1594,3	776,6
78	− 7,4	1483,2	2370,9	
79	1475,8	1524,5		
80	3000,3	1568,4		
81	4568,7	1611,3		
82	6180,0			

TABLE V. — For an Ellipsis.

This table is to be used in finding the true anomaly v, corresponding to the time t from the perihelion, in a very excentrical ellipsis ; the excentricity e and the perihelion distance D being given. In point of accuracy, it is not restricted to the first and second powers of $1-e$, like Table IV., but includes all the powers of that quantity. This table is nearly in the same form as it was first given by Professor Gauss.

Rule. From e find $\alpha = 1-e$, $\alpha'^2 = 0{,}1 + 0{,}9\,.\,e$, and then find the approximate value of log. t', by the following formula;

$$\text{Approx. log. } t' = \text{log. } t + \text{log. } \alpha' - \tfrac{3}{2}\,.\,\text{log. } D.$$

With this value of t' find the corresponding value of U in Table III. ; also,

$$\text{log. } \beta = \text{log. } \alpha + \text{arith. log. co. } \alpha'^2 + 9{,}6989700 - 10{,}0000000\,;$$

$$\text{Approx. log. } \mathcal{A} = \text{log. } \beta + 2 \text{ log. tang. } \tfrac{1}{2}\,U.$$

Enter Table V., with the natural number corresponding to this value of log. $\mathcal{A}$, and find the corresponding log. B, which is to be subtracted from the approximate log. t' to obtain the corrected value of log. t'. With this corrected value find, in Table III., the corrected value of U, and for the sake of distinction, we shall represent it by w; then the corrected value of log. $\mathcal{A}$ is found by the following formula, which is similar to the preceding one, changing U into w;

$$\text{Correct. log. } \mathcal{A} = \text{log. } \beta + 2 \text{ log. tang.} \tfrac{1}{2}\,w.$$

It will very rarely be necessary to repeat again this operation to get a more accurate value of $\mathcal{A}$; we may therefore, with this value of $\mathcal{A}$, find the correct value of C, in Table V., and then,

$$\text{tang.}^2\,\tfrac{1}{2}\,v = \frac{\mathcal{A}}{\mathrm{C} - 0{,}8\,.\,\mathcal{A}}\,.\,\frac{1+e}{1-e}\,; \qquad \text{[Anomaly } v\text{]}.$$

$$r = \frac{\mathrm{C} - 0{,}8\,.\,\mathcal{A}}{\mathrm{C} + 0{,}2\,.\,\mathcal{A}}\,.\,D\,.\,\text{sec.}^2\,\tfrac{1}{2}\,v. \qquad \text{[Radius vector } r\text{]}.$$

We may observe that in computing a large number of observations, it will frequently happen that the value of B is very nearly known, at the commencement of the operation ; in this case the correction B, may be applied to the first process, in finding the approximate value of t'.

EXAMPLE.

Given the excentricity $e = 0{,}96764567$; log. perihelion distance $D = 9{,}7656500$; $t = 63^{\text{days}}{,}544$; to find v and r.
From the value of e we get, $\alpha = 1 - e = 0{,}03235433$; $\alpha'^2 = 0{,}1 + 0{,}9\,e = 0{,}970881103$.

Approximate Operation.			*Corrected Operation.*		
$\alpha'^2 = 0{,}1 + 0{,}9\,.\,e$	log.	9,9871661			
α'	log.	9,9935830			
D	log. co.	0,2343500			
	its half	0,1171750			
$t = 63^{\text{days}}{,}544$	log.	1,8030745			
	Approx. log. t'	2,1481825	Subtract log. $B = 0{,}0000040$ gives correct log. t'		2,1481785
Hence $U = 99^d\,6^m$, in Table III.			Hence U or $w = 99^d\,6^m\,13^s{,}4$ in Table III.		
$\alpha = 1 - e$	log.	8,5099325	. .		
α'^2	log. co.	0,0128339	. .		
	Constant log.	9,6989700	. .		
Sum gives β	log.	8,2217364	. . . .	same	8,2217364
$\tfrac{1}{2}\,U = 49^d\,33^m$	tang.	0,06927	$\tfrac{1}{2}\,w = 49^d\,33^m\,6^s{,}7$	tang.	0,0692972
	same	0,06927		same	0,0692972
Approx. $\mathcal{A} = 0{,}022923$	log.	8,36027	Corrected $\mathcal{A} = 0{,}0229261$	log.	8,3603308
Corresponding log. $B = 0{,}0000040$, Table V.			$0{,}8\,\mathcal{A} = 0{,}0183409$		. .
			$\mathrm{C} = 1{,}0000242$		. .
			$\mathrm{C} - 0{,}8\,\mathcal{A} = 0{,}9816833$	log. co.	0,0080286
			$1 + e = 1{,}96764547$	log.	0,2939469
			$1 - e = \alpha$	log. co.	1,4900675
				Sum is 2 log. tang. $\tfrac{1}{2}\,v$	0,1523738
			$\tfrac{1}{2}\,v = 50^d\,0^m\,0^s{,}1$	tang.	0,0761869
			$v = 100^d\,0^m\,0^s{,}2$		
			$\mathrm{C} + 0{,}2\,\mathcal{A} = 1{,}0046094$	log. co.	9,9980028
			$\mathrm{C} - 0{,}8\,\mathcal{A} = 0{,}9816833$	log.	9,9919714
			D	log.	9,7656500
			$\tfrac{1}{2}\,v = 50^d\,0^m\,0^s{,}1$	sec.	0,1919327
				same	0,1919327
			r	log.	0,1394896

TABLE V. — For an Ellipsis.

In the inverse problem, we have given, the true anomaly v, the perihelion distance D, and the excentricity e, to find the time t from the perihelion in days. This is obtained by the following rule.

Rule. With e and v find $T = \frac{1-e}{1+e} \cdot \text{tang.}^2 \tfrac{1}{2} v$, and then by Table V., the corresponding value of C. Also,

$$\text{Log. } A = \text{log. } T + \text{log. } C + \text{arith. comp. log. } (1 + 0{,}8 \, . \, T);$$

from which we find log. B, by means of Table V. Then we find,

$$\text{log. } t_1 = 2{,}0654486 + \tfrac{3}{2}\text{ log. } D + \tfrac{1}{2}\text{ log. } A + \text{log. } B - \tfrac{1}{2}\text{ log. } (1-e);$$

$$\text{log. } t_2 = \text{log. } t_1 + 8{,}8239087 + \text{log. } A + \text{log. } (1 + 9\,e) - \text{log. } (1-e);$$

$$t = t_1 + t_2 .$$

EXAMPLE.

Given as before $e = 0{,}96764567$; log. perihelion distance $D = 9{,}7656500$; and the true anomaly $v = 100^d\, 0^m\, 0^s{,}2$; to find the time t from the perihelion in days.

$1-e$	log.	8,5099325
$1+e$	log. co.	9,7060531
$\frac{1}{2} v = 50^d\, 0^m\, 0^s{,}1$	tang.	0,0761869
	same tang.	0,0761869
$T = 0{,}0233539$	log.	8,3683594
Hence $C = 1{,}0000242$ Table V.	log.	0,0000105
$1 + 0{,}8\,T = 1{,}0186831$	log. co.	9,9919609
$A = 0{,}0229261$	log.	8,3603308
Corresponding log. B in Table V.		0,0000040

	Constant log.	2,0654486
	$\frac{3}{2}$ log. D	9,6484750
	$\frac{1}{2}$ log. A	9,1801654
	log. B	0,0000040
	$\frac{1}{2}$ log. $(1-e)$ arith. co.	0,7450337
$t_1 = 43^{\text{days}}{,}564$	log.	1,6391267
. .	Constant	8,8239087
. .	A log.	8,3603308
. .	$1 + 9\,e = 9{,}7088110$ log.	0,9871661
. .	$(1-e)$ log. co.	1,4900675
$t_2 = 19^{\text{days}}{,}980$	log.	1,3005998
$t_1 + t_2 = 63^{\text{days}}{,}544 = t.$		

A	Log. B	C	T	A	Log. B	C	T	A	Log. B	C	T
0,000	0,0000000	1,0000000	0,00000	0,040	0,0000120	1,0000741	0,041319	0,080	0,0000485	1,0003006	0,085443
001	000	1,0000000	0,00100	041	126	1,0000779	0,042387	081	498	1,0003083	0,086584
002	000	1,0000002	0,00200	042	133	1,0000818	0,043457	082	510	1,0003160	0,087727
003	001	1,0000004	0,00301	043	139	1,0000858	0,044528	083	523	1,0003239	0,088872
004	001	1,0000007	0,00401	044	146	1,0000898	0,045601	084	535	1,0003319	0,090019
0,005	0,0000002	1,0000011	0,00502	0,045	0,0000152	1,0000940	0,046676	0,085	0,0000548	1,0003399	0,091168
006	003	1,0000016	0,00603	046	159	1,0000982	0,047753	086	561	1,0003481	0,092319
007	004	1,0000022	0,00704	047	166	1,0001026	0,048831	087	575	1,0003564	0,093472
008	005	1,0000029	0,00805	048	173	1,0001070	0,049911	088	588	1,0003647	0,094627
009	006	1,0000037	0,00907	049	181	1,0001116	0,050993	089	602	1,0003732	0,095784
0,010	0,0000007	1,0000046	0,01008	0,050	0,0000188	1,0001162	0,052077	0,090	0,0000615	1,0003818	0,096943
011	009	1,0000056	0,01110	051	196	1,0001210	0,053163	091	629	1,0003904	0,098104
012	011	1,0000066	0,01212	052	204	1,0001258	0,054250	092	643	1,0003992	0,099266
013	013	1,0000078	0,01314	053	212	1,0001307	0,055339	093	658	1,0004081	0,100431
014	015	1,0000090	0,01416	054	220	1,0001358	0,056430	094	672	1,0004170	0,101598
0,015	0,0000017	1,0000103	0,01518	0,055	0,0000228	1,0001409	0,057523	0,095	0,0000687	1,0004261	0,102766
016	019	1,0000118	0,01621	056	236	1,0001461	0,058618	096	701	1,0004353	0,103937
017	022	1,0000133	0,01723	057	245	1,0001514	0,059714	097	716	1,0004446	0,105110
018	024	1,0000149	0,01826	058	254	1,0001568	0,060812	098	731	1,0004539	0,106284
019	027	1,0000166	0,01929	059	263	1,0001623	0,061912	099	746	1,0004634	0,107461
0,020	0,0000030	1,0000184	0,02032	0,060	0,0000272	1,0001679	0,063014	0,100	0,0000762	1,0004730	0,108640
021	033	1,0000203	0,02136	061	281	1,0001736	0,064118	101	777	1,0004826	0,109820
022	036	1,0000223	0,02239	062	290	1,0001794	0,065223	102	793	1,0004924	0,111003
023	040	1,0000244	0,02343	063	300	1,0001853	0,066331	103	809	1,0005023	0,112188
024	043	1,0000265	0,02447	064	309	1,0001913	0,067440	104	825	1,0005123	0,113375
0,025	0,0000047	1,0000288	0,02551	0,065	0,0000319	1,0001974	0,068551	0,105	0,0000841	1,0005224	0,114563
026	051	1,0000312	0,02655	066	329	1,0002036	0,069664	106	857	1,0005325	0,115754
027	055	1,0000336	0,02760	067	339	1,0002099	0,070779	107	873	1,0005428	0,116947
028	059	1,0000362	0,02864	068	350	1,0002163	0,071896	108	890	1,0005532	0,118142
029	063	1,0000388	0,02969	069	360	1,0002228	0,073014	109	907	1,0005637	0,119339
0,030	0,0000067	1,0000416	0,03074	0,070	0,0000371	1,0002294	0,074135	0,110	0,0000924	1,0005743	0,120538
031	072	1,0000444	0,03179	071	381	1,0002360	0,075257	111	941	1,0005850	0,121739
032	077	1,0000473	0,03284	072	392	1,0002428	0,076381	112	958	1,0005958	0,122942
033	082	1,0000503	0,03389	073	403	1,0002497	0,077507	113	975	1,0006067	0,124148
034	087	1,0000535	0,03495	074	415	1,0002567	0,078635	114	993	1,0006177	0,125355
0,035	0,0000092	1,0000567	0,03601	0,075	0,0000426	1,0002638	0,079765	0,115	0,0001011	1,0006288	0,126564
036	097	1,0000600	0,03707	076	437	1,0002709	0,080897	116	1029	1,0006400	0,127776
037	103	1,0000634	0,03813	077	449	1,0002782	0,082030	117	1047	1,0006513	0,128989
038	108	1,0000669	0,03919	078	461	1,0002856	0,083166	118	1065	1,0006627	0,130205
039	114	1,0000704	0,04025	079	473	1,0002930	0,084303	119	1083	1,0006742	0,131423
0,040	0,0000120	1,0000741	0,04132	0,080	0,0000485	1,0003006	0,085443	0,120	0,0001102	1,0006858	0,132643

TABLE V. — For an Ellipsis.

To find the true anomaly in a very excentric ellipsis, by the method of Gauss.

A	Log. *B*	C	T	*A*	Log. *B*	C	T	*A*	Log. *B*	C	T
0,120	0,0001102	1,0006858	0,132643	0,180	0,0002515	1,0015764	0,209894	0,240	0,0004537	1,0028644	0,295980
121	1121	1,0006976	0,133865	181	2543	1,0015945	0,211253	241	4576	1,0028894	0,297498
122	1139	1,0007094	0,135089	182	2572	1,0016128	0,212614	242	4615	1,0029145	0,299018
123	1158	1,0007213	0,136315	183	2601	1,0016311	0,213977	243	4654	1,0029397	0,300542
124	1178	1,0007334	0,137543	184	2630	1,0016496	0,215343	244	4694	1,0029651	0,302068
0,125	0,0001197	1,0007455	0,138774	0,185	0,0002660	1,0016682	0,216712	0,245	0,0004734	1,0029905	0,303597
126	1217	1,0007577	0,140007	186	2689	1,0016868	0,218083	246	4774	1,0030161	0,305129
127	1236	1,0007701	0,141241	187	2719	1,0017057	0,219456	247	4814	1,0030418	0,306664
128	1256	1,0007825	0,142478	188	2749	1,0017246	0,220832	248	4854	1,0030676	0,308202
129	1276	1,0007951	0,143717	189	2779	1,0017436	0,222211	249	4894	1,0030935	0,309743
0,130	0,0001296	1,0008077	0,144959	0,190	0,0002809	1,0017627	0,223592	0,250	0,0004935	1,0031196	0,311286
131	1317	1,0008205	0,146202	191	2839	1,0017820	0,224975	251	4976	1,0031458	0,312833
132	1337	1,0008334	0,147448	192	2870	1,0018013	0,226361	252	5017	1,0031721	0,314382
133	1358	1,0008463	0,148695	193	2900	1,0018208	0,227750	253	5058	1,0031985	0,315935
134	1378	1,0008594	0,149945	194	2931	1,0018404	0,229141	254	5099	1,0032250	0,317490
0,135	0,0001399	1,0008726	0,151197	0,195	0,0002962	1,0018601	0,230535	0,255	0,0005141	1,0032517	0,319048
136	1421	1,0008859	0,152452	196	2993	1,0018799	0,231931	256	5182	1,0032784	0,320610
137	1442	1,0008993	0,153708	197	3025	1,0018998	0,233329	257	5224	1,0033053	0,322174
138	1463	1,0009128	0,154967	198	3056	1,0019198	0,234731	258	5266	1,0033323	0,323741
139	1485	1,0009264	0,156228	199	3088	1,0019400	0,236135	259	5309	1,0033595	0,325312
0,140	0,0001507	1,0009401	0,157491	0,200	0,0003120	1,0019602	0,237541	0,260	0,0005351	1,0033867	0,326885
141	1529	1,0009539	0,158756	201	3152	1,0019806	0,238950	261	5394	1,0034141	0,328461
142	1551	1,0009678	0,160024	202	3184	1,0020011	0,240361	262	5436	1,0034416	0,330041
143	1573	1,0009819	0,161294	203	3216	1,0020217	0,241776	263	5479	1,0034692	0,331623
144	1596	1,0009960	0,162566	204	3249	1,0020424	0,243192	264	5522	1,0034970	0,333208
0,145	0,0001618	1,0010102	0,163840	0,205	0,0003282	1,0020632	0,244612	0,265	0,0005566	1,0035248	0,334797
146	1641	1,0010246	0,165116	206	3315	1,0020842	0,246034	266	5609	1,0035528	0,336388
147	1664	1,0010390	0,166395	207	3348	1,0021052	0,247458	267	5653	1,0035809	0,337983
148	1687	1,0010536	0,167676	208	3381	1,0021264	0,248885	268	5697	1,0036091	0,339580
149	1710	1,0010683	0,168959	209	3414	1,0021477	0,250315	269	5741	1,0036375	0,341181
0,150	0,0001734	1,0010830	0,170245	0,210	0,0003448	1,0021690	0,251748	0,270	0,0005785	1,0036659	0,342785
151	1757	1,0010979	0,171533	211	3482	1,0021905	0,253183	271	5829	1,0036945	0,344392
152	1781	1,0011129	0,172823	212	3516	1,0022122	0,254620	272	5874	1,0037232	0,346002
153	1805	1,0011280	0,174115	213	3550	1,0022339	0,256061	273	5919	1,0037521	0,347615
154	1829	1,0011432	0,175410	214	3584	1,0022557	0,257504	274	5964	1,0037810	0,349231
0,155	0,0001854	1,0011585	0,176707	0,215	0,0003618	1,0022777	0,258950	0,275	0,0006009	1,0038101	0,350850
156	1878	1,0011739	0,178006	216	3653	1,0022998	0,260398	276	6054	1,0038393	0,352473
157	1903	1,0011894	0,179308	217	3688	1,0023220	0,261849	277	6100	1,0038686	0,354098
158	1927	1,0012051	0,180612	218	3723	1,0023443	0,263303	278	6145	1,0038981	0,355727
159	1952	1,0012208	0,181918	219	3758	1,0023667	0,264759	279	6191	1,0039277	0,357359
0,160	0,0001977	1,0012366	0,183226	0,220	0,0003793	1,0023892	0,266218	0,280	0,0006237	1,0039573	0,358994
161	2003	1,0012526	0,184537	221	3829	1,0024119	0,267680	281	6283	1,0039872	0,360632
162	2028	1,0012686	0,185850	222	3865	1,0024347	0,269145	282	6330	1,0040171	0,362274
163	2054	1,0012848	0,187166	223	3900	1,0024576	0,270612	283	6376	1,0040472	0,363918
164	2080	1,0013011	0,188484	224	3936	1,0024806	0,272082	284	6423	1,0040774	0,365566
0,165	0,0002106	1,0013175	0,189804	0,225	0,0003973	1,0025037	0,273555	0,285	0,0006470	1,0041077	0,367217
166	2132	1,0013340	0,191127	226	4009	1,0025269	0,275031	286	6517	1,0041381	0,368871
167	2158	1,0013506	0,192452	227	4046	1,0025502	0,276509	287	6564	1,0041687	0,370526
168	2184	1,0013673	0,193779	228	4082	1,0025737	0,277990	288	6612	1,0041994	0,372186
169	2211	1,0013841	0,195109	229	4119	1,0025973	0,279474	289	6660	1,0042302	0,373853
0,170	0,0002238	1,0014010	0,196441	0,230	0,0004156	1,0026210	0,280960	0,290	0,0006708	1,0042611	0,375521
171	2265	1,0014181	0,197775	231	4194	1,0026448	0,282450	291	6756	1,0042922	0,377191
172	2292	1,0014352	0,199112	232	4231	1,0026687	0,283942	292	6804	1,0043233	0,378865
173	2319	1,0014525	0,200451	233	4269	1,0026928	0,285437	293	6852	1,0043547	0,380542
174	2347	1,0014699	0,201793	234	4306	1,0027169	0,286935	294	6901	1,0043861	0,382222
0,175	0,0002374	1,0014873	0,203137	0,235	0,0004344	1,0027412	0,288435	0,295	0,0006950	1,0044177	0,383906
176	2402	1,0015049	0,204484	236	4382	1,0027656	0,289939	296	6999	1,0044493	0,385593
177	2430	1,0015226	0,205832	237	4421	1,0027901	0,291445	297	7048	1,0044812	0,387283
178	2458	1,0015404	0,207184	238	4459	1,0028148	0,292954	298	7097	1,0045131	0,388977
179	2486	1,0015583	0,208538	239	4498	1,0028395	0,294466	299	7147	1,0045452	0,390673
0,180	0,0002515	1,0015764	0,209894	0,240	0,0004537	1,0028644	0,295980	0,300	0,0007196	1,0045775	0,392374

TABLE VI. — For an Hyperbola.

This table is used in finding the true anomaly v of a comet, moving in a hyperbolic orbit, which approaches very nearly to the form of a parabola; the excentricity e, the perihelion distance D, and the time t from passing the perihelion being given. Like the preceding table, it is not restricted to the first and second powers of $e-1$, but includes all the powers of that quantity.

Rule. From e find $\alpha'^2 = 0{,}1 + 0{,}9\,.\,e$; and then the approximate value of log. t' from the formula,

$$\text{Approx. log. } t' = \text{log. } t + \text{log. } \alpha' - \tfrac{3}{2}\,\text{log. } D.$$

With this value of t', find the corresponding value of U, in Table III., also,

$$\text{log. } \beta = \text{log. } (e-1) + \text{arith. co. log. } \alpha'^2 + 9{,}6989700 - 10{,}0000000\,;$$

$$\text{Approx. log. } A = \text{log. } \beta + 2\text{ log. tang.} \tfrac{1}{2}\,U.$$

Enter Table VI., with the natural number, corresponding to this value of log. A, and find in it the corresponding log. B; which is to be subtracted from the approximate log. t', to obtain the corrected value of log. t'. With this corrected value, find in Table III., the corrected value of log. U; and for distinction, we shall call it w; then the corrected value of log. A is found by the following formula, which is similar to the preceding;

$$\text{Correct. log. } A = \text{log. } \beta + 2\text{ log. tang.} \tfrac{1}{2}\,w.$$

It will very rarely be necessary to repeat the operation, to get a more accurate value of A; we shall therefore use it, in finding the correct value of C, in Table VI., and then,

$$\text{tang.}^2 \tfrac{1}{2}\,v = \frac{A}{C + 0{,}8\,.\,A} \cdot \frac{e+1}{e-1}\,; \qquad [\text{Anomaly } v].$$

$$r = \frac{C + 0{,}8\,.\,A}{C - 0{,}2\,.\,A} \cdot D\,.\,\text{sec.}^2 \tfrac{1}{2}\,v. \qquad [\text{Radius vector } r].$$

In computing a large number of observations, it will frequently happen that the value of B is very nearly known, at the commencement of the operation; in this case, the correction B, may be applied in the first process, for finding the approximate value of t'.

EXAMPLE.

Given the excentricity $e = 1{,}261882$; log. perihelion distance $= 0{,}0201657$; $t = 65^{\text{days}}{,}41236$; to find v and r.

Approximate Operation.

$\alpha'^2 = 0{,}1 + 0{,}9\,.\,e = 1{,}2356938$	log.	0,0919108
α'	log.	0,0459554
D	log. co.	9,9798343
	its half	9,9899171
$t = 65^{\text{days}}{,}41236$	log.	1,8156598
	Approx. log. t'	1,8313666
Hence $U = 70^d\,32^m$, nearly, in Table III.		
$e - 1 = 0{,}261882$	log.	9,4181056
α'^2	log. co.	9,9080892
	Constant log.	9,6989700
	Sum gives β log.	9,0251648
$\frac{1}{2}\,U = 35^d\,16^m$	tang.	9,84952
	same	9,84952
Approx. $A = 0{,}05299$	log.	8,72420
Corresponding log. $B = 0{,}0000207$, Table VI.		

Corrected Operation.

Subtract log. $B = 0{,}0000207$ gives correct log. $t' = 1{,}8313459$		
Hence U or $w = 70^d\,31^m\,37^s{,}0$ in Table III.		
. . . .	same	9,0251648
$\frac{1}{2}\,w = 35^d\,15^m\,48^s{,}5$	tang.	9,8494702
	same	9,8494702
Corrected $A = 0{,}0529792$	log.	8,7241052
$0{,}8\,A = 0{,}0423834$		
$C = 1{,}0001261$		
$C + 0{,}8\,A = 1{,}0425095$	log. co.	9,9819200
$e + 1 = 2{,}261882$	log.	0,3544699
$e - 1 = 0{,}261882$	log. co.	0,5818944
Sum is 2 log. tang. $\frac{1}{2}\,v$		9,6423895
$\frac{1}{2}\,v = 33^d\,31^m\,30^s$	tang.	9,8211947
$v = 67^d\,03^m\,00^s$		
$C - 0{,}2\,A = 0{,}9895303$	log. co.	0,0045709
$C + 0{,}8\,A = 1{,}0425095$	log.	0,0180800
D	log.	0,0201657
$\frac{1}{2}\,v = 33^d\,31^m\,30^s$	sec.	0,0790189
	same	0,0790189
r	log.	0,2008544

TABLE VI. — For an Hyperbola.

In the inverse problem, we have given, the true anomaly v, the perihelion distance D, and the excentricity e, to find the time t from the perihelion in days. This is obtained by the following rule, which is similar to that for an ellipsis, in the last table.

Rule. With e and v find $T = \frac{e-1}{e+1} \cdot \text{tang.}^2 \tfrac{1}{2} v$, and then by Table VI., the corresponding value of C. Also,

$$\text{Log. } A = \text{log. T} + \text{log. C} + \text{arith. co. log. } (1 - 0{,}8 \cdot \text{T});$$

and the corresponding log. B, in Table VI. Then find,

$$\text{log. } t_1 = 2{,}0654486 + \tfrac{3}{2} \text{log. } D + \tfrac{1}{2} \text{log. } A + \text{log. } B - \tfrac{1}{2} \text{log. } (e-1);$$

$$\text{log. } t_2 = \text{log. } t_1 + 8{,}8239087 + \text{log. } A + \text{log. } (1 + 9e) - \text{log. } (e-1);$$

$$t = t_1 + t_2.$$

EXAMPLE.

Given as before $e = 1{,}261882$; log. perihelion distance $D = 0{,}0201657$; and the true anomaly $v = 67^d\, 03^m\, 0^s$; to find the time from the perihelion t.

$e - 1 = a = 0{,}261882$ log. 9,4181056
$e + 1 = 2{,}261882$ log. co. 9,6455301
$\tfrac{1}{2} v = 33^d\, 31^m\, 30^s$ tang. 9,8211946
same tang. 9,8211946

T $= 0{,}0508189$ log. 8,7060249
Hence C $= 1{,}0001261$ Table VI. log. 0,0000548
$1 - 0{,}8\,\text{T} = 0{,}9593449$ log. co. 0,0180252

$A = 0{,}0529791$ log. 8,7241049

Corresponding log. B in Table VI. 0,0000207

Constant log. 2,0654486
$\tfrac{3}{2}$ log. D 0,0302485
$\tfrac{1}{2}$ log. A 9,3620524
log. B 0,0000207
$\tfrac{1}{2}$ log. $(e-1)$ arith. co. 0,2909472

$t_1 = 56^{\text{days}},0830$ log. 1,7487174
Constant 8,8239087
A log. 8,7241049
$1 + 9e = 12{,}356938$ log. 1,0919108
$(e-1)$ log. co. 0,5818944

$t_2 = 9^{\text{days}},3440$ log. 0,9705362

$t_1 + t_2 = 65^{\text{days}},41237 = t.$

TABLE VI.

A	Log. B	C	T	A	Log. B	C	T	A	Log. B	C	T
0,000	0,0000000	1,0000000	0,00000	0,040	0,0000118	1,0000722	0,038757	0,080	0,0000468	1,0002850	0,075168
001	000	1,0000000	0,00100	041	124	1,0000758	0,039695	081	480	1,0002921	0,076050
002	000	1,0000002	0,00200	042	130	1,0000796	0,040632	082	492	1,0002992	0,076930
003	001	1,0000004	0,00299	043	136	1,0000835	0,041567	083	504	1,0003065	0,077810
004	001	1,0000007	0,00399	044	143	1,0000872	0,042500	084	516	1,0003138	0,078688
0,005	0,0000002	1,0000011	0,00498	0,045	0,0000149	1,0000912	0,043432	0,085	0,0000528	1,0003212	0,079564
006	003	1,0000016	0,00597	046	156	1,0000953	0,044363	086	540	1,0003287	0,080439
007	004	1,0000022	0,00696	047	163	1,0000994	0,045292	087	553	1,0003363	0,081313
008	005	1,0000029	0,00795	048	170	1,0001037	0,046220	088	566	1,0003440	0,082186
009	006	1,0000037	0,00894	049	177	1,0001080	0,047147	089	578	1,0003517	0,083057
0,010	0,0000007	1,0000046	0,00992	0,050	0,0000184	1,0001124	0,048072	0,090	0,0000591	1,0003595	0,083927
011	009	1,0000055	0,01090	051	191	1,0001169	0,048995	091	604	1,0003674	0,084796
012	011	1,0000066	0,01189	052	199	1,0001215	0,049917	092	618	1,0003754	0,085663
013	013	1,0000077	0,01287	053	207	1,0001262	0,050838	093	631	1,0003835	0,086529
014	015	1,0000089	0,01384	054	215	1,0001310	0,051757	094	645	1,0003917	0,087394
0,015	0,0000017	1,0000102	0,01482	0,055	0,0000223	1,0001338	0,052675	0,095	0,0000658	1,0003999	0,088257
016	019	1,0000116	0,01580	056	231	1,0001407	0,053592	096	672	1,0004083	0,089119
017	021	1,0000131	0,01677	057	239	1,0001458	0,054507	097	686	1,0004167	0,089980
018	024	1,0000147	0,01774	058	247	1,0001509	0,055420	098	700	1,0004252	0,090840
019	027	1,0000164	0,01872	059	256	1,0001561	0,056332	099	714	1,0004338	0,091698
0,020	0,0000030	1,0000182	0,01968	0,060	0,0000265	1,0001614	0,057243	0,100	0,0000728	1,0004424	0,092555
021	033	1,0000200	0,02065	061	273	1,0001667	0,058152	101	743	1,0004512	0,093410
022	036	1,0000220	0,02162	062	282	1,0001722	0,059060	102	758	1,0004600	0,094265
023	039	1,0000240	0,02258	063	291	1,0001777	0,059967	103	772	1,0004689	0,095118
024	043	1,0000261	0,02355	064	301	1,0001833	0,060872	104	787	1,0004779	0,095969
0,025	0,0000046	1,0000283	0,02451	0,065	0,0000310	1,0001891	0,061776	0,105	0,0000802	1,0004870	0,096820
026	050	1,0000306	0,02547	066	320	1,0001949	0,062678	106	817	1,0004962	0,097669
027	054	1,0000330	0,02643	067	329	1,0002007	0,063579	107	833	1,0005055	0,098517
028	058	1,0000355	0,02739	068	339	1,0002067	0,064479	108	848	1,0005148	0,099364
029	062	1,0000381	0,02834	069	349	1,0002128	0,065377	109	864	1,0005242	0,100209
0,030	0,0000067	1,0000407	0,02930	0,070	0,0000359	1,0002189	0,066274	0,110	0,0000880	1,0005337	0,101053
031	071	1,0000435	0,03025	071	370	1,0002251	0,067170	111	895	1,0005432	0,101896
032	076	1,0000463	0,03120	072	382	1,0002314	0,06806[illegible]	112	911	1,0005529	0,102738
033	080	1,0000492	0,03215	073	390	1,0002378	0,068957	113	928	1,0005626	0,103578
034	085	1,0000523	0,03310	074	401	1,0002443	0,0698[illegible]	114	944	1,0005724	0,104417
0,035	0,0000091	1,0000554	0,03404	0,075	0,0000412	1,0002509	0,070738	0,115	0,0000960	1,0005823	0,105255
036	096	1,0000585	0,03499	076	423	1,0002575	0,071627	116	0977	1,0005923	0,106092
037	101	1,0000618	0,03593	077	434	1,0002643	0,072514	117	0994	1,0006024	0,106927
038	107	1,0000652	0,03688	078	445	1,0002711	0,073400	118	1010	1,0006125	0,107761
039	112	1,0000686	0,03782	079	457	1,0002780	0,074285	119	1027	1,0006228	0,108594
0,040	0,0000118	1,0000722	0,03876	0,080	0,0000468	1,0002850	0,075168	0,120	0,0001045	1,0006331	0,109426

TABLE VI. — For an Hyperbola.

To find the true anomaly in a hyperbolic orbit, which is nearly of a parabolic form, by the method of Gauss.

A	Log. *B*	C	T	*A*	Log. *B*	C	T	*A*	Log. *B*	C	T
0,120	0,0001045	1,0006331	0,109426	0,180	0,0002321	1,0013978	0,157151	0,240	0,0004076	1,0024396	0,200931
121	1062	1,0006435	0,110256	181	2346	1,0014129	0,157911	241	4110	1,0024592	0,201630
122	1079	1,0006539	0,111085	182	2372	1,0014281	0,158671	242	4143	1,0024789	0,202328
123	1097	1,0006645	0,111913	183	2398	1,0014434	0,159429	243	4176	1,0024987	0,203025
124	1114	1,0006751	0,112740	184	2423	1,0014588	0,160187	244	4210	1,0025185	0,203721
0,125	0,0001132	1,0006858	0,113566	0,185	0,0002449	1,0014742	0,160943	0,245	0,0004244	1,0025384	0,204416
126	1150	1,0006966	0,114390	186	2475	1,0014898	0,161698	246	4277	1,0025584	0,205110
127	1168	1,0007075	0,115213	187	2502	1,0015054	0,162453	247	4311	1,0025785	0,205803
128	1186	1,0007185	0,116035	188	2528	1,0015210	0,163206	248	4346	1,0025986	0,206495
129	1205	1,0007295	0,116855	189	2554	1,0015368	0,163958	249	4380	1,0026188	0,207186
0,130	0,0001223	1,0007406	0,117675	0,190	0,0002581	1,0015526	0,164709	0,250	0,0004414	1,0026391	0,207876
131	1242	1,0007518	0,118493	191	2608	1,0015685	0,165458	251	4449	1,0026594	0,208565
132	1261	1,0007631	0,119310	192	2634	1,0015845	0,166207	252	4483	1,0026799	0,209254
133	1280	1,0007745	0,120126	193	2661	1,0016005	0,166955	253	4518	1,0027004	0,209941
134	1299	1,0007859	0,120940	194	2688	1,0016167	0,167702	254	4553	1,0027209	0,210627
0,135	0,0001318	1,0007974	0,121754	0,195	0,0002716	1,0016329	0,168447	0,255	0,0004588	100,27416	0,211313
136	1337	1,0008090	0,122566	196	2743	1,0016491	0,169192	256	4623	1,0027623	0,211997
137	1357	1,0008207	0,123377	197	2771	1,0016655	0,169935	257	4658	1,0027830	0,212681
138	1376	1,0008325	0,124186	198	2798	1,0016819	0,170678	258	4644	1,0028039	0,213364
139	1396	1,0008443	0,124995	199	2826	1,0016984	0,171419	259	4729	1,0028248	0,214045
0,140	0,0001416	1,0008562	0,125802	0,200	0,0002854	1,0017150	0,172159	0,260	0,0004765	1,0028458	0,214726
141	1436	1,0008682	0,126609	201	2882	1,0017317	0,172899	261	4801	1,0028669	0,215406
142	1456	1,0008803	0,127414	202	2910	1,0017484	0,173637	262	4838	1,0028880	0,216085
143	1476	1,0008925	0,128217	203	2938	1,0017652	0,174374	263	4873	1,0029092	0,216763
144	1497	1,0009047	0,129020	204	2967	1,0017821	0,175110	264	4909	1,0029305	0,217440
0,145	0,0001517	1,0009170	0,129822	0,205	0,0002995	1,0017991	0,175845	0,265	0,0004945	1,0029519	0,218116
146	1538	1,0009294	0,130622	206	3024	1,0018161	0,176579	266	4981	1,0029733	0,218791
147	1559	1,0009419	0,131421	207	3053	1,0018332	0,177312	267	5018	1,0029948	0,219465
148	1580	1,0009545	0,132219	208	3082	1,0018504	0,178044	268	5055	1,0030164	0,220138
149	1601	1,0009671	0,133016	209	3111	1,0018677	0,178775	269	5091	1,0030380	0,220811
0,150	0,0001622	1,0009798	0,133812	0,210	0,0003140	1,0018850	0,179505	0,270	0,0005128	1,0030597	0,221482
151	1643	1,0009926	0,134606	211	3169	1,0019024	0,180234	271	5165	1,0030815	0,222153
152	1665	1,0010055	0,135399	212	3199	1,0019199	0,180962	272	5202	1,0031033	0,222822
153	1686	1,0010185	0,136191	213	3228	1,0019375	0,181688	273	5240	1,0031253	0,223491
154	1708	1,0010315	0,136982	214	3258	1,1019551	0,182414	274	5277	1,0031473	0,224159
0,155	0,0001730	1,0010446	0,137772	0,215	0,0003288	1,0019728	0,183139	0,275	0,0005315	1,0031693	0,224826
156	1752	1,0010578	0,138561	216	3318	1,0019906	0,183863	276	5352	1,0031915	0,225492
157	1774	1,0010711	0,139349	217	3348	1,0020084	0,184585	277	5390	1,0032137	0,226157
158	1797	1,0010844	0,140135	218	3378	1,0020264	0.185307	278	5428	1,0032359	0,226821
159	1819	1,0010978	0,140920	219	3409	1,0020444	0,186028	279	5466	1.0032583	0,227484
0,160	0,0001842	1,0011113	0,141704	0,220	0,0003439	1,0020625	0,186747	0,280	0,0005504	1.0032807	9,228147
161	1864	1,0011249	0,142487	221	3470	1,0020806	0,187466	281	5542	1,0033032	0,228808
162	1887	1,0011386	0,143269	222	3500	1,0020988	0,188184	282	5581	1,0033257	0,229469
163	1910	1,0011523	0,144050	223	3531	1,0021172	0,188900	283	5619	1,0033484	0,230128
164	1933	1,0011661	0,144829	224	3562	1,0021355	0,189616	284	5658	1,0033711	0,230787
0,165	0,0001956	1,0011800	0,145608	0,225	0,0003594	1,0021540	0.190331	0,285	0,0005697	1,0033938	0,231445
166	1980	1,0011940	0,146385	226	3625	1,0021725	0,191044	286	5736	1,0034167	0,232102
167	2003	1,0012081	0,147161	227	3656	1,0021911	0,191757	287	5775	1,0034396	0,232758
168	2027	1,0012222	0,147937	228	3688	1,0022098	0,192468	288	5814	1,0034626	0,233413
169	2051	1,0012364	0,148710	229	3719	1,0022285	0,193179	289	5853	1,0034856	0,234068
0,170	0,0002075	1,0012507	0,149483	0,230	0,0003751	1,0022473	0,193889	0,290	0,0005893	1,0035087	0,234721
171	2099	1,0012651	0,150255	231	3783	1,0022662	0,194597	291	5932	1,0035319	0,235374
172	2123	1,0012795	0,151026	232	3815	1,0022852	0,195305	292	5972	1,0035552	0,236025
173	2147	1,0012940	0,151795	233	3847	1,0023042	0,196012	293	6012	1,0035785	0,236676
174	2172	1,0013086	0,152564	234	3880	1,0023234	0,196717	294	6052	1,0036019	0,237326
0,175	0,0002196	1,0013233	0,153331	0,235	0,0003912	1,0023425	0.197422	0,295	0,0006092	1,0036253	0,237975
176	2221	1,0013380	0,154097	236	3945	1,0023618	0,198126	296	6132	1,0036489	0,238623
177	2246	1,0013529	0,154862	237	3977	1,0023811	0,198829	297	6172	1,0036725	0,239271
178	2271	1,0013678	0,155626	238	4010	1.0024005	0,199530	298	6213	1,0036961	0,239917
179	2296	1,0013827	0,156389	239	4043	1,0024200	0.200231	299	6253	1,0037199	0.240563
0,180	0,0002321	1,0013978	0,157151	0.240	0,0004076	1,0024396	0,200931	0,300	0,0006294	1,0037437	0.241207

TABLE VII. — For a Parabola.

This table is for computing the time t in days, for a comet to describe, in a parabolic orbit, an arc of the true anomaly, represented by $v'-v=2f$. This arc $2f$ being given, together with the extreme radii r, r'.

Rule. Put $\text{tang. } z=\sqrt{\frac{r'}{r}}$; $\cos. y=\cos. f.\sin. 2z$.

With this value of y, find in Table VII. the corresponding log. C; then we have,

$$\log. t=\log. \mathrm{C}+\log. \sin. \tfrac{1}{2}y+3.\log. \left(\frac{\sqrt{r}}{\cos. z}\right).$$

EXAMPLE.

Given log. $r=9{,}9115140$, log. $r'=9{,}7902520$; $2f=11^d\,44^m\,22^s$; to find t in days.

	½ log. r' 9,8951260		
	½ log. r 9,9557570	. . . .	½ log. r 9,9557570
$z=41^d\ 0^m\ 48^s{,}5$	tang. 9,9393690	. . . .	z cos. 9,8776911
$2z=82^d\ 1^m\ 37^s$	sin. 9,9957814		$\frac{\sqrt{r}}{\cos. z}$ log. 0,0780659
$f=5^d\ 52^m\ 11^s$	cos. 9,9977170		Multiplied by 3 . log. 0,2341977
$y=9^d\ 53^m\ 22^s$	cos. 9,9934984		Table VII. log. C. 1,7622613
			$\frac{1}{2}y=4^d\ 56^m\ 41^s$ sin. 8,9354800
			$t=8^{\text{days}},5494$ log. 0,9319390

TABLE VII. — For a Parabola.

With the two radii r, r', and the included arc $v'-v=2f$, to find the time t in days, for a comet to describe that arc, in a parabolic orbit.

y	Log. C	Diff.	y	Log. C	Diff.	y	Log. C	Diff.	y	Log. C	Diff.
d m		neg.	d m		neg.	d m		neg.	d m		neg.
0,00	1,7644177	6	5,00	1,7638665	373	10,00	1,7622129	741	15,00	1,7594568	1109
0,10	1,7644171	18	5,10	1,7638292	386	10,10	1,7621388	754	15,10	1,7593459	1121
0,20	1,7644153	31	5,20	1,7637906	398	10,20	1,7620634	765	15,20	1,7592338	1133
0,30	1,7644122	43	5,30	1,7637508	411	10,30	1,7619869	778	15,30	1,7591205	1145
0,40	1,7644079	55	5,40	1,7637097	422	10,40	1,7619091	790	15,40	1,7590060	1157
0,50	1,7644024	67	5,50	1,7636675	435	10,50	1,7618301	803	15,50	1,7588903	1170
1,00	1,7643957	80	6,00	1,7636240	447	11,00	1,7617498	814	16,00	1,7587733	1182
1,10	1,7643877	92	6,10	1,7635793	459	11,10	1,7616684	827	16,10	1,7586551	1194
1,20	1,7643785	104	6,20	1,7635334	472	11,20	1,7615857	840	16,20	1,7585357	1207
1,30	1,7643681	116	6,30	1,7634862	484	11,30	1,7615017	851	16,30	1,7584150	1219
1,40	1,7643565	129	6,40	1,7634378	496	11,40	1,7614166	863	16,40	1,7582931	1231
1,50	1,7643436	141	6,50	1,7633882	508	11,50	1,7613303	876	16,50	1,7581700	1243
2,00	1,7643295	153	7,00	1,7633374	521	12,00	1,7612427	888	17,00	1,7580457	1256
2,10	1,7643142	165	7,10	1,7632853	533	12,10	1,7611539	901	17,10	1,7579201	1268
2,20	1,7642977	178	7,20	1,7632320	545	12,20	1,7610638	912	17,20	1,7577933	1280
2,30	1,7642799	189	7,30	1,7631775	558	12,30	1,7609726	925	17,30	1,7576653	1292
2,40	1,7642610	202	7,40	1,7631217	569	12,40	1,7608801	937	17,40	1,7575361	1304
2,50	1,7642408	215	7,50	1,7630648	582	12,50	1,7607864	949	17,50	1,7574057	1317
3,00	1,7642193	226	8,00	1,7630066	595	13,00	1,7606915	962	18,00	1,7572740	1329
3,10	1,7641967	239	8,10	1,7629471	606	13,10	1,7605953	973	18,10	1,7571411	1341
3,20	1,7641728	251	8,20	1,7628865	618	13,20	1,7604980	986	18,20	1,7570070	1354
3,30	1,7641477	264	8,30	1,7628247	631	13,30	1,7603994	999	18,30	1,7568716	1365
3,40	1,7641213	275	8,40	1,7627616	643	13,40	1,7602995	1010	18,40	1,7567351	1378
3,50	1,7640938	288	8,50	1,7626973	655	13,50	1,7601985	1023	18,50	1,7565973	1390
4,00	1,7640650	300	9,00	1,7626318	668	14,00	1,7600962	1035	19,00	1,7564583	1403
4,10	1,7640350	313	9,10	1,7625650	680	14,10	1,7599927	1047	19,10	1,7563180	1415
4,20	1,7640037	324	9,20	1,7624970	692	14,20	1,7598880	1060	19,20	1,7561765	1427
4,30	1,7639713	337	9,30	1,7624278	704	14,30	1,7597820	1072	19,30	1,7560338	1439
4,40	1,7639376	349	9,40	1,7623574	716	14,40	1,7596748	1084	19,40	1,7558899	1451
4,50	1,7639027	362	9,50	1,7622858	729	14,50	1,7595664	1096	19,50	1,7557448	1464
5,00	1,7638665	373	10,00	1,7622129	741	15,00	1,7594568	1109	20,00	1,7555984	

USES OF TABLES VIII. IX. AND X.

TABLE VIII. combined with Table IX., for an elliptical orbit, and with Table X., for a hyperbolic orbit, are used in finding the elements of the orbit; when we have given, the two radii r, r', the included heliocentric arc $v'-v=2f$, and the time t of describing that arc, expressed in days. These tables are limited to the most useful values of h, H, which do not exceed 0,6; and to values of x, z, which do not exceed 0,3. These limits include the most common cases; and in observations which do not fall within them, we can use the indirect solutions explained in this appendix. *When h or H exceeds 0,040, and log. yy, or log. YY, is required to be correct in the seventh decimal place, we must use the second differences.*

PRECEPTS FOR TABLES VIII., IX., IN AN ELLIPTICAL ORBIT.

The particular object of these tables is to facilitate the computation of the value of $2g=E'-E$, representing the difference between the two excentric anomalies E', E; corresponding respectively to the true anomalies v', v; which is an important part of the preliminary process, in computing the elements of the orbit. After g has been found, the elements may be computed by the methods, given in this appendix; we shall not however enter here upon this subject, but shall restrict our remarks to the mere explanation of the method of computing the value of g, by means of the tables.

In the calculation of g, there are two separate cases; the one when f is *acute*, or $v'-v$ between 0^d and 180^d; the other when f is *obtuse*, or $v'-v$ between 180^d and 360^d. We shall give the precepts, in both these cases, at full length, for convenience of reference; remarking, however, that the case of f being acute, is that which occurs most frequently in practice, and is that for which these tables are particularly designed.

When f is acute.

We must find w, l, mm, h, by the following formulas;

$$\text{tang.}\ (45^d+w)=\sqrt[4]{\frac{r'}{r}}\,;$$

$$l=\frac{\sin.^2 \frac{1}{2}f}{\cos. f}+\frac{\text{tang.}^2\, 2\, w}{\cos. f}\,;$$

$$\log.\ mm=5{,}5680729+2\log.\ t-3\log.\ \cos. f-\tfrac{3}{2}\log.\ (rr')\,;$$

$$\text{Approx. log. } h=\log.\ mm-\log.\ (l+\tfrac{5}{6})\,.$$

With this approximate value of h, find, in Table VIII., the coresponding approximate value of log. yy, also,

$$\text{Approx. value of } x=\frac{mm}{yy}-l\,.$$

With this approximate value of x, find, in Table, IX., the corresponding approximate value of ξ, and then the corrected value of h, from the formula,

$$\text{corrected log. } h=\log.\ mm-\log.\ (l+\tfrac{5}{6}+\xi)\,.$$

With this corrected value of h, find a new value of log. yy, in Table VIII., which is to be used in finding a corrected value of x, by the formula used above,

$$\text{corrected value of } x=\frac{mm}{yy}-l\,.$$

If necessary, we may repeat the operation until the assumed and computed values of ξ agree; then we have,

$$x=\sin.^2 \tfrac{1}{2}\,g=\sin.^2 \tfrac{1}{4}\,(E'-E)\,;$$

from which we easily obtain g or $E'-E$.

When f is obtuse.

We must find w, L, MM, H, by the following formulas;

$$\text{tang.}\ (45^d+w)=\sqrt[4]{\frac{r'}{r}}\,;$$

$$L=-\frac{\sin.^2 \frac{1}{2}f}{\cos. f}-\frac{\text{tang.}^2\, 2\, w}{\cos. f}\,;$$

$$\log.\ MM=5{,}5680729+2\log.\ t-3\log.\ (-\cos. f)-\tfrac{3}{2}\log.\ (rr')\,;$$

$$\text{Approx. log. } H=\log.\ MM-\log.\ (L-\tfrac{5}{6})\,.$$

With this approximate value of H, find, in Table VIII., the corresponding approximate value of log. YY, also,

$$\text{Approx. value of } x=L-\frac{MM}{YY}\,.$$

With this approximate value of x, find, in Table IX., the corresponding approximate value of ξ, and the corrected value of H, from the formula,

$$\text{corrected log. } H=\log.\ MM-\log.\ (L-\tfrac{5}{6}-\xi)\,.$$

With this corrected value of H, find a new value of log. YY, in Table VIII., which is to be used in finding a corrected value of x, by the formula used above,

$$\text{corrected value of } x=L-\frac{MM}{YY}\,.$$

If necessary, we may repeat the operation until the assumed and computed values of ξ agree; then we have,

$$x=\sin.^2 \tfrac{1}{2}\,g=\sin.^2 \tfrac{1}{4}\,(E'-E)\,;$$

from which we easily obtain g or $E'-E$.

EXAMPLE.

Given log. $r=0{,}3307640$; log. $r'=0{,}3222239$: $v'-v=2f=7^d\,34^m\,53^s{,}73$; $t=21^{\text{days}}{,}93391$; to find $2g=E'-E$, or rather $x=\sin.^2 \frac{1}{2}\,g$.

r'	log.	0,3222239
r	log.	0,3307640
$\frac{r'}{r}=\text{tang.}^4\,(45^d+w)$	log.	9,9914599
$45^d+w=44^d\,51^m\,33^s$	tang.	9,9978650
$w=-\ 8^m\,27^s$		

	0,3222239
	0,3307640
sum	0,6529879
half	0,3264940
$(rr')^{\frac{3}{2}}$ log.	0,9794819
arith. co.	9,0205181

$2\,w=-16^m\,54^s$	tang.	$7{,}6916163_n$
	same	$7{,}6916163_n$
	f cos. arith. co.	0,0009512
$\frac{\text{tang.}^2\,2\,w}{\cos. f}=0{,}0000242205$	log.	5,3841838
	constant log.	5,5680729
$t=21^{\text{days}}{,}93391$	log.	1,3411160
	same	1,3411160
(arith. co. log. cos. f) $\times$ 3		0,0028536
$\frac{3}{2}$ log. rr'	arith. co.	9,0205181
	mm log.	7,2736766

$f=3^d\,47^m\,26^s{,}865$	cos. arith. co.	0,0009512
$\frac{1}{2}f=1^d\,53^m\,43^s{,}4325$	sin.	8,5194986
	same	8,5194986
$\frac{\sin.^2 \frac{1}{2}f}{\cos. f}=0{,}0010963480$	log.	7,0399484
$\frac{\text{tang.}^2\,2\,w}{\cos. f}=0{,}0000242205$		
$l=0{,}0011205685$		
$\frac{5}{6}=0{,}8333333$		
$l+\frac{5}{6}=0{,}8344539$	log.	9,9214023
	mm log.	7,2736766
Approx. $h=0{,}00225047$	log.	7,3522743
Corresponds in Table VIII., to approx.	log. yy =	0,0021633
	mm log.	7,2736766
$\frac{mm}{yy}=0{,}0018685871$	log.	7,2715133
$l=0{,}0011205685$		
Approx. $x=0{,}0007480186$		

The correction, Table IX., corresponding to this value of x is insensible, therefore, we may assume this value of x for the true value of $\sin.^2 \frac{1}{2}\,g=0{,}0007480186$.

PRECEPTS FOR TABLES VIII. AND X., IN A HYPERBOLIC ORBIT.

The process for calculating the elements of a hyperbolic orbit, by means of $r, r', v'-v=2f$ and t, varies but very little from that in an elliptical orbit, which we have just explained. The formulas for the computation of w, l, m, L, M, are identically the same. The formulas for h, H, are the same, with the exception of using ζ Table X, instead of ξ Table IX; moreover x is changed into $-z$. For convenience in reference we shall here give the formulas, for the hyperbola, arranged in the same order as for the ellipsis,

When f is acute.

$$\text{tang.}\,(45^d+w)=\sqrt[4]{\frac{r'}{r}}\;;$$

$$l=\frac{\sin.^2\frac{1}{2}f}{\cos.f}+\frac{\text{tang.}^2\,2w}{\cos.f}\;;$$

$$\log.\,mm=5{,}5680729+2\,\log.\,t-3\,\log.\,\cos.f-\tfrac{3}{2}\,\log.\,(r\,r')\;;$$

$$\text{approximate}\ \log.\,h=\log.\,mm-\log.\,(l+\tfrac{5}{6}).$$

With this approximate value of h, find in Table VIII. the corresponding approximate value of $\log.\,yy$, also

$$\text{approximate value of}\ z=l-\frac{mm}{yy}.$$

With this approximate value of z, find in Table X. the corresponding value of ζ, and then the corrected value of h, from the formula,

$$\text{corrected}\ \log.\,h=\log.\,mm-\log.\,(l+\tfrac{5}{6}+\zeta).$$

With this corrected value of h, find a new value of $\log.\,yy$, in Table VIII., which is to be used in finding a corrected value of z, by the formula used above, namely,

$$\text{corrected value of}\ z=l-\frac{mm}{yy}.$$

If necessary we may repeat the operation, until the assumed and computed value of ζ agree; and this must be taken for the true value of ζ, to be used in computing the elements of the orbit, by the formulas given in this appendix.

When f is obtuse.

$$\text{tang.}\,(45^d+w)=\sqrt[4]{\frac{r'}{r}}\;;$$

$$L=-\frac{\sin.^2\frac{1}{2}f}{\cos.f}-\frac{\text{tang.}^2\,2w}{\cos.f}\;;$$

$$\log.\,MM=5{,}5680729+2\log.\,t-3\log.\,(-\cos.f)-\tfrac{3}{2}\log.\,(r\,r')$$

$$\text{approximate}\ \log.\,H=\log.\,MM-\log.\,(L-\tfrac{5}{6}).$$

With this approximate value of H, find in Table VIII. the corresponding approximate value of $\log.\,YY$, also

$$\text{approximate value of}\ z=\frac{MM}{YY}-L.$$

With this approximate value of z, find in Table X. the corresponding value of ζ, and then the corrected value of H from the formula,

$$\text{corrected}\ \log.\,H=\log.\,MM-\log.\,(L-\tfrac{5}{6}-\zeta).$$

With this corrected value of H, find a new value of $\log.\,YY$, in Table VIII., which is to be used in finding a corrected value of z, by the formula given above, namely,

$$\text{corrected value of}\ z=\frac{MM}{YY}-L.$$

If necessary we may repeat the operation, until the assumed and computed value of ζ agree; and this must be taken for the true value of ζ, to be used in computing the elements of the orbit, by the formulas given in this appendix.

EXAMPLE.

Given $\log.\,r=0{,}0333585$; $\log.\,r'=0{,}2008541$; $v'-v=2f=48^d\ 12^m$; $t=51,^{\text{days}}\,49788$; to find z.

r' log. 0,2008541
r log. 0,0333585

$\frac{r'}{r}$ 0,1674956

$45^d+w=47^d\,45^m\,28^s,47$ tang. 0,0418739

$w=\ 2^d\,45^m\,28^s,47$

0,2008541
0,0333585
sum 0,2342126
half 0,1171063
$(r\,r')^{\frac{3}{2}}$ log. 0,3513189
arith. co. 9,6486811

$2w=\ 5^d\,30^m\,56^s,94$ tang. 8,9848318
same 8,9848318
f arith. co. cos. 0,0396081

$\frac{\text{tang.}^2\,2w}{\cos.f}=0{,}010215784$ log. 8,0092717

constant 5,5680729
$t=51^{\text{days}},49788$ log. 1,7117894
same 1,7117894
(arith. co. log. cos. f) $\times$ 3 0,1188243
$\frac{3}{2}$ log. $r\,r'$ arith. co. 9,6486811

mm log. 8,7591571

$f=24^d\ 6^m$ cos. arith. co. 0,0396081
$\frac{1}{2}f=12^d\ 3^m$ sin. 9,3196581
same 9,3196581

$\frac{\sin.^2\frac{1}{2}f}{\cos.f}=0{,}04774604$ log. 8,6789243

$\frac{\text{tang.}^2\,2w}{\cos.f}=0{,}010215784$

$l=0{,}057960388$
$\frac{5}{6}=0{,}8333333$

$l+\frac{5}{6}=0{,}8912937$ log. 9,9500208
mm log. 8,7591571

Approx. $h=0{,}0644371$ log. 8,8091363

Corresponds in Table VIII. to approx. log. $yy=0{,}0560848$
mm log. 8,7591571

$\frac{mm}{yy}=0{,}05047454$ log. 8,7030723
$l=0{,}05796039$

Approx. $z=0{,}00748585\quad=l-\frac{mm}{yy}$

Corresponding to this in Table X. is $\zeta=0{,}0000032$

Hence, $l+\frac{5}{6}+\zeta=0{,}8912969$ log. 9,9500224
mm log. 8,7591571

corrected $h=0{,}0644369$ log. 8,8091347

Corresponds in Table VIII. to corrected log. $yy=0{,}0560846$
mm log. 8,7591571

$\frac{mm}{yy}=0{,}05047456$ log. 8,7030725
$l=0{,}05796039$

Corrected $z=0{,}00748583$ which agrees with the assumed value.

TABLE VIII. — For an Ellipsis or Hyperbola.

This table, with Tables IX., X., are for computing the elements of the orbit, when there are given the two radii r, r'; the included heliocentric arc $v' - v = 2f$, and the time t of describing that arc, expressed in days.

h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.
0,0000	0,0000000	965	0,0060	0,0057298	945	0,0120	0,0113417	926	0,0180	0,0168412	907
0001	0965	965	0061	58243	944	0121	114343	925	0181	169319	907
0002	1930	964	0062	59187	944	0122	115268	925	0182	170226	907
0003	2894	964	0063	60131	944	0123	116193	925	0183	171133	906
0004	3858	963	0064	61075	944	0124	117118	925	0184	172039	906
0,0005	0,0004821	963	0,0065	0,0062019	943	0,0125	0,0118043	924	0,0185	0,0172945	906
0006	5784	963	0066	62962	943	0126	118967	923	0186	173851	906
0007	6747	963	0067	63905	942	0127	119890	924	0187	174757	905
0008	7710	962	0068	64847	943	0128	120814	923	0188	175662	905
0009	8672	962	0069	65790	942	0129	121737	923	0189	176567	904
0,0010	0,0009634	961	0,0070	0,0066732	941	0,0130	0,0122660	922	0,0190	0,0177471	905
0011	10595	962	0071	67673	941	0131	123582	923	0191	178376	904
0012	11557	960	0072	68614	941	0132	124505	922	0192	179280	903
0013	12517	961	0073	69555	941	0133	125427	921	0193	180183	904
0014	13478	960	0074	70496	940	0134	126348	921	0194	181087	903
0,0015	0,0014438	960	0,0075	0,0071436	940	0,0135	0,0127269	921	0,0195	0,0181990	903
0016	15398	959	0076	72376	940	0136	128190	921	0196	182893	903
0017	16357	959	0077	73316	939	0137	129111	921	0197	183796	902
0018	17316	959	0078	74255	939	0138	130032	920	0198	184698	902
0019	18275	959	0079	75194	939	0139	130952	919	0199	185600	901
0,0020	0,0019234	958	0,0080	0,0076133	938	0,0140	0,0131871	920	0,0200	0,0186501	902
0021	20192	958	0081	77071	938	0141	132791	919	0201	187403	901
0022	21150	957	0082	78009	938	0142	133710	919	0202	188304	901
0023	22107	957	0083	78947	937	0143	134629	918	0203	189205	900
0024	23064	957	0084	79884	937	0144	135547	919	0204	190105	900
0,0025	0,0024021	956	0,0085	0,0080821	937	0,0145	0,0136466	917	0,0205	0,0191005	900
0026	24977	956	0086	81758	936	0146	137383	918	0206	191905	900
0027	25933	956	0087	82694	936	0147	138301	917	0207	192805	899
0028	26889	956	0088	83630	936	0148	139218	917	0208	193704	899
0029	27845	955	0089	84566	936	0149	140135	917	0209	194603	899
0,0030	0,0028800	955	0,0090	0,0085502	935	0,0150	0,0141052	916	0,0210	0,0195502	899
0031	29755	954	0091	86437	935	0151	141968	916	0211	196401	898
0032	30709	954	0092	87372	934	0152	142884	916	0212	197299	898
0033	31663	954	0093	88306	934	0153	143800	916	0213	198197	897
0034	32617	953	0094	89240	934	0154	144716	915	0214	199094	898
0,0035	0,0033570	953	0,0095	0,0090174	934	0,0155	0,0145631	915	0,0215	0,0199992	897
0036	34523	953	0096	91108	933	0156	146546	914	0216	200889	896
0037	35476	952	0097	92041	933	0157	147460	915	0217	201785	897
0038	36428	953	0098	92974	932	0158	148375	913	0218	202682	896
0039	37381	951	0099	93906	933	0159	149288	914	0219	203578	896
0,0040	0,0038332	952	0,0100	0,0094839	931	0,0160	0,0150202	913	0,0220	0,0204474	895
0041	39284	951	0101	95770	932	0161	151115	913	0221	205369	895
0042	40235	951	0102	96702	931	0162	152028	913	0222	206264	895
0043	41186	950	0103	97633	931	0163	152941	913	0223	207159	895
0044	42136	950	0104	98564	931	0164	153854	912	0224	208054	894
0,0045	0,0043086	950	0,0105	0,0099495	930	0,0165	0,0154766	912	0,0225	0,0208948	895
0046	44036	949	0106	100425	931	0166	155678	911	0226	209843	893
0047	44985	949	0107	101356	929	0167	156589	911	0227	210736	894
0048	45934	949	0108	102285	930	0168	157500	911	0228	211630	893
0049	46883	949	0109	103215	929	0169	158411	911	0229	212523	893
0,0050	0,0047832	948	0,0110	0,0104144	929	0,0170	0,0159322	910	0,0230	0,0213416	893
0051	48780	948	0111	105073	928	0171	160232	910	0231	214309	892
0052	49728	947	0112	106001	928	0172	161142	910	0232	215201	892
0053	50675	947	0113	106929	928	0173	162052	909	0233	216093	892
0054	51622	947	0114	107857	928	0174	162961	909	0234	216985	891
0,0055	0,0052569	946	0,0115	0,0108785	927	0,0175	0,0163870	909	0,0235	0,0217876	892
0056	53515	947	0116	109712	927	0176	164779	909	0236	218768	891
0057	54462	945	0117	110639	926	0177	165688	908	0237	219659	890
0058	55407	946	0118	111565	926	0178	166596	908	0238	220549	891
0059	56353	945	0119	112491	926	0179	167504	908	0239	221440	890
0,0060	0,0057298	945	0,0120	0,0113417	926	0,0180	0,0168412	907	0,0240	0,0222330	890

TABLE VIII.—For an Ellipsis or Hyperbola.

This table, with Table IX., X., are for computing the elements of the orbit, when there are given the two radii r, r'; the included heliocentric arc $v' - v = 2f$, and the time t of describing that arc, expressed in days.

y H	Log. yy Log. YY	Diff.	y H	Log. yy Log. YY	Diff.	h H	Log. yy Log. YY	Diff.	h H	Log. yy Log. YY	Diff.
0,0240	0,0222330	890	0,0300	0,0275218	873	0,0360	0,0327120	856	0,060	0,0525626	7976
0241	223220	889	0301	276091	873	0361	327976	857	061	533602	7954
0242	224109	889	0302	276964	872	0362	328833	856	062	541556	7932
0243	224998	889	0303	277836	872	0363	329689	857	063	549488	7909
0244	225887	889	0304	278708	872	0364	330546	855	064	557397	7888
0,0245	0,0226776	888	0,0305	0,0279580	872	0,0365	0,0331401	856	0,065	0,0565285	7865
0246	227664	888	0306	280452	871	0366	332257	855	066	573150	7844
0247	228552	888	0307	281323	871	0367	333112	855	067	580994	7823
0248	229440	888	0308	282194	871	0368	333967	855	068	588817	7801
0249	230328	887	0309	283065	871	0369	334822	855	069	596618	7780
0,0250	0,0231215	887	0,0310	0,0283936	870	0,0370	0,0335677	854	0,070	0,0604398	7759
0251	232102	886	0311	284806	870	0371	336531	854	071	612157	7738
0252	232988	887	0312	285676	870	0372	337385	854	072	619895	7717
0253	233875	886	0313	286546	869	0373	338239	853	073	627612	7696
0254	234761	886	0314	287415	869	0374	339092	854	074	635308	7676
0,0255	0,0235647	885	0,0315	0,0288284	869	0,0375	0,0339946	853	0,075	0,0642984	7655
0256	236532	885	0316	289153	869	0376	340799	852	076	650639	7635
0257	237417	885	0317	290022	868	0377	341651	853	077	658274	7614
0258	238302	885	0318	290890	868	0378	342504	852	078	665888	7595
0259	239187	884	0319	291758	868	0379	343356	852	079	673483	7574
0,0260	0,0240071	885	0,0320	0,0292626	868	0,0380	0,0344208	851	0,080	0,0681057	7555
0261	240956	883	0321	293494	867	0381	345059	852	081	688612	7534
0262	241839	884	0322	294361	867	0382	345911	851	082	696146	7515
0263	242723	883	0323	295228	867	0383	346762	851	083	703661	7496
0264	243606	883	0324	296095	866	0384	347613	851	084	711157	7476
0,0265	0,0244489	883	0,0325	0,0296961	866	0,0385	0,0348464	850	0,085	0,0718633	7457
0266	245372	882	0326	297827	866	0386	349314	850	086	726090	7437
0267	246254	882	0327	298693	866	0387	350164	850	087	733527	7418
0268	247136	882	0328	299559	865	0388	351014	850	088	740945	7400
0269	248018	882	0329	300424	866	0389	351864	849	089	748345	7380
0,0270	0,0248900	881	0,0330	0,0301290	864	0,0390	0,0352713	849	0,090	0,0755725	7362
0271	249781	881	0331	302154	865	0391	353562	849	091	763087	7343
0272	250662	881	0332	303019	864	0392	354411	848	092	770430	7324
0273	251543	880	0333	303883	864	0393	355259	849	093	777754	7306
0274	252423	881	0334	304747	864	0394	356108	848	094	785060	7288
0,0275	0,0253304	879	0,0335	0,0305611	864	0,0395	0,0356956	848	0,095	0,0792348	7269
0276	254183	880	0336	306475	863	0396	357804	847	096	799617	7251
0277	255063	879	0337	307338	863	0397	358651	848	097	806868	7233
0278	255942	880	0338	308201	863	0398	359499	847	098	814101	7215
0279	256822	878	0339	309064	862	0399	360346	846	099	821316	7197
0,0280	0,0257700	879	0,0340	0,0309926	862	0,0400	0,0361192	8454	0,100	0,0828513	7180
0281	258579	878	0341	310788	862	0,041	369646	8429	101	835693	7161
0282	259457	878	0342	311650	862	042	378075	8403	102	842854	7145
0283	260335	878	0343	312512	861	043	386478	8378	103	849999	7126
0284	261213	877	0344	313373	861	044	394856	8353	104	857125	7110
0,0285	0,0262090	877	0,0345	0,0314234	861	0,045	0,0403209	8328	0,105	0,0864235	7092
0286	262967	877	0346	315095	861	046	411537	8304	106	871327	7074
0287	263844	877	0347	315956	860	047	419841	8280	107	878401	7058
0288	264721	876	0348	316816	860	048	428121	8255	108	885459	7041
0289	265597	876	0349	317676	860	049	436376	8231	109	892500	7023
0,0290	0,0266473	876	0,0350	0,0318536	860	0,050	0,0444607	8207	0,110	0,0899523	7007
0291	267349	875	0351	319396	859	051	452814	8184	111	906530	6990
0292	268224	875	0352	320255	859	052	460998	8159	112	913520	6974
0293	269099	875	0353	321114	859	053	469157	8137	113	920494	6957
0294	269974	875	0354	321973	858	054	477294	8113	114	927451	6940
0,0295	0,0270849	874	0,0355	0,0322831	858	0,055	0,0485407	8089	0,115	0,0934391	6924
0296	271723	874	0356	323689	858	056	493496	8067	116	941315	6908
0297	272597	874	0357	324547	858	057	501563	8044	117	948223	6891
0298	273471	874	0358	325405	857	058	509607	8021	118	955114	6876
0299	274345	873	0359	326262	858	059	517628	7998	119	961990	6859
0,0300	0,0275218	873	0,0360	0,0327120	856	0,060	0,0525626	7976	0,120	0,0968849	6843

TABLE VIII. — For an Ellipsis or Hyperbola.

This table, with Table IX., X., are for computing the elements of the orbit, when there are given the two radii r, r'; the included heliocentric arc $v' - v = 2f$, and the time t of describing that arc, expressed in days.

h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.
0,120	0,0968849	6843	0,180	0,1353804	6014	0,240	0,1695092	5378	0,300	0,2002285	4872
121	975692	6828	181	1359818	6003	241	1700470	5368	301	2007157	4864
122	982520	6811	182	1365821	5990	242	1705838	5359	302	2012021	4857
123	989331	6796	183	1371811	5978	243	1711197	5350	303	2016878	4849
124	996127	6780	184	1377789	5966	244	1716547	5340	304	2021727	4842
0,125	0,1002907	6765	0,185	0,1383755	5955	0,245	0,1721887	5331	0,305	0,2026569	4834
126	1009672	6749	186	1389710	5943	246	1727218	5322	306	2031403	4827
127	1016421	6733	187	1395653	5932	247	1732540	5313	307	2036230	4820
128	1023154	6719	188	1401585	5919	248	1737853	5303	308	2041050	4812
129	1029873	6703	189	1407504	5908	249	1743156	5295	309	2045862	4805
0,130	0,1036576	6688	0,190	0,1413412	5897	0,250	0,1748451	5285	0,310	0,2050667	4797
131	1043264	6672	191	1419309	5885	251	1753736	5277	311	2055464	4790
132	1049936	6658	192	1425194	5874	252	1759013	5267	312	2060254	4783
133	1056594	6643	193	1431068	5863	253	1764280	5258	313	2065037	4776
134	1063237	6628	194	1436931	5851	254	1769538	5250	314	2069813	4768
0,135	0,1069865	6613	0,195	0,1442782	5840	0,255	0,1774788	5241	0,315	0,2074581	4761
136	1076478	6598	196	1448622	5828	256	1780029	5232	316	2079342	4754
137	1083076	6584	197	1454450	5818	257	1785261	5222	317	2084096	4747
138	1089660	6569	198	1460268	5806	258	1790483	5215	318	2088843	4739
139	1096229	6554	199	1466074	5795	259	1795698	5205	319	2093582	4733
0,140	0,1102783	6540	0,200	0,1471869	5784	0,260	0,1800903	5197	0,320	0,2098315	4725
141	1109323	6526	201	1477653	5774	261	1806100	5188	321	2103040	4719
142	1115849	6511	202	1483427	5762	262	1811288	5179	322	2107759	4711
143	1122360	6497	203	1489189	5751	263	1816467	5171	323	2112470	4704
144	1128857	6483	204	1494940	5741	264	1821638	5162	324	2117174	4697
0,145	0,1135340	6469	0,205	0,1500681	5730	0,265	0,1826800	5153	0,325	0,2121871	4691
146	1141809	6455	206	1506411	5719	266	1831953	5145	326	2126562	4683
147	1148264	6440	207	1512130	5708	267	1837098	5137	327	2131245	4676
148	1154704	6427	208	1517838	5697	268	1842235	5128	328	2135921	4670
149	1161131	6413	209	1523535	5687	269	1847363	5120	329	2140591	4662
0,150	0,1167544	6399	0,210	0,1529222	5677	0,270	0,1852483	5111	0,330	0,2145253	4656
151	1173943	6386	211	1534899	5665	271	1857594	5102	331	2149909	4649
152	1180329	6372	212	1540564	5656	272	1862696	5095	332	2154558	4642
153	1186701	6358	213	1546220	5645	273	1867791	5086	333	2159200	4635
154	1193059	6345	214	1551865	5634	274	1872877	5078	334	2163835	4629
0,155	0,1199404	6331	0,215	0,1557499	5624	0,275	0,1877955	5069	0,335	0,2168464	4621
156	1205735	6318	216	1563123	5614	276	1883024	5061	336	2173085	4615
157	1212053	6304	217	1568737	5603	277	1888085	5053	337	2177700	4608
158	1218357	6292	218	1574340	5593	278	1893138	5045	338	2182308	4602
159	1224649	6278	219	1579933	5583	279	1898183	5037	339	2186910	4595
0,160	0,1230927	6265	0,220	0,1585516	5573	0,280	0,1903220	5029	0,340	0,2191505	4588
161	1237192	6252	221	1591089	5563	281	1908249	5020	341	2196093	4582
162	1243444	6238	222	1596652	5552	282	1913269	5012	342	2200675	4575
163	1249682	6226	223	1602204	5543	283	1918281	5005	343	2205250	4568
164	1255908	6213	224	1607747	5532	284	1923286	4996	344	2209818	4562
0,165	0,1262121	6200	0,225	0,1613279	5523	0,285	0,1928282	4989	0,345	0,2214380	4555
166	1268321	6187	226	1618802	5513	286	1933271	4980	346	2218935	4548
167	1274508	6175	227	1624315	5502	287	1938251	4973	347	2223483	4543
168	1280683	6162	228	1629817	5493	288	1943224	4964	348	2228026	4535
169	1286845	6149	229	1635310	5483	289	1948188	4957	349	2232561	4530
0,170	1,1292994	6137	0,230	0,1640793	5474	0,290	0,1953145	4949	0,350	0,2237091	4522
171	1299131	6124	231	1646267	5463	291	1958094	4941	351	2241613	4517
172	1305255	6112	232	1651730	5454	292	1963035	4933	352	2246130	4510
173	1311367	6099	233	1657184	5444	293	1967968	4926	353	2250640	4503
174	1317466	6087	234	1662628	5435	294	1972894	4917	354	2255143	4497
0,175	0,1323553	6075	0,235	0,1668063	5425	0,295	0,1977811	4910	0,355	0,2259640	4491
176	1329628	6062	236	1673488	5415	296	1982721	4903	356	2264131	4484
177	1335690	6050	237	1678903	5406	297	1987624	4894	357	2268615	4479
178	1341740	6038	238	1684309	5396	298	1992518	4888	358	2273094	4471
179	1347778	6026	239	1689705	5387	299	1997406	4879	359	2277565	4466
0,180	0,1353804	6014	0,240	0,1695092	5378	0,300	0,2002285	4872	0,360	0,2282031	4459

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h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.	h / H	Log. yy / Log. YY	Diff.
0,360	0,2282031	4459	0,420	0,2539153	4116	0,480	0,2777272	3824	0,540	0,2999178	3574
361	2286490	4453	421	2543269	4110	481	2781096	3820	541	3002752	3571
362	2290943	4447	422	2547379	4106	482	2784916	3816	542	3006323	3565
363	2295390	4441	423	2551485	4099	483	2788732	3811	543	3009888	3564
364	2299831	4434	424	2555584	4095	484	2792543	3806	544	3013452	3559
0,365	0,2304265	4429	0,425	0,2559679	4090	0,485	0,2796349	3803	0,545	0,3017011	3555
366	2308694	4422	426	2563769	4084	486	2800152	3797	546	3020566	3551
367	2313116	4416	427	2567853	4079	487	2803949	3794	547	3024117	3547
368	2317532	4410	428	2571932	4074	488	2807743	3789	548	3027664	3544
369	2321942	4404	429	2576006	4069	489	2811532	3784	549	3031208	3540
0,370	0,2326346	4397	0,430	0,2580075	4064	0,490	0,2815316	3780	0,550	0,3034748	3536
371	2330743	4392	431	2584139	4059	491	2819096	3776	551	3038284	3532
372	2335135	4386	432	2588198	4054	492	2822872	3772	552	3041816	3528
373	2339521	4379	433	2592252	4048	493	2826644	3767	553	3045344	3525
374	2343900	4374	434	2596300	4044	494	2830411	3762	554	3048869	3521
0,375	0,2348274	4368	0,435	0,2600344	4038	0,495	0,2834173	3759	0,555	0,3052390	3517
376	2352642	4361	436	2604382	4033	496	2837932	3754	556	3055907	3513
377	2357003	4356	437	2608415	4029	497	2841686	3750	557	3059420	3510
378	2361359	4350	438	2612444	4023	498	2845436	3745	558	3062930	3506
379	2365709	4344	439	2616467	4019	499	2849181	3742	559	3066436	3502
0,380	0,2370053	4338	0,440	0,2620486	4013	0,500	0,2852923	3737	0,560	0,3069938	3499
381	2374391	4332	441	2624499	4008	501	2856660	3732	561	3073437	3494
382	2378723	4327	442	2628507	4004	502	2860392	3729	562	3076931	3491
383	2383050	4320	443	2632511	3998	503	2864121	3724	563	3080422	3488
384	2387370	4315	444	2636509	3994	504	2867845	3720	564	3083910	3484
0,385	0,2391685	4308	0,445	0,2640503	3989	0,505	0,2871565	3716	0,565	0,3087394	3480
386	2395993	4303	446	2644492	3983	506	2875281	3711	566	3090874	3476
387	2400296	4298	447	2648475	3979	507	2878992	3708	567	3094350	3473
388	2404594	4291	448	2652454	3974	508	2882700	3703	568	3097823	3469
389	2408885	4286	449	2656428	3969	509	2886403	3699	569	3101292	3466
0,390	0,2413171	4280	0,450	0,2660397	3965	0,510	0,2890102	3695	0,570	0,3104758	3462
391	2417451	4274	451	2664362	3959	511	2893797	3690	571	3108220	3458
392	2421725	4269	452	2668321	3955	512	2897487	3687	572	3111678	3455
393	2425994	4263	453	2672276	3950	513	2901174	3682	573	3115133	3451
394	2430257	4257	454	2676226	3945	514	2904856	3679	574	3118584	3447
0,395	0,2434514	4252	0,455	0,2680171	3940	0,515	0,2908535	3674	0,575	0,3122031	3444
396	2438766	4246	456	2684111	3935	516	2912209	3670	576	3125475	3440
397	2443012	4240	457	2688046	3931	517	2915879	3666	577	3128915	3437
398	2447252	4235	458	2691977	3926	518	2919545	3662	578	3132352	3433
399	2451487	4229	459	2695903	3921	519	2923207	3657	579	3135785	3430
0,400	0,2455716	4224	0,460	0,2699824	3917	0,520	0,2926864	3654	0,580	0,3139215	3426
401	2459940	4218	461	2703741	3911	521	2930518	3650	581	3142641	3423
402	2464158	4213	462	2707652	3907	522	2934168	3645	582	3146064	3419
403	2468371	4207	463	2711559	3903	523	2937813	3642	583	3149483	3415
404	2472578	4201	464	2715462	3898	524	2941455	3637	584	3152898	3412
0,405	0,2476779	4196	0,465	0,2719360	3893	0,525	0,2945092	3634	0,585	0,3156310	3409
406	2480975	4191	466	2723253	3888	526	2948726	3629	586	3159719	3405
407	2485166	4185	467	2727141	3884	527	2952355	3626	587	3163124	3401
408	2489351	4180	468	2731025	3879	528	2955981	3621	588	3166525	3398
409	2493531	4174	469	2734904	3874	529	2959602	3618	589	3169923	3395
0,410	0,2497705	4169	0,470	0,2738778	3870	0,530	0,2963220	3613	0,590	0,3173318	3391
411	2501874	4164	471	2742648	3865	531	2966833	3610	591	3176709	3387
412	2506038	4158	472	2746513	3861	532	2970443	3606	592	3180096	3385
413	2510196	4153	473	2750374	3856	533	2974049	3601	593	3183481	3380
414	2514349	4147	474	2754230	3852	534	2977650	3598	594	3186861	3378
0,415	0,2518496	4142	0,475	0,2758082	3847	0,535	0,2981248	3594	0,595	0,3190239	3373
416	2522638	4137	476	2761929	3842	536	2984842	3590	596	3193612	3371
417	2526775	4131	477	2765771	3838	537	2988432	3586	597	3196983	3367
418	2530906	4126	478	2769609	3834	538	2992018	3582	598	3200350	3364
419	2535032	4121	479	2773443	3829	539	2995600	3578	599	3203714	3360
0,420	0,2539153	4116	0,480	0,2777272	3824	0,540	0,2999178	3574	0,600	0,3207074	

TABLE IX. — For an Elliptical Orbit.

This table is used in connexion with Table VIII., in finding the elements of the orbit, by means of the true radii r', r; the included heliocentric arc $v' - v = 2f$, and the time t of describing that arc, in days.

x	ξ	Diff.	x	ξ	Diff.	x	ξ	Diff.	x	ξ	Diff.	x	ξ	Diff.
0,000	0,0000000	1	0,060	0,0002131	73	0,120	0,0008845	154	0,180	0,0020685	244	0,240	0,0038289	346
001	0001	1	061	2204	74	121	8999	155	181	20929	246	241	38635	348
002	0002	3	062	2278	76	122	9154	157	182	21175	247	242	38983	350
003	0005	4	063	2354	77	123	9311	158	183	21422	249	243	39333	352
004	0009	5	064	2431	78	124	9469	159	184	21671	251	244	39685	354
0,005	0,0000014	7	0,065	0,0002509	79	0,125	0,0009628	161	0,185	0,0021922	252	0,245	0,0040039	355
006	0021	7	066	2588	81	126	09789	162	186	22174	254	246	40394	358
007	0028	9	067	2669	82	127	09951	164	187	22428	255	247	40752	359
008	0037	10	068	2751	83	128	10115	165	188	22683	258	248	41111	361
009	0047	10	069	2834	84	129	10280	167	189	22941	258	249	41472	363
0,010	0,0000057	13	0,070	0,0002918	86	0,130	0,0010447	168	0,190	0,0023199	261	0,250	0,0041835	364
011	0070	13	071	3004	87	131	10615	169	191	23460	262	251	42199	367
012	0083	14	072	3091	89	132	10784	171	192	23722	263	252	42566	368
013	0097	16	073	3180	89	133	10955	173	193	23985	266	253	42934	371
014	0113	17	074	3269	91	134	11128	173	194	24251	267	254	43305	372
0,015	0,0000130	18	0,075	0,0003360	93	0,135	0,0011301	176	0,195	0,0024518	268	0,255	0,0043677	374
016	0148	19	076	3453	93	136	11477	177	196	24786	270	256	44051	376
017	0167	20	077	3546	95	137	11654	178	197	25056	272	257	44427	377
018	0187	22	078	3641	97	138	11832	180	198	25328	274	258	44804	380
019	0209	22	079	3738	97	139	12012	181	199	25602	275	259	45184	382
0,020	0,0000231	24	0,080	0,0003835	99	0,140	0,0012193	183	0,200	0,0025877	277	0,260	0,0045566	383
021	0255	25	081	3934	100	141	12376	184	201	26154	279	261	45949	385
022	0280	26	082	4034	102	142	12560	185	202	26433	280	262	46334	387
023	0306	28	083	4136	103	143	12745	188	203	26713	282	263	46721	390
024	0334	28	084	4239	104	144	12933	188	204	26995	283	264	47111	391
0,025	0,0000362	30	0,085	0,0004343	105	0,145	0,0013121	190	0,205	0,0027278	286	0,265	0,0047502	392
026	0392	31	086	4448	107	146	13311	192	206	27564	287	266	47894	395
027	0423	32	087	4555	108	147	13503	193	207	27851	288	267	48289	397
028	0455	34	088	4663	110	148	13696	195	208	28139	290	268	48686	399
029	0489	34	089	4773	111	149	13891	196	209	28429	293	269	49085	400
0,030	0,0000523	36	0,090	0,0004884	112	0,150	0,0014087	198	0,210	0,0028722	293	0,270	0,0049485	403
031	0559	37	091	4996	113	151	14285	199	211	29015	296	271	49888	404
032	0596	38	092	5109	115	152	14484	200	212	29311	297	272	50292	407
033	0634	40	093	5224	117	153	14084	202	213	29608	299	273	50699	408
034	0674	40	094	5341	117	154	14886	204	214	29907	300	274	51107	410
0,035	0,0000714	42	0,095	0,0005458	119	0,155	0,0015090	205	0,215	0,0030207	302	0,275	0,0051517	413
036	0756	43	096	5577	120	156	15295	207	216	30509	305	276	51930	414
037	0799	45	097	5697	122	157	15502	208	217	30814	305	277	52344	416
038	0844	45	098	5819	123	158	15710	210	218	31119	308	278	52760	418
039	0889	47	099	5942	124	159	15920	211	219	31427	309	279	53178	420
0,040	0,0000936	48	0,100	0,0006066	126	0,160	0,0016131	213	0,220	0,0031736	311	0,280	0,0053598	422
041	0984	49	101	6192	127	161	16344	215	221	32047	312	281	54020	424
042	1033	51	102	6319	129	162	16559	216	222	32359	315	282	54444	426
043	1084	51	103	6448	130	163	16775	217	223	32674	316	283	54870	428
044	1135	53	104	6578	131	164	16992	219	224	32990	318	284	55298	430
0,045	0,0001188	54	0,105	0,0006709	133	0,165	0,0017211	221	0,225	0,0033308	319	0,285	0,0055728	432
046	1242	56	106	6842	134	166	17432	222	226	33627	322	286	56160	434
047	1298	56	107	6976	135	167	17654	224	227	33949	323	287	56594	436
048	1354	58	108	7111	137	168	17878	225	228	34272	325	288	57030	438
049	1412	59	109	7248	138	169	18103	227	229	34597	327	289	57468	440
0,050	0,0001471	61	0,110	0,0007386	140	0,170	0,0018330	228	0,230	0,0034924	328	0,290	0,0057908	442
051	1532	61	111	7526	141	171	18558	230	231	35252	330	291	58350	445
052	1593	63	112	7667	142	172	18788	232	232	35582	332	292	58795	446
053	1656	64	113	7809	144	173	19020	233	233	35914	334	293	59241	448
054	1720	65	114	7953	145	174	19253	234	234	36248	336	294	59689	450
0,055	0,0001785	67	0,115	0,0008098	147	0,175	0,0019487	237	0,235	0,0036584	337	0,295	0,0060139	452
056	1852	68	116	8245	148	176	19724	237	236	36921	339	296	60591	454
057	1920	69	117	8393	149	177	19961	240	237	37260	341	297	61045	457
058	1989	71	118	8542	151	178	20201	241	238	37601	343	298	61502	458
059	2060	71	119	8693	152	179	20442	243	239	37944	345	299	61960	461
0,060	0,0002131	73	0,120	0,0008845	154	0,180	0,0020685	244	0,240	0,0038289	346	0,300	0,0062421	

TABLE X. — For a Hyperbolic Orbit.

This table is used in connexion with Table VIII., in finding the elements of the orbit by means of the two radii r', r; the included heliocentric arc $v' - v = 2f$, and the time t of describing that arc, in days.

z	ζ	Diff.	z	ζ	Diff.	z	ζ	Diff.	z	ζ	Diff.	z	ζ	Diff.
0,000	0,0000000	1	0,060	0,0001988	66	0,120	0,0007698	124	0,180	0,0016782	178	0,240	0,0028939	227
001	0001	1	061	2054	67	121	7822	126	181	16960	179	241	29166	228
002	0002	3	062	2121	68	122	7948	126	182	17139	180	242	29394	229
003	0005	4	063	2189	68	123	8074	128	183	17319	181	243	29623	229
004	0009	5	064	2257	70	124	8202	128	184	17500	181	244	29852	231
0,005	0,0000014	6	0,065	0,0002327	71	0,125	0,0008330	129	0,185	0,0017681	183	0,245	0,0030083	231
006	0020	8	066	2398	72	126	8459	131	186	17864	183	246	30314	231
007	0028	8	067	2470	73	127	8590	131	187	18047	184	247	30545	233
008	0036	10	068	2543	74	128	8721	132	188	18231	185	248	30778	233
009	0046	11	069	2617	74	129	8853	133	189	18416	186	249	31011	234
0,010	0,0000057	12	0,070	0,0002691	76	0,130	0,0008986	134	0,190	0,0018602	187	0,250	0,0031245	235
011	0069	13	071	2767	77	131	9120	135	191	18789	187	251	31480	236
012	0082	14	072	2844	78	132	9255	135	192	18976	189	252	31716	236
013	0096	15	073	2922	79	133	9390	137	193	19165	189	253	31952	237
014	0111	16	074	3001	80	134	9527	138	194	19354	190	254	32189	238
0,015	0,0000127	18	0,075	0,0003081	81	0,135	0,0009665	138	0,195	0,0019544	191	0,255	0,0032427	239
016	0145	19	076	3162	82	136	09803	140	196	19735	191	256	32666	239
017	0164	19	077	3244	83	137	09943	140	197	19926	193	257	32905	241
018	0183	21	078	3327	84	138	10083	141	198	20119	193	258	33146	241
019	0204	22	079	3411	85	139	10224	142	199	20312	195	259	33387	241
0,020	0,0000226	23	0,080	0,0003496	86	0,140	0,0010366	143	0,200	0,0020507	195	0,260	0,0033628	243
021	0249	24	081	3582	87	141	10509	144	201	20702	195	261	33871	243
022	0273	25	082	3669	88	142	10653	145	202	20897	197	262	34114	244
023	0298	27	083	3757	89	143	10798	146	203	21094	198	263	34358	245
024	0325	27	084	3846	90	144	10944	147	204	21292	198	264	34603	245
0,025	0,0000352	29	0,085	0,0003936	91	0,145	0,0011091	147	0,205	0,0021490	199	0,265	0,0034848	246
026	0381	29	086	4027	92	146	11238	149	206	21689	200	266	35094	247
027	0410	31	087	4119	93	147	11387	149	207	21889	201	267	35341	248
028	0441	32	088	4212	94	148	11536	151	208	22090	201	268	35589	249
029	0473	33	089	4306	95	149	11687	151	209	22291	203	269	35838	249
0,030	0,0000506	33	0,090	0,0004401	95	0,150	0,0011838	152	0,210	0,0022494	203	0,270	0,0036087	250
031	0539	36	091	4496	97	151	11990	153	211	22697	204	271	36337	250
032	0575	36	092	4593	98	152	12143	153	212	22901	205	272	36587	252
033	0611	37	093	4691	99	153	12296	155	213	23106	205	273	36839	252
034	0648	38	094	4790	100	154	12451	156	214	23311	207	274	37091	253
0,035	0,0000686	40	0,095	0,0004890	101	0,155	0,0012607	156	0,215	0,0023518	207	0,275	0,0037344	254
036	0726	40	096	4991	101	156	12763	158	216	23725	207	276	37598	254
037	0766	41	097	5092	103	157	12921	158	217	23932	210	277	37852	255
038	0807	43	098	5195	104	158	13079	159	218	24142	210	278	38107	256
039	0850	44	099	5299	104	159	13238	160	219	24352	210	279	38363	257
0,040	0,0000894	44	0,100	0,0005403	106	0,160	0,0013398	161	0,220	0,0024562	212	0,280	0,0038620	257
041	0938	46	101	5509	107	161	13559	162	221	24774	212	281	38877	258
042	0984	47	102	5616	107	162	13721	162	222	24986	213	282	39135	259
043	1031	48	103	5723	109	163	13883	164	223	25199	213	283	39394	260
044	1079	49	104	5832	109	164	14047	164	224	25412	215	284	39654	260
0,045	0,0001128	50	0,105	0,0005941	111	0,165	0,0014211	166	0,225	0,0025627	215	0,285	0,0039914	261
046	1178	51	106	6052	111	166	14377	166	226	25842	216	286	40175	262
047	1229	52	107	6163	112	167	14543	167	227	26058	217	287	40437	263
048	1281	53	108	6275	114	168	14710	168	228	26275	218	288	40700	263
049	1334	55	109	6389	114	169	14878	169	229	26493	218	289	40963	264
0,050	0,0001389	55	0,110	0,0006503	115	0,170	0,0015047	169	0,230	0,0026711	220	0,290	0,0041227	264
051	1444	56	111	6618	116	171	15216	171	231	26931	220	291	41491	266
052	1500	58	112	6734	117	172	15387	171	232	27151	220	292	41757	266
053	1558	58	113	6851	118	173	15558	172	233	27371	222	293	42023	267
054	1616	59	114	6969	119	174	15730	173	234	27593	223	294	42290	267
0,055	0,0001675	61	0,115	0,0007088	120	0,175	0,0015903	174	0,235	0,0027816	223	0,295	0,0042557	269
056	1736	62	116	7208	121	176	16077	175	236	28039	224	296	42826	269
057	1798	62	117	7329	122	177	16252	176	237	28263	224	297	43095	269
058	1860	64	118	7451	123	178	16428	176	238	28487	226	298	43364	271
059	1924	64	119	7574	124	179	16604	178	239	28713	226	299	43635	271
0,060	0,0001988	66	0,120	0,0007698	124	0,180	0,0016782	178	0,240	0,0028939	227	0,300	0,0043906	

PRECEPTS FOR THE USES OF TABLES XI. AND XII.

These Tables are inserted for the purpose of changing the arcs of the centesimal division of the quadrant into sexagesimals.

Table XI., is divided into three distinct parts. The *first part* gives the degrees and minutes, in sexagesimals, for every degree of the centesimal division, from 0° to 399°; the *tens* being in the *side* column, and the units at the top. Thus we see by inspection, that $260° = 234^d\,00^m$; $261° = 234^d\,54^m$; &c. The *second part* gives the minutes and seconds in sexagesimals, corresponding to the centesimal division from 0′ to 99′; the *tens* of minutes being at the *side*, and the *units* at the *top*; thus $60' = 32^m\,24^s$; $61' = 32^m\,56^s,4$; &c. The *third* part gives the seconds and decimals in sexagesimals, corresponding to the centesimal division, from 0″ to 99″; the *tens* of seconds being at the *side* and the *units* at the *top*; thus $40'' = 12^s,960$; $41'' = 13^s,284$; &c. The two following examples, show its use in more complicated cases; they require no particular explanation.

EXAMPLE I.

Change $293°\,21'\,17''$ into sexagesimals.

$$\begin{array}{lrcr} \text{Table XI.} & 293° & = & 263^d\,42^m\,00^s \\ & 21' & = & 11\;\;20,4 \\ & 17'' & = & 5,508 \\ \hline & 293^d\,21'\,17'' & = & 263^d\,53^m\,25^s,908 \end{array}$$

EXAMPLE II.

Change $263^d\,53^m\,25^s,908$ into centesimals.

$$\begin{array}{llcr} \text{Table XI.} & 263^d\,42^m & = & 293°\;00'\;00'' \\ \text{Remainder,} & 11^m\,25^s,908 & & \\ \text{Table XI.} & 11^m\,20^s,4 & = & 0\;\;21\;\;00 \\ \text{Remainder Table XI.} & 5^s,508 & = & 17 \\ \hline & 263^d\,53^m\,25^s\,908 & = & 293°\;21'\;17'' \end{array}$$

Table XII., gives the seconds and decimals, in sexagesimals, for every second of the centesimal division, from 0″ to 999″; the *tens* being in the *side* column and the units at the top. It is computed by the rule $s = 0,324.\,c$; s being the number of sexagesimal seconds corresponding to c in centesimal seconds. Hence we have by inspection $570'' = 184^s,680$; $571'' = 185^s,004$; &c. If we change the decimal point, three places to the left, we shall get, from the table, by inspection, the value of every thousandth part of 1″ from 0″,001 to 0″,999. Thus we have, by using the same numbers as before, $0'',570 = 0^s,184680$; $0'',571 = 0^s,185004$; &c. In like manner, by changing the decimal point to the left 6 units we get the values from 0″,000001 to 0″,000999; &c. We may also change the decimal point to the right, if larger numbers are wanted.

EXAMPLE III.

Change 327″, 345 into sexagesimals.

$$\begin{array}{rcr} 327'',000 & = & 105^s,948 \\ ,345 & & 111780 \\ \hline 327'',345 & = & 106^s,059780 \end{array}$$

EXAMPLE IV.

Change $106^s,059780$ into centesimal seconds.

$$\begin{array}{lrcr} \text{Table XII.} & 105^s,948 & = & 327'',000 \\ \text{Table XII.} & 111780 & = & 345 \\ \hline & 106^s,059780 & = & 327'',345 \end{array}$$

EXAMPLE V.

Change 327345″ into sexagesimals.

$$\begin{array}{rcr} 327000'' & = & 105948^s \\ 345'' & = & 111^s,780 \\ \hline 327345'' & = & 106059^s,780 \end{array}$$

EXAMPLE VI.

Change $106059^s,780$ into centesimal seconds.

$$\begin{array}{lrcr} \text{Table XII.} & 105948^s & = & 327000'' \\ \text{Table XIII.} & 111^s,780 & = & 345'' \\ \hline & 106059^s,780 & = & 327345'' \end{array}$$

EXAMPLE VII.

Change 0″, 644302 into sexagesimals.

$$\begin{array}{rcr} 0'',644 & = & 0^s,208656 \\ 0'',000302 & = & 98 \\ \hline 0'',644302 & = & 0^s,208754 \end{array}$$

EXAMPLE VIII.

Change 0″, 076897 into sexagesimals.

$$\begin{array}{rcr} 0'',076 & = & 0^s,024624 \\ 0'',000897 & = & 291 \\ \hline 0'',076897 & = & 0^s,024915 \end{array}$$

Table XII., has been found very convenient in making the reductions of the planetary inequalities, in this volume, from centesimal to sexagesimal seconds, to six places of decimals, as in the two last examples. Since it is easy to obtain the sum of the two parts of the fraction, without the trouble of writing them down separately; the last part of the fraction being generally so small that it is easy to add it to the large tabular number corresponding to the first part. Thus in example VII., the number 98 is easily added to 0,208656, to obtain 0,208754, by mere inspection. The numbers given in this volume were in the first place computed from the table, and then verified by a numerical calculation; found by putting $s' = 0,3\,c$ and $s = s' + 0,08\,s'$. So that instead of writing down the number c and then multiplying it by 0,324; we may write down, in the first instance $0,3\,c$; and then multiply it by 0,08, which gives $0,024\,c$; whose sum is $s = 0,324\,c$. This method, applied to the preceding examples VII., VIII., produce the following results:

EXAMPLE IX.

Change 0″, 644302 into sexagesimals.

$$\begin{array}{lcr} 0,3\,c & = & 0^s,1932906 \\ \text{Multiply by } 0,08 & & 15463248 \\ \hline & & 0^s,208753848 \end{array}$$

EXAMPLE X.

Change 0″, 076897 into sexagesimals.

$$\begin{array}{lcr} 0,3\,c & = & 0^s,0230691 \\ \text{Multiply by } 0,08 & & 1845528 \\ \hline & & 0^s,0249114628 \end{array}$$

TABLE XI.

To convert centesimal degrees, minutes, and seconds, into sexagesimals.

I. — To convert centesimal degrees into sexagesimals.

Centes.	0	1	2	3	4	5	6	7	8	9
°	d m	d m	d m	d m	d m	d m	d m	d m	d m	d m
0	0,00	0,54	1,48	2,42	3,36	4,30	5,24	6,18	7,12	8,06
1	9,00	9,54	10,48	11,42	12,36	13,30	14,24	15,18	16,12	17,06
2	18,00	18,54	19,48	20,42	21,36	22,30	23,24	24,18	25,12	26,06
3	27,00	27,54	28,48	29,42	30,36	31,30	32,24	33.18	34,12	35,06
4	36,00	36,54	37,48	38,42	39,36	40,30	41,24	42,18	43,12	44,06
5	45,00	45,54	46,48	47,42	48,36	49,30	50,24	51,18	52,12	53,06
6	54,00	54,54	55,48	56,42	57,36	58,30	59,24	60,18	61,12	62,06
7	63,00	63,54	64,48	65,42	66,36	67,30	68,24	69,18	70,12	71,06
8	72,00	72,54	73,48	74,42	75,36	76,30	77,24	78,18	79,12	80,06
9	81,00	81,54	82,48	83,42	84,36	85,30	86,24	87,18	88,12	89,06
10	90,00	90,54	91,48	92,42	93,36	94,30	95,24	96,18	97,12	98,06
11	99,00	99,54	100,48	101,42	102,36	103,30	104,24	105,18	106,12	107,06
12	108,00	108,54	109,48	110,42	111,36	112,30	113,24	114,18	115,12	116,06
13	117,00	117,54	118,48	119,42	120,36	121,30	122,24	123,18	124,12	125,06
14	126,00	126,54	127,48	128,42	129,36	130,30	131,24	132,18	133,12	134,06
15	135,00	135,54	136,48	137,42	138,36	139,30	140,24	141,18	142,12	143,06
16	144,00	144,54	145,48	146,42	147,36	148,30	149,24	150,18	151,12	152,06
17	153,00	153,54	154,48	155,42	156,36	157,30	158,24	159,18	160,12	161,06
18	162,00	162,54	163,48	164,42	165,36	166,30	167,24	168,18	169,12	170,06
19	171,00	171,54	172,48	173,42	174,36	175,30	176,24	177,18	178,12	179,06
20	180,00	180,54	181,48	182,42	183,36	184,30	185,24	186,18	187,12	188,06
21	189,00	189,54	190,48	191,42	192,36	193,30	194,24	195,18	196,12	197,06
22	198,00	198,54	199,48	200,42	201,36	202,30	203,24	204,18	205,12	206,06
23	207,00	207,54	208,48	209,42	210,36	211,30	212,24	213,18	214,12	215,06
24	216,00	216,54	217,48	218,42	219,36	220,30	221,24	222,18	223,12	224,06
25	225,00	225,54	226,48	227,42	228,36	229,30	230,24	231,18	232,12	233,06
26	234,00	234,54	235,48	236,42	237,36	238,30	239,24	240,18	241,12	242,06
27	243,00	243,54	244,48	245,42	246,36	247,30	248,24	249,18	250,12	251,06
28	252,00	252,54	253,48	254,42	255,36	256,30	257,24	258,18	259,12	260,06
29	261,00	261,54	262,48	263,42	264,36	265,30	266,24	267,18	268,12	269,06
30	270,00	270,54	271,48	272,42	273,36	274,30	275,24	276,18	277,12	278,06
31	279,00	279,54	280,48	281,42	282,36	283,30	284,24	285,18	286,12	287,06
32	288,00	288,54	289,48	290,42	291,36	292,30	293,24	294,18	295,12	296,06
33	297,00	297,54	298,48	299,42	300,36	301,30	302,24	303,18	304,12	305,06
34	306,00	306,54	307,48	308,42	309,36	310,30	311,24	312,18	313,12	314,06
35	315,00	315,54	316,48	317,42	318,36	319,30	320,24	321,18	322,12	323,06
36	324,00	324,54	325,48	326,42	327,36	328,30	329,24	330,18	331,12	332,06
37	333,00	333,54	334,48	335,42	336,36	337,30	338,24	339,18	340,12	341,06
38	342,00	342,54	343,48	344,42	345,36	346,30	347,24	348,18	349,12	350,06
39	351,00	351,54	352,48	353,42	354,36	355,30	356,24	357,18	358,12	359,06

II. — To convert centesimal minutes into sexagesimals.

Centes.	0	1	2	3	4	5	6	7	8	9
′	m s	m s	m s	m s	m s	m s	m s	m s	m s	m s
0	0,00,0	0,32,4	1,04,8	1,37,2	2,09,6	2,42,0	3,14,4	3,46,8	4,19,2	4,51,6
1	5,24,0	5,56,4	6,28,8	7,01,2	7,33,6	8,06,0	8,38,4	9,10,8	9,43,2	10,15,6
2	10,48,0	11,20,4	11,52,8	12,25,2	12,57,6	13,30,0	14,02,4	14,34,8	15,07,2	15,39,6
3	16,12,0	16,44,4	17,16,8	17,49,2	18,21,6	18,54,0	19,26,4	19,58,8	20,31,2	21,03,6
4	21,36,0	22,08,4	22,40,8	23,13,2	23,45,6	24,18,0	24,50,4	25,22,8	25,55,2	26,27,6
5	27,00,0	27,32,4	28,04,8	28,37,2	29,09,6	29,42,0	30,14,4	30,46,8	31,19,2	31,51,6
6	32,24,0	32,56,4	33,28,8	34,01,2	34,33,6	35,06,0	35,38,4	36,10,8	36,43,2	37,15,6
7	37,48,0	38,20,4	38,52,8	39,25,2	39,57,6	40,30,0	41,02,4	41,34,8	42,07,2	42,39,6
8	43,12,0	43,44,4	44,16,8	44,49,2	45,21,6	45,54,0	46,26,4	46,58,8	47,31,2	48,03,6
9	48,36,0	49,08,4	49,40,8	50,13,2	50,45,6	51,18,0	51,50,4	52,22,8	52,55,2	53,27,6

III. — To convert centesimal seconds into sexagesimals.

Centes.	0	1	2	3	4	5	6	7	8	9
″	s	s	s	s	s	s	s	s	s	s
0	0,000	0,324	0,648	0,972	1,296	1,620	1,944	2,268	2,592	2,916
1	3,240	3,564	3,888	4,212	4,536	4,860	5,184	5,508	5,832	6,156
2	6,480	6,804	7,128	7,452	7,776	8,100	8,424	8,748	9,072	9,396
3	9,720	10,044	10,368	10,692	11,016	11,340	11,664	11,988	12,312	12,636
4	12,960	13,284	13,608	13,932	14,256	14,580	14,904	15,228	15,552	15,876
5	16,200	16,524	16,848	17,172	17,496	17,820	18,144	18,468	18,792	19,116
6	19,440	19,764	20,088	20,412	20,736	21,060	21,384	21,708	22,032	22,356
7	22,680	23,004	23,328	23,652	23,976	24,300	24,624	24,948	25,272	25,596
8	25,920	26,244	26,568	26,892	27,216	27,540	27,864	28,188	28,512	28,836
9	29,160	29,484	29,808	30,132	30,456	30,780	31,104	31,428	31,752	32,076

TABLE XII.

To convert centesimal seconds into sexagesimals.

Centes.	0	1	2	3	4	5	6	7	8	9	Centes.
″		s	s	s	s	s	s	s	s	s	″
0	s	0,324	0,648	0,972	1,296	1,620	1,944	2,268	2,592	2,916	0
1	3,240	3,564	3,888	4,212	4,536	4,860	5,184	5,508	5,832	6,156	1
2	6,480	6,804	7,128	7,452	7,776	8,100	8,424	8,748	9,072	9,396	2
3	9,720	10,044	10,368	10,692	11,016	11,340	11,664	11,988	12,312	12,636	3
4	12,960	13,284	13,608	13,932	14,256	14,580	14,904	15,228	15,552	15,876	4
5	16,200	16,524	16,848	17,172	17,496	17,820	18,144	18,468	18,792	19,116	5
6	19,440	19,764	20,088	20,412	20,736	21,060	21,384	21,708	22,032	22,356	6
7	22,680	23,004	23,328	23,652	23,976	24,300	24,624	24,948	25,272	25,596	7
8	25,920	26,244	26,568	26,892	27,216	27,540	27,864	28,188	28,512	28,836	8
9	29,160	29,484	29,808	30,132	30,456	30,780	31,104	31,428	31,752	32,076	9
10	32,400	32,724	33,048	33,372	33,696	34,020	34,344	34,668	34,992	35,316	10
11	35,640	35,964	36,288	36,612	36,936	37,260	37,584	37,908	38,232	38,556	11
12	38,880	39,204	39,528	39,852	40,176	40,500	40,824	41,148	41,472	41,796	12
13	42,120	42,444	42,768	43,092	43,416	43,740	44,064	44,388	44,712	45,036	13
14	45,360	45,684	46,008	46,332	46,656	46,980	47,304	47,628	47,952	48,276	14
15	48,600	48,924	49,248	49,572	49,896	50,220	50,544	50,868	51,192	51,516	15
16	51,840	52,164	52,488	52,812	53,136	53,460	53,784	54,108	54,432	54,756	16
17	55,080	55,404	55,728	56,052	56,376	56,700	57,024	57,348	57,672	57,996	17
18	58,320	58,644	58,968	59,292	59,616	59,940	60,264	60,588	60,912	61,236	18
19	61,560	61,884	62,208	62,532	62,856	63,180	63,504	63,828	64,152	64,476	19
20	64,800	65,124	65,448	65,772	66,096	66,420	66,744	67,068	67,392	67,716	20
21	68,040	68,364	68,688	69,012	69,336	69,660	69,984	70,308	70,632	70,956	21
22	71,280	71,604	71,928	72,252	72,576	72,900	73,224	73,548	73,872	74,196	22
23	74,520	74,844	75,168	75,492	75,816	76,140	76,464	76,788	77,112	77,436	23
24	77,760	78,084	78,408	78,732	79,056	79,380	79,704	80,028	80,352	80,676	24
25	81,000	81,324	81,648	81,972	82,296	82,620	82,944	83,268	83,592	83,916	25
26	84,240	84,564	84,888	85,212	85,536	85,860	86,184	86,508	86,832	87,156	26
27	87,480	87,804	88,128	88,452	88,776	89,100	89,424	89,748	90,072	90,396	27
28	90,720	91,044	91,368	91,692	92,016	92,340	92,664	92,988	93,312	93,636	28
29	93,960	94,284	94,608	94,932	95,256	95,580	95,904	96,228	96,552	96,876	29
30	97,200	97,524	97,848	98,172	98,496	98,820	99,144	99,468	99,792	100,116	30
31	100,440	100,764	101,088	101,412	101,736	102,060	102,384	102,708	103,032	103,356	31
32	103,680	104,004	104,328	104,652	104,976	105,300	105,624	105,948	106,272	106,596	32
33	106,920	107,244	107,568	107,892	108,216	108,540	108,864	109,188	109,512	109,836	33
34	110,160	110,484	110,808	111,132	111,456	111,780	112,104	112,428	112,752	113,076	34
35	113,400	113,724	114,048	114,372	114,696	115,020	115,344	115,668	115,992	116,316	35
36	116,640	116,964	117,288	117,612	117,936	118,260	118,584	118,908	119,232	119,556	36
37	119,880	120,204	120,528	120,852	121,176	121,500	121,824	122,148	122,472	122,796	37
38	123,120	123,444	123,768	124,092	124,416	124,740	125,064	125,388	125,712	126,036	38
39	126,360	126,684	127,008	127,332	127,656	127,980	128,304	128,628	128,952	129,276	39
40	129,600	129,924	130,248	130,572	130,896	131,220	131,544	131,868	132,192	132,516	40
41	132,840	133,164	133,488	133,812	134,136	134,460	134,784	135,108	135,432	135,756	41
42	136,080	136,404	136,728	137,052	137,376	137,700	138,024	138,348	138,672	138,996	42
43	139,320	139,644	139,968	140,292	140,616	140,940	141,264	141,588	141,912	142,236	43
44	142,560	142,884	143,208	143,532	143,856	144,180	144,504	144,828	145,152	145,476	44
45	145,800	146,124	146,448	146,772	147,096	147,420	147,744	148,068	148,392	148,716	45
46	149,040	149,364	149,688	150,012	150,336	150,660	150,984	151,308	151,632	151,956	46
47	152,280	152,604	152,928	153,252	153,576	153,900	154,224	154,548	154,872	155,196	47
48	155,520	155,844	156,168	156,492	156,816	157,140	157,464	157,788	158,112	158,436	48
49	158,760	159,084	159,408	159,732	160,056	160,380	160,704	161,028	161,352	161,676	49
	0	1	2	3	4	5	6	7	8	9	

TABLE XII.

To convert centesimal seconds into sexagesimals.

Centes	0	1	2	3	4	5	6	7	8	9	Centes.
"	s	s	s	s	s	s	s	s	s	s	"
50	162,000	162,324	162,648	162,972	163,296	163,620	163,944	164,268	164,592	164,916	50
51	165,240	165,564	165,888	166,212	166,536	166,860	167,184	167,508	167,832	168,156	51
52	168,480	168,804	169,128	169,452	169,776	170,100	170,424	170,748	171,072	171,396	52
53	171,720	172,044	172,368	172,692	173,016	173,340	173,664	173,988	174,312	174,636	53
54	174,960	175,284	175,608	175,932	176,256	176,580	176,904	177,228	177,552	177,876	54
55	178,200	178,524	178,848	179,172	179,496	179,820	180,144	180,468	180,792	181,116	55
56	181,440	181,764	182,088	182,412	182,736	183,060	183,384	183,708	184,032	184,356	56
57	184,680	185,004	185,328	185,652	185,976	186,300	186,624	186,948	187,272	187,596	57
58	187,920	188,244	188,568	188,892	189,216	189,540	189,864	190,188	190,512	190,836	58
59	191,160	191,484	191,808	192,132	192,456	192,780	193,104	193,428	193,752	194,076	59
60	194,400	194,724	195,048	195,372	195,696	196,020	196,344	196,668	196,992	197,316	60
61	197,640	197,964	198,288	198,612	198,936	199,260	199,584	199,908	200,232	200,556	61
62	200,880	201,204	201,528	201,852	202,176	202,500	202,824	203,148	203,472	203,796	62
63	204,120	204,444	204,768	205,092	205,416	205,740	206,064	206,388	206,712	207,036	63
64	207,360	207,684	208,008	208,332	208,656	208,980	209,304	209,628	209,952	210,276	64
65	210,600	210,924	211,248	211,572	211,896	212,220	212,544	212,868	213,192	213,516	65
66	213,840	214,164	214,488	214,812	215,136	215,460	215,784	216,108	216,432	216,756	66
67	217,080	217,404	217,728	218,052	218,376	218,700	219,024	219,348	219,672	219,996	67
68	220,320	220,644	220,968	221,292	221,616	221,940	222,264	222,588	222,912	223,236	68
69	223,560	223,884	224,208	224,532	224,856	225,180	225,504	225,828	226,152	226,476	69
70	226,800	227,124	227,448	227,772	228,096	228,420	228,744	229,068	229,392	229,716	70
71	230,040	230,364	230,688	231,012	231,336	231,660	231,984	232,308	232,632	232,956	71
72	233,280	233,604	233,928	234,252	234,576	234,900	235,224	235,548	235,872	236,196	72
73	236,520	236,844	237,168	237,492	237,816	238,140	238,464	238,788	239,112	239,436	73
74	239,760	240,084	240,408	240,732	241,056	241,380	241,704	242,028	242,352	242,676	74
75	243,000	243,324	243,648	243,972	244,296	244,620	244,944	245,268	245,592	245,916	75
76	246,240	246,564	246,888	247,212	247,536	247,860	248,184	248,508	248,832	249,156	76
77	249,480	249,804	250,128	250,452	250,776	251,100	251,424	251,748	252,072	252,396	77
78	252,720	253,044	253,368	253,692	254,016	254,340	254,664	254,988	255,312	255,636	78
79	255,960	256,284	256,608	256,932	257,256	257,580	257,904	258,228	258,552	258,876	79
80	259,200	259,524	259,848	260,172	260,496	260,820	261,144	261,468	261,792	262,116	80
81	262,440	262,764	263,088	263,412	263,736	264,060	264,384	264,708	265,032	265,356	81
82	265,680	266,004	266,328	266,652	266,976	267,300	267,624	267,948	268,272	268,596	82
83	268,920	269,244	269,568	269,892	270,216	270,540	270,864	271,188	271,512	271,836	83
84	272,160	272,484	272,808	273,132	273,456	273,780	274,104	274,428	274,752	275,076	84
85	275,400	275,724	276,048	276,372	276,696	277,020	277,344	277,668	277,992	278,316	85
86	278,640	278,964	279,288	279,612	279,936	280,260	280,584	280,908	281,232	281,556	86
87	281,880	282,204	282,528	282,852	283,176	283,500	283,824	284,148	284,472	284,796	87
88	285,120	285,444	285,768	286,092	286,416	286,740	287,064	287,388	287,712	288,036	88
89	288,360	288,684	289,008	289,332	289,656	289,980	290,304	290,628	290,952	291,276	89
90	291,600	291,924	292,248	292,572	292,896	293,220	293,544	293,868	294,192	294,516	90
91	294,840	295,164	295,488	295,812	296,136	296,460	296,784	297,108	297,432	297,756	91
92	298,080	298,404	298,728	299,052	299,376	299,700	300,024	300,348	300,672	300,996	92
93	301,320	301,644	301,968	302,292	302,616	302,940	303,264	303,588	303,912	304,236	93
94	304,560	304,884	305,208	305,532	305,856	306,180	306,504	306,828	307,152	307,476	94
95	307,800	308,124	308,448	308,772	309,096	309,420	309,744	310,068	310,392	310,716	95
96	311,040	311,364	311,688	312,012	312,336	312,660	312,984	313,308	313,632	313,956	96
97	314,280	314,604	314,928	315,252	315,576	315,900	316,224	316,548	316,872	317,196	97
98	317,520	317,844	318,168	318,492	318,816	319,140	319,464	319,788	320,112	320,436	98
99	320,760	321,084	321,408	321,732	322,056	322,380	322,704	323,028	323,352	323,676	99
	0	1	2	3	4	5	6	7	8	9	